Exercises 22, 26, 27, 28 from Boolean Algebra notes

Exercise 22) We will break this down into 4 cases: when the lattice has 0, 1, 2, 3, or 4 elements.

<u>0 elements:</u> vacuously true

<u>1 element:</u>

$$A = \{a\}$$

$$a \wedge (a \vee a) \stackrel{?}{=} (a \wedge a) \vee (a \wedge a)$$

Left side: a (idempotency)

Right side: a (idempotency)

Left side = right side

<u>2 elements:</u>

$$A = \{a, b\}$$

3 cases:

<u>1.</u>

$$a \wedge (b \vee b) \stackrel{?}{=} (a \wedge b) \vee (a \wedge b)$$

Left side: $(a \wedge b)$ (idempotency)

Right side: $(a \wedge b)$ (idempotency)

<u>2.</u>

$$a \wedge (b \vee a) \stackrel{?}{=} (a \wedge b) \vee (a \wedge a)$$

Left side: a (lattice law)

Right side: $(a \wedge b) \vee a = a$ (idempotency and lattice law)

Left side = right side

<u>3.</u>

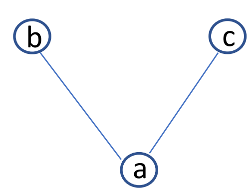
$$a \wedge (a \vee b) \stackrel{?}{=} (a \wedge a) \vee (a \wedge b)$$

Equivalent to case 2 because of the commutative property

<u>3 elements:</u>

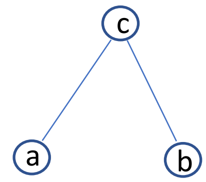
$$A = \{a, b, c\}; a \leq b \leq c$$

And we have to realize that for A to be a lattice, it can't look like this



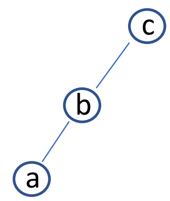
Because it doesn't have $b \vee c$.

And it also can't look like this



Because it doesn't have $a \wedge b$.

A as a lattice can only look like this



So with this image in mind, we can begin to analyze.
6 cases:
<u>1.</u>

$$a \wedge (b \vee c) \stackrel{?}{=} (a \wedge b) \vee (a \wedge c)$$
Left side: $a \wedge c = a$
Right side: $a \vee a = a$
Left side = right side

<u>2.</u>

$$a \wedge (c \vee b) \stackrel{?}{=} (a \wedge c) \vee (a \wedge b)$$
Equivalent to case 1 via the commutative property

<u>3.</u>

$$b \wedge (a \vee c) \stackrel{?}{=} (b \wedge a) \vee (b \wedge c)$$
Left side: $b \wedge c = b$
Right side: $a \vee b = b$
Left side = right side

<u>4.</u>

$$b \wedge (c \vee a) \stackrel{?}{=} (b \wedge c) \vee (b \wedge a)$$
Equivalent to case 3 via the commutative property

<u>5.</u>

$$c \wedge (a \vee b) \stackrel{?}{=} (c \wedge a) \vee (c \wedge b)$$
Left side: $c \wedge b = b$
Right side: $a \vee b = b$

Left side = right side

6.

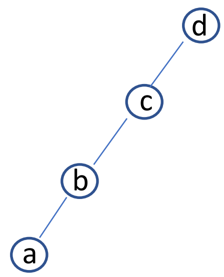
$$c \wedge (b \vee a) \stackrel{?}{=} (c \wedge b) \vee (c \wedge a)$$

Equivalent to case 5 via the commutative property

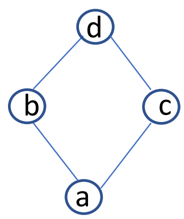
4 elements:

$$A = \{a, b, c, d\}; a \leq b \leq c \leq d$$

And we have to realize that for A to be a lattice, it has to look like this (call it lattice L)



Or like this (call it lattice S)



So with these images in mind, we can begin to analyze the 24 cases.

1.

$$a \wedge (b \vee c) \stackrel{?}{=} (a \wedge b) \vee (a \wedge c)$$

2.

$$a \wedge (c \vee b) \stackrel{?}{=} (a \wedge c) \vee (a \wedge b)$$

3.

$$a \wedge (b \vee d) \stackrel{?}{=} (a \wedge b) \vee (a \wedge d)$$

4.

$$a \wedge (d \vee b) \stackrel{?}{=} (a \wedge d) \vee (a \wedge b)$$

5.

$$a \wedge (c \vee d) \stackrel{?}{=} (a \wedge d) \vee (a \wedge c)$$

6.

$$a \wedge (d \vee c) \stackrel{?}{=} (a \wedge d) \vee (a \wedge c)$$

These six cases can all be proven by noticing the fact that $a \wedge \text{anything} = a$, and then applying idempotency.

<u>7.</u>

$$b \wedge (a \vee c) \stackrel{?}{=} (b \wedge a) \vee (b \wedge c)$$

Lattice L:
Left side: $b \wedge c = a$
Right side: $a \vee a = a$
Left side = right side

Lattice S:
Left side: $b \wedge c = b$
Right side: $a \vee b = b$
Left side = right side

<u>8.</u>

$$b \wedge (c \vee a) \stackrel{?}{=} (b \wedge c) \vee (b \wedge a)$$

Commutative with case 7

<u>9.</u>

We have to notice that for a join or meet operation, unless it's $(b \wedge c)$ or $(b \vee c)$, the two lattices give the same result.

$$b \wedge (a \vee d) \stackrel{?}{=} (b \wedge a) \vee (b \wedge d)$$

Both lattices:
Left side: $b \wedge d = b$
Right side: $a \vee b = b$
Left side = right side

<u>10.</u>

$$b \wedge (d \vee a) \stackrel{?}{=} (b \wedge d) \vee (b \wedge a)$$

Commutative with case 9

<u>11.</u>

$$b \wedge (c \vee d) \stackrel{?}{=} (b \wedge c) \vee (b \wedge d)$$

Lattice L:
Left side: $b \wedge d = b$
Right side: $a \vee b = b$
Left = right

Lattice S:
Left side: $b \wedge d = b$
Right side: $b \vee b = b$
Left = right

<u>12.</u>

$$b \wedge (d \vee c) \stackrel{?}{=} (b \wedge d) \vee (b \wedge c)$$

Commutative with case 11

13.

$$c \wedge (a \vee b) \stackrel{?}{=} (c \wedge a) \vee (c \wedge b)$$

Lattice S:

Left side: $c \wedge b = a$

Right side: $a \vee a = a$

Left = right

Lattice L:

Left side: $c \wedge b = b$

Right side: $a \vee b = b$

Left = right

14.

$$c \wedge (b \vee a) \stackrel{?}{=} (c \wedge b) \vee (c \wedge a)$$

Commutative with case 13

15.

$$c \wedge (a \vee d) \stackrel{?}{=} (c \wedge a) \vee (c \wedge d)$$

Left side: $c \wedge d = c$

Right side: $a \vee c = c$

Left = right

16.

$$c \wedge (d \vee a) \stackrel{?}{=} (c \wedge d) \vee (c \wedge a)$$

Commutative with case 15

17.

$$c \wedge (b \vee d) \stackrel{?}{=} (c \wedge b) \vee (c \wedge d)$$

Lattice S:

Left side: $c \wedge d = c$

Right side: $a \vee c = c$

Left = right

Lattice L:

Left side: $c \wedge d = c$

Right side: $b \vee c = c$

Left = right

18.

$$c \wedge (d \vee b) \stackrel{?}{=} (c \wedge d) \vee (c \wedge b)$$

Commutative with case 17

19.

$$d \wedge (a \vee b) \stackrel{?}{=} (d \wedge a) \vee (d \wedge b)$$

Left side: $d \wedge b = b$

Right side: $a \vee b = b$

Left = right

20.

$$d \wedge (b \vee a) \stackrel{?}{=} (d \wedge b) \vee (d \wedge a)$$

Commutative with case 19

21.

$$d \wedge (a \vee c) \stackrel{?}{=} (d \wedge a) \vee (d \wedge c)$$

Left side: $d \wedge c = c$

Right side: $a \vee c = c$

Left = right

22.

$$d \wedge (c \vee a) \stackrel{?}{=} (d \wedge c) \vee (d \wedge a)$$

Commutative with case 21

23.

$$d \wedge (b \vee c) \stackrel{?}{=} (d \wedge b) \vee (d \wedge c)$$

Lattice S:

Left side: $d \wedge d = d$

Right side: $b \vee c = d$

Left = right

Lattice L:

Left side: $d \wedge c = c$

Right side: $b \vee c = c$

Left = right

24.

$$d \wedge (c \vee a) \stackrel{?}{=} (d \wedge c) \vee (d \wedge a)$$

Commutative with case 23

Exercise 26) We need to prove that $(A, \leq)$ is a Boolean algebra, and the operations of this Boolean algebra coincide with those given above in the notes.

From exercise 18, it is known that $(A, \leq)$ is a lattice in which the operations are the lattice operations. Since a Boolean Algebra is a distributive lattice with a zero and a unit, and every element has a complement, we now need to prove that $(A, \leq)$ as a lattice is distributive, has a zero and a unit, and that every element has a complement.

We know that it's distributive because it is given that it follows all the equational axioms for a Boolean Algebra, which includes distributivity. And because of the convention mentioned earlier, we know it has a unit and a zero. Because of that, we know each element has a complement because the zero and unit are defined by the meet and join of an element with itself.

Exercise 27)

a. $a \leq b$ is defined as $a \wedge b = a$ and equivalently $a \vee b = b$. If we negate $a \wedge b = a$, we get $\neg(a \wedge b) = \neg a$. Which through De Morgan's laws is $\neg b \vee \neg a = \neg a$, which means $\neg b \leq \neg a$.

   And since this is an if and only if statement, we need to prove the other way too. Given $\neg b \leq \neg a$, that means $\neg b \wedge \neg a = \neg b$, and equivalently, $\neg b \vee \neg a = \neg a$. If we negate $\neg b \vee \neg a = \neg a$, we get $\neg(\neg b \vee \neg a = \neg a)$, which is $a \wedge b = a$, which means $a \leq b$.
b. The zero is unique law of Boolean Algebras states
   $$\forall a, b \in A, (a \wedge \neg a) = (b \wedge \neg b)$$
   Let's set $b = \neg a$, then
   $$(a \wedge \neg a) = (\neg a \wedge \neg\neg a)$$
   Thus, we see $a = \neg\neg a$.

Exercise 28)

a. If it's closed under $\wedge$,$\vee$, and $\neg$, that means $\forall a \in B', \neg a \in B'$. And $\forall a, b \in B'$, $a \wedge b \in B'$, $a \vee b \in B'$. For B' to be a subalgebra, it must be a distributive lattice with a unit and a zero, and every element must have a complement.

   We know that every element in B' has a compliment because it's closed under $\neg$. And because of that, we know B' has a unit and a zero because of $a \vee \neg a$ and $a \wedge \neg a$. We know that B' is distributive because if three elements $a, b, c \in B'$, then $a, b, c \in B$, and the laws of distributivity don't change.
b. To prove this set S of all finite or cofinite subsets of D is a Boolean Algebra, all we need to do is proving it's a subalgebra. To do that, we need to prove S is closed under $\vee$,$\wedge$, and $\neg$.

   For operator $\vee$ on sets:
   - The union of two finite sets is finite, thus in S.
   - The union of a finite and cofinite set is cofinite, thus also in S.
   - The union of two cofinite sets is also cofinite, thus also in S

   Thus S is closed under $\vee$.

   For operator $\wedge$ on sets:
   - The intersection of two finite sets is finite, thus in S.
   - The intersection of a finite and cofinite set is finite, thus also in S.
   - The intersection of two cofinite sets is also cofinite, thus also in S

   Thus S is closed under $\wedge$.

The complement of a finite set is cofinite, and the complement of a cofinite set is finite. Thus, S is closed under $\neg$. Hence, S is a subalgebra of D, and so a Boolean algebra.

c. Again, to prove this set S of all countable or cocountable subsets of D is a Boolean Algebra, all we need to do is proving it's a subalgebra. To do that, we need to prove it's closed under $\vee$,$\wedge$, and $\neg$.

   For operator $\vee$ on sets:
   - The union of two countable sets is countable, thus in S.
   - The union of a countable and cocountable set is cocountable, thus also in S.
   - The union of two cocountable sets is also cocountable, thus also in S

   Thus S is closed under $\vee$.

   For operator $\wedge$ on sets:
   - The intersection of two countable sets is countable, thus in S.
   - The intersection of a countable and cocountable set is countable, thus also in S.
   - The intersection of two cocountable sets is also cocountable, thus also in S

   Thus S is closed under $\wedge$.

The complement of a countable set is cocountable, and the complement of a cocountable set is countable. Thus, S is closed under $\neg$. Hence, S is a subalgebra of D, and so a Boolean algebra.