Exercises 43, 46, 51, 52

## Exercise 43

Claim: $\forall a, b \in A, \exists c \in A$ such that $a \wedge b = c$

Proof: The greatest lower bound between $a, b \in A$, i.e. $a \wedge b$, is the greatest common divisor of $a$ and $b$, $\gcd(a, b)$, because $A$ is ordered by divisibility, the greatest $c \in A$ such that $c \leq a$ and $c \leq b$ is by definition the greatest $c \in A$ such that $c$ divides $a$ and $c$ divides $b$ – the definition of $\gcd(a, b)$. Such a c exists because (not sure if I have to prove this but this is elementary school math) $\gcd(a, b)$ is the product of every element in the intersection of the prime factors of $a$ and of $b$, which are included in $A$ because $A$ is the set of all the positive integer divisors of $n$.

Claim: $\forall a, b \in A, \exists c \in A$ such that $a \vee b = c$

Proof: The lowest upper bound between $a$ and $b$, i.e. $a \vee b$, is the least common multiple of $a$ and $b$, $\operatorname{lcm}(a, b)$, because $A$ is ordered by divisibility, the smallest $c \in A$ such that $c \geq a$ and $c \geq b$ is by definition the smallest $c \in A$ such that $a$ and $b$ both divide $c$ – the definition of $\operatorname{lcm}(a, b)$. Such a $c$ exists because (again, elementary school math so I hope I don't gotta prove it), $\operatorname{lcm}(a, b)$ is the product of every element in the union of the prime factors of $a$ and $b$, which are included in $A$ because $A$ is the set of all the positive integer divisors of $n$.

Claim: $n$ is the one (or unit) of $A$

Proof: $n$ is the least upper bound of all the elements of $A$ because by definition of $A$, every element in $A$ divides $n$, so $\forall c \in A, c \leq n$.

Claim: $1$ is the zero of $A$

Proof: $1$ is the greatest lower bound of all the elements of $A$ because $1$ divides every positive integer, so it divides everything in $A$, so $\forall c \in A$, $1 \leq c$.

Claim: $(A, \leq)$ is distributive, i.e. $\forall x, y, z \in A$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Proof: $x \wedge (y \vee z) = \gcd\big(x, lcm(y, z)\big)$. Let $X$ be the set of all prime factors of $x$ (we know we aren't "losing" any due to repetition because it is given that $n$ has no square factors $f > 1$). Let $Y$ be the set of all prime factors of $y$. Let $Z$ be the set of all prime factors of $z$. $\gcd\big(x, \operatorname{lcm}(y, z)\big) = X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z) = \operatorname{lcm}(\gcd(x, y), \gcd(x, y)) = (x \wedge y) \vee (x \wedge z)$.

Claim: if $\forall a \in A, \neg a = n \div a$, then $a \vee \neg a = one = n$ and $a \wedge \neg a = zero = 1$

Proof: $a \vee \neg a = \operatorname{lcm}(a, \neg a) = n$, because firstly, $n$ divides both $a$ and $\frac{n}{a}$, and secondly, $a$ and $\frac{n}{a}$ don't share any prime factors (i.e. the intersection between their prime factors is the empty set) because of the fact that $n$ doesn't have any square factors $f > 1$; so, if $X$ is the set of all of $a$'s prime factors, then $\frac{n}{a}$ is the product of all elements of the set $P \setminus X$, where $P$ is the set of all of $n$'s prime factors. Thus, the least common multiple must be their product: $a \times \frac{n}{a} = n$.

$a \wedge \neg a = \gcd(a, \neg a) = 1$ because as we said in the previous paragraph, $a$ and $\frac{n}{a}$ don't share any prime factors.

Claim: the prime divisors of $n$ are the atoms of $(A, \leq)$.
Proof: For each prime divisor $p$ of $n$, there's no smaller non-zero element because any element $s \in A$ such that $s < p$ means $s \neq p$ and $s$ divides $p$, and because of the definition of prime, whose only positive divisors are $1$ and $p$, $s$ has to be $1$, which is the zero of $A$.

Claim: the theorem given earlier in the notes is the unique factorization theorem for $n$.
Proof: The theorem given earlier is, "in a finite Boolean algebra, each element is the lub of a (unique) finite set of atoms". $(A, \leq)$ is finite. $n$ is the product of all of its prime factors, i.e. if we define $S$ to be the set of all atoms of $(A, \leq)$ (i.e. all prime factors of $n$), then $n = \prod_{s \in S} s$. This is equal to $\mathrm{lcm}_{s \in S}\, s$, or $\bigvee_{s \in S} s$.

## Exercise 46

Claim: the Boolean algebra of finite unions of left closed right open intervals in the rationals is atomless
Proof: We already proved this in the last homework, in Exercise 40 part b.

Claim: the Boolean algebra of finite unions of left closed right open intervals in the rationals is not complete
Proof: The least upper bound of all left closed right open intervals $[n, n+1[$ where $\exists k \in \mathbb{N}$ such that $n = 2k$ (i.e. left closed right open intervals that start at an even number and are of length one) does not exist because that would be the union of all of them, but there are an infinite number of them, so it would be an infinite union, but the set we're dealing with only has finite unions.

## Exercise 51

a) There's at least one element in $A$ between $v_0$ and $v_1$, and $v_0$ is less than $v_1$
b) $v_0$ is not strictly less than $v_1$
c) There's at least one element $v_2$ in $A$ such that $v_0$ is strictly less than $v_1$ and that $v_2$ is greater than $v_1$. Though, if $v_0$ is less than $v_1$ for one element in $A$, then it's true

in all instances, because the statement $v_0 < v_1$ has nothing to do with $v_2$. So, the statement can be reduced to $\boxed{v_0 \text{ is less than } v_1 \text{ and there's at least one element in } A \text{ greater than } v_1}$.

d) :

   a. There's at least one non-negative integer between the two non-negative integers $v_0$ and $v_1$, where $v_0 < v_1$
   b. The non-negative integer $v_0$ is not strictly less than the non-negative integer $v_1$
   c. The non-negative integer $v_0$ is less than the non-negative integer $v_1$ and at least one non-negative integer is greater than $v_1$, but every non-negative integer is less than some other non-negative integer, so, this statement can be reduced to $\boxed{\text{the non-negative integer } v_0 \text{ is less than the non-negative integer } v_1}$

e) :

   a. There's at least one subset $S \subseteq \{a, b, c\}$ such that it properly includes subset $v_0$ and is properly included by subset $v_1$
   b. Subset $v_0$ is not properly included by subset $v_1$
   c. Subset $v_0$ is properly included by subset $v_1$, and there's at least one subset $S \subseteq \{a, b, c\}$ such that it properly includes $v_1$

f) :

   a. there's at least one positive divisor of $30$ that's not $v_0$ or $v_1$ (who are also positive divisors of $30$), and it is divisible by $v_0$ and is a divisor of $v_1$
   b. $v_0$ is not a divisor of $v_1$, and both are positive divisors of $30$
   c. $v_0$ does not equal $v_1$, and $v_0$ is a divisor of $v_1$, and there's at least one positive divisor of $30$ that's not $v_0$ or $v_1$ such that it is divisible by $v_1$

## Exercise 52

a) First direction: assuming $(\forall v_i)\big(\phi(v_i)\big)$, want to prove $\neg(\exists v_i)\big(\neg\phi(v_i)\big)$.
Consider an arbitrary sequence $x \in A^\omega$, then $x$ satisfies $(\forall v_i)\big(\phi(v_i)\big)$ by assumption. Since it satisfies it, it means $\forall y \in A^\omega$ such that $(\forall j \neq i)\big(y_j = x_j\big)$, $y$ satisfies $\phi(v_i)$.

We prove by structural induction on the length of the formula $\phi(v_i)$. The inductive hypothesis $P(\psi)$ will mean "if $(\forall v_i)(\psi)$, then $\neg(\exists v_i)(\neg\psi)$".

Base case:
In the base case, the least complex $\phi(v_i)$ will ever be is an atomic formula $a$, which means for all variables, the atomic formula $a$ is true. Then that means isn't a formula for which $a$ is ever false: $\neg(\exists v_i)(\neg a) = \neg(\exists v_i)\big(\neg\phi(v_i)\big)$.

Inductive case:
Since $\phi(v_i)$ is a long formula made up of shorter formulas glued together via operators, there's several possibilities we need to examine:

1. $\phi(v_i) = \neg w$ for some formula $w$. The structural inductive hypothesis in this case is
$$\text{if } (\forall v_i)(w), \text{ then } \neg(\exists v_i)(\neg w)$$
Well, we just add a negation sign in front of the two appearances of $w$ here, so it becomes
$$\text{if } (\forall v_i)(\neg w), \text{ then } \neg(\exists v_i)(\neg\neg w)$$
And if we take out the double negation,
$$\text{"if } (\forall v_i)(\neg w), \text{ then } \neg(\exists v_i)(w)\text{"}$$
Which is
$$\text{if } (\forall v_i)(\phi(v_i)), \text{ then } \neg(\exists v_i)\big(\neg\phi(v_i)\big)$$

2. $\phi(v_i) = \alpha \vee \beta$ for some formulas $\alpha$ and $\beta$. The two structural inductive hypotheses in this case are
$$\text{if } (\forall v_i)(\alpha), \text{ then } \neg(\exists v_i)(\neg\alpha)$$
And
$$\text{if } (\forall v_i)(\beta), \text{ then } \neg(\exists v_i)(\neg\beta)$$
$\alpha$ and $\beta$ are joined together in the "if" by $\vee$, so their negations are joined together by $\wedge$ in the "then" due to de Morgan's laws:
$$\text{If } (\forall v_i)(\alpha \vee \beta), \text{ then } \neg(\exists v_i)(\neg\alpha \wedge \neg\beta)$$
Which is
$$\text{if } (\forall v_i)(\phi(v_i)), \text{ then } \neg(\exists v_i)\big(\neg\phi(v_i)\big)$$

3. $\phi(v_i) = \alpha \wedge \beta$ for some formulas $\alpha$ and $\beta$. The two structural inductive hypotheses are the same as the previous one. And since the two formulas are this time joined together in the "if" by $\wedge$, their negations are joined together by $\vee$ in the "then", again due to de Morgan's:
$$\text{If } (\forall v_i)(\alpha \wedge \beta), \text{ then } \neg(\exists v_i)(\neg\alpha \vee \neg\beta)$$
Which is
$$\text{if } (\forall v_i)(\phi(v_i)), \text{ then } \neg(\exists v_i)\big(\neg\phi(v_i)\big)$$

4. $\phi(v_i) = (\exists v_j)(\alpha)$ for some variable $v_j$ and some formula $\alpha$. I don't think we should consider this case because $(\exists v_j)(\alpha)$ is not shorter than $\phi(v_i)$, it's the same length, so we can't break it down into several smaller parts like we did with $\alpha \vee \beta$, for example.
5. $\phi(v_i) = (\forall v_j)(\alpha)$ for some variable $v_j$ and some formula $\alpha$. I don't think we should consider this case either for the same reason as the previous one.

Other direction: assume $\neg(\exists v_i)\big(\neg\phi(v_i)\big)$, want to prove $(\forall v_i)\big(\phi(v_i)\big)$.
Structural inductive hypothesis: "if $\neg(\exists v_i)(\neg\psi)$, then $(\forall v_i)(\psi)$"

Base case:
$\phi(v_i)$ is an atomic formula $a$, which means "there's no variable for which $a$ is false", which is equivalent to "for all variables, $a$ is true", i.e. $(\forall v_i)(a) = (\forall v_i)\big(\phi(v_i)\big)$.

Inductive case:

1. $\phi(v_i) = \neg w$ for some formula $w$. The structural induction hypothesis in this case is

$$\text{if } \neg(\exists v_i)(\neg w)\text{, then } (\forall v_i)(w)$$

Well, if we add a negation in front of each of the two appearances of $w$, it's

$$\text{if } \neg(\exists v_i)(\neg\neg w)\text{, then } (\forall v_i)(\neg w)$$

Taking out the double negation,

$$\text{if } \neg(\exists v_i)(w)\text{, then } (\forall v_i)(\neg w)$$

Which is

$$\text{if } \neg(\exists v_i)\big(\neg\phi(v_i)\big)\text{, then } (\forall v_i)\big(\phi(v_i)\big)$$

2. $\phi(v_i) = \alpha \vee \beta$ for some formulas $\alpha$ and $\beta$. Then the two structural induction hypotheses in this case are

$$\text{if } \neg(\exists v_i)(\neg\alpha)\text{, then } (\forall v_i)(\alpha)$$

and

$$\text{if } \neg(\exists v_i)(\neg\beta)\text{, then } (\forall v_i)(\beta)$$

$\alpha$ and $\beta$ are joined together in the "if" by $\vee$, so their negations are joined together by $\wedge$ in the "then" due to de Morgan's laws:

$$\text{If } \neg(\exists v_i)(\alpha \vee \beta)\text{, then } (\forall v_i)(\neg\alpha \wedge \neg b)$$

Which is

$$\text{if } \neg(\exists v_i)\big(\neg\phi(v_i)\big)\text{, then } (\forall v_i)\big(\phi(v_i)\big)$$

3. $\phi(v_i) = \alpha \wedge \beta$ for some formulas $\alpha$ and $\beta$. The two structural induction hypotheses in this case are the same as in the previous case. $\alpha$ and $\beta$ are joined together in the "if" by $\wedge$, so their negations are joined together by $\vee$ in the "then" due to de Morgan's laws:

$$\text{If } \neg(\exists v_i)(\alpha \wedge \beta)\text{, then } (\forall v_i)(\neg\alpha \vee \neg b)$$

Which is

$$\text{if } \neg(\exists v_i)\big(\neg\phi(v_i)\big)\text{, then } (\forall v_i)\big(\phi(v_i)\big)$$

QED.

b) Assume $(\exists v_i)\big(\phi(v_i)\big)$, want to prove $\neg(\forall v_i)\big(\neg\phi(v_i)\big)$.
Structural induction hypothesis: "if $(\exists v_i)(\psi)$, then $\neg(\forall v_i)(\neg\psi)$"

Base case:
$\phi(v_i)$ is an atomic formula $a$, which means there's at least one variable for which $a$ is true, which means it's not the case that for all variables $a$ is false, i.e. $\neg(\forall v_i)(\neg a) = \neg(\forall v_i)\big(\neg\phi(v_i)\big)$.

Inductive case:

1. $\phi(v_i) = \neg w$ for some formula $w$. The structural inductive hypothesis here is
$$\text{If } (\exists v_i)(w), \text{ then } \neg(\forall v_i)(\neg w)$$
Again, we put a negation in front of every instance of $w$:
$$\text{If } (\exists v_i)(\neg w), \text{ then } \neg(\forall v_i)(\neg\neg w)$$
Get rid of the double negative:
$$\text{If } (\exists v_i)(\neg w), \text{ then } \neg(\forall v_i)(w)$$
Now it's equivalent to
$$\text{If } (\exists v_i)\big(\phi(v_i)\big), \text{ then } \neg(\forall v_i)(\neg\phi(v_i))$$

2. $\phi(v_i) = \alpha \vee \beta$ for some formulas $\alpha$ and $\beta$. The two structural inductive hypotheses here are
$$\text{If } (\exists v_i)(\alpha), \text{ then } \neg(\forall v_i)(\neg\alpha)$$
And
$$\text{If } (\exists v_i)(\beta), \text{ then } \neg(\forall v_i)(\neg\beta)$$
Combining them yields
$$\text{If } (\exists v_i)(\alpha \vee \beta), \text{ then } \neg(\forall v_i)(\neg\alpha \wedge \neg\beta)$$
Now it's equivalent to
$$\text{If } (\exists v_i)\big(\phi(v_i)\big), \text{ then } \neg(\forall v_i)(\neg\phi(v_i))$$
3. $\phi(v_i) = \alpha \wedge \beta$ for some formulas $\alpha$ and $\beta$. The same two structural inductive hypotheses as the previous case. Combining them this time yields
$$\text{If } (\exists v_i)(\alpha \wedge \beta), \text{ then } \neg(\forall v_i)(\neg\alpha \vee \neg\beta)$$
Which is equivalent to
$$\text{If } (\exists v_i)\big(\phi(v_i)\big), \text{ then } \neg(\forall v_i)(\neg\phi(v_i))$$

Other direction: assume $\neg(\forall v_i)\big(\neg\phi(v_i)\big)$, want to prove $(\exists v_i)\big(\phi(v_i)\big)$.
Structural inductive hypothesis: "if $\neg(\forall v_i)\big(\neg\phi(v_i)\big)$, then $(\exists v_i)\big(\phi(v_i)\big)$"

Base case:
$\phi(v_i)$ is an atomic formula $a$. So the assumption means "it's not the case that for every variable, $a$ is false", which means "there's at least one variable for which $a$ is true", i.e. $(\exists v_i)(a) = (\exists v_i)\big(\phi(v_i)\big)$

Inductive case:

1. $\phi(v_i) = \neg w$ for some formula $w$. The inductive hypothesis here:
$$\text{If } \neg(\forall v_i)(\neg w), \text{ then } (\exists v_i)(w)$$
Put a negation in front of each $w$:
$$\text{If } \neg(\forall v_i)(\neg\neg w), \text{ then } (\exists v_i)(\neg w)$$
Take out double negative:
$$\text{If } \neg(\forall v_i)(w), \text{ then } (\exists v_i)(\neg w)$$
Which is equivalent to
$$\text{If } \neg(\forall v_i)(\neg\phi(v_i)), \text{ then } (\exists v_i)\big(\phi(v_i)\big)$$

2. $\phi(v_i) = \alpha \vee \beta$. The induction hypotheses:

$$\text{If } \neg(\forall v_i)(\neg a)\text{, then } (\exists v_i)(a)$$

And

$$\text{If } \neg(\forall v_i)(\neg \beta)\text{, then } (\exists v_i)(\beta)$$

Combining them yields

$$\text{If } \neg(\forall v_i)(\alpha \vee \beta)\text{, then } (\exists v_i)(\neg a \wedge \neg b)$$

Which is equivalent to

$$\text{If } \neg(\forall v_i)(\neg \phi(v_i))\text{, then } (\exists v_i)\big(\phi(v_i)\big)$$

3. $\phi(v_i) = \alpha \wedge \beta$. Combining the same induction hypotheses from the previous case:

$$\text{If } \neg(\forall v_i)(\alpha \wedge \beta)\text{, then } (\exists v_i)(\neg a \vee \neg b)$$

Which is equivalent to

$$\text{If } \neg(\forall v_i)(\neg \phi(v_i))\text{, then } (\exists v_i)\big(\phi(v_i)\big)$$

QED.