Exercises 22, 26, 27, 28 from Boolean Algebra notes

Exercise 22) We will break this down into 4 cases: when the lattice has 0, 1, 2, 3, or 4 elements.

<u>0 elements:</u> vacuously true

<u>1 element:</u>

$$A = \{a\}$$

$$a \wedge (a \vee a) \stackrel{?}{=} (a \wedge a) \vee (a \wedge a)$$

Left side: a (idempotency)

Right side: a (idempotency)

Left side = right side

<u>2 elements:</u>

$$A = \{a, b\}$$

3 cases:

<u>1.</u>

$$a \wedge (b \vee b) \stackrel{?}{=} (a \wedge b) \vee (a \wedge b)$$

Left side: $(a \wedge b)$ (idempotency)

Right side: $(a \wedge b)$ (idempotency)

<u>2.</u>

$$a \wedge (b \vee a) \stackrel{?}{=} (a \wedge b) \vee (a \wedge a)$$

Left side: a (lattice law)

Right side: $(a \wedge b) \vee a = a$ (idempotency and lattice law)

Left side = right side

<u>3.</u>

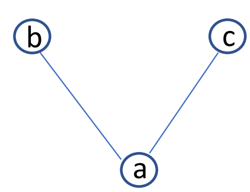
$$a \wedge (a \vee b) \stackrel{?}{=} (a \wedge a) \vee (a \wedge b)$$

Equivalent to case 2 because of the commutative property

<u>3 elements:</u>

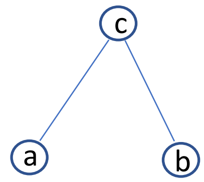
$$A = \{a, b, c\}; a \leq b \leq c$$

And we have to realize that for A to be a lattice, it can't look like this



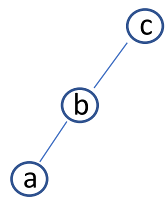
Because it doesn't have $b \vee c$.

And it also can't look like this



Because it doesn't have $a \wedge b$.

A as a lattice can only look like this



So with this image in mind, we can begin to analyze.
6 cases:
<u>1.</u>

$$a \wedge (b \vee c) \stackrel{?}{=} (a \wedge b) \vee (a \wedge c)$$
Left side: $a \wedge c = a$
Right side: $a \vee a = a$
Left side = right side

<u>2.</u>

$$a \wedge (c \vee b) \stackrel{?}{=} (a \wedge c) \vee (a \wedge b)$$
Equivalent to case 1 via the commutative property

<u>3.</u>

$$b \wedge (a \vee c) \stackrel{?}{=} (b \wedge a) \vee (b \wedge c)$$
Left side: $b \wedge c = b$
Right side: $a \vee b = b$
Left side = right side

<u>4.</u>

$$b \wedge (c \vee a) \stackrel{?}{=} (b \wedge c) \vee (b \wedge a)$$
Equivalent to case 3 via the commutative property

<u>5.</u>

$$c \wedge (a \vee b) \stackrel{?}{=} (c \wedge a) \vee (c \wedge b)$$
Left side: $c \wedge b = b$
Right side: $a \vee b = b$

Left side = right side

6.

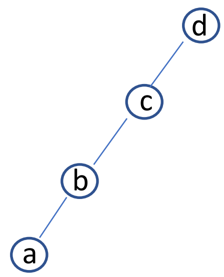
$$c \wedge (b \vee a) \stackrel{?}{=} (c \wedge b) \vee (c \wedge a)$$

Equivalent to case 5 via the commutative property

4 elements:

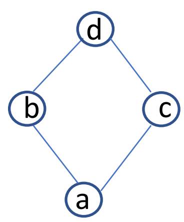
$$A = \{a, b, c, d\}; a \leq b \leq c \leq d$$

And we have to realize that for A to be a lattice, it has to look like this



But since we already analyzed this straight line shape in our 3-element case, we will abandon this.

Or it looks like this



We will use this shape instead. So with this images in mind, we can begin to analyze the 24 cases.

1.

$$a \wedge (b \vee c) \stackrel{?}{=} (a \wedge b) \vee (a \wedge c)$$

2.

$$a \wedge (c \vee b) \stackrel{?}{=} (a \wedge c) \vee (a \wedge b)$$

3.

$$a \wedge (b \vee d) \stackrel{?}{=} (a \wedge b) \vee (a \wedge d)$$

4.

$$a \wedge (d \vee b) \stackrel{?}{=} (a \wedge d) \vee (a \wedge b)$$

5.

$$a \wedge (c \vee d) \stackrel{?}{=} (a \wedge d) \vee (a \wedge c)$$

6.

$$a \wedge (d \vee c) \stackrel{?}{=} (a \wedge d) \vee (a \wedge c)$$

These six cases can all be proven by noticing the fact that $a \wedge \text{anything} = a$, and then applying idempotency.

7.

$$b \wedge (a \vee c) \stackrel{?}{=} (b \wedge a) \vee (b \wedge c)$$

Left side: $b \wedge c = a$

Right side: $a \vee a = a$

Left side = right side

8.

$$b \wedge (c \vee a) \stackrel{?}{=} (b \wedge c) \vee (b \wedge a)$$

Commutative with case 7

9.

$$b \wedge (a \vee d) \stackrel{?}{=} (b \wedge a) \vee (b \wedge d)$$

Left side: $b \wedge d = b$

Right side: $a \vee b = b$

Left side = right side

10.

$$b \wedge (d \vee a) \stackrel{?}{=} (b \wedge d) \vee (b \wedge a)$$

Commutative with case 9

11.

$$b \wedge (c \vee d) \stackrel{?}{=} (b \wedge c) \vee (b \wedge d)$$

Left side: $b \wedge d = b$

Right side: $a \vee b = b$

Left = right

12.

$$b \wedge (d \vee c) \stackrel{?}{=} (b \wedge d) \vee (b \wedge c)$$

Commutative with case 11

13.

$$c \wedge (a \vee b) \stackrel{?}{=} (c \wedge a) \vee (c \wedge b)$$

Left side: $c \wedge b = a$

Right side: $a \vee a = a$

Left = right

14.

$$c \wedge (b \vee a) \stackrel{?}{=} (c \wedge b) \vee (c \wedge a)$$

Commutative with case 13

15.

$$c \wedge (a \vee d) \stackrel{?}{=} (c \wedge a) \vee (c \wedge d)$$

Left side: $c \wedge d = c$

Right side: $a \vee c = c$

Left = right

16.

$$c \wedge (d \vee a) \stackrel{?}{=} (c \wedge d) \vee (c \wedge a)$$

Commutative with case 15

17.

$$c \wedge (b \vee d) \stackrel{?}{=} (c \wedge b) \vee (c \wedge d)$$

Left side: $c \wedge d = c$

Right side: $a \vee c = c$

Left = right

18.

$$c \wedge (d \vee b) \stackrel{?}{=} (c \wedge d) \vee (c \wedge b)$$

Commutative with case 17

19.

$$d \wedge (a \vee b) \stackrel{?}{=} (d \wedge a) \vee (d \wedge b)$$

Left side: $d \wedge b = b$

Right side: $a \vee b = b$

Left = right

20.

$$d \wedge (b \vee a) \stackrel{?}{=} (d \wedge b) \vee (d \wedge a)$$

Commutative with case 19

21.

$$d \wedge (a \vee c) \stackrel{?}{=} (d \wedge a) \vee (d \wedge c)$$

Left side: $d \wedge c = c$

Right side: $a \vee c = c$

Left = right

22.

$$d \wedge (c \vee a) \stackrel{?}{=} (d \wedge c) \vee (d \wedge a)$$

Commutative with case 21

23.

$$d \wedge (b \vee c) \stackrel{?}{=} (d \wedge b) \vee (d \wedge c)$$

Left side: $d \wedge d = d$

Right side: $b \vee c = d$

Left = right

24.

$$d \wedge (c \vee a) \stackrel{?}{=} (d \wedge c) \vee (d \wedge a)$$

Commutative with case 23

Exercise 26) Since the lattice laws weren't given, we will prove them instead, because we will need it to prove $(A, \leq)$ is a Boolean Algebra. There were no identity laws given in the notes, and I don't know how to prove that from the ones given, but I think we need it for this problem, so I'm just gonna assume it.

$$a \vee (a \wedge b)$$
$$= (a \wedge 1) \vee (a \wedge b) \text{ (identity)}$$
$$= a \wedge (1 \vee b) \text{ (distributivity)}$$
$$= a \wedge 1 \text{ (law for one)}$$
$$= a \text{ (identity)}$$

And because of the distributive property:

$$a \vee (a \wedge b) = (a \vee a) \wedge (a \vee b)$$
$$a \vee (a \wedge b) = a \wedge (a \vee b) = a$$

Anyway, if two elements a and b have the relation either $a \leq b$ or $b \geq a$, then we know what their least upper bound and greatest lower bound are because it's given what this relation implies. If there's no relation between the two elements, then we have to look to the lattice laws:

$$a \vee (a \wedge b) = a$$

Which means

$$(a \wedge b) \leq a$$

And

$$a \wedge (a \vee b) = a$$

Which means

$$a \leq (a \vee b)$$

Putting the two inequalities together,

$$(a \wedge b) \leq a \leq (a \vee b)$$

If we switch a and b in the above steps, we get

$$b \vee (b \wedge a) = b, \qquad b \wedge (b \vee a) = b$$

$$(b \wedge a) \leq b \leq (b \vee a)$$

Putting it all together, we get

$$(a \wedge b) \leq a, b \leq (a \vee b)$$

Which means any two elements are bounded by their meet and join from below and above, respectively, and thus $(A, \leq)$ is a lattice.

We know that it's distributive because it is given that it follows all the equational axioms for a Boolean Algebra, which includes distributivity. We know each element has a complement because there's the unary operator $\neg$. Because of that, we know there's a unit and a zero because of laws e through h.

Exercise 27)

a. $a \leq b$ is defined as $a \wedge b = a$ and equivalently $a \vee b = b$. If we negate $a \wedge b = a$, we get $\neg(a \wedge b) = \neg a$. Which through De Morgan's laws is $\neg b \vee \neg a = \neg a$, which means $\neg b \leq \neg a$.

   And since this is an if and only if statement, we need to prove the other way too. Given $\neg b \leq \neg a$, that means $\neg b \wedge \neg a = \neg b$, and equivalently, $\neg b \vee \neg a = \neg a$. If we negate $\neg b \vee \neg a = \neg a$, we get $\neg(\neg b \vee \neg a = \neg a)$, which is $a \wedge b = a$, which means $a \leq b$.

b. The zero is unique law of Boolean Algebras states

   $$\forall a, b \in A, (a \wedge \neg a) = (b \wedge \neg b)$$

   Let's set $b = \neg a$, then

   $$(a \wedge \neg a) = (\neg a \wedge \neg\neg a)$$

   Thus, we see $a = \neg\neg a$.

Exercise 28)

a. If it's closed under $\wedge$,$\vee$, and $\neg$, that means $\forall a \in B', \neg a \in B'$. And $\forall a, b \in B', a \wedge b \in B', a \vee b \in B'$. For B' to be a subalgebra, it must be a distributive lattice with a unit and a zero, and every element must have a complement.

   We know that every element in B' has a compliment because it's closed under $\neg$. And because of that, we know B' has a unit and a zero because of $a \vee \neg a$ and $a \wedge \neg a$. We know that B' is distributive because if three elements $a, b, c \in B'$, then $a, b, c \in B$, and the laws of distributivity don't change.

b. To prove this set S of all finite or cofinite subsets of D is a Boolean Algebra, all we need to do is proving it's a subalgebra. To do that, we need to prove S is closed under $\vee$,$\wedge$, and $\neg$.

   For operator $\vee$ on sets:

   - The union of two finite sets is finite, thus in S.

- The union of a finite and cofinite set is cofinite, thus also in S.
- The union of two cofinite sets is also cofinite, thus also in S

Thus S is closed under ∨.

For operator ∧ on sets:

- The intersection of two finite sets is finite, thus in S.
- The intersection of a finite and cofinite set is finite, thus also in S.
- The intersection of two cofinite sets is also cofinite, thus also in S

Thus S is closed under ∧.

The complement of a finite set is cofinite, and the complement of a cofinite set is finite. Thus, S is closed under ¬. Hence, S is a subalgebra of D, and so a Boolean algebra.

c. Again, to prove this set S of all countable or cocountable subsets of D is a Boolean Algebra, all we need to do is proving it's a subalgebra. To do that, we need to prove it's closed under ∨,∧, and ¬.

For operator ∨ on sets:

- The union of two countable sets is countable, thus in S.
- The union of a countable and cocountable set is cocountable, thus also in S.
- The union of two cocountable sets is also cocountable, thus also in S

Thus S is closed under ∨.

For operator ∧ on sets:

- The intersection of two countable sets is countable, thus in S.
- The intersection of a countable and cocountable set is countable, thus also in S.
- The intersection of two cocountable sets is also cocountable, thus also in S

Thus S is closed under ∧.

The complement of a countable set is cocountable, and the complement of a cocountable set is countable. Thus, S is closed under ¬. Hence, S is a subalgebra of D, and so a Boolean algebra.