Exercises 32, 34a, 39, 40, 41 from Boolean algebra notes

Exercise 32)

1. We have to show the Cartesian product of partially ordered sets and is reflexive, transitive, and antisymmetric.
2. Reflexive: want to prove .

We know that and are partial orders, so they must be reflexive, so

and

And by the definition of partial order on Cartesian products,

1. Transitive: want to prove and if and , then .

From the assumption and givens, we know that and and and , which means

and

And thus .

1. Antisymmetry: want to prove and , if and , then .

Once again, from the assumptions and the givens, this means and , and and , which means

and

Thus, .

This was the base case. We can prove that the Cartesian product of any number of partially ordered sets is also a partial order through induction on the number of sets (and thus the number of elements in each tuple).

1. To show that the Cartesian product of Boolean algebras and is also a Boolean algebra, we need to prove that there’s a meet and join for every two elements, that there’s a unit and zero, that there’s a complement for every element, and that it’s distributive,.
2. Want to prove and , .

Since and are Boolean algebras, then we know

and

Then obviously,

and

because for and , there’s no upper bound smaller than and no lower bound greater than . Let’ assume for the same of contradiction that there’s some least upper bound such that

Then that means either

Or

or both.

We know that there’s only one least upper bound in a Boolean algebra, so it must be that

and

But means strict inequality. We have reached a contradiction.

We can similarly prove our claim for the meet.

1. Want to prove there’s a zero and a unit in .

Once again, obviously,

And

because in there’s no upper or lower bound smaller than or greater than , respectively, because of the definition of partial orders on Cartesian products.

1. Want to prove and , .

We know that

and

So

By the definition of complement, we must prove that

Well,

And we also must prove that

Well,

1. Want to prove distributivity, i.e. that and , . Exercise 21 showed that proving one law essentially proves the other, so we won’t have to prove the other one too. Throughout the proof, we employ the definition of meet and join on two Cartesian product pairs, and the fact that and are distributive.

Left side:

Right side:

Left side = right side

This was the base case. We can prove that the Cartesian product of any number of Boolean algebras is also a Boolean algebra through induction on the number of Boolean Algebras (and thus the number of elements in each tuple).

Exercise 34a)

1. . This is just part 3 of this question but have and switched, and XOR is commutative. Thus:
2. .
3. . This is just part 5 of this question with and switched, and is commutative. So it’s .

Exercise 39) Want to prove that there doesn’t exist any smaller (by inclusion) non-zero elements than one-element subsets in a power set. The zero element in a power set is the null set . The power set of any one-element subset is . And between and , only is strictly smaller than . So, any elements in a power set smaller than a one-element subset must only be the null set.

Exercise 40)

* 1. We need to prove has a zero, a unit, is distributive, every element has a complement, and that every two elements have a meet and join. Since this is a set, we use inclusion as the partial order and union and intersection as join and meet, respectively.
     + The meet of two elements is , which is also in because it’s either or a new left closed right open interval.
     + The join of two elements is , which is also in because of ’s definition of being the set of all finite unions of left closed right open intervals. The union of unions is still a union.
     + The zero is the empty interval because it lower-bounds everything in
     + The unit is because it upper-bounds everything in .
     + It is distributive because because of the distributive laws of set intersection and union.
  2. Note that is infinite, so such that .

So, let’s break down the three cases for “every non-zero element of ” and for each one, show the disjoint union of two smaller non-zero elements:

* + - 1. It is a left closed right open interval . Then it can be decomposed into the disjoint union of , where
      2. It is a left closed interval . Then it can be decomposed into the disjoint union of , where and is a left closed interval
      3. It is a right open interval . Then it can be decomposed into the disjoint union of , where and is a right open interval

Thus, has no atoms.

Exercise 41) For all three of these, we want to prove both directions because they’re if and only if statements.

1. Want to prove if that if a non-zero in a Boolean algebra is an atom, then , or equivalently, . Let’s consider cases of :
   1. : then is true
   2. Then has to be because is an atom, and by definition there’s nothing smaller than that isn’t Then because the join of with anything is

The other way around, we have to prove implies is an atom.

Overall, we want to prove

To prove this is true, we only have to prove that when is not an atom, then has to also be false (since is the only time an if-then statement evaluates to false). So, if is not an atom, then the following could happen: both and are much greater than atoms, and and .

1. Want to prove that if a non-zero in a Boolean algebra is an atom, then , equivalently,

Altogether, we need to prove is an atom, then

Let’s again consider by cases:

1. : then is true
2. : then is true
3. and : then and have to be because is an atom, and by definition there’s nothing smaller than that isn’t . Then , which is smaller than

Now we want to prove the other way, that

As with part a, we prove that when isn’t an atom, then is not always true. That is, it could be the case that . For example, this could happen: are far greater than atoms, and (which makes it by definition and ).

1. Want to prove that if is an atom, then

Let’s again consider by cases:

1. : then is true
2. : then is true
3. : then that means is , because is an atom, so , which could make either or true
4. : then that means is , so , which could make either or true.
5. : then , so so is true

As with the previous 2 parts of this exercise, we prove the other direction, that is,

Again, we prove that when isn’t an atom, then could be false. We look at one specific case where this is false: are far greater than atoms, and and so .