Exercises 32, 34a, 39, 40, 41 from Boolean algebra notes

Exercise 32)

a. We have to show the Cartesian product of partially ordered sets A and B is reflexive, transitive, and antisymmetric.

   1. Reflexive: want to prove $\forall a \in A, b \in B, (a,b) \leq (a,b)$.
      We know that $(A, \leq)$ and $(B, \leq)$ are partial orders, so they must be reflexive, so
      $$a \leq a \text{ and } b \leq b$$
      And by the definition of partial order on Cartesian products, $(a,b) \leq (a,b)$

   2. Transitive: want to prove $\forall a, a', a'' \in A$, and $b, b', b'' \in B$, if $(a,b) \leq (a',b')$ and $(a',b') \leq (a'',b'')$, then $(a,b) \leq (a'',b'')$.
      From the assumption and givens, we know that $a \leq a'$ and $a' \leq a''$ and $b \leq b'$ and $b' \leq b''$, which means
      $$a \leq a'' \text{ and } b \leq b''$$
      And thus $(a,b) \leq (a'',b'')$.
   3. Antisymmetry: want to prove $\forall a, a' \in A$, and $b, b' \in B$, if $(a,b) \leq (a',b')$ and $(a',b') \leq (a,b)$, then $(a,b) = (a',b')$.
      Once again, from the assumptions and the givens, this means $a \leq a'$ and $a' \leq a$, and $b \leq b'$ and $b' \leq b$, which means
      $$a = a' \text{ and } b = b'$$
      Thus, $(a,b) = (a',b')$.

   This was the base case. We can prove that the Cartesian product of any number of partially ordered sets is also a partial order through induction on the number of sets (and thus the number of elements in each tuple).

b. To show that the Cartesian product of Boolean algebras $(A, \leq)$ and $(B, \leq)$ is also a Boolean algebra, we need to prove that there's a meet and join for every two elements, that there's a unit and zero, that there's a complement for every element, and that it's distributive,.

   1. Want to prove $\forall a, a' \in A$ and $b, b' \in B, \exists (a,b) \wedge (a',b'), (a,b) \vee (a',b') \in (A \times B, \leq)$.
      Since $(A, \leq)$ and $(B, \leq)$ are Boolean algebras, then we know
      $$\exists a \wedge a', a \vee a' \in A \text{ and } b \wedge b', b \vee b' \in B$$
      Then obviously,
      $$(a,b) \wedge (a',b') = (a \wedge a', b \wedge b') \in A \times B$$

and

$$(a, b) \vee (a', b') = (a \vee a', b \vee b') \in A \times B$$

because for (a, b) and (a', b'), there's no upper bound smaller than $(a \vee a', b \vee b')$ and no lower bound greater than $(a \wedge a', b \wedge b')$. Let' assume for the same of contradiction that there's some least upper bound (a'', b'') such that

$$(a'', b'') < (a, b) \vee (a', b')$$

Then that means either

$$a'' < a \vee a'$$

Or

$$b'' < b \vee b'$$

or both.

We know that there's only one least upper bound in a Boolean algebra, so it must be that

$$a'' = a \vee a'$$

and

$$b'' = b \vee b'$$

But $<$ means strict inequality. We have reached a contradiction.

We can similarly prove our claim for the meet.

2. Want to prove there's a zero and a unit in $(A \times B, \leq)$.
   Once again, obviously,

$$0_{A \times B} = (0_A, 0_B)$$

And

$$1_{A \times B} = (1_A, 1_B)$$

because in $(A \times B, \leq)$, there's no upper or lower bound $(a, b) \in A \times B$ smaller than $(0_A, 0_B)$ or greater than $(1_A, 1_B)$, respectively, because of the definition of partial orders on Cartesian products.

3. Want to prove $\forall a \in A$ and $b \in B$, $\exists \neg(a, b) \in A \times B$.
   We know that

$$\exists \neg a \in A \text{ and } \neg b \in B$$

So

$$\neg(a, b) = (\neg a, \neg b) \in A \times B$$

By the definition of complement, we must prove that

$$(a, b) \vee (\neg a, \neg b) = 1_{A \times B} = (1_A, 1_B)$$

Well,

$$(a, b) \vee (\neg a, \neg b) = (a \vee \neg a, b \vee \neg b) = (1_A, 1_B)$$

And we also must prove that

$$(a, b) \wedge (\neg a, \neg b) = 0_{A \times B} = (0_A, 0_B)$$

Well,
$$(a,b) \wedge (\neg a, \neg b) = (a \wedge \neg a, b \wedge \neg b) = (0_A, 0_B)$$

4. Want to prove distributivity, i.e. that $\forall a, a', a'' \in A$ and $b, b', b'' \in B$, $(a,b) \wedge \big((a',b') \vee (a'',b'')\big) = \big((a,b) \wedge (a',b')\big) \vee \big((a,b) \wedge (a'',b'')\big)$. Exercise 21 showed that proving one law essentially proves the other, so we won't have to prove the other one too. Throughout the proof, we employ the definition of meet and join on two Cartesian product pairs, and the fact that $(A, \leq)$ and $(B, \leq)$ are distributive.

$$\begin{aligned}\text{Left side: } &(a,b) \wedge \big((a',b') \vee (a'',b'')\big)\\ &= (a,b) \wedge \big((a' \vee a'', b' \vee b'')\big)\\ &= \big(a \wedge (a' \vee a''), b \wedge (b' \vee b'')\big)\\ &= \big((a \wedge a') \vee (a \wedge a''), (b \wedge b') \vee (b \wedge b'')\big)\end{aligned}$$

$$\begin{aligned}\text{Right side: } &\big((a,b) \wedge (a',b')\big) \vee \big((a,b) \wedge (a'',b'')\big)\\ &= \big((a \wedge a', b \wedge b') \vee (a \wedge a'', b \wedge b'')\big)\\ &= ((a \wedge a') \vee (a \wedge a''), (b \wedge b') \vee (b \wedge b''))\end{aligned}$$

Left side = right side

This was the base case. We can prove that the Cartesian product of any number of Boolean algebras is also a Boolean algebra through induction on the number of Boolean Algebras (and thus the number of elements in each tuple).

Exercise 34a)

1) $a_1 \vee (a_2 \wedge \neg a_1) = (a_1 \vee a_2) \wedge (a_1 \vee \neg a_1) = (a_1 \vee a_2) \wedge 1 = \boxed{a_1 \vee a_2}$
2) $\neg\big((a \Rightarrow b) \Rightarrow c\big) = \neg\big((\neg a \vee b) \Rightarrow c\big) = \neg(\neg(\neg a \vee b) \vee c) = \neg\big((a \wedge \neg b) \vee c\big) = (\neg(a \wedge \neg b) \wedge \neg c) = \big((\neg a \vee b) \wedge c\big) = (\neg a \wedge c) \vee (b \wedge c) = (\neg a \wedge c \wedge b) \vee (\neg a \wedge c \wedge \neg b) \vee (a \wedge b \wedge c) \vee (\neg a \wedge b \wedge c) = \boxed{(\neg a \wedge c \wedge b) \vee (\neg a \wedge c \wedge \neg b) \vee (a \wedge b \wedge c)}$
3) $a + (b + c) = a + \big((\neg b \wedge c) \vee (b \wedge \neg c)\big) = \Big(\neg a \wedge \big((\neg b \wedge c) \vee (b \wedge \neg c)\big)\Big) \vee \Big(a \wedge \big((\neg b \wedge c) \vee (b \wedge \neg c)\big)\Big) = \boxed{(\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge b \wedge \neg c) \vee (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c)}$
4) $(a + b) + c$. This is just part 3 of this question but have a and c switched, and XOR is commutative. Thus: $\boxed{(\neg c \wedge \neg b \wedge a) \vee (\neg c \wedge b \wedge \neg a) \vee (c \wedge \neg b \wedge a) \vee (c \wedge b \wedge \neg a)}$
5) $(a|b)|c = (\neg a \vee \neg b)|c = \neg(\neg a \vee \neg b) \vee \neg c = (a \wedge b) \vee \neg c = (a \vee \neg c) \wedge (b \vee \neg c) = (a \vee b \vee \neg c) \wedge (a \vee \neg b \vee \neg c) \wedge (a \vee b \vee \neg c) \wedge (\neg a \vee b \vee \neg c) = \boxed{(a \vee b \vee \neg c) \wedge (a \vee \neg b \vee \neg c) \wedge (\neg a \vee b \vee \neg c)}$.

6) $a|(b|c)$. This is just part 5 of this question with a and c switched, and $NAND$ is commutative. So it's $\boxed{(c \vee b \vee \neg a) \wedge (c \vee \neg b \vee \neg a) \wedge (\neg c \vee b \vee \neg a).}$

Exercise 39) Want to prove that there doesn't exist any smaller (by inclusion) non-zero elements than one-element subsets in a power set. The zero element in a power set is the null set $\emptyset$. The power set of any one-element subset $\{a\}$ is $\{\{a\}, \emptyset\}$. And between $\{a\}$ and $\emptyset$, only $\emptyset$ is strictly smaller than $\{a\}$. So, any elements in a power set smaller than a one-element subset must only be the null set.

Exercise 40)

a. We need to prove B has a zero, a unit, is distributive, every element has a complement, and that every two elements have a meet and join. Since this is a set, we use inclusion as the partial order and union and intersection as join and meet, respectively.
   - The meet of two elements $a, b \in B$ is $a \cap b$, which is also in B because it's either $\emptyset$ or a new left closed right open interval.
   - The join of two elements $a, b \in B$ is $a \cup b$, which is also in B because of B's definition of being the set of all finite unions of left closed right open intervals. The union of unions is still a union.
   - The zero is the empty interval because it lower-bounds everything in B
   - The unit is $\mathbb{Q}$ because it upper-bounds everything in B.
   - It is distributive because $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ because of the distributive laws of set intersection and union.

b. Note that $\mathbb{Q}$ is infinite, so $\forall a, c \in \mathbb{Q}, \exists b \in \mathbb{Q}$ such that $a < b < c$.
   So, let's break down the three cases for "every non-zero element of B" and for each one, show the disjoint union of two smaller non-zero elements:
   1) It is a left closed right open interval $[a, c[$. Then it can be decomposed into the disjoint union of $[a, b[\cup [b, c[$, where $a < b < c$
   2) It is a left closed interval $[a, \infty[$. Then it can be decomposed into the disjoint union of $[a, b[\cup [b, \infty[$, where $a < b$ and $[b, \infty[$ is a left closed interval
   3) It is a right open interval $[-\infty, b[$. Then it can be decomposed into the disjoint union of $[-\infty, a[\cup [a, b[$, where $a < b$ and $[-\infty, a[$ is a right open interval

Thus, B has no atoms.

Exercise 41) For all three of these, we want to prove both directions because they're if and only if statements.

a. Want to prove if that if a non-zero x in a Boolean algebra is an atom, then $(x \wedge y \neq 0) \Longrightarrow (x \leq y)$, or equivalently, $(x \wedge y = 0) \vee (x \leq y)$. Let's consider 2 cases of y:

1) $x \leq y$: then $(x \leq y)$ is true
2) $x > y$: Then y has to be 0 because x is an atom, and by definition there's nothing smaller than x that isn't 0. Then $x \wedge y = 0$, because the join of 0 with anything is 0

The other way around, we have to prove $\big((x \wedge y \neq 0) \Longrightarrow x \leq y\big)$ implies x is an atom.

$$\big((x \wedge y \neq 0) \Longrightarrow x \leq y\big) = \big(\neg(x \wedge y \neq 0) \vee (x \leq y)\big)$$
$$= \big((x \wedge y = 0) \vee (x \leq y)\big)$$

Overall, we want to prove

$$\big((x \wedge y = 0) \vee (x \leq y)\big) \Longrightarrow (x \text{ is an atom})$$

To prove this is true, we only have to prove that when x is not an atom, then $\big((x \wedge y = 0) \vee (x \leq y)\big)$ has to also be false (since $True \Longrightarrow False$ is the only time an if-then statement evaluates to false). So, if x is not an atom, then the following could happen: both x and y are much greater than atoms, and $x > y$, and $x \wedge y = y \neq 0$.

b. Want to prove that if a non-zero x in a Boolean algebra is an atom, then $x \leq (y \vee z) \Longrightarrow (x \leq y \text{ or } x \leq z)$, equivalently,

$$\big(x > (y \vee z)\big) \vee (x \leq y) \vee (x \leq z)$$

Altogether, we need to prove x is an atom, then $\big(x > (y \vee z)\big) \vee (x \leq y) \vee (x \leq z)$

Let's again consider by cases:
1) $x \leq y$: then $(x \leq y)$ is true
2) $x \leq z$: then$(x \leq z)$ is true
3) $x > y$ and $x > z$: then y and z have to be 0 because x is an atom, and by definition there's nothing smaller than x that isn't 0. Then $(y \vee z) = 0$, which is smaller than x

Now we want to prove the other way, that

$$\big(x > (y \vee z)\big) \vee (x \leq y) \vee (x \leq z) \Longrightarrow x \text{ is an atom}$$

As with part a, we prove that when x isn't an atom, then $\big(x > (y \vee z)\big) \vee (x \leq y) \vee (x \leq z)$ is not always true. That is, it could be the case that $\big(x \leq (y \vee z)\big) \wedge (x > y) \wedge (x > z)$. For example, this could happen: x, y, z are far greater than atoms, and $y \vee z = x$ (which makes it by definition $x > y$ and $x > z$).

c. Want to prove that if x is an atom, then $(x = y \vee z \text{ and } y \wedge z = 0) \Longrightarrow (y = x \text{ or } z = x)$

$$(x = y \vee z \text{ and } y \wedge z = 0) \Longrightarrow (y = x \text{ or } z = x) =$$
$$\neg(x = y \vee z \text{ and } y \wedge z = 0) \vee (y = x \text{ or } z = x)$$
$$= (x \neq y \vee z) \vee (y \wedge z \neq 0) \vee (y = x) \vee (z = x)$$

Let's again consider by cases:

1) $x = y$: then $(y = x)$ is true
2) $x = z$: then $(z = x)$ is true
3) $x > y$: then that means y is 0, because x is an atom, so $y \vee z = z$, which could make either $(x \neq y \vee z)$ or $(z = x)$ true
4) $x < y, x > z$: then that means z is 0, so $y \vee z = y$, which could make either $(x \neq y \vee z)$ or $(y = x)$ true.
5) $x < y, x < z$: then $y \vee z > x$, so $x < y \vee z$, so $(x \neq y \vee z)$ is true

As with the previous 2 parts of this exercise, we prove the other direction, that is,

$$(x \neq y \vee z) \vee (y \wedge z \neq 0) \vee (y = x) \vee (z = x) \Longrightarrow (x \text{ is an atom})$$

Again, we prove that when x isn't an atom, then $(x \neq y \vee z) \vee (y \wedge z \neq 0) \vee (y = x) \vee (z = x)$ could be false. We look at one specific case where this is false: x, y, z are far greater than atoms, and $y > x$, $z > x$, and $y \wedge z = x$, so $y \vee z > x$.