Finding EFL and EQL Allocations of Indivisible Goods

Wei Huang
Guangxi Key Laboratory of Trusted Software, Guilin University of Electronic Technology, Guilin 541004, China
huangwei@guet.edu.cn

Wen Huang*
Guilin University of Electronic Technology
1803303012@mails.guet.edu.cn

Dong Cai
Guilin University of Electronic Technology
1803201001@mails.guet.edu.cn

***ABSTRACT*—Fair resource allocation has become an emerging research topic in Computer Science and Artificial Intelligence. We can judge whether the allocation is "fair" from two aspects. First, if each agent prefers his own set of bundle, that is, envy-free, then we say that the allocation is fair; If each agent's valuation of his own item set is equal to other agents' valuation of its own item-equitability (EQ), then we also say that this allocation is fair. Solving the problem of envy-free (EF) or equitability (EQ) fair allocation is proved to be NP-hard, so this paper mainly studies the problem of approximate fair allocation i.e., approximate envy-free and approximate equitability. Our contribution are as follows. 1. We proved that equitable up to one less-preferred good (EQL) allocation always exists and can be found in polynomial time; 2. We proved that the allocation that satisfies both equitable up to one less-preferred good (EQL) and Pareto optimality (PO) exists and provides a pseudo-polynomial time algorithm that can find the allocation when the valuation function is additive with strictly positive; 3. We proved that when the allocation with specific valuation that satisfies equitable up to one less-preferred good (EQL), envy-free up to one less-preferred good (EFL) and Pareto optimality (PO) exists then it can be found in polynomial time.**

Keywords—Computational Economics; Fair allocation; Optimal allocation

I. INTRODUCTION

The fair allocation problem is that the central planner allocates a set of goods among a group of agents, and each self-interested agent has its own preference sequence for the goods. The literature has two different views on whether allocation is fair. The first is that each agent only likes his own collection and does not envy others. The second is that each agent gets the same utility value. This article focuses on fair allocation. [1] has been proved through experiments that fairness as a fairness standard is more extensive in practical applications.

In the problem of fair allocation of divisible items (such as the cake cutting problem), fair allocation exists [2, 3], but it cannot be found by polynomial time. In order to solve this problem, people have proposed a solution that can solve the approximate fair allocation in polynomial time, namely -fair split [4,5]. Unfortunately, in the issue of indivisible resource allocation, fair allocation does not exist. It can be easily seen from the simple example of two agents splitting a product. Therefore, people shift their research focus to approximate fair allocation. [6] proposed the concept of approximately envy-free, that is, for a pair of agents, by removing the goods in the bundle of agents with higher valuations to eliminate envy. In addition, [7] studied the relationship between Pareto optimality and maximized fair distribution of social welfare. A related issue is the development of an approximation algorithm for maximizing NSW. In recent years, this issue has received considerable attention [8, 9]. In this article, we similarly relax the fairness. We mainly study equitable up to one less-preferred good (EQL). We have studied the existence of equitable up to one less-preferred good (EQL) and Pareto optimality (PO), given a pseudo-polynomial algorithm under strictly positive valuation function to obtain EQL + PO allocation. At the same time, combined with the concept of envy-free relaxation, it is given whether the equitable up to one less preferred good (EQL), envy-free up to one good (EFL) and Pareto optimality (PO) allocation problem is usually a strong NP-hard problem. But under the binary valuation function, this problem is solvable.

II. MODEL

A problem finds a fair allocation of a set of indivisible goods $[m] = \{1, \dots, m\}$, among a group of agents $[n] = \{1, \dots, n\}$. We define an allocation as an n-partition $\Pi = \{\pi_1, \pi_2, \dots, \pi_n\}$, where π_i is the set of goods allocated to agent i. The utility of agent $i \in [n]$ for the bundle is u_i or $u_i(\pi_i)$. We use valuations $U = (u_1, u_2, \dots, u_n)$ to represent all agents$'$ utility for own bundle. For convenience, we will use $u_{i,j}$ to represent $u_i(\{j\})$. The valuation is additive i.e., $u_i(\pi_i) = \sum_{j \in \pi_i} u_{i,j}$. In this paper, we assume the valuation for every good with $i \in [n]$ and $j \in [m]$ to be non-negative i.e., $u_i(\{j\}) \geq 0$.

$\boldsymbol{Definition\ 1}$ (Equitable allocations) An allocation $\Pi = (\pi_1, \pi_2, \dots, \pi_n)$ is defined as equitable (EQ) if for every pair of agents $i, k \in [n]$, we have $u_i(\pi_i) = u_k(\pi_k)$.

$\boldsymbol{Definition\ 2}$ (Envy-free) An allocation $\Pi = (\pi_1, \pi_2, \dots, \pi_n)$ is defined as envy-free (EF) if for every pair of agents $i, k \in [n]$, we have $u_i(\pi_i) \geq u_i(\pi_k)$.

$\boldsymbol{Definition\ 3}$ (Envy-free up to one good) An allocation $\Pi = (\pi_1, \pi_2, \dots, \pi_n)$ is said to be envy-free up to one good (EF1) if for every pair of agents $i, k \in [n]$, there exits some good $j \in \pi_k$ such that $u_i(\pi_i) \geq u_i(\pi_k \setminus \{j\})$.

$\boldsymbol{Definition 4}$ (Envy-free up to one less-preferred good) An allocation $\Pi = (\pi_1, \pi_2, \dots, \pi_n)$ is said to be envy-free up to one less-preferred good (EFL) if for every pair of agents $i, k \in [n]$ at least one of the following conditions hold:

978-1-7281-9481-3/20/$31.00 ©2020 IEEE
DOI 10.1109/CVIDL51233.2020.00-53

1. There exists a good $j \in \pi_k$ such that $u_i(\pi_i) \geq u_i(\pi_k \setminus \{j\})$ and $u_i(\pi_i) \geq u_{i,j}$;

2. π_k contains at most one good which is positively valued by i (i.e., $|\pi_k \cap \{j | u_{i,j} > 0\}| \leq 1$).

***Definition* 5** (Equitable up to one less-preferred good) An allocation $\Pi = (\pi_1, \pi_2, \dots, \pi_n)$ is said to be Equitable up to one less-preferred good (EQL) if for every pair of agents $i, k \in [n]$ at least one of the following conditions hold:

1. There exists a good $j \in \pi_k$ such that $u_i(\pi_i) \geq u_k(\pi_k \setminus \{j\})$ and $u_i(\pi_i) \geq u_{k,j}$;

2. π_k contains at most one good which is positively (i.e., $|\{j | j \in \pi_k, u_{k,j} > 0\}| \leq 1$).

***Definition* 6** (Pareto optimality) An allocation $(A_1, A_2, \dots, A_n)$ is Pareto dominated by another allocation $(B_1, B_2, \dots, B_n)$ if $u_i(B_i) \geq u_i(A_i)$ for every agent $i \in [n]$ with at least one of the inequalities being strict. A Pareto optimal (PO) allocation is one that is not Pareto dominated by any other allocation.

Next, we consider the concept of equitability and its relaxations. At the same time, considering the existence and Computation of Pareto optimality. Finally, add the concepts of envy-freeness and relaxation to consider.

***Theorem* 1** An EQL allocation always exists and can be computed in polynomial time.

Proof: The proof of Theorem 1 is inspired by the greedy algorithm in [6]. In the algorithm, each round allocates the lowest valued commodity to the source node in the envy graph [13]. Therefore, through this algorithm, each agent will get the least favorite goods. Since each new good is assigned to an agent with the least utility, an allocation that is equitable up to one less-preferred good (EQL) prior to the assignment continues to be so after it (up to the removal of the most recently assigned good). By running the algorithm, the existence of the claim can be guaranteed. ■

***Theorem* 2** Determining whether a given instance with additive valuation has an equitable up to one less preferred good (EQL) and Pareto optimal (PO) allocation is strongly NP-hard. *Proof*: According to [17] which prove the relevant theorem, we can reduce the proof of Theorem 2 to the problem of 3-PARTITION which is known to be strongly NP-hard [15]. We give a $3-PARTITION$ example. We give a set of $\chi = \{x_1, x_2, \dots, x_{3m}\}$ elements of $3m$, and divide the set into m subsets such as $\chi_1, \chi_2, \dots, \chi_m$, that is, each subset contains the three elements of the original set, and the elements in each subset satisfy $\tau = \frac{1}{m}\sum_{x_i \in \chi} x_i$. The specific certification process will be divided into two steps to prove its adequacy and necessity.

Before specific proofs, we will construct a fair division instance with $r+1$ agents $a_1, a_2, \dots a_{r+1}$ and $3r+2$ goods $g_1, g_2, \dots, g_{3r+1}, g_{3r+2}$. And the valuation function of the agent is as follows:

for agent $i \in [r]$:

$$u_i(g_j) = \begin{cases} x_j, & j \in [3r] \\ 0, & j \in \{3r+1, 3r+2\} \end{cases}$$

and agent $r+1$:

$$u_i(g_j) = \begin{cases} 0, & j \in [3r] \\ \tau, & j \in \{3r+1, 3r+2\} \end{cases}$$

$(Sufficiency)$ An example $\chi = \{x_1, x_2, \dots, x_{3m}\}$ is now given. We consider the following allocation $\Pi = (\pi_1, \pi_2, \dots, \pi_{r+1})$ which $\pi_i = \{g_j | u_i(g_j) = x_j, x_j \in \chi_i\}$ with agent $i \in [r]$ and $\pi_{r+1} = \{g_{3r+1}, g_{3r+2}\}$. For agent $i \in [r]$, we can get $u_i(\pi_i) \geq u_{r+1}(\pi_{r+1} \setminus \{g_{3r+1}\})$ and $u_i(\pi_i) \geq u_{r+1}(g_{3r+1})$. For $i, k \in [r]$, this conclusion is naturally true. Because we can get $u_i(\pi_i) \geq u_k(\pi_k \setminus \{g_j\})$ and $u_i(\pi_i) \geq u_k(g_j)$ with $g_j \in \pi_k$. For agent a_{r+1}, we can also get $u_{r+1}(\pi_{r+1}) \geq u_i(\pi_i \setminus \{g\})$ and $u_{r+1}(\pi_{r+1}) \geq u_i(g)$ with $i \in [r]$. Based on the above conclusions, it can be concluded that this allocation is an EQL allocation. At the same time, we can easily see that each agent gets the best valuation, so the sub-allocation also satisfies PO. In summary, the sufficiency proof is completed.

$(Necessity)$ Next consider the necessity. Suppose we consider that allocation Π satisfies EQL and PO. Because this allocation is PO, then each agent must get the allocation with the best valuation, then π_{r+1} must be $\{g_{3r+1}, g_{3r+2}\}$. In order to make this allocation satisfy EQL, then for agent $i \in [r]$, we should be $u_i(\pi_i) \geq u_{r+1}(\pi_{r+1} \setminus \{g_{3r+1}\})$ and $u_i(\pi_i) \geq u_{r+1}(g_{3r+1})$ with established and the valuation should be τ. Then Π could be $\pi_i = \{g_i | u_i(g_j) = x_j\}$ with agent $i \in [r]$ and $\pi_{r+1} = \{g_{3r+1}, g_{3r+2}\}$. As can be seen from the above, this allocation satisfies the 3-PARTITION allocation. ■

***Theorem* 3** An allocation that is equitable up to one less-preferred good (EQL) and Pareto optimal (PO) always exists and can be computed in pseudo-polynomial time when the given fair division instance with strictly positive and additive valuations.

We define the $envy-graph\ G(\Pi)$. The vertices of the directed graph are given by the set of agents $[n]$.There is an arc from i to j if and only if i envies j. For each directed cycle $C = (i_1, i_2, \dots, i_k)$ of the envy graph $G(\Pi)$ where i_t envies i_{t+1} for each $t \in [k]$ and $i_k = i_1$. The algorithm uses the framework of the Fisher market, which is a fully studied model of a group of buyers spending budget virtual currency to maximize utility bundles. The standard welfare theorem in economics ensures that the equilibrium (i.e., market liquidation) results of these markets have economic benefits. Now we give the definition of relevant symbols used in the algorithm. The algorithm uses the framework of the Fisher market, which is a fully studied model of a group of buyers spending budget virtual currency to maximize utility bundles. The standard welfare theorem in economics ensures that the equilibrium (i.e., market liquidation) results of these markets have economic benefits. Now we give the definition of relevant symbols used in the algorithm, following:

1. ε -rounded instance: Given any $\varepsilon > 0$, an ε-rounded instance refers to a fair division instance $([n], [m], U)$ in which the valuations are either zero or a non-negative integral power of $(1+\varepsilon)$. That is, for every agent $i \in [n]$ and every good $g \in [m]$, we have $u_i(g) \in \{0, (1+\varepsilon)^t\}$ for some $t \in \mathbb{N} \cup \{0\}$. $P =$

$(p_1, p_2, \dots, p_m)$ is a price vector that associates a price $p_j \geq 0$ with every good $j \in [m]$.

2. Accessibility set: Given a price vector $P = (p_1, p_2, \dots, p_m)$, define the bang-per-buck ratio of agent i for good j as $\alpha_{i,j} = u_{i,j}/p_j$. The maximum bang-per-buck ratio (or MBB ratio) of agent i is $\alpha_i = max_{j \in [m]} \alpha_{i,j}$ and i.e., $MBB_i = \{ j \in [m]: u_{i,j}/p_j = \alpha_i \}$ .An MBB-allocation graph is an undirected bipartite graph G with vertex set $[n] \cup [m]$ and an edge between agent $i \in [n]$ and good $j \in [m]$ if either $j \in \pi_i$ (called an allocation edge) or $j \in MBB_i$ (called an MBB edge). Notice that if Π is MBB-consistent (i.e., $j \in \pi_i \Longrightarrow j \in MBB_i$), then the allocation edges are a subset of MBB edges. For an MBB-allocation graph, define an alternating path $P = (i, j_1, i_1, j_2, i_2, \dots, i_{k-1}, j_k, b)$ from agent i to agent b (and involving the agents $i_1, i_2, \dots, i_{k-1}$ and the goods $j_1, j_2, \dots, j_k$) as a series of alternating MBB and allocation edges such that $j_1 \in MBB_i \cup \pi_{i_1}, j_2 \in MBB_{i_1} \cup \pi_{i_2}, \dots, j_k \in MBB_{k-1} \cup \pi_b$. We say that agent b is reachable from agent i via the alternating path $P = (i, j_1, i_1, j_2, i_2, \dots, i_{k-1}, j_k, b)$. Pick a source agent $i \in [n]$ from G. Define the level of an agent $b \in [n]$ as half the length of the shortest alternating path from i to b if one exists (i.e., if b is reachable from i), otherwise set the level of b to be n. The accessibility set $\mathcal{A}_i$ of agent i is defined as a level-wise collection of all agents that are reachable from i i.e., $\mathcal{A}_i = (\mathcal{A}_i^0, \mathcal{A}_i^1, \mathcal{A}_i^2, \dots)$, where $\mathcal{A}_i^x$ denotes the set of agents that are at level x with respect to agent i. Given accessibility set $\mathcal{A}_i$, we can redefine an alternating path as a set of alternating MBB and allocation edges connecting agents at a lower level to those at a higher level. Formally, we will call a path $P = (i, j_1, i_1, j_2, i_2, \dots, i_{k-1}, j_k, b)$ alternating if (1) $j_1 \in MBB_i \cup \pi_{i_1}, j_2 \in MBB_{i_1} \cup \pi_{i_2}, \dots, j_k \in MBB_{k-1} \cup \pi_b$ and (2) $leavel(i) < leavel(i_1) < leavel(i_2) <, \dots, < leavel(i_{k-1}) < leavel(b)$. Thus, an alternating path cannot have edges between agents at the same level.

3. Path-violators: Given any $\varepsilon > 0$, an agent $s \in [n]$ is an ε-violator if $\pi_s \neq \emptyset$ and for every good $j \in \pi_s$, we have $u_s(\pi_s \setminus \{j\}) > (1+\varepsilon) u_i(\pi_i)$. An allocation Π is ε-EQ1 if and only if no ε-violator. An allocation Π is ε-EQL if and only if $u_s(\pi_s \setminus \{j\}) \leq (1+\varepsilon) u_i(\pi_i)$ and $u_{s,j} \leq (1+\varepsilon) u_i(\pi_i)$. An agent $b \in \mathcal{A}_i$ is an ε-path-violator with respect to the alternating path $P = (i, j_1, i_1, j_2, i_2, \dots, i_{k-1}, j_k, b)$ if $u_b(\pi_b \setminus \{j_k\}) > (1+\varepsilon) u_i(\pi_i)$.

$Proof$: (Sketch)Firstly, according to the algorithm, we prove that when the allocation product contains only one good i.e., $u_i(g) > 0$ for all agent $i, s \in [n]$, we can easily get this allocation to satisfy π_s contains at most one good which is positively: $|\{g | g \in u_s, u_s(g) > 0\}| \leq 1$. The allocation is satisfied EQL in the lines 1-12 of the algorithm, it can be concluded that the allocation obtained by this algorithm satisfies EQL.

If this allocation does not meet the EQL, it can adjust it by the algorithm below line 13. Secondly, according to the [10] revelation, it is possible to prove the allocation obtained by this algorithm to meet the PO using the counterproof method. Suppose there is Π dominated by an allocation A i.e., $u_i(A_i) \geq (1+\varepsilon) u_i(\pi_i)$ for all agent $i \in [n]$ and $\exists t \in [n], u_t(A_t) \geq (1+\varepsilon) u_t(\pi_t)$. At the same time, according to $u_{i,j} \leq v_{i,j} = (1+\varepsilon)^{\lceil log_{1+\varepsilon} u_{i,j} \rceil} \leq (1+\varepsilon) u_{i,j}$ for every agent i and every good j, we can conclude that $v_i(A_i) \geq v_i(\pi_i)$ for agent $i \in [n]$ and $u_t(A_t) \geq u_t(\pi_t)$ for some agent $t \in [n]$, so a Pareto dominates Π which is obviously contrary to the premise of the article. According to the algorithm proof tips in the [10, 17], it can be concluded that this algorithm can be obtained in $O(poly(m, n, max_{i,j} u_{i,j}))$ time. ■

***Algorithm* 1**: Finding an $EQL + PO$ Allocation

Input: An ε-rounded instance $I = ([n], [m], U)$

Output: An $EQL + PO$ Allocation Π

1. Initialize $\Pi = (\pi_1, \pi_2, \dots, \pi_n)$ with $\pi_i = \emptyset$;
2. Initialize $G(\Pi) = ([n], E)$ where $E = \emptyset$;
3. ***while*** $[m] \neq \emptyset$ ***do***
4. Pick an agent i with no incoming edge in the graph $G(\Pi)$;
5. Set $v_{i,j} \leftarrow (1+\varepsilon)^{\lceil log_{1+\varepsilon} u_{i,j} \rceil}$;
6. Choose $g \in arg\,max_{t \in [m]} v_{i,t}$
7. Set $p_j \leftarrow v_{i,j}$ if $j \in \pi_i$ for every good $j \in [m]$;
8. Update $\pi_i \leftarrow \pi_i \cup \{g\}$ and $[m] \leftarrow [m] \setminus \{g\}$;
9. ***while*** current envy graph $G(\Pi)$ contains a cycle C ***do***
10. Update $\pi_i \leftarrow \pi_{i+1}$ with $i \in C$;
11. ***if*** Π is ε-EQL ***then***
12. ***return*** Π;
13. Pick an agent $i \in arg\,min_{s \in [n]} u_s(\pi_s)$;
14. Set $\mathcal{A}_i = (\mathcal{A}_i^0, \mathcal{A}_i^1, \mathcal{A}_i^2, \dots)$ and $k \leftarrow 1$;
15. ***while*** $\mathcal{A}_i^k \neq \emptyset$ and Π is not ε-EQL ***do***
16. ***if*** $b \in \mathcal{A}_i^k$ is an ε-path-violator along the alternating path $P = (i, j_1, i_1, j_2, i_2, \dots, i_{k-1}, j_k, b)$ ***then***
17. ***if*** $u_b(\pi_b \setminus \{j\}) \geq u_{b_{k-1}}(j)$ ***then***
18. Update $\pi_b \leftarrow \pi_b \setminus \{j\}$ and $\pi_{b_{k-1}} \leftarrow \pi_{b_{k-1}} \cup \{j\}$;
19. Start from line 13;
20. ***else***
21. $k \leftarrow k + 1$;
22. ***if*** Π is ε-EQL ***then***
23. ***return*** Π;
24. $\partial \leftarrow min_{b \in \mathcal{A}_i, j \in [m] \setminus (\cup_{b \in \mathcal{A}_i} \pi_b)} \alpha_b / (v_{i,j}/p_j)$;
25. ***for*** $j \in \cup_{b \in \mathcal{A}_i} \pi_b$ ***do***
26. $p_j \leftarrow \partial \cdot p_j$;
27. Start from line 13;

Envy-free and the equitability allocations has its own advantages. If the existence of a certain allocation is considered separately, the allocation may be insufficient or unconvincing in some respects. In order to make the allocation fairer and more efficient, we will now consider a combination of three attributes.

***Theorem* 4** When given any instance of fair division with additive and binary valuation and the equitable up to one less-preferred good (EQL), Pareto optimal (PO) and envy-free up to one less-preferred good (EFL) allocation exists it can be found in polynomial time.

Proof: [16] proved that each Nash optimal allocation satisfies the allocation of EF1 + PO. In [14], we come to this conclusion that $EFX \Rightarrow EFL \Rightarrow EF1$. According to the concepts of definition, this conclusion can also be easily drawn that the EFL allocation also meets the $EF1$ allocation. Obviously, combining these two points, we can conclude that the optimal Nash allocation satisfies the allocation of $EFL + PO$. In addition, when the valuation function is binary, we can find the optimal Nash allocation in polynomial time. These conclusions can be found in [10, 11]. To prove the correctness of this theorem, we now need to prove that if the allocation of $EQL + PO$ exists, this allocation is also the Nash optimal allocation. Next, we give proof. According to the revelation of [10], we can use the hypothesis method to prove this theorem. Suppose A is the allocation of $EQL + PO$ but not the Nash optimal allocation, and B is the Nash optimal allocation. According to Corollary Three which proposed in [10], we can use the transfer path in the transfer graph to get that if A is the Pareto optimal allocation, we can get the allocation through the exchange path. But the allocation obtained through this path does not satisfy the EQL allocation. This conclusion is contrary to the assumption, so A must be the optimal Nash allocation. ■

III. DISCUSSION

Our main work studies the concept of fairness relaxation and the relationship between fairness and envy-free. Our contribution are as follows.

Table 1. Summary of results.

Allocation	Existence		Computational	
	General valuation	Special valuation	General valuation	Special valuation
EQ	fail	fail	strongly NP-c	strongly NP-c
EQ1	exist	exist	Poly-time	Poly-time
EQL	exist	exist	Poly-time	Poly-time
EQ1+PO	fail	exist	strongly NP-h	Pseudo-poly
EQL+PO	-	exist	strongly NP-h	Pseudo-poly
EQL+PO +EFL	-	-	-	Poly-time

With reference to the similar technique of [18] We studied equitable up to one less-preferred good (EQL) and Pareto optimality (PO) allocation. But for a general valuation function, it is difficult to find this allocation. Unfortunately, no solution has been found for equitable up to any good (EQX). In our setting, one can define an allocation to be EQX if for every pair of agents $i, k \in [n]$ such that $\pi_k \neq \emptyset$ and for every good $j \in \pi_k$ such that $u_{i,j} \geq 0$ we have $u_i(\pi_i) \geq u_k(\pi_k \setminus \{j\})$. In future research, we can extend this result to the allocation of debris. If the commodity and sundries are mixed, it is more widely used in practice. In addition, the research on public decision-making model is also very enlightening. [12, 19] show that contiguous allocations that meet certain fairness concepts -MMS must exist in certain restricted domains. It is still unknown whether similar results can be obtained from indivisible goods and fair allocation of chores.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China under grants No.61105039 and No.61662011. Supported by Guangxi Key Laboratory of Trusted Software (No kx201725) and Innovation Project of GUET Graduate Education (Issue Number 2019YCXS055, 2020YCXS054).

REFERENCES

[1] Dorothea K Herreiner and Clemens D Puppe. 2009. Envy Freeness in Experimental Fair Division Problems. Theory and Decision, 67(1):65C100.

[2] Lester E Dubins and Edwin H Spanier. 1961. How to Cut a Cake Fairly. The American Mathematical Monthly, 68(1P1):1C17.

[3] Noga Alon. Splitting Necklaces. 1987. Advances in Mathematics, 63(3):247C253.

[4] Katarína Cechlárová and Eva Pillárová. 2012. On the Computability of Equitable Divisions. Discrete Optimization, 9(4):249C257.

[5] Katarína Cechlárová and Eva Pillárová. 2012. A Near Equitable 2-Person Cake Cutting Algorithm. Optimization, 61(11):1321C1330.

[6] Laurent Gourvès, Jérôme Monnot, and Lydia Tlilane. 2014. Near Fairness in Matroids. In Proceedings of the Twenty-First European Conference on Artificial Intelligence, pages 393C398.

[7] Steven J Brams, Michal Feldman, John K Lai, Jamie Morgenstern, and Ariel D Procaccia. 2012. On Maxsum Fair Cake Divisions. In Proceedings of the Twenty-Sixth AAAI Conference on Artificial Intelligence, pages 1285C1291.

[8] Nima Anari, Shayan Oveis Gharan, Tung Mai, and Vijay V Vazirani. 2018. Nash Social Welfare for Indivisible Items under Separable, Piecewise-Linear Concave Utilities. In Proceedings of the 29th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA). 2274C2290.

[9] Richard Cole, Nikhil Devanur, Vasilis Gkatzelis, Kamal Jain, Tung Mai, Vijay V Vazirani, and Sadra Yazdanbod. 2017. Convex Program Duality, Fisher Markets, and Nash Social Welfare. In Proceedings of the 2017 ACM Conference on Economics and Computation (EC). 459C460.

[10] Siddharth Barman, Arpita Biswas, Sanath Kumar Krishnamurthy and Yadati Narahari. 2018 Groupwise Maximin Fair Allocation of Indivisible Goods. The Thirty-Second AAAI Conference on Artificial Intelligence (AAAI-18).

[11] Andreas Darmann and Joachim Schauer. 2015. Maximizing Nash Product Social Welfare in Allocating Indivisible Goods. European Journal of Operational Research, 247(2):548–559. (Cited on pages 8 and 27).

[12] Bouveret, S. Cechlaova, K. Elkind, E. Igarashi, A. and Peters, D. 2017. Fair division of a graph. In Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI-17, 135C141.

[13] Lipton, R. J. Markakis, E. Mossel, E. and Saberi, A. 2004. On approximately fair allocations of indivisible goods. In Proceedings of the 5th ACM conference on Electronic commerce, 125C131. ACM.

[14] Haris Aziz, Ioannis Caragiannis, and Ayumi Igarashi. 2018. Fair Allocation of Combinations of Indivisible Goods and Chores. arXiv preprint arXiv:1807.10684.

[15] Michael R. Garey and David S. Johnson.1979 Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman & Co., 1979.

[16] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D Procaccia, Nisarg Shah, and Junxing Wang 2016 The Unreasonable Fairness of Maximum Nash Welfare. In Proceedings of the 2016 ACM Conference on Economics and Computation, pages 305–322.

[17] Freeman, R. , Sikdar, S. , Vaish, R. , & Xia, L. 2019. Equitable allocations of indivisible goods. Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence (IJCAI-19).

[18] S. Barman and S. Kumar Krishna Murthy. 2017. Approximation algorithms for maximin fair division. In Proc. of the 18th ACM-EC Conference, pages 647C664.

[19] V. Bilo, I. Caragiannis, M. Flammini, A. Igarashi, G. Monaco, D. Peters, C. Vinci, and W. S. Zwicker. 2018. Almost Envy-Free Allocations with

Connected Bundles. In Avrim Blum, editor, 10th Innovations in Theoretical Computer Science Conference (ITCS 2019) volume 124 of Leibniz International Proceedings in Informatics (LIPIcs), pages 14:1C14:21, Dagstuhl, Germany.