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MATH 4810 HW 2

1) We have to prove it's injective, surjective, and monotonically non-decreasing (preserves partial order)
    a. Want to prove it's injective. Assume we have $b, b' \in B$ and $b = b'$, and $f(a) = b$ and $f(a') = b'$, want to prove $a = a'$.

       Due to the definition of initial segment, a has to be the biggest element in b (i.e. the outermost, most inclusive, set in b, because B is partially ordered by inclusion ($\subseteq$)). Similarly, a' has to be the biggest element in b'. If $b = b'$, then consequently $a = a'$.

    b. Want to prove it's surjective. Want to prove that, $\forall b \in B, \exists a \in A$ such that $f(a) = b$. Again, due to the definition of initial segment and the definition of the mapping f, each initial segment $b \in B$ is mapped from its most inclusive set in A, so it cannot be *not* mapped from something in A.
    c. Want to prove it preserves the partial order.
        i. Want to prove if $a, a' \in A$ and $a \leq a'$ and $f(a) = b$ and $f(a') = b'$, then $b \subseteq b'$. This is proven by the fact that by definition of initial segment, $b = [y \in A | y \leq a]$, and $b' = [y' \in A | y' \leq a']$. Since $a \leq a'$, then all of b has to be included in b', i.e. $b \subseteq b'$.
        ii. Want to prove if $a, a' \in A$ and $f(a) = b$ and $f(a') = b'$ and $b \subseteq b'$, then $a \leq a'$. As we said before, a is by definition the most inclusive set in b and a' the most inclusive set in b'. Since $b \subseteq b'$, the biggest thing in b has to be included in the biggest thing in b', thus, $a \leq a'$.

2) :
    a. To prove any strict partial order can be extended to partial order via $x \leq y$ iff $x < y$ or $x = y$, we need to prove this definition satisfies transitivity, reflexivity, and antisymmetry.
        i. Transitivity: if $x \leq y$ and $y \leq z$, then $x \leq z$. We translate this to if ($x < y$ or $x = y$) and ($y < z$ or $y = z$), then ($x < z$ or $x = z$). This will be broken down into four cases:
            1. ($x < y$ and $y < z$), then we know $x < z$ because strict partial ordering is transitive.
            2. ($x < y$ and $y = z$), then we know $x < z$
            3. ($x = y$ and $y < z$), then we know $x < z$
            4. ($x = y$ and $y = z$), then we know $x = z$
        ii. Reflexivity: $x \leq x$. We can translate this to ($x < x$ or $x = x$). $x = x$ is true.
        iii. Antisymmetry: if ($x \leq y$ and $y \leq x$), then $x = y$. We translate this to if ($x < y$ or $x = y$) and ($y < x$ or $y = x$), then $x = y$. This is broken down into four cases again:

1. ($x < y$ and $y < x$), then it's vacuously true that $x = y$.
2. ($x < y$ and $y = x$), again, vacuously true that $x = y$.
3. ($x = y$ and $y < x$), again, vacuously true that $x = y$.
4. ($x = y$ and $y = x$), it's the obviously true that $x = y$.

b. To prove that any partial order can be turned into a strict partial order via $x < y$ iff $x \leq y$ and $x \neq y$, we need to prove this definition satisfies transitivity and irreflexivity.
   i. Transitivity: If ($x < y$ and $y < z$), then $x < z$. We translate this to if ($x < y$ and $x \neq y$) and ($y < z$ and $y \neq z$), then ($x < z$ and $x \neq z$). This is obviously true. We can see that $x < y < z$ and they're all unequal to each other.
   ii. Irreflexivity: ($x < x$) is false. We translate this to ($x \leq x$ and $x \neq x$) is false. Equivalently, this means $x \leq x$ is false **or** $x \neq x$ is false. Obviously, $x \neq x$ is false. Therefore, $x < x$ is false.

3) :

a. To prove that any strict simple order can be extended to a simple order via the definition $x \leq y$ iff $x < y$ or $x = y$, we need to prove this definition satisfies $\forall x, y \in A$, either $x \leq y$ or $y \leq x$. This condition can be translated to $\forall x, y \in A$, either ($x < y$ or $x = y$) or ($y < x$ or $y = x$). We can regroup the or's: $\forall x, y \in A$, either ($x < y$ or $y < x$) or ($x = y$ or $y = x$). We know from strict simple order's definition that ($x < y$ or $y < x$) is true.

b. To prove that any simple order can be extended to a strict simple order via the definition $x < y$ iff $x \leq y$ and $x \neq y$, we need to prove this definition satisfies $\forall x, y \in A$, either $x < y$ or $y < x$. This condition can be translated to $\forall x, y \in A$, either ($x \leq y$ and $x \neq y$) or ($y \leq x$ and $y \neq x$). This can be converted into conjunctive normal form via distributing the and's: $\forall x, y \in A$, ($x \leq y$ or $y \leq x$) and ($x \leq y$ or $y \neq x$) and ($x \neq y$ or $y \leq x$) and ($x \neq y$ or $y \neq x$).
   i. ($x \leq y$ or $y \leq x$) is true because of the definition of simple order
   ii. ($x \leq y$ or $y \neq x$) is true because if $x \nleq y$, then that means $x > y$, which implies $y \neq x$.
   iii. ($x \neq y$ or $y \leq x$) follows the same reasoning for ii, just exchange x and y.
   iv. ($x \neq y$ or $y \neq x$) is obviously true because $x \neq y$ is the same thing as $y \neq x$

4) :

a. We will prove the intersection of two transitive relations R_1 and R_2 on set A is still a transitive relation. After this base case is proven, you can use this to inductively prove the intersection of any number of transitive relations on set A is still a transitive relation.

   We take two arbitrary relations from $R_1 \cap R_2$: (a, b) and (b, c). Because they're from the intersection of the two relations, they're each in R_1 and R_2:

$$(a, b) \in R_1, (a, b) \in R_2, (b, c) \in R_1, (b, c) \in R_2$$

And because R_1 and R_2 are both transitive relations, we know

$$(a,c) \in R_1$$
$$(a,c) \in R_2$$

Thus, by the definition of intersection,

$$(a,c) \in R_1 \cap R_2$$

QED.

b. Take three arbitrary elements $a, b, c \in A$. By the definition of Cartesian product, we know $(a,b), (b,c), (a,c) \in A \times A$. By the definition of transitivity, if $(a,b), (b,c) \in A \times A$, then so should $(b,c) \in A \times A$. And it *is* there!! QED.

c. The transitive closure on R would be obtained by this simple algorithm: $\forall (a,b), (b,c) \in R$ such that $(a,c) \notin R$, add (a,c) to R. The result returned by this algorithm is the smallest transitive relation on A that contains R because first of all, it is obviously transitive, and second of all, the smallest because if you take away any of the new ordered pairs added by this algorithm, there would exist $(a',b'), (b',c') \in R$ such that $(a',c') \notin R$.

5) :

a. Since this is an iff statement, we need to prove both directions:

   i. Want to prove that if there is such a sequence as described, then (x,y) is in the transitive closure of R.

      Let's look at the algorithm I gave in question 4c. The algorithm would, for each x_i such that $i \leq n-1$, add the relations $(x_i, x_{i+1}), (x_i, x_{i+2}), \ldots, (x_i, x_{n-1}), (x_i, x_n)$, if they aren't already in R. $(x,y) = (x_1, x_n)$ is among these.

   ii. Want to prove the other way now, that if (x,y) is in the transitive closure of R, then there is such a sequence as described.

      We prove the contrapositive of this statement, that if there's no such sequence, then there's no corresponding (x,y) in R. For there to be no such sequence, $\exists i \leq n-1$ such that $(x_i, x_{i+1}) \notin R$ (i.e. the sequence is "broken" in at least one place). Now, (x,y) won't exist in the transitive closure anymore because there will be no more (x_a, x_c) where $a \leq i$ and $c \geq i+1$ for the said i. And of course, there will be no more $(x,y) = (x_1, x_n)$.

b. The transitive closure of the relation of successor on the set of integers is $[(x, x+i) | x \in \mathbb{Z} \text{ } and \text{ } i \in +\mathbb{Z}\,]$.

6) :

a. Since this is an iff statement, we need to prove both directions:

   i. Want to prove that if R's transitive closure is a strict partial order, then R can be extended to a strict partial order.

If its transitive closure is a strict partial order, then we can just extend R to be its transitive closure. This is a valid extension because R's transitive closure is a bigger subset of $A \times A$ by inclusion than R (i.e. it includes R).

ii. Want to prove that if R can be extended to a strict partial order, then its transitive closure is a strict partial order.

    If R can be extended to R_2, a strict partial order, that means $R_2 = R' \cup R$ such that $R' \subseteq A \times A$. This means that R and R' are both already irreflexive and asymmetric, because if they weren't, then R_2 (their union) wouldn't be irreflexive and asymmetric (2 of 3 requirements for a strict partial order). R might not be transitive, but R_2 by definition of a strict partial order is (the last of the 3 requirements).

    By definition of a transitive closure, it adds all necessary pairs such that it becomes transitive, the same thing the extension is doing. We now also know that this transitive closure is also a strict partial order because we just proved R and R' are irreflexive and asymmetric.

b. Since this is an iff statement, we need to prove both directions:
    i. Want to prove if the transitive closure of the union is a partial order, then R can be extended to a partial order.

       The extension of R to a partial order can just be the transitive closure of the union. This is a valid extension because the transitive closure is a bigger subset of $A \times A$ by inclusion than R (i.e. it includes R).

    ii. Want to prove if R can be extended to a partial order, then the transitive closure of the described union is a partial order.

       Extending R to a partial order means making it reflexive, transitive, and antisymmetric (if it wasn't already). Making it reflexive means $\forall x \in A$, add (x, x), the transitive closure in the problem does this because of the union; making it transitive is by definition also done by the transitive closure; and making it antisymmetric is also already done by making it transitive, because if $(a, b), (b, a) \in R$, then by transitivity $(a, a) \in R$, and also $(b, b) \in R$. QED.

7) :

a. By the definition of countable set, there exists a bijection from a subset of natural numbers to A. Among all possible such bijections, there is one bijection f that is an isomorphism. In this one bijection, we can determine whether each $x, y \in A$ is in the order of $x \leq y$ or $y \leq x$ by looking at if $f(x) \leq f(y)$ or $f(y) \leq f(x)$.

Alternate answer: $\forall x, y \in A$ such that $(x, y) \notin R$ and $(y, x) \notin R$, add (x, y) to R'. And $\forall (x', y') \in R$, add (x', y') to R' also. Now make R' its transitive closure. R' by definition is a simple ordering because $\forall a, b \in A$, either $(a, b) \in R'$ or $(b, a) \in R'$, or both.

b. We prove something stronger, that every partial order R on set A is the intersection of all of R's simple order extensions. We just proved in part a of this question that every partial order on set A has a simple order extension. In fact, it has many simple orders extensions: $\forall x, y \in A$ such that $(x, y) \notin R$ and $(y, x) \notin R$, we can add either (x, y), (y, x), or both to the simple order. Thus, for all the different simple order extensions of R, they include all ordered pairs in R, but every ordered pair not in R do not exist in all of these extensions, so they would get cancelled out through intersection. Thus, the intersection of all simple order extensions of R equals R. QED.

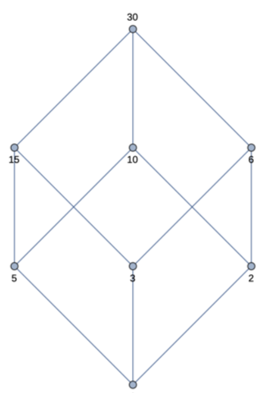
c. I don't know them):

8) :

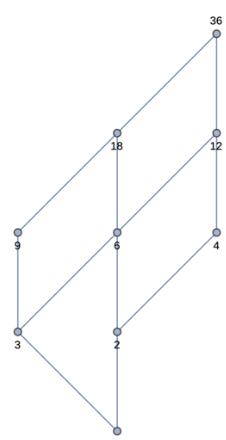
a. x, by the definition of least upper bound for B, is an upper bound and $\forall a$ that is an upper bound for B, $x \leq a$. This means $x \leq y$. Through the same reasoning and exchanging x with y, $y \leq x$. By antisymmetry, $x = y$.

b. a is a lower bound for B if $\forall b \in B$, $b \geq a$. a is a greatest lower bound for B if a is a lower bound for B and whenever x is a lower bound for B, we have $x \geq a$.

9) :

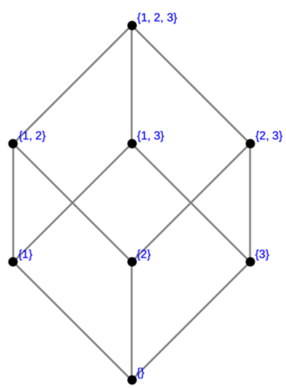


a.



b.

c. 

d. 

(just subtract one from all the numbers except in the null set please!)

e.