CMSC 5718 INTRODUCTION TO COMPUTATIONAL FINANCE

Lecture 9

Outline

- Hedging an equity portfolio with futures contract
- Hedging the risks of an option position
- Reference: John Hull, Options, Futures, and other Derivatives, Chaps. 3 and 17 (7th edition) or Chaps. 3 and 18 (8th edition)

Concept of hedging

- Definition: **no profit or loss** when certain parameters move ("the risks have been hedged")

- We need to refer to hedging against certain parameters, e.g. stock price, interest rates, (volatility?)
  - A portfolio could usually not be completely hedged; there would be some residual risks

- In order to make money from a position that would benefit from a certain view, the other risks are supposed to be hedged

Concept of hedging

- Optimal hedge of a portfolio
  - If we are not uncertain of the directional movement of the underlying variables, we may want to hedge the portfolio "approximately"
  - This strategy is often used in managing a portfolio where the aim is to achieve outperformance of the underlying basket against an index

- Optimal hedge of an option position
  - According to the pricing theory, the option price is equal to the hedging cost
  - We want to find the correct strategy so that the option position would result in no profit / loss
  - The "unhedged" position is the view taken by the trader

Hedging Using Index Futures

- To hedge the risk in a portfolio the number of contracts that should be shorted is

  $$\beta \frac{V_A}{V_F}$$

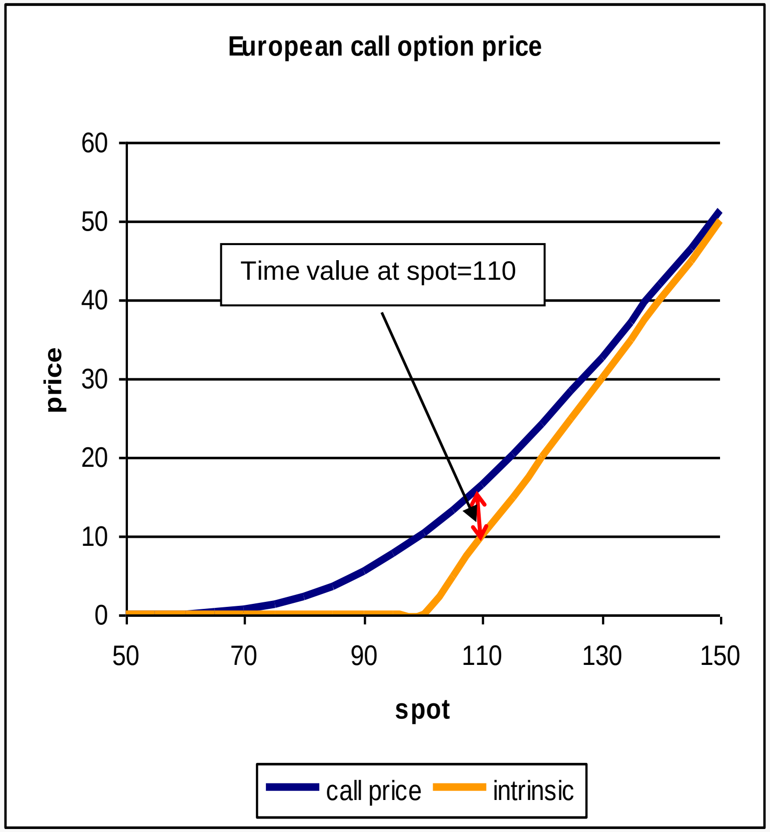
  where V_A is the value of the portfolio, β is its beta, and V_F is the value of one futures contract

- Note that β is only a statistical measure; even if we hedge with this ratio, in the short term, the futures contract may not move synchronously with the portfolio which would result in fluctuations in profit or loss

Example

- Hang Seng Index futures price is 23000, value of portfolio is HK$15 million, beta of portfolio is 1.5
- Each futures contract has a value of 23000 x $50 = $1150000 ($50 being the value of 1 point)
- The theoretical number of futures contracts to hedge the portfolio is thus: 1.5 x 15000000 / 1150000 = 19.56
- Since we can trade only a round number of contracts, 19 or 20 contracts would be acceptable

Option characteristics



Strike 100, maturity 3 months, r=2%, volatility 50%

- Note that the relationship between the option price and the underlying spot price is not linear
- We can calculate the sensitivities of the option price to various parameters, commonly known as option "Greeks"

Common option sensitivities (Greeks)

- Delta Δ = Change in option price / change in share price
- Vega ν = Change in option price / change in implied volatility
- Theta Θ = Change in option price / change in time to maturity
- Rho ρ = Change in option price / change in interest rate

- Gamma Γ = Change in delta / change in share price

- Shares are linear instruments, therefore only delta is non-zero (delta = 1); gamma=vega=rho=theta=0

- Options are sensitive to all the other parameters

Black-Scholes Greeks

- For a European call option on a non-dividend paying stock, under the Black-Scholes framework we have the following formulas:

$$\Delta = N(d_1)$$

$$\nu = S_0\sqrt{T}N'(d_1)$$

$$\Theta = -\frac{S_0N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

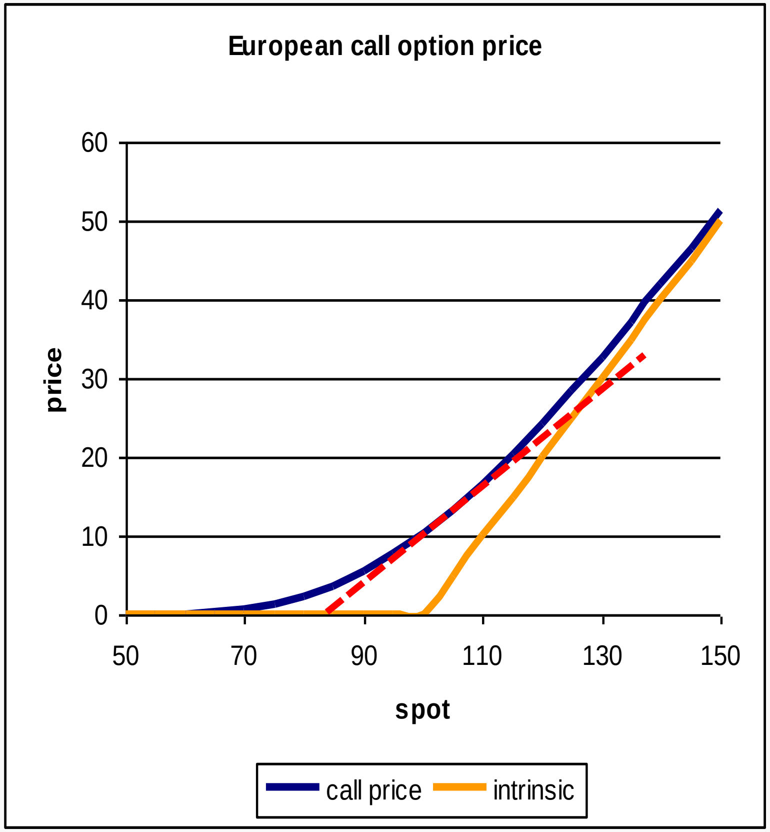
$$\rho = KTe^{-rT}N(d_2)$$

$$\Gamma = \frac{N'(d_1)}{S_0\sigma\sqrt{T}}$$

- Using the put-call parity $C - P = S_0 - Ke^{-rT}$

  we can easily obtain the corresponding formulas for a put

What is option delta?



Strike 100, maturity 3 months, r=2%, volatility 50%

- Delta
  - = sensitivity of option price to share price
  - = dP/dS where P is the option price and S is the underlying share price
- E.g. say delta = 0.74
- If share price moves up by \$1, dS = 1, therefore dP = 0.74 x 1 = \$0.74, i.e. option price would go up by \$0.74
- This is the slope of the tangent in the Option price–underlying price diagram
- Can be seen as the “share equivalent” of the option position

What is the Black-Sholes delta?

- Under the Black-Scholes framework, supposedly if we keep the portfolio to be delta neutral everyday, the final amount in our account is the same as the option price
  - Definition of "delta neutral": delta of the portfolio = 0
- A few strong assumptions have been made
  - Continuous re-balancing of the portfolio
  - Interest rate is constant and is known for sure
  - Volatility is constant and is known for sure
- If we use a wrong volatility to calculate delta, we may not be able to recover the original option premium even though other conditions have been satisfied
  - But we never know volatility for sure!

Example of delta calculation

- The delta of a European call option (Δ) is:

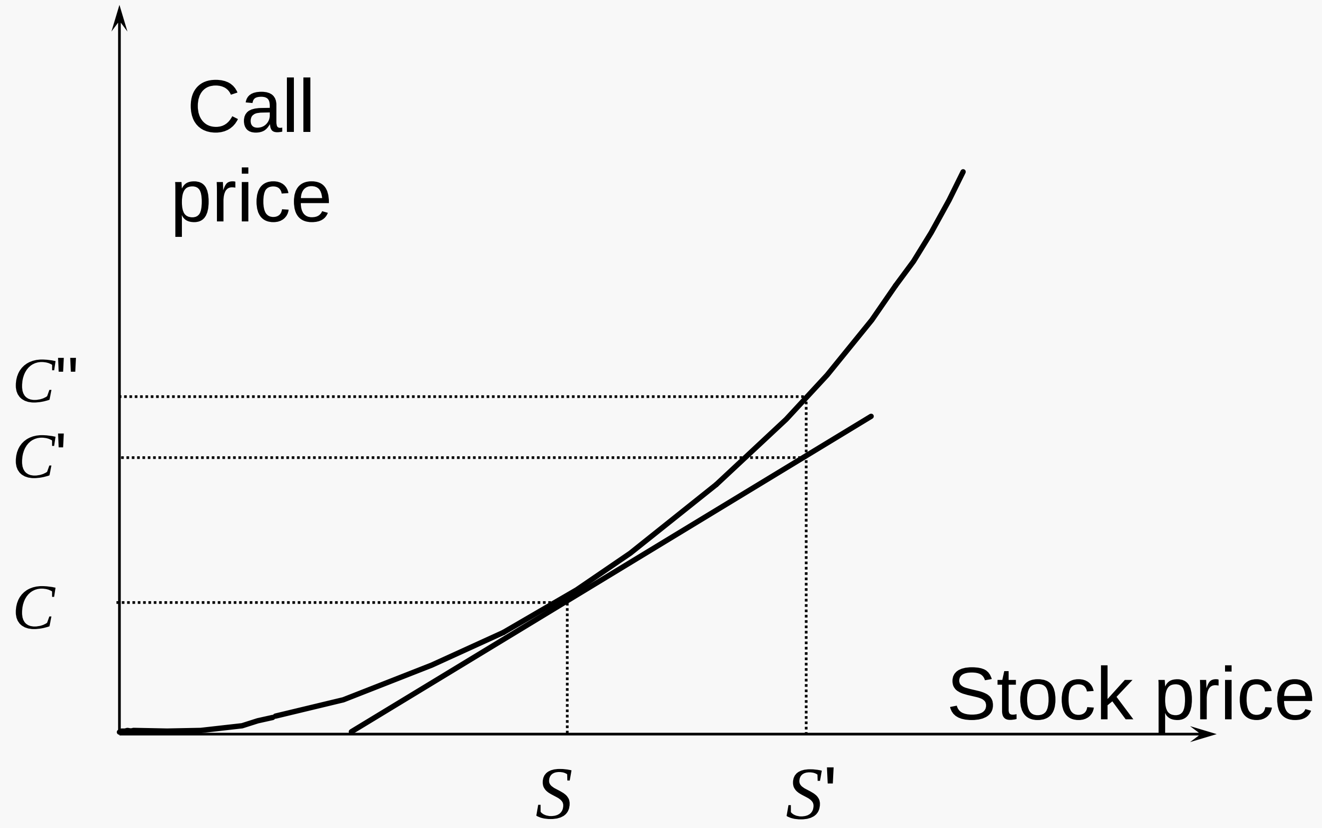
$$\Delta = \exp(-dt)N(d_1)$$

- For example, if spot price = 100, strike = 100, time = 1, volatility = 30%, r = 3%, d = 1%, we could calculate Δ = 0.5799
  - d is the dividend yield; note the notation as in Lecture 7, p.9

- When spot price moves to 120, Δ becomes 0.7872

- **Note that the delta for a vanilla option is always between -1 and 0 (for a put), 0 and 1 (for a call)**

The reason for Gamma



- Gamma measures the delta hedging errors caused by curvature

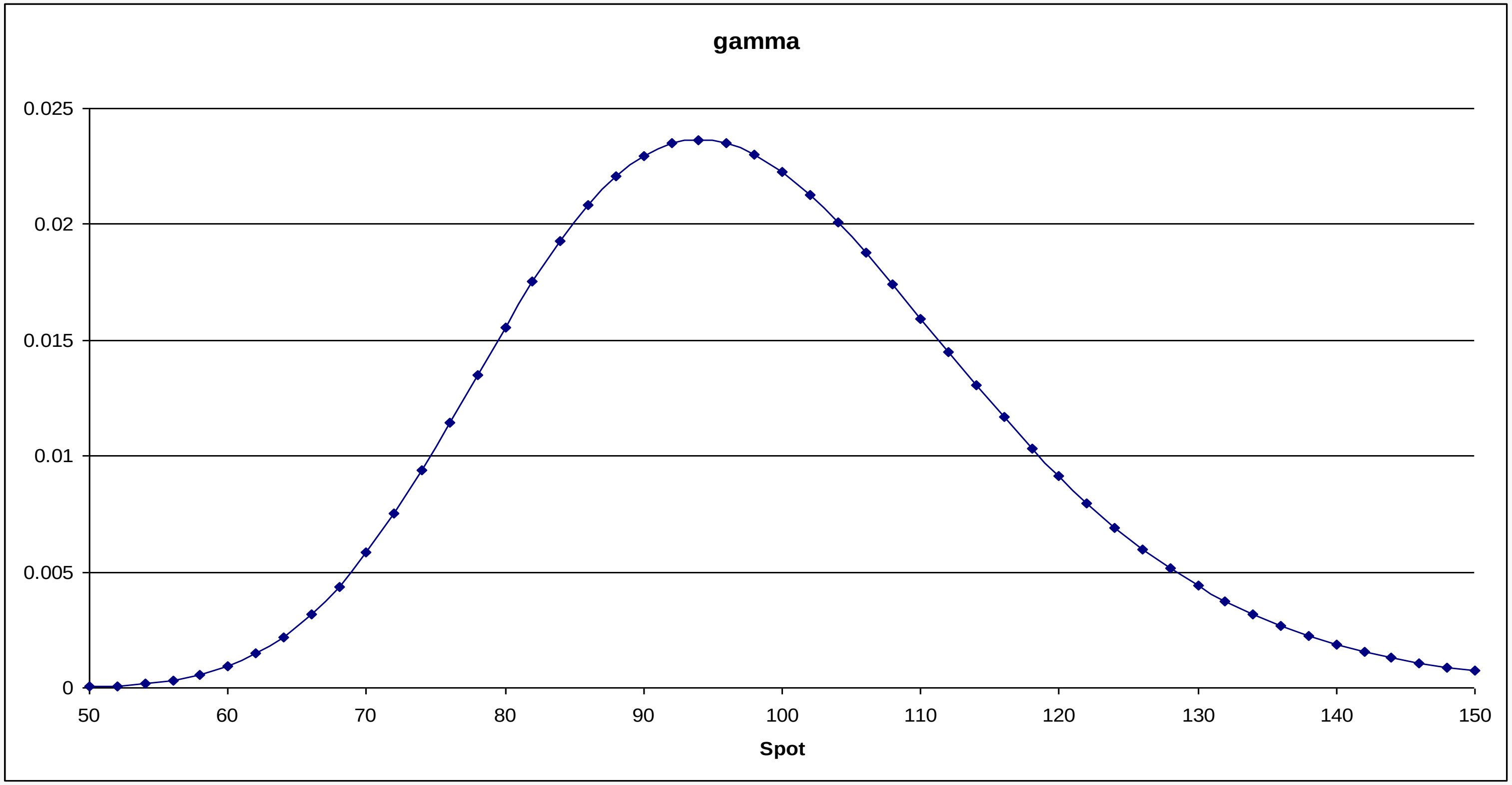
Gamma profile

Spot	Delta	Gamma
40	2.5	0.66
60	71.2	7.7
80	301.6	14.7
100	580.8	12.9
120	780.5	7.7
140	887.8	3.7
160	936.9	1.6

- Long 1000 European call option
- Maturity 1 year, interest rate 3%, strike 100, volatility 30%

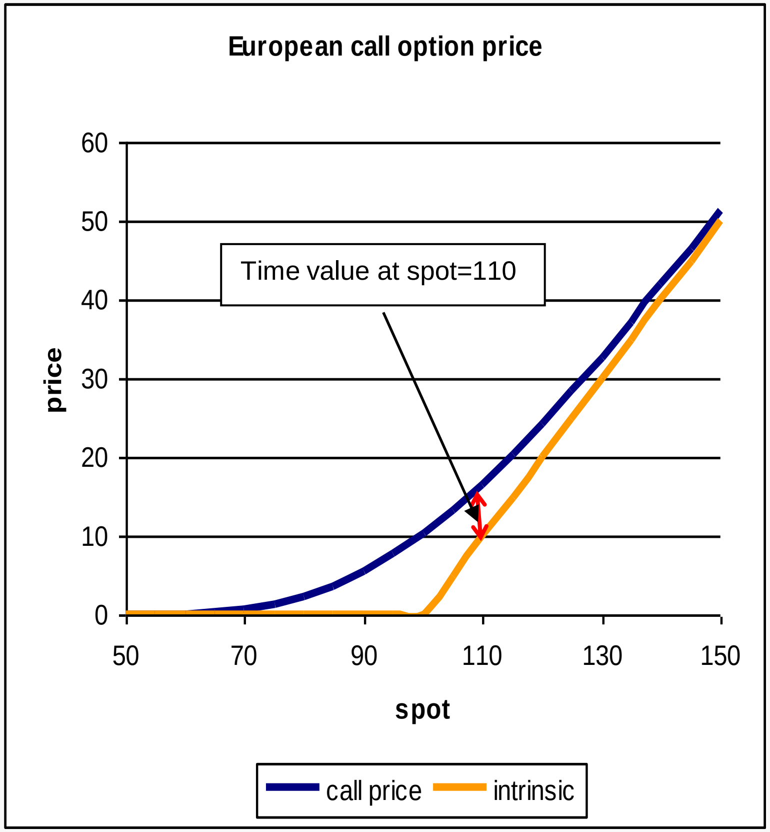
- When stock price goes up, delta becomes higher, i.e. the position is equivalent to holding more shares
- When stock price goes down, delta becomes smaller
- This is a desirable position to have, as it limits the downside risk but will fully participate in the upside
  - Higher delta means higher gain when stock price goes up, and vice versa
- There is a cost to enter into this position

Gamma profile



- European call, Maturity 6 months, volatility 25%, strike 100, rate 3%
- Gamma is highest when spot price is near to at-the-money

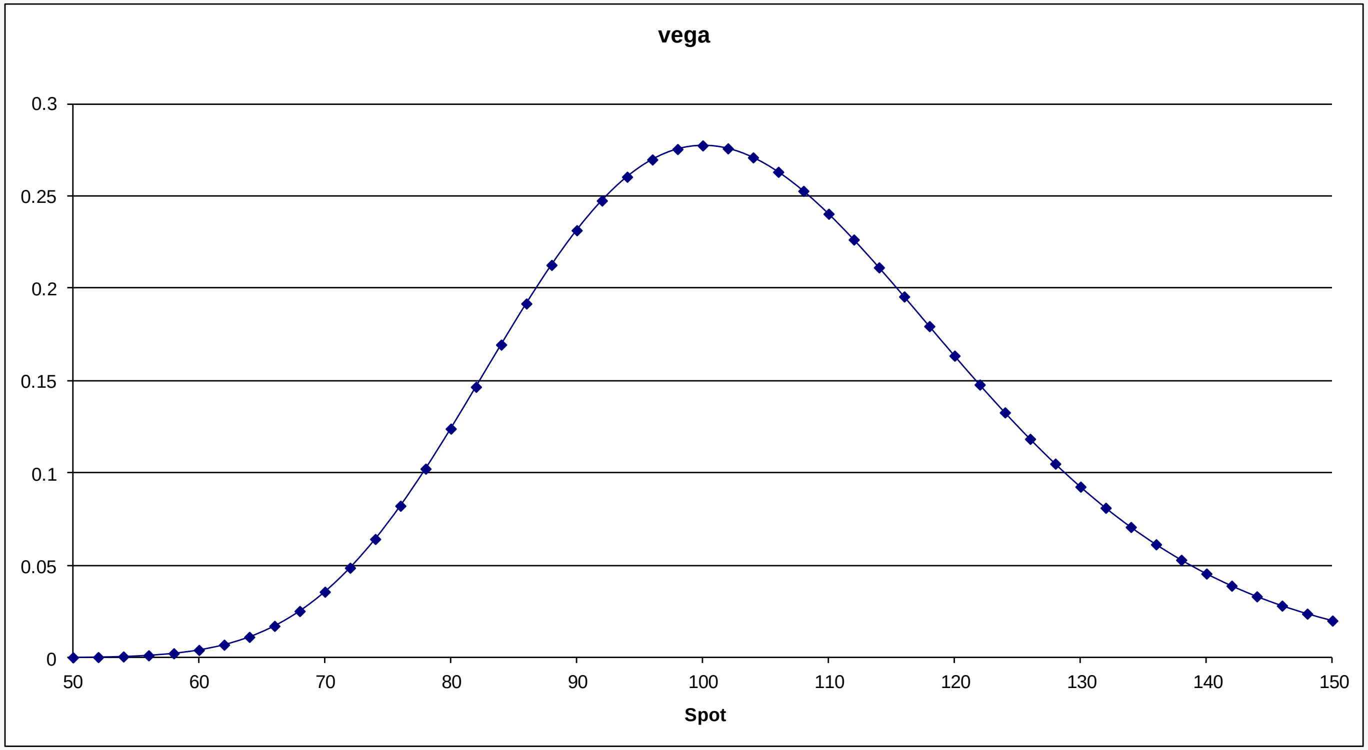
Why would options have vega/theta?



Strike 100, maturity 3 months, r=2%, volatility 50%

- Vega
  - Non-symmetric payoff pattern
  - If volatility increases, there is more chance of higher payoff, but the maximum downside remains to be 0 (when option is out-of-the-money), therefore leading to a higher option price
- Theta
  - The blue line would move towards the orange line as the option approaches maturity
  - Less chance of moving substantially beyond the current spot price
  - Reduction of the time value of the option

Vega profile



- European call, Maturity 6 months, volatility 25%, strike 100, rate 3%
- Vega is highest when spot price is near to at-the-money

Components of option theta

- On p.16, we note that an option has theta because the time value would change when the time to maturity is reduced
  - Less possibility of the spot price moving higher because of less time to maturity

- Another effect comes from the holding cost caused by the interest rate
  - This is known as funding cost

- Question: stock closing price yesterday was $100; today's close is $100. We are long 1000 shares. What is the P&L?
  - Funding cost issues

Components of option theta

- Example:
  - Spot price = 100, strike = 50, rate = 5%, volatility = 10%, dividend = 0, European call
  - Maturity 365 days, price = 52.3810 (from Black-Scholes model)
    - Roughly = {100 x (1+0.05) – 50 } / (1+0.05) = 100 – 50/1.05
  - Maturity 364 days, price = 52.3747 (from Black-Scholes model)
    - Roughly = 100 – 50/(1+0.05x364/365)
  - Therefore theta comes from the 1 day discounting effect of the strike, if option is deep-in-the-money

- Implication
  - *Long deep-in-the-money call, short shares at 100% delta*
  - *Would take a P&L hit everyday if funding cost is not taken into account*

Another look at theta

- A common definition of theta
  - What a trader would see as P&L between last night's close and first thing in the morning (before market opens)

- Let's say yesterday was January 25, 2017 (Weds) in Hong Kong and January 26 (Thurs) was not a holiday. How many days of theta (due to discounting) would we see?
  - In a trading system, the correct date schedule would be used, i.e. cashflows would be discounted from settlement day to settlement day
  - On Jan 25, assuming settlement is T+2, settlement day was Jan 27. On Jan 26, because Jan 30 and 31 are holidays in Hong Kong, the settlement day was Feb 1. If we use the example in the previous page, there is a 5-day theta between Jan 25 and 26!

Another look at theta

- Theta comes from two components
  - Change of forward, derived from the calendar days *between current date and maturity*
  - Change of discounting effect, derived from the calendar days *between settlement dates*

- In the example in the previous slide, the first effect would account for a 1-day theta move of the change of forward, but a 5-day move of the change of discounting effect

- Usually the first effect is more prominent, for options not deep-in-the-money

Components of option rho

- Look at Black (1976) again
- European call $C = \exp(-rt)\left[FN(d_1) - KN(d_2)\right]$

$$d_1 = \frac{\ln(F/K) + \sigma^2 t/2}{\sigma\sqrt{t}}, d_2 = d_1 - \sigma\sqrt{t}$$

- Two effects of r
  - Affect the calculation of the forward F
  - Affect the discounting of the payoff in $\exp(-rt)$

- If interest rate goes up (assume "parallel shift")
  - F goes higher
  - Discounting factor is smaller, but the impact is less
  - Therefore call price is more expensive

Vanilla option characteristics

	Long call	Long put	Short call	Short put
Delta	+	–	–	+
Gamma	+	+	–	–
Vega	+	+	–	–
Theta	–	–	+	+
Rho	+	–	–	+

- Note that these are general properties for a single option; the characteristics of a portfolio with a mix of options with different strikes and maturities can be quite complex

Calculating the Greeks in practice

- Remember the definition of the Greeks
  - E.g. vega = Change in option price / Change in volatility
- If we have access to a pricing model, we could calculate all the Greeks manually
- Example:
  - Spot price = 100, strike = 100, rate = 5%, maturity = 1 year, dividend = 0, European call
  - Volatility = 30%, price = 14.1730 (from Black-Scholes model)
  - Volatility = 31%, price = 14.5529
  - Vega = 14.5529 – 14.1730 = 0.3799 (approximately)
- More accurate: center-difference approach
  - Vega = Price(volatiliy+0.5%) – Price(volatility–0.5%)

Should we "hedge" our portfolios?

- **By definition, if the portfolio is perfectly "hedged", we could NOT make any money!**
- As traders, we want to enter into trades such that we could earn money when our views of the market are correct
- We should only hedge when we are not sure of market conditions
  - E.g. say we have bought some shares and are bullish in the long run, but there may be some short term uncertainty, we may consider shorting some futures as a hedge
- Volatility traders may only want to express a view in volatility but not the direction of the market
  - We should hedge the direction move (delta neutral portfolio) but leave the volatility risk (vega) unhedged

Static and dynamic hedge

- Static hedge (rare): once the portfolio is hedged, there is no need for any re-balancing before maturity of contract

- Dynamic hedge (the usual case): the hedging portfolio has to be adjusted depending on market conditions
  - Buying/selling shares would create profit/loss; theoretically this hedging cost is equal to the option premium

Hedging issues

- Only instruments with gamma could hedge gamma exposures
- Only instruments with vega could hedge vega exposures
- **Therefore, we could hedge the delta with shares, but could only hedge vega and gamma with (any) option**
- **Gamma and vega profiles could change dramatically with time, spot price**
- Need to consider practical issues when engaging in hedge activities, e.g. liquidity, bid/ask spread, transaction costs

Option hedging

- **Delta-neutral**: No P&L movement when spot price moves
- **Gamma-neutral**: Delta is unchanged when spot price moves
- **Vega-neutral**: No P&L movement when implied volatility changes
- Option portfolios are rarely neutral to all these parameters
  - E.g. if we think that the volatility would increase but are unsure whether the market would go up or down, we could keep the portfolio to be delta neutral but vega positive

Dynamic delta hedging example

- Issuer sells 100,000 HSBC call options. Spot price 90, delta 55% (calculated from Black-Scholes formula)
- Issuer buys 55,000 shares (= 100000 x 0.55) @90 or better when option is traded
- On day 2, spot price becomes 93, new delta is 61% (calculated using the new parameters). Therefore Issuer needs to buy another 6,000 shares @93 or better
  - The new delta is 61,000 shares but Issuer already has 55,000 shares; therefore only another 6,000 shares need to be bought
- On day 3, spot price drops to 88, delta is now 51% (again re-calculated from the formula). Issuer needs to sell 10,000 shares @ 88 or better to maintain a delta-neutral position
  - 61,000 – 51,000 = 10,000 shares
- And so on

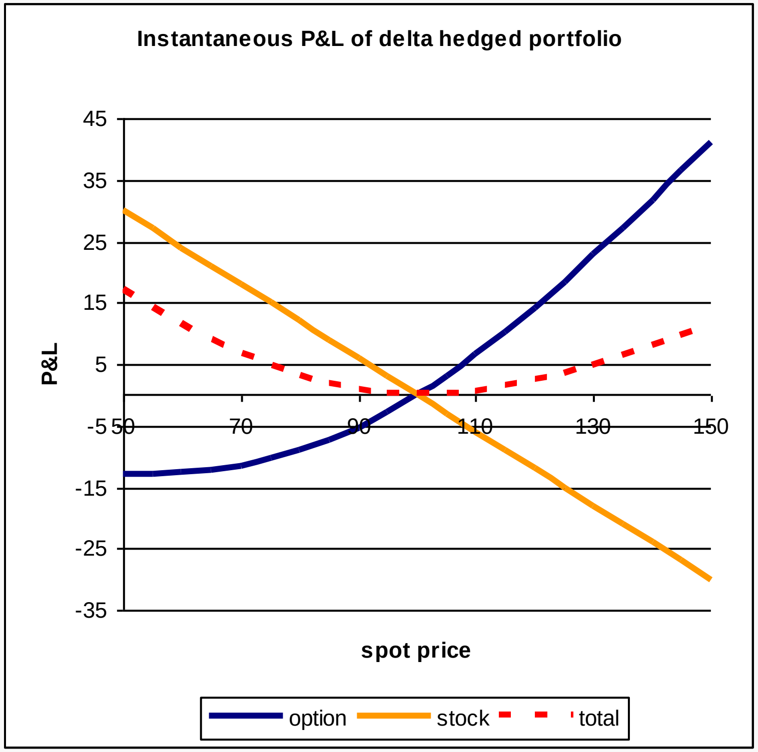
P&L of delta-hedged position

- Using a similar example as on the previous slide, on day 1 Issuer sold 1000 options and received some option premium
- To delta-hedge the position, Issuer needs to buy and sell shares (according to the delta, odd lots ignored here); the actual P&L comes from its cash position

	Initial Position	Share Price	Delta Required	New Trade	Amount	Account Balance
	(A)	(B)	(C)	(D)=(C) - (A)	(E) = – (D) x (B)	(F) = (F)previous + (E)
Day 1	0	90	550	550	-49500	-49500
Day 2	550	93	610	60	-5580	-55080
Day 3	610	88	510	-100	8800	-46280

- If the Black-Scholes theory is correct and the parameters are stable, *the final amount in the cash account should equal to the initial option price, no matter whether the option is out-of-the-money or in-the-money at maturity*

P&L profile of delta-hedged position



strike 100, maturity 1 yr, r=2%, volatility = 30%, delta = 0.6

- This diagram shows the instantaneous P&L of a delta-hedged position when underlying spot price moves
- Portfolio consists of:
  - Long 1 European call option, short Δ shares
- Almost always make a profit (!) if stock has suddenly moved, because for this option, the price is a convex function of spot
- No free lunch – if stock remains unchanged tomorrow, there would be a loss on time value (i.e. maturity is reduced everyday)

What could go wrong in hedging?

- Wrong estimate of volatility
- Transaction cost
- Intra-day movement
- Gap movement in underlying price
- Liquidity constraints
- Second order changes in the risk parameters

Concept check

- At time T_1, Investment Bank A sold a European call option on HSBC to Hedge Fund B

- After the transaction, share price of HSBC has increased gradually; now it is 25% above its level at time T_1

- What is the likely P&L of
  - i) Hedge Fund B?
  - ii) Investment Bank A

P&L attribution

Instrument	Position	Stock Price	Volatility	MTM Price	Delta (shares)	Vega ($)	P&L ($)
Call option	10000	100	25%	11.344	5788	3878	0
Shares	-8000	100		100	-8000	0	0
Total					-2212	3878	0
Call option	10000	105	26%	14.894	6527	3797	35500
Shares	-8000	105			-8000	0	-40000
Total					-1473	3797	-4500

- In this example, initially we are short delta (-2212) and long vega (3878)
- When stock price and volatility both go up, it depends on whether the delta effect or the vega effect is more dominant
- Note that delta changes rapidly
- Rough estimate of P&L: delta = –1800 x 5 = – 9000, vega = 3800, total = –5200 (compared to the actual P&L of –4500)
  - – 1800 is roughly the average of – 2212 and – 1473

Hedging strategies in practice

- It is rare to hedge a single position – portfolio risks would be managed together
  - Keep within trading limits (delta, gamma, vega, theta)
  - However, note how these can be aggregated

- Mostly commonly, traders would need to apply some kind of hedge to the first order price risk (i.e. delta)
  - C.f. Volatility trading strategies

- Hedge is only applied for short term; seldom throughout the life of an option

Aggregation of option risks

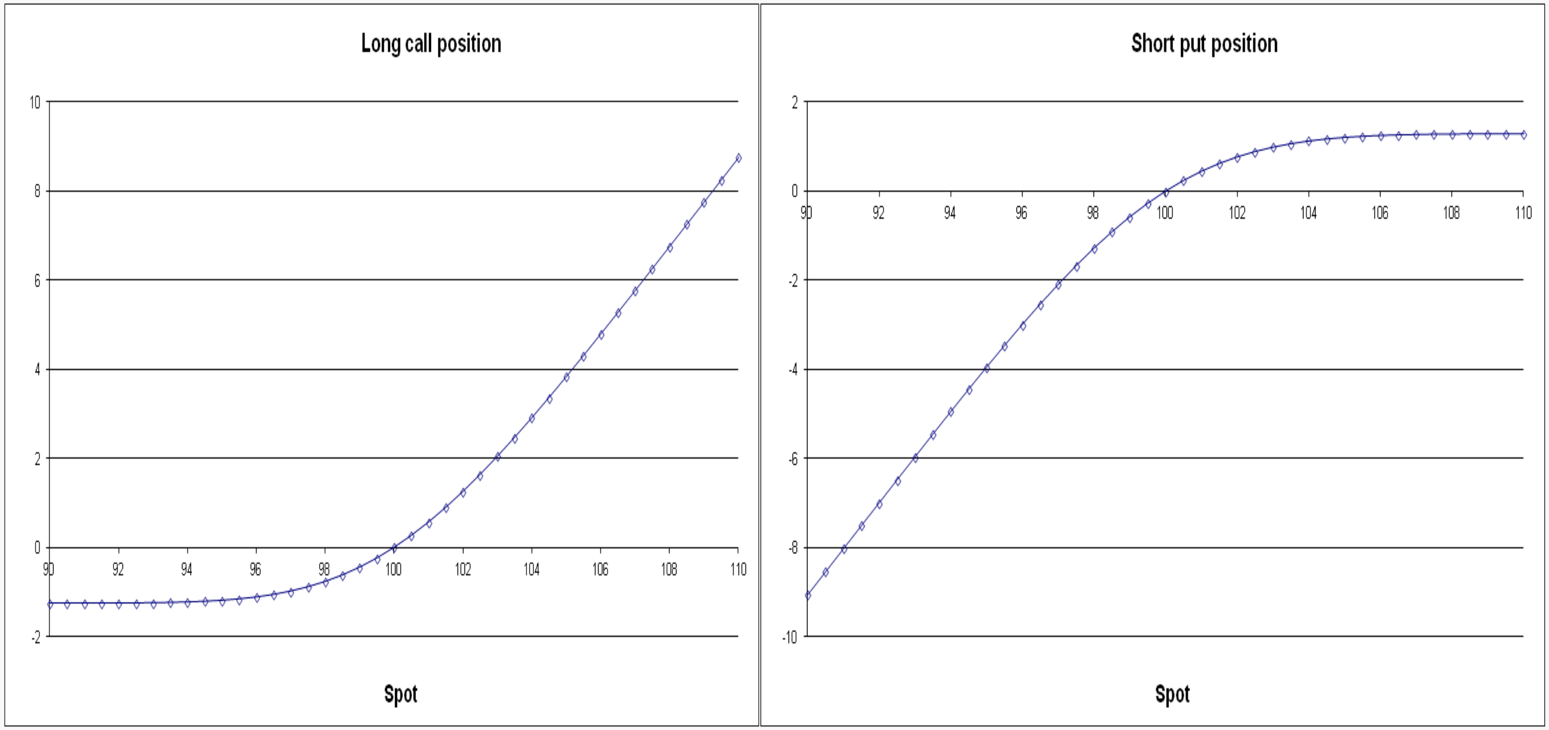
- In trading a stock portfolio, there are two levels of aggregation

- Aggregation for a single underlying, for example, a stock index
  - The risks of options at different strikes / maturities are summed together
  - Risk parameters can change rapidly when conditions change

- Aggregation across all underlyings, normally from a single country
  - The inherent assumption is that the same shock is applied to all the underlyings

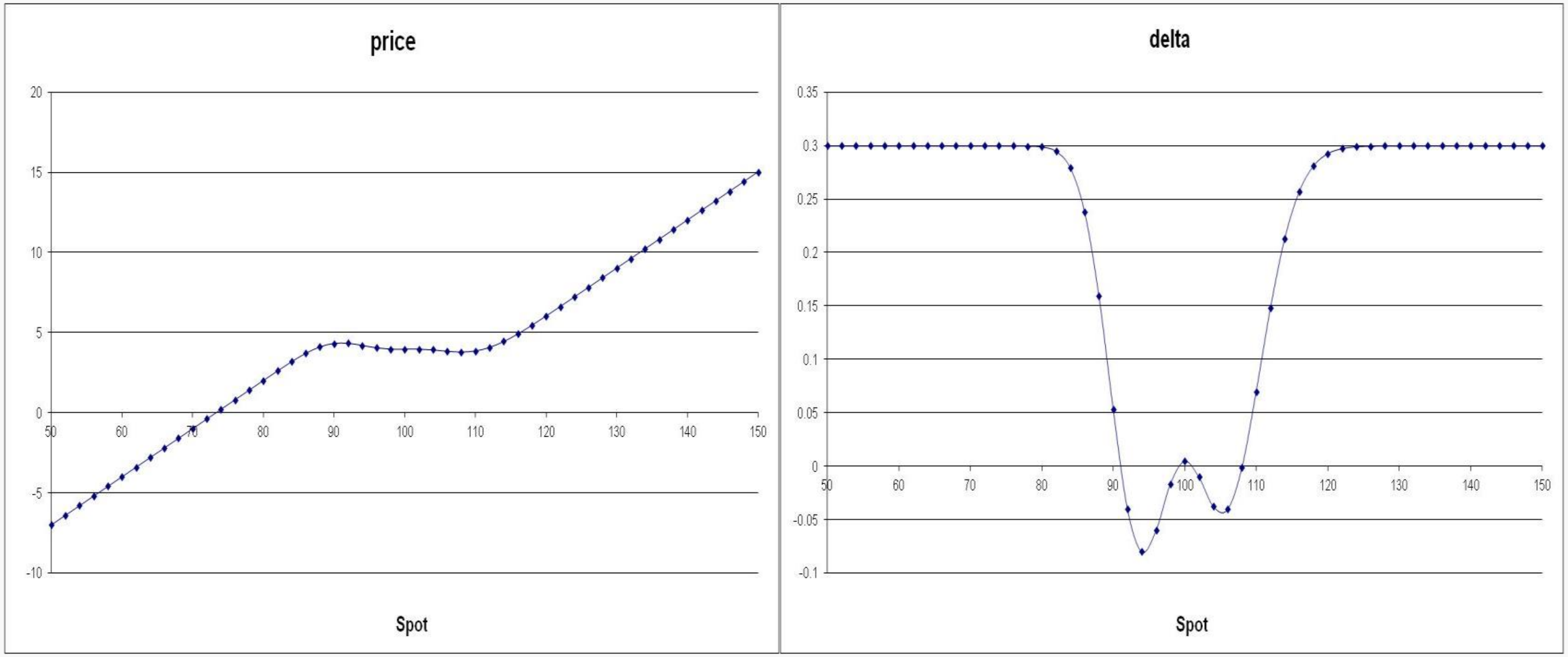
- Because of the complexity of the option characteristics, these aggregate numbers can only serve as approximate indicators

Using delta to represent position risk



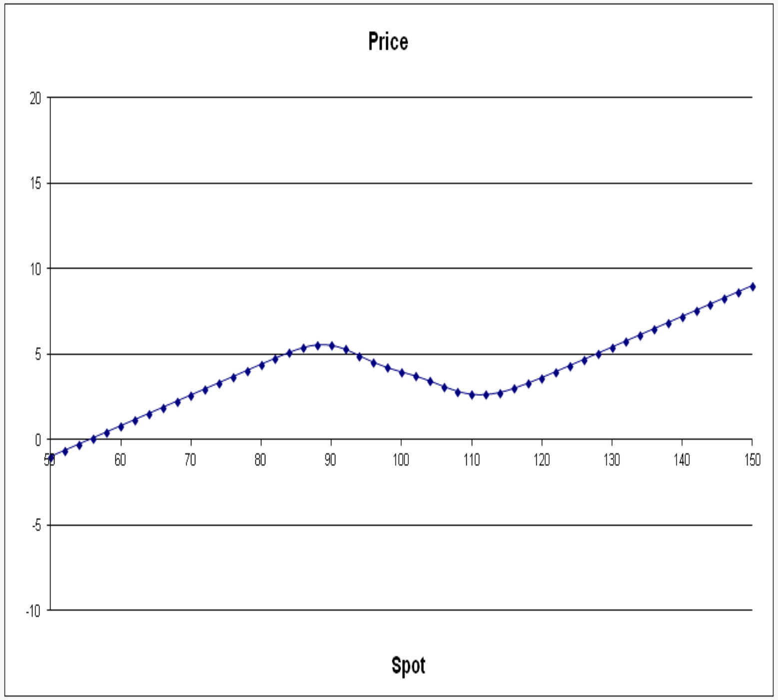
- In both these examples, the delta at the current spot price (assume = $100) is the same; it is obvious that their risk profiles are very different
  - Separate between upside and downside risks

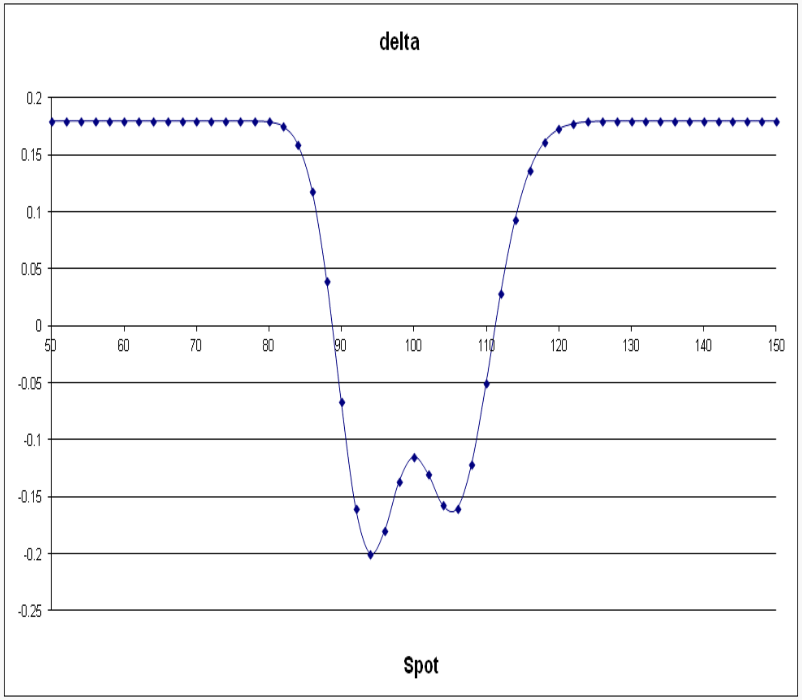
Delta hedging example



- Consider the following positions
- Short 92 strike put, Long 96 strike call, short 104 strike call, long 108 strike call, current spot price = \$100, delta at spot = 0.7
- Hedge with a delta of –0.7 to make the position delta neutral
- Other parameters: r=1%, σ=16%, T=0.083 (1 month)

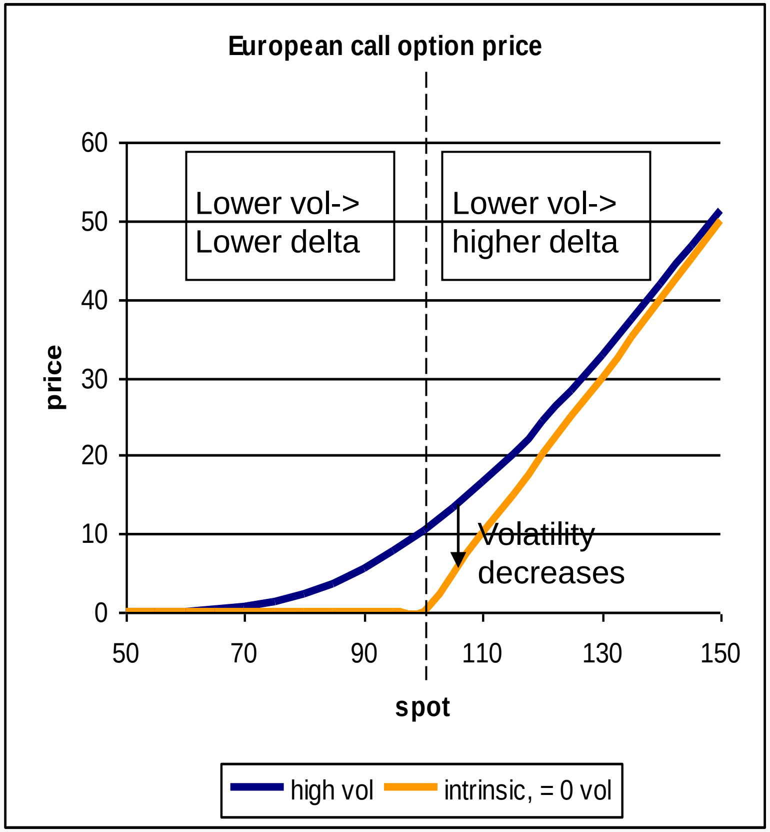
A modified delta hedge





- Instead of hedging with a delta of –0.7 to make the position delta neutral, the hedge applied is –0.82
- It is seen that while there may be some residual P&L around the current spot, the overall P&L is limited across a wider range of spot movement; this is useful especially when the spot is volatile

Volatility and delta



Strike 100, maturity 3 months, r=2%

- When we change the volatility marking, how would delta move?
- Depends on whether option is in-the-money or out-of-the-money
- As volatility decreases, the option profile would move towards the intrinsic value
- Remember that delta is the slope of the tangent of the option payoff

Second order changes

- Spot price is 100; we have the following position:
  - European call, strike 100, maturity 1 year, volatility 25%, rate 4%; vega = 0.3831
- We would like to hedge so that the portfolio is vega neutral; the following instruments are available:
  - A) European call, strike 100, maturity 1 month, volatility 30%, rate 4%; vega = 0.1148
  - B) European call, strike 100, maturity 2 years, volatility 23%, rate 4%; vega = 0.5194
- We could hedge the position with either:
  - 0.3831 / 0.1148 = 3.337 of option A
  - or 0.3831 / 0.5194 = 0.7376 of option B
- Which one should we choose?
  - A is usually more liquid and is preferable from the trading point of view

Second order changes

- If spot price stays at 100, both A and B seem to work fine
- If spot price moves to 110, the vegas would become:
  - Vega of original position = 0.3519
  - Vega of A = 0.0400; net vega = 0.3519 – 3.337x0.04 = 0.2184
  - Vega of B = 0.4869; net vega = 0.3519 – 0.7376x0.4869 = –0.0072
- If the volatility moves down while the spot is at 110, the first portfolio would suffer a bigger loss
- The aggregate Greeks (vega in this example) only show part of the story; ideally we need to break it down into maturity buckets
- Otherwise, be aware that Greeks could change when certain underlying market parameters change
  - Hence it is advisable to perform some scenario analysis to check whether the hedge parameters change much when certain other parameters move

The importance of scenario analysis

- The risk parameters only represent a snapshot of the risks at the current parameters

- Even if the risks are well hedged, there may be higher moment exposures and there may not be enough time to adjust the hedges (gap moves)

- For complex portfolios, different scenario reports have to be generated at least on a daily basis
  - Forward shift in time
  - Shift in underlying spot levels
  - Shift in volatilities of the underlyings