Exercises 32, 34a, 39, 40, 41 from Boolean algebra notes

Exercise 32)

1. We have to show the Cartesian product of partially ordered sets and is reflexive, transitive, and antisymmetric.
2. Reflexive: want to prove .

We know that and are partial orders, so they must be reflexive, so

and

And by the definition of partial order on Cartesian products,

1. Transitive: want to prove and if and , then .

From the assumption and givens, we know that and and and , which means

and

And thus .

1. Antisymmetry: want to prove and , if and , then .

Once again, from the assumptions and the givens, this means and , and and , which means

and

Thus, .

This was the base case. We can prove that the Cartesian product of any number of partially ordered sets is also a partial order through induction.

1. To show that the Cartesian product of Boolean algebras and is also a Boolean algebra, we need to prove that there’s a meet and join for every two elements, that there’s a unit and zero, that there’s a complement for every element, and that it’s distributive,.
2. Want to prove and , .

Since and are Boolean algebras, then we know

and

Then obviously,

and

because for and , there’s no upper bound smaller than and no lower bound greater than . Let’ assume for the same of contradiction that there’s some least upper bound such that

Then that means either

Or

or both.

We know that there’s only one least upper bound in a Boolean algebra, so it must be that

and

But means strict inequality. We have reached a contradiction.

We can similarly prove our claim for the meet.

1. Want to prove there’s a zero and a unit in .

Once again, obviously,

And

because in there’s no upper or lower bound smaller than or greater than , respectively, because of the definition of partial orders on Cartesian products.

1. Want to prove and , .

We know that

and

So

By the definition of complement, we must prove that

Well,

And we also must prove that

Well,

1. Want to prove distributivity, i.e. that and , . Exercise 21 showed that proving one law essentially proves the other, so we won’t have to prove the other one too. Throughout the proof, we employ the definition of meet and join on two Cartesian product pairs, and the fact that and are distributive.

Left side:

Right side:

Left side = right side

This was the base case. We can prove that the Cartesian product of any number of Boolean algebras is also a Boolean algebra through induction.

Exercise 34a)

1. . This is just part 3 of this question but have and switched, and XOR is commutative. Thus:
2. .
3. . This is just part 5 of this question with and switched, and is commutative. So it’s .

Exercise 39) Want to prove that there doesn’t exist any smaller (by inclusion) non-zero elements than one-element subsets in a power set. The zero element in a power set is the null set. Any elements in a power set smaller than a one-element subset must be the null set.

Exercise 40)

* 1. We need to prove has a zero, a unit, is distributive, every element has a complement, every two elements have a meet and join.
     + The meet of two elements is
     + The join of two elements is
     + The zero is the empty interval
     + The unit is
     + It is distributive because because of the laws of set intersection and union
  2. Note that is infinite, so such that .

So, let’s break down the three cases for “every non-zero element of ” and for each one, show the disjoint union of two smaller non-zero elements:

* + - 1. It is a left closed right open interval . Then it can be decomposed into the disjoint union of , where
      2. It is a left closed interval . Then it can be decomposed into the disjoint union of , where and is a left closed interval
      3. It is a right open interval . Then it can be decomposed into the disjoint union of , where and is a right open interval

Thus, has no atoms.

Exercise 41)

alternative answer that I spent way too long on without knowing if it’s correct it’s probably really incorrect: We can prove is a Boolean algebra by proving it’s a Boolean subalgebra, that is, it’s closed under union, intersection, and complement. It’s already closed under union without proving anything because of the definition of being the set of all finite unions of left closed right open intervals.

* It’s closed under union, intersection, and complement because of these 6 exhaustive cases:
  + two left closed right open intervals , where
    - : this means there’s no overlap, then their union is just these two intervals, which is by definition in
    - : then their union is , which is itself a left closed right open interval, so it’s of course in
  + A left closed right open interval and left closed interval
    - * : this means there’s no overlap, then their union is just these two intervals, which is by definition in
      * : then their union is , which is itself a left closed interval, so it’s in
    - : then is completely within , so their union is just , which is already in
  + A left closed right open interval and right open interval
    - : then is completely within , so their union is just , which is already in
    - such that
      * : then their union is , which in itself is a right open interval, so it’s already in
      * : then there’s no overlap, then their union is just these two intervals, which is by definition in
  + Two left closed intervals and such that
    - : then their union is , which is already in
    - : then their union is , which is already in
  + A left closed interval and a right open interval
    - : then their union is by definition in
    - : this means there’s no overlap, so their union is just these two intervals combined, which is in
  + Two right open intervals and
    - : then their union is , which is already in the set
    - : then their union is