Exercises 43, 46, 51, 52

# Exercise 43

The wording of the question makes me not sure what the question givens are. I asked a question (@10) on Piazza and at the time of writing this, it hasn’t been answered. I think the question givens are a partial order by divisibility, where is the set of positive integral divisors of any positive integer with no square factors , and that we need to show this is a Boolean algebra where the prime divisors are the atoms and that the theorem earlier is the unique factorization theorem for .

Claim: such that

Proof: The greatest lower bound between , i.e. , is the greatest common divisor of and , , because is ordered by divisibility, the greatest such that and is by definition the greatest such that divides and divides – the definition of . Such a exists because (not sure if I have to prove this but this is elementary school math) is the product of every element in the intersection of the prime factors of and of , which are included in because is the set of all the positive integer divisors of .

Claim: such that

Proof: The lowest upper bound between and , i.e. , is the least common multiple of and , , because is ordered by divisibility, the smallest such that and is by definition the smallest such that and both divide – the definition of . Such a exists because (again, elementary school math so I hope I don’t gotta prove it), is the product of every element in the union of the prime factors of and , which are included in because is the set of all the positive integer divisors of .

Claim: is the one (or unit) of

Proof: is the least upper bound of all the elements of because by definition of , every element in divides , so .

Claim: is the zero of

Proof: is the greatest lower bound of all the elements of because divides every positive integer, so it divides everything in , so , .

Claim: is distributive, i.e. ,

Proof: . Let be the set of all prime factors of (we know we aren’t “losing” any due to repetition because it is given that has no square factors ). Let be the set of all prime factors of . Let be the set of all prime factors of . .

Claim: if , then and

Proof: , because firstly, divides both and , and secondly, and don’t share any prime factors (i.e. the intersection between their prime factors is the empty set) because of the fact that doesn’t have any square factors ; so, if is the set of all of ’s prime factors, then is the product of all elements of the set , where is the set of all of ’s prime factors. Thus, the least common multiple must be their product: .

because as we said in the previous paragraph, and don’t share any prime factors.

Claim: the prime divisors of are the atoms of .

Proof: For each prime divisor of , there’s no smaller non-zero element because any element such that means and divides , and because of the definition of prime, whose only positive divisors are and , has to be , which is the zero of .

Claim: the theorem given earlier in the notes is the unique factorization theorem for .

Proof: The theorem given earlier is, “in a finite Boolean algebra, each element is the lub of a (unique) finite set of atoms”. is finite. is the product of all of its prime factors, i.e. if we define to be the set of all atoms of (i.e. all prime factors of ), then . This is equal to , or .