Exercises 43, 46, 51, 52

# Exercise 43

Claim: such that

Proof: The greatest lower bound between , i.e. , is the greatest common divisor of and , , because is ordered by divisibility, the greatest such that and is by definition the greatest such that divides and divides – the definition of . Such a exists because (not sure if I have to prove this but this is elementary school math) is the product of every element in the intersection of the prime factors of and of , which are included in because is the set of all the positive integer divisors of .

Claim: such that

Proof: The lowest upper bound between and , i.e. , is the least common multiple of and , , because is ordered by divisibility, the smallest such that and is by definition the smallest such that and both divide – the definition of . Such a exists because (again, elementary school math so I hope I don’t gotta prove it), is the product of every element in the union of the prime factors of and , which are included in because is the set of all the positive integer divisors of .

Claim: is the one (or unit) of

Proof: is the least upper bound of all the elements of because by definition of , every element in divides , so .

Claim: is the zero of

Proof: is the greatest lower bound of all the elements of because divides every positive integer, so it divides everything in , so , .

Claim: is distributive, i.e. ,

Proof: . Let be the set of all prime factors of (we know we aren’t “losing” any due to repetition because it is given that has no square factors ). Let be the set of all prime factors of . Let be the set of all prime factors of . .

Claim: if , then and

Proof: , because firstly, divides both and , and secondly, and don’t share any prime factors (i.e. the intersection between their prime factors is the empty set) because of the fact that doesn’t have any square factors ; so, if is the set of all of ’s prime factors, then is the product of all elements of the set , where is the set of all of ’s prime factors. Thus, the least common multiple must be their product: .

because as we said in the previous paragraph, and don’t share any prime factors.

Claim: the prime divisors of are the atoms of .

Proof: For each prime divisor of , there’s no smaller non-zero element because any element such that means and divides , and because of the definition of prime, whose only positive divisors are and , has to be , which is the zero of .

Claim: the theorem given earlier in the notes is the unique factorization theorem for .

Proof: The theorem given earlier is, “in a finite Boolean algebra, each element is the lub of a (unique) finite set of atoms”. is finite. is the product of all of its prime factors, i.e. if we define to be the set of all atoms of (i.e. all prime factors of ), then . This is equal to , or .

# Exercise 46

Claim: the Boolean algebra of finite unions of left closed right open intervals in the rationals is atomless

Proof: We already proved this in the last homework, in Exercise 40 part b.

Claim: the Boolean algebra of finite unions of left closed right open intervals in the rationals is not complete

Proof: The least upper bound of all left closed right open intervals where such that (i.e. left closed right open intervals that start at an even number and are of length one) does not exist because that would be the union of all of them, but there are an infinite number of them, so it would be an infinite union, but the set we’re dealing with only has finite unions.

# Exercise 51

1. There’s at least one element in between and , and is less than
2. is not strictly less than
3. There’s at least one element in such that is strictly less than and that is greater than . Though, if is less than for one element in , then it’s true in all instances, because the statement has nothing to do with . So, the statement can be reduced to is less than and there’s at least one element in greater than .
4. :
   1. There’s at least one non-negative integer between the two non-negative integers and , where
   2. The non-negative integer is not strictly less than the non-negative integer
   3. The non-negative integer is less than the non-negative integer and at least one non-negative integer is greater than , but every non-negative integer is less than some other non-negative integer, so, this statement can be reduced to the non-negative integer is less than the non-negative integer
5. :
   1. There’s at least one subset such that it properly includes subset and is properly included by subset
   2. Subset is not properly included by subset
   3. Subset is properly included by subset , and there’s at least one subset such that it properly includes
6. :
   1. there’s at least one positive divisor of that’s not or (who are also positive divisors of ), and it is divisible by and is a divisor of
   2. is not a divisor of , and both are positive divisors of
   3. does not equal , and is a divisor of , and there’s at least one positive divisor of that’s not or such that it is divisible by

# Exercise 52

1. Assume , want to prove . Consider an arbitrary sequence , then satisfies by assumption. Since it satisfies it, it also means we can choose a sequence such that and satisfies for all (clauses and ). If we do this inductively and start from , then, Then when you move onto , you have . We can see from this that . And thus becomes , or equivalently, . For to be true, has to not satisfy , that is, iff there’s no sequence such that and does not satisfy , and we just proved that all sequences such that satisfy !

Assume , want to prove .

Consider an arbitrary sequence , then satisfies by assumption. Since it satisfies it, it also means we cannot choose a sequence such that and does not satisfy, because such a doesn’t exist. For to be true, it means we can choose a sequence such that and satisfies for all . And since we just proved that there’s no such that that doesn’t satisfy , it means all such that satisfy .