Exercises 32, 34a, 39, 40, 41 from Boolean algebra notes

Exercise 32)

1. We have to show the Cartesian product of partially ordered sets and is reflexive, transitive, and antisymmetric.
2. Reflexive: want to prove .

We know that and are partial orders, so they must be reflexive, so

and

And by the definition of partial order on Cartesian products,

1. Transitive: want to prove and if and , then .

From the assumption and givens, we know that and and and , which means

and

And thus .

1. Antisymmetry: want to prove and , if and , then .

Once again, from the assumptions and the givens, this means and , and and , which means

and

Thus, .

This was the base case. We can prove that the Cartesian product of any number of partially ordered sets is also a partial order through induction.

1. To show that the Cartesian product of Boolean algebras and is also a Boolean algebra, we need to prove that there’s a meet and join for every two elements, that there’s a unit and zero, that there’s a complement for every element, and that it’s distributive,.
2. Want to prove and , .

Since and are Boolean algebras, then we know

and

Then obviously,

and

because for and , there’s no upper bound smaller than and no lower bound greater than . Let’ assume for the same of contradiction that there’s some least upper bound such that

Then that means either

Or

or both.

We know that there’s only one least upper bound in a Boolean algebra, so it must be that

and

But means strict inequality. We have reached a contradiction.

We can similarly prove our claim for the meet.

1. Want to prove there’s a zero and a unit in .

Once again, obviously,

And

because in there’s no upper or lower bound smaller than or greater than , respectively, because of the definition of partial orders on Cartesian products.

1. Want to prove and , .

We know that

and

So

By the definition of complement, we must prove that

Well,

And we also must prove that

Well,

1. Want to prove distributivity, i.e. that and , . Exercise 21 showed that proving one law essentially proves the other, so we won’t have to prove the other one too. Throughout the proof, we employ the definition of meet and join on two Cartesian product pairs, and the fact that and are distributive.

Left side:

Right side:

Left side = right side

This was the base case. We can prove that the Cartesian product of any number of Boolean algebras is also a Boolean algebra through induction.

Exercise 34a)

1. . This is just part 3 of this question but have and switched, and XOR is commutative. Thus:
2. .
3. . This is just part 5 of this question with and switched, and is commutative. So it’s .