1. .
   1. Von Neumann’s definition of ordinal: An *ordinal* is a set *y* that is transitive, connected, and well founded.
   2. Claim: the ordinals are well ordered by .

Proof: For a set to be well ordered, every non-empty subset has to have a least element, which in our case is an -minimal element. This is well foundedness. Consider any nonempty set of ordinals. Let be an arbitrary element of . If , then is an -minimal element of and we’re done; if not, we need to show has an -minimal element. We consider , where is the immediate successor of . We know that is also an ordinal (Proposition 9.1). We know that there’s an -minimal element in because is an ordinal, and we claim that is also the -minimal element in . So, we need to prove that , either or . Regarding and , we know that it’s one of three cases: , , or . If , then we know , so either or because is the -minimal element of . For other two cases of if or , then due to transitivity of an ordinal, and because , it is so that . QED.

Claim: the ordinals are not a set.

Proof: Assume for the sake of contradiction that there were a set of all ordinals, then would be an ordinal itself (Theorem 8.5), so , violating the well foundedness of ordinals, which is a contradiction.

1. .
   1. Axiom of choice: For every set of nonempty sets there is a function with domain such that , .
   2. The 1-1 mapping , where is an ordinal and is any set. Let us set to some set not in . By the Axiom of Choice, there is a choice function (where is the power set of ) such that, for all nonempty , . The functional property will be defined inductively (transfinite induction).

Base case: .

Inductive case: Suppose has been defined for all If is empty, then ; otherwise . It is clear that , there exists at most one ordinal such that . Thus, is 1-1.

1. Consider if ’s domain was the collection of all ordinals, then would be a 1-1 mapping from the collection of all ordinals into . If there were no such that , then (the inverse of ) would be a mapping from into the collection of all ordinals, making the collection of all ordinals a set by the Replacement Axiom, contradicting what we proved in question 1 part b.
2. Thus, there must exist an who is the least ordinal such that . We already proved in question 2 that (restriction of to ) is 1-1. We now claim that ’s range is all of as required to prove the initial claim that every set is well orderable. If ’s range does not include all of (assuming that for the sake of contradiction), then is not empty, and according to how we defined earlier, , which is an element of , a contradiction.

Since there’s some ordinal mapped 1-1 onto , and ordinals are well ordered can be well ordered. QED.