Exercises 43, 46, 51, 52

# Exercise 43

Claim: such that

Proof: The greatest lower bound between , i.e. , is the greatest common divisor of and , , because is ordered by divisibility, the greatest such that and is by definition the greatest such that divides and divides – the definition of . Such a exists because (not sure if I have to prove this but this is elementary school math) is the product of every element in the intersection of the prime factors of and of , which are included in because is the set of all the positive integer divisors of .

Claim: such that

Proof: The lowest upper bound between and , i.e. , is the least common multiple of and , , because is ordered by divisibility, the smallest such that and is by definition the smallest such that and both divide – the definition of . Such a exists because (again, elementary school math so I hope I don’t gotta prove it), is the product of every element in the union of the prime factors of and , which are included in because is the set of all the positive integer divisors of .

Claim: is the one (or unit) of

Proof: is the least upper bound of all the elements of because by definition of , every element in divides , so .

Claim: is the zero of

Proof: is the greatest lower bound of all the elements of because divides every positive integer, so it divides everything in , so , .

Claim: is distributive, i.e. ,

Proof: . Let be the set of all prime factors of (we know we aren’t “losing” any due to repetition because it is given that has no square factors ). Let be the set of all prime factors of . Let be the set of all prime factors of . .

Claim: if , then and

Proof: , because firstly, divides both and , and secondly, and don’t share any prime factors (i.e. the intersection between their prime factors is the empty set) because of the fact that doesn’t have any square factors ; so, if is the set of all of ’s prime factors, then is the product of all elements of the set , where is the set of all of ’s prime factors. Thus, the least common multiple must be their product: .

because as we said in the previous paragraph, and don’t share any prime factors.

Claim: the prime divisors of are the atoms of .

Proof: For each prime divisor of , there’s no smaller non-zero element because any element such that means and divides , and because of the definition of prime, whose only positive divisors are and , has to be , which is the zero of .

Claim: the theorem given earlier in the notes is the unique factorization theorem for .

Proof: The theorem given earlier is, “in a finite Boolean algebra, each element is the lub of a (unique) finite set of atoms”. is finite. is the product of all of its prime factors, i.e. if we define to be the set of all atoms of (i.e. all prime factors of ), then . This is equal to , or .

# Exercise 46

Claim: the Boolean algebra of finite unions of left closed right open intervals in the rationals is atomless

Proof: We already proved this in the last homework, in Exercise 40 part b.

Claim: the Boolean algebra of finite unions of left closed right open intervals in the rationals is not complete

Proof: The least upper bound of all left closed right open intervals where such that (i.e. left closed right open intervals that start at an even number and are of length one) does not exist because that would be the union of all of them, but there are an infinite number of them, so it would be an infinite union, but the set we’re dealing with only has finite unions.

# Exercise 51

1. There’s at least one element in between and , and is less than
2. is not strictly less than
3. There’s at least one element in such that is strictly less than and that is greater than . Though, if is less than for one element in , then it’s true in all instances, because the statement has nothing to do with . So, the statement can be reduced to is less than and there’s at least one element in greater than .
4. :
   1. There’s at least one non-negative integer between the two non-negative integers and , where
   2. The non-negative integer is not strictly less than the non-negative integer
   3. The non-negative integer is less than the non-negative integer and at least one non-negative integer is greater than , but every non-negative integer is less than some other non-negative integer, so, this statement can be reduced to the non-negative integer is less than the non-negative integer
5. :
   1. There’s at least one subset such that it properly includes subset and is properly included by subset
   2. Subset is not properly included by subset
   3. Subset is properly included by subset , and there’s at least one subset such that it properly includes
6. :
   1. there’s at least one positive divisor of that’s not or (who are also positive divisors of ), and it is divisible by and is a divisor of
   2. is not a divisor of , and both are positive divisors of
   3. does not equal , and is a divisor of , and there’s at least one positive divisor of that’s not or such that it is divisible by

# Exercise 52

1. First direction: assuming , want to prove .

Consider an arbitrary sequence , then satisfies by assumption. Since it satisfies it, it means such that , satisfies .

We prove by structural induction on the length of the formula . The inductive hypothesis will mean “if , then ”.

Base case:

In the base case, the least complex will ever be is an atomic formula , which means for all variables, the atomic formula is true. Then that means isn’t a formula for which is ever false: .

Inductive case:

Since is a long formula made up of shorter formulas glued together via operators, there’s several possibilities we need to examine:

* + 1. for some formula . The structural inductive hypothesis in this case is

if , then

Well, we just add a negation sign in front of the two appearances of here, so it becomes

if , then

And if we take out the double negation,

“if , then ”

Which is

if , then

* + 1. for some formulas and . The two structural inductive hypotheses in this case are

if , then

And

if , then

and are joined together in the “if” by , so their negations are joined together by in the “then” due to de Morgan’s laws:

If , then

Which is

if , then

* + 1. for some formulas and . The two structural inductive hypotheses are the same as the previous one. And since the two formulas are this time joined together in the “if” by , their negations are joined together by in the “then”, again due to de Morgan’s:

If , then

Which is

if , then

* + 1. for some variable and some formula . I don’t think we should consider this case because is not shorter than , it’s the same length, so we can’t break it down into several smaller parts like we did with , for example.
    2. for some variable and some formula . I don’t think we should consider this case either for the same reason as the previous one.

Other direction: assume , want to prove .

Structural inductive hypothesis: “if , then ”

Base case:

is an atomic formula , which means “there’s no variable for which is false”, which is equivalent to “for all variables, is true”, i.e. .

Inductive case:

1. for some formula . The structural induction hypothesis in this case is

if , then

Well, if we add a negation in front of each of the two appearances of , it’s

if , then

Taking out the double negation,

if , then

Which is

if , then

1. for some formulas and . Then the two structural induction hypotheses in this case are

if , then

and

if , then

and are joined together in the “if” by , so their negations are joined together by in the “then” due to de Morgan’s laws:

If , then

Which is

if , then

1. for some formulas and . The two structural induction hypotheses in this case are the same as in the previous case. and are joined together in the “if” by , so their negations are joined together by in the “then” due to de Morgan’s laws:

If , then

Which is

if , then

QED.

1. Assume , want to prove .

Structural induction hypothesis: “if , then ”

Base case:

is an atomic formula , which means there’s at least one variable for which is true, which means it’s not the case that for all variables is false, i.e. .

Inductive case:

1. for some formula . The structural inductive hypothesis here is

If , then

Again, we put a negation in front of every instance of :

If , then

Get rid of the double negative:

If , then

Now it’s equivalent to

If , then

1. for some formulas and . The two structural inductive hypotheses here are

If , then

And

If , then

Combining them yields

If , then

Now it’s equivalent to

If , then

1. for some formulas and . The same two structural inductive hypotheses as the previous case. Combining them this time yields

If , then

Which is equivalent to

If , then

Other direction: assume , want to prove .

Structural inductive hypothesis: “if , then ”

Base case:

is an atomic formula . So the assumption means “it’s not the case that for every variable, is false”, which means “there’s at least one variable for which is true”, i.e.

Inductive case:

1. for some formula . The inductive hypothesis here:

If , then

Put a negation in front of each :

If , then

Take out double negative:

If , then

Which is equivalent to

If , then

1. . The induction hypotheses:

If , then

And

If , then

Combining them yields

If , then

Which is equivalent to

If , then

1. . Combining the same induction hypotheses from the previous case:

If , then

Which is equivalent to

If , then

QED.