CMSC 5718

INTRODUCTION TO COMPUTATIONAL FINANCE

Lecture 2 & 3

Modern Portfolio Theory

- Modern Portfolio Theory (MPT) was pioneered by Harry Markowitz in a paper published in 1952
- Tradeoff between risk and return
- Measurement of risk via variance of returns
- **Mean-variance efficient portfolio**
- Portfolio selection through an optimization process

Statement of the problem

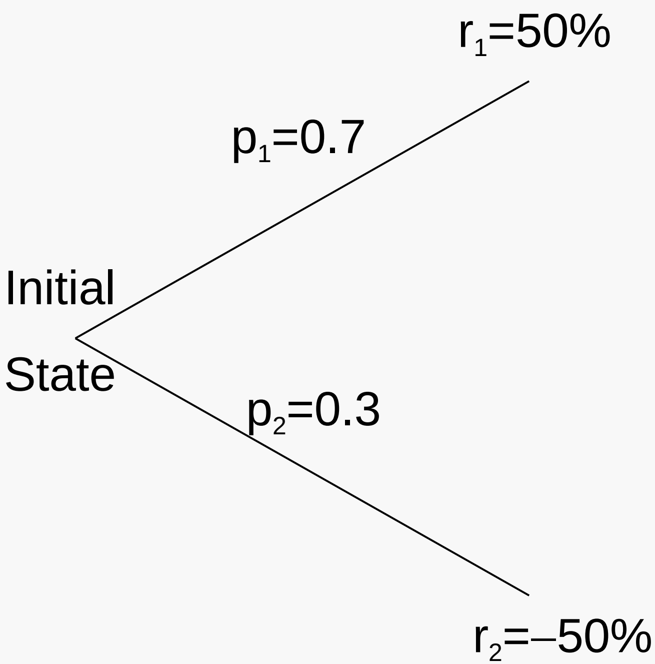
- Given the characteristics of the assets (e.g. expected mean and variance of return), we want to either:
  - Maximize return for a given level of risk
  - Minimize risk for a given level of return
- Question: how do we determine the optimum portfolio allocation, i.e. how should we allocate the assets to achieve the goal?

A definition of “risk”

- Presence of risk -> more than one outcome is possible and there is a chance that one can suffer a loss

- With an initial investment of $1000, consider the following alternatives:
  - Buy a 1-year government bill, yielding 2%, i.e. would get back $1020 after 1 year
    - Outcome is almost certain -> risk free (or “riskless”)
  - Invest in stocks, which you think there is a probability of 0.7 that the stock would worth $1500 and a probability of 0.3 that it would become $500
    - While there is a higher probability that you will make money, you can lose money -> risky

A measure of risk



- Expected return

$$E(r) = \bar{r} = \sum_{i=1}^{2} p_i r_i = 20\%$$

- p_i is the probability of each outcome; r_i is the return
- Standard deviation of return

$$\sigma = \sqrt{\sum_{i=1}^{2} p_i \left(r_i - \bar{r}\right)^2}$$

$$= 45.8\%$$

- A higher σ means higher risk
- **Risk free investment: σ = 0**

Risk premium

- Compare to the risk free investment (return = 2%), there is an additional return of 18%; this is known as the risk premium
  - Defined as the extra return over a risk free investment

- Risk premium is a compensation for the risk of an investment

- Is this enough to entice you into the investment opportunity?

- What if there is an investment with an expected return of 10%, but σ=20%?

Types of investors

- Risk averse: although the return is high, these investors would "penalize" high risk investments; a higher risk premium is required for investments with higher risk (although expected return may be higher)

- Risk neutral: investors who are happy to receive the expected return, ignoring risk factors; the higher the return the better

- Risk seeking: investors who like risk, so they are happy to receive a lower than expected return to enter the game (e.g. in a casino)

- Most people are risk averse when they consider investments

Types of investors: example

- There are five investment choices

  i) E(r) = 2%, σ = 0

  ii) E(r) = 20%, σ = 0.46

  iii) E(r) = 21%, σ = 0.90

  iv) E(r) = 10%, σ = 0.20

  v) E(r) = 1%, σ = 1.25; small probability of extra return = 200%

- A risk seeking investor would choose the option with the highest risk but possible highest return, i.e. (v)
- A risk neutral investor would choose the option with the highest expected return, i.e. (iii)
- A risk averse investor may choose (i), (ii), or (iv), depending on the individual's risk preference; (iii) is still possible, but unlikely because the big increase in risk is only compensated by a small increase in return

Types of investors: example

Investment choice	Risk averse	Risk seeking	Risk neutral
1. E(r) = 2%, $\sigma = 0$	Possible	X	X
2. E(r) = 20%, $\sigma = 0.46$	Possible	X	X
3. E(r) = 21%, $\sigma = 0.90$	Unlikely	X	√
4. E(r) = 10%, $\sigma = 0.20$	Possible	X	X
5. E(r) = 1%, $\sigma = 1.25$ small probability of extra return = 200%	X	√	X

How do we quantify risk aversion?

- Utility function: a means to rank portfolios with different risk/return characteristics
- many valid forms exist
- One popular choice is a *quadratic function*

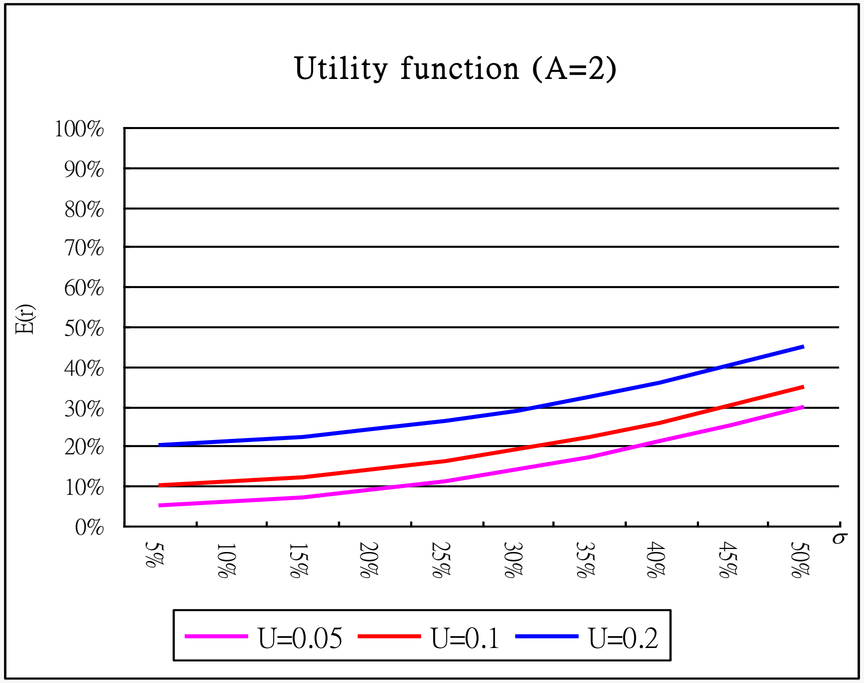
$$U = E(r) - \frac{1}{2} A\sigma^2$$

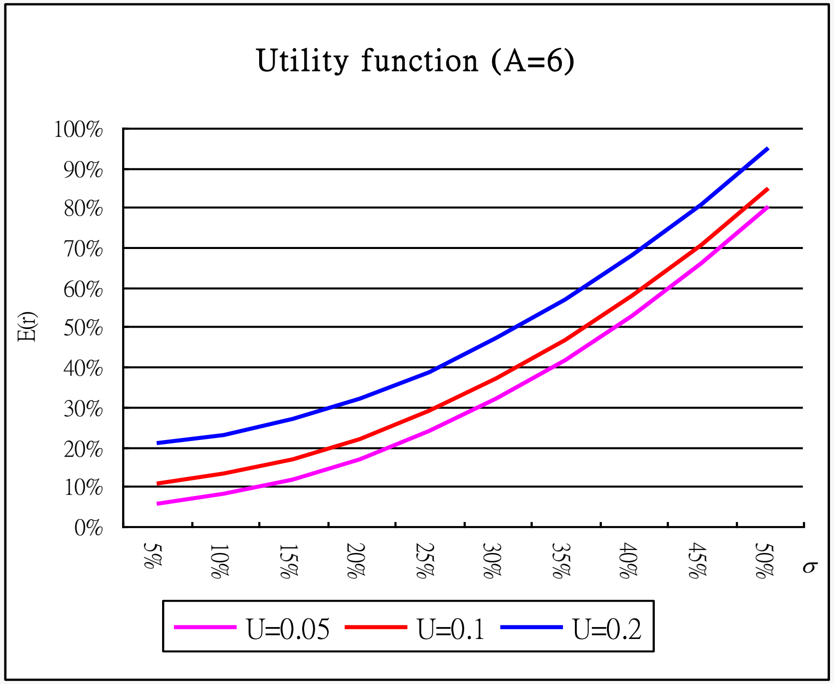
- A is a measure of the risk tolerance of the investor
- "Non-satiation": more money is preferred to less money
  - The higher the utility the better

Some properties of utility function

- A>0: risk averse
  - If σ is higher, then $E(r)$ has to be even higher to compensate for the risk
- A =0: risk neutral
  - σ does not affect the utility; $E(r)$ determines U
- A <0: risk seeking
  - The higher σ is the better

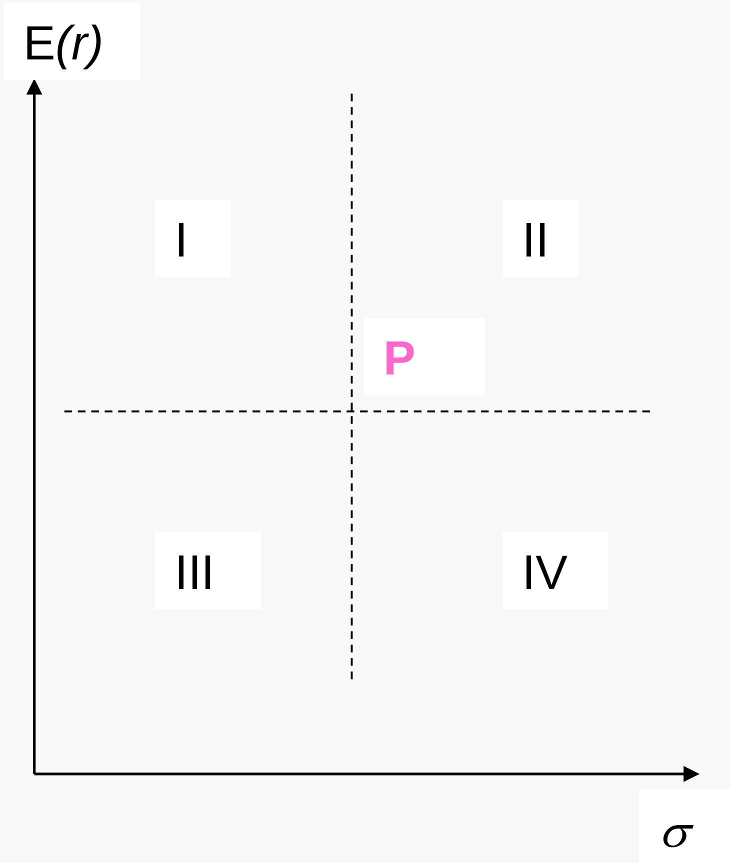
Indifference curves





- Keeping U constant, plot $E(r)$ against σ
- Curve moves towards the North-West direction for higher U's
- Curve is steeper if A is higher (more risk averse)

The mean-variance criterion



- Portfolios in quadrant I are preferable to P, which is preferable to those in quadrant IV, because $E(r)$ is higher and σ is lower
- Formally speaking, if

$$E(r_M) \geq E(r_N)$$
$$\text{and } \sigma_M \leq \sigma_N$$

  and at least one inequality is strict (i.e. without the equal sign), then we say M **dominates** N; M is more preferable to N
- **Whether portfolios in quadrants II and III are preferable to P depends on the investor's risk aversion**

Steps in the asset allocation process

- **Step 1**: select a combination of risky assets in a portfolio
- **Step 2**: combine the risky portfolio with a risk free asset (if the risk free asset is available)
- **Step 3**: select the optimal portfolio according to the individual investor's preference

Step 1: combining risky assets

- Assume that we have already selected some risky assets to be put into the portfolio, and know the expected return $E(r)$ and standard deviation σ of each asset
  - This is a big assumption
- We form a portfolio P using N risky assets

$$P = \sum_{i=1}^{N} w_i S_i$$

  where w_i and S_i are the weight and price of asset i and P is the portfolio's value
- By convention, we could set $\sum_{i=1}^{N} w_i = 1$

Portfolio characteristics

- Return of the portfolio = weighted average of the return on the individual assets
- Expected return of the portfolio = weighted average of the expected return on individual assets $\overline{r_p} = \sum_{i=1}^{n} w_i \overline{r_i}$
- Covariance between two assets: $\sigma_{ij} = \sum_{k=1}^{N} \left[\left(r_{ik} - \overline{r_i} \right) \left(r_{jk} - \overline{r_j} \right) \right]$
- Correlation between two assets: $\rho_{ij} = \sigma_{ij} \big/ \sigma_i \sigma_j$
- Portfolio variance: $\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} w_j w_k \sigma_{jk}$

Portfolio variance calculation

- We have the following data

			Covariance with		
Security	Weight	s.d. (σ)	Cheung Kong	HSBC	MTR
Cheung Kong	1/3	36%		0.0648	0.04968
HSBC	1/2	20%			0.023
MTR	1/6	23%			

- The portfolio variance is:

$$\sigma_p^2 = \left(\frac{1}{3}\right)^2 \times 0.36^2 + \left(\frac{1}{2}\right)^2 \times 0.20^2 + \left(\frac{1}{6}\right)^2 \times 0.23^2 + 2 \times \left(\frac{1}{3}\right) \times \left(\frac{1}{2}\right) \times 0.0648$$

$$+ 2 \times \left(\frac{1}{3}\right) \times \left(\frac{1}{6}\right) \times 0.04968 + 2 \times \left(\frac{1}{2}\right) \times \left(\frac{1}{6}\right) \times 0.023$$

$$= 0.0568 \quad \text{or} \quad \sigma_P = 23.8\%$$

Matrix convention

- Portfolio weights: $\boldsymbol{w} = (w_1 \ldots w_N)^T$
- Expected returns of assets: $\bar{\boldsymbol{r}} = (\bar{r}_1, \ldots, \bar{r}_N)^T$
- NxN covariance matrix Ω, where Ω is symmetric and $\Omega_{ij} = \sigma_{ij} = \text{covar}(r_i, r_j)$
- Portfolio expected return: $\bar{r}_{\boldsymbol{p}} = \boldsymbol{w}^T \bar{\boldsymbol{r}}$
- Portfolio variance: $\sigma_p^2 = \boldsymbol{w}^T \Omega \boldsymbol{w}$

Matrix convention

- For N=2, we have

$$r_p = w_1 r_1 + w_2 r_2$$

$$\sigma_p^2 = [w_1 \quad w_2] \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= [w_1\sigma_{11} + w_2\sigma_{21} \quad w_1\sigma_{12} + w_2\sigma_{22}] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= w_1^2\sigma_{11} + w_2^2\sigma_{22} + 2w_1w_2\sigma_{12}$$

$$= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2$$

- Note that we assume Ω to be symmetric and positive definite, so that Ω^{-1} always exist

The effect of diversification

- Assume equal weighting, i.e. $w_i = 1/N$
- Portfolio variance

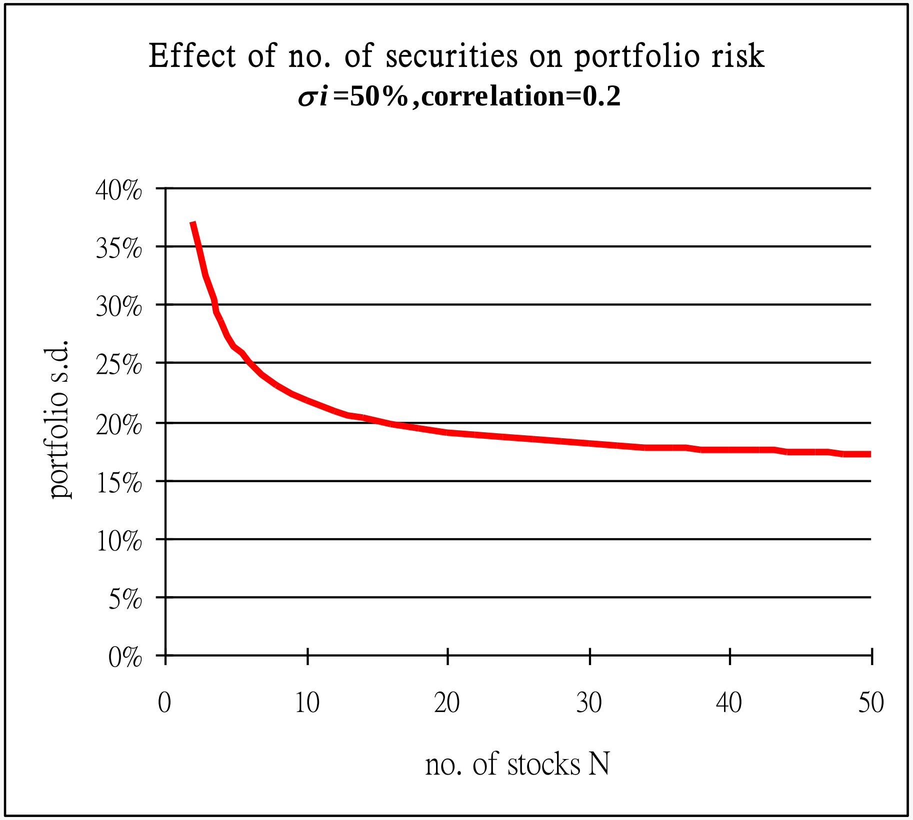
$$\sigma_p^2 = \sum_{i=1}^{N} (1/N)^2 \sigma_i^2 + \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} (1/N)^2 \sigma_{jk}$$

$$= \frac{1}{N} \sum_{i=1}^{N} \frac{\sigma_i^2}{N} + \frac{N-1}{N} \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} \left[\frac{\sigma_{jk}}{N(N-1)} \right]$$

$$= \frac{1}{N} \overline{\sigma_i^2} + \frac{N-1}{N} \overline{\sigma_{jk}}$$

- $N \to \infty$, portfolio variance -> average of the covariance of the assets in the portfolio; therefore *covariance is important*
- Individual asset variance is “irrelevant”

The effect of diversification



- Covariance between assets are sometimes attributed to market factors (e.g. country)
- First term in the previous equation is known as "**diversifiable risk**", i.e. risk that can be eliminated by diversification
- Second term is sometimes known as "**non-diversifiable risk**" or "**systematic risk**"

Two-asset portfolio example

- Assume there are two assets, with $\overline{r_1}$ and $\overline{r_2}$ as the expected returns, and σ_1 and σ_2 are the standard deviation of returns
- A portfolio P can be formed by combining these two assets in different proportions (w_1 and w_2 being the weights assigned to each asset), effectively creating a new "asset"

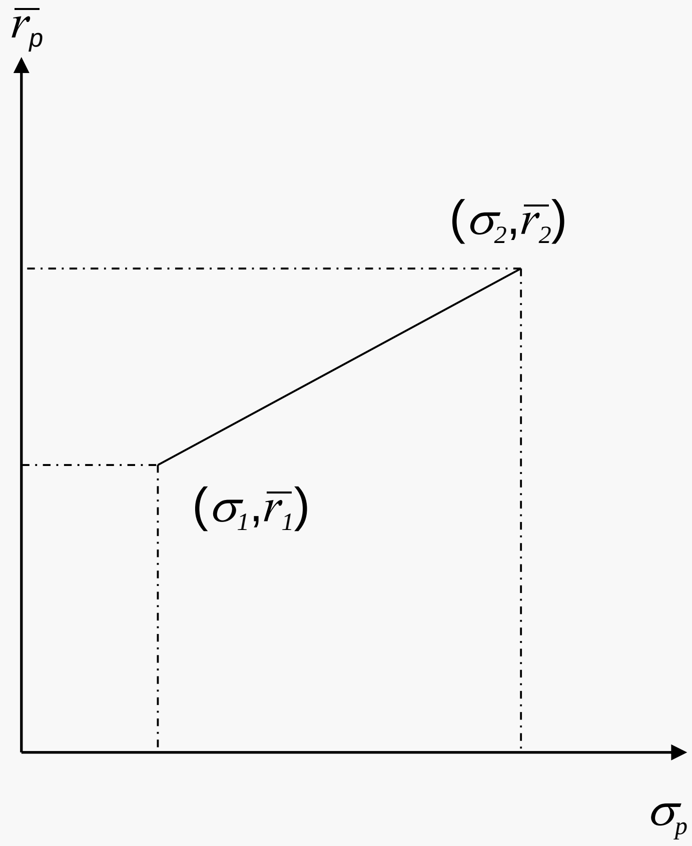
$$w_1 + w_2 = 1$$

$$\overline{r_p} = w_1 \overline{r_1} + w_2 \overline{r_2} = w_1 \overline{r_1} + (1 - w_1)\overline{r_2}$$

$$\sigma_p = \sqrt{w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2\rho w_1 w_2 \sigma_1 \sigma_2}$$

- Depending on the covariance of the two assets, the portfolio characteristics can be represented on an Expected return-standard deviation diagram

Two-asset portfolio: $\rho = 1$



$$\overline{r_p} = w_1\overline{r_1} + (1-w_1)\overline{r_2}$$

$$\sigma_p = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2}$$
$$= w_1\sigma_1 + (1-w_1)\sigma_2$$

- Eliminating w_1, we find

$$\overline{r_p} = \frac{\overline{r_2}\sigma_1 - \overline{r_1}\sigma_2}{\sigma_1-\sigma_2} + \frac{\overline{r_1}-\overline{r_2}}{\sigma_1-\sigma_2}\sigma_p$$

- If $1 > w_1 > 0$, the straight line must end at $(\sigma_1,\overline{r}_1)$ and $(\sigma_2,\overline{r}_2)$

Two-asset portfolio: case $\rho = -1$

- The portfolio variance is

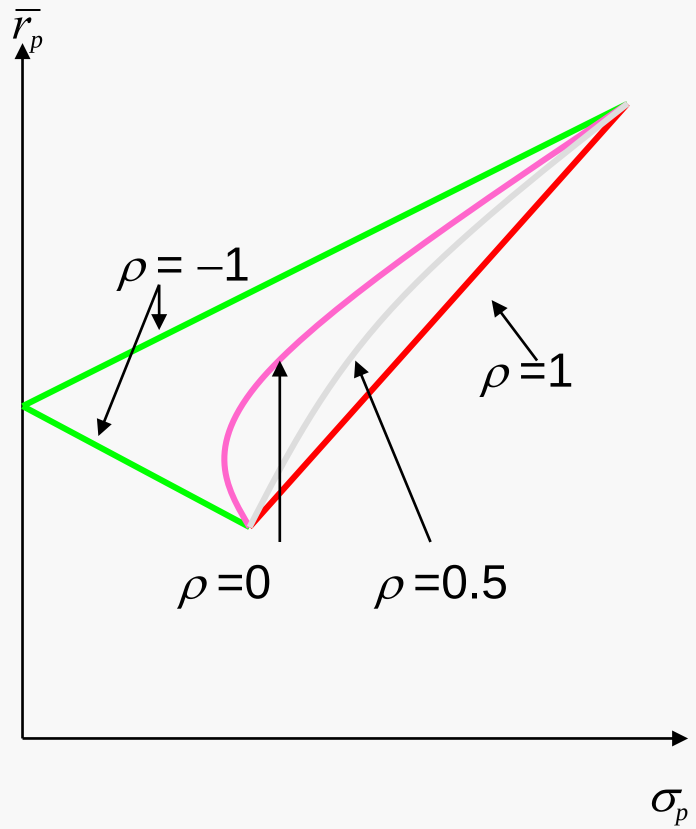
$$\sigma_p = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 - 2w_1w_2\sigma_1\sigma_2}$$
$$= \left| w_1\sigma_1 - (1-w_1)\sigma_2 \right|$$

- This is equivalent to the following two straight lines

$$\sigma_p = w_1\sigma_1 - (1-w_1)\sigma_2,\ \ w_1 \geq \frac{\sigma_2}{\sigma_1+\sigma_2}$$
$$= (1-w_1)\sigma_2 - w_1\sigma_1,\ \ w_1 < \frac{\sigma_2}{\sigma_1+\sigma_2}$$

- Note that when $w_1 = \frac{\sigma_2}{\sigma_1+\sigma_2}$, the portfolio variance becomes 0

Two-asset portfolio: other cases



- The diagram shows combinations of the portfolio opportunity set for different ρ (only valid when $w_i > 0$)
- Clearly, the lower the ρ, the more risk reduction through diversification could be achieved (by having a smaller σ_p)

Two-asset portfolio example

	Asset A	Asset B
Mean return (%)	10	20
Variance (%)	10	15

weight		$\rho = -1$		$\rho = -0.5$		$\rho = 0.5$		$\rho = 1$	
w_A	$w_B = 1 - w_A$	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance
1.0	0.0	10.0	10.00	10.0	10.00	10.0	10.00	10.0	10.00
0.8	0.2	12.0	3.08	12.0	5.04	12.0	8.96	12.0	10.92
0.5	0.5	15.0	0.13	15.0	3.19	15.0	9.31	15.0	12.37
0.2	0.8	18.0	6.08	18.0	8.04	18.0	11.96	18.00	13.92
0.0	1.0	20.0	15.00	20.0	15.00	20.0	15.00	20.0	15.00

$$\overline{r_p} = w_A \times 10 + (1 - w_A) \times 20$$

$$\sigma_p = \sqrt{w_A^2 \times 10 + (1 - w_A)^2 \times 15 + 2\rho w_A (1 - w_A) \times \sqrt{10} \times \sqrt{15}}$$

- A lower variance is achieved for a given mean when the correlation of the pair of assets' returns becomes more negative

Two asset-portfolio: minimum variance portfolio

$$\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho w_1 w_2 \sigma_1\sigma_2$$

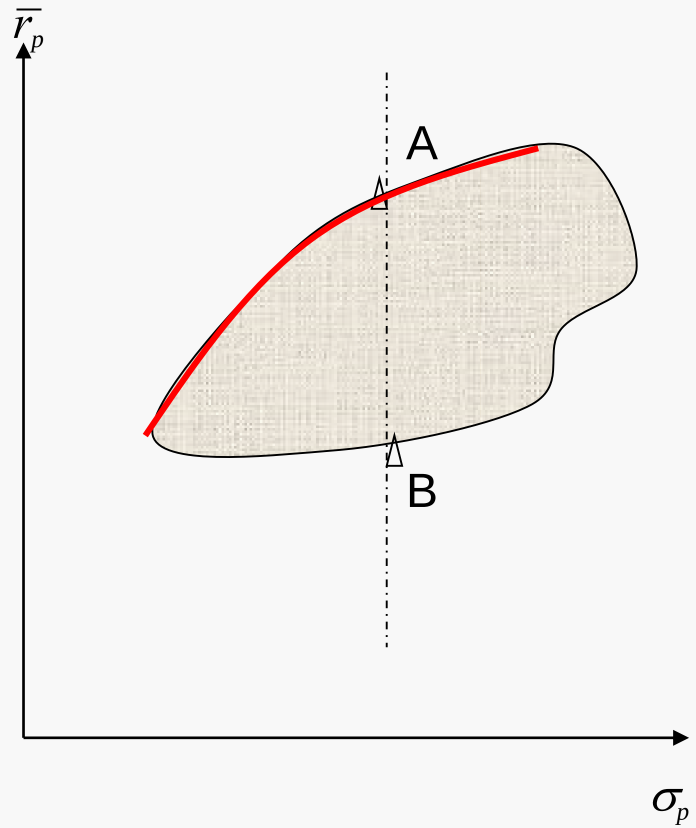
$$\frac{\partial(\sigma_P^2)}{\partial w_1} = 2w_1\sigma_1^2 - 2(1-w_1)\sigma_2^2 + 2\rho(1-2w_1)\sigma_1\sigma_2 = 0$$

$$\Rightarrow w_1 = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$$

- When $\rho = 0$, this expression is reduced to

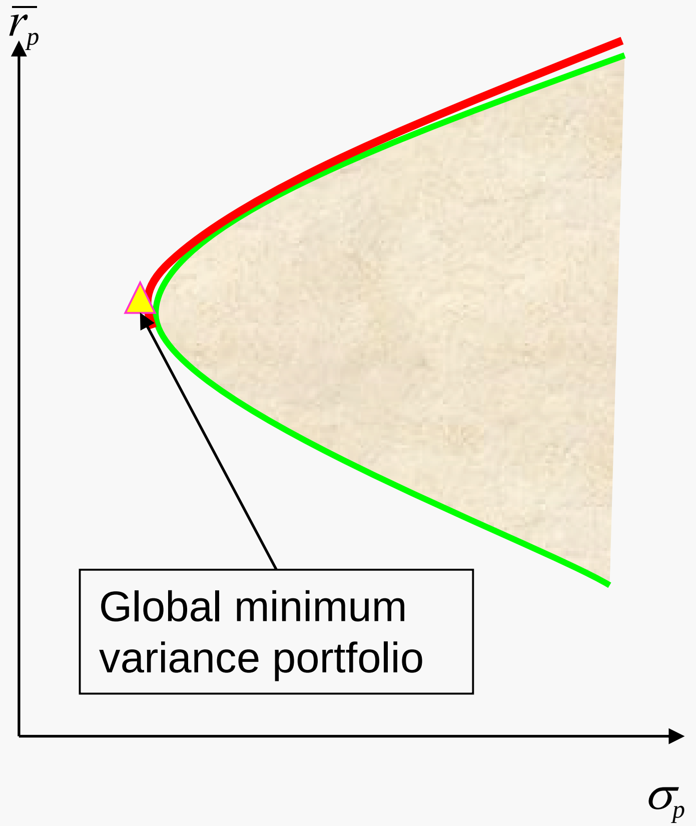
$$w_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

The feasible set for many assets



- Only portfolios within the shaded area are possible; those outside the area are not achievable
- Portfolio A "dominates" portfolio B (same risk but higher return), therefore we should always select A
- The red line is the efficient frontier of risky assets

Minimum variance frontier



- The green line is the **minimum variance frontier**: at each $\bar{r}_p$, the portfolio on the line is the portfolio with the smallest σ_p
- The red line is the **efficient frontier** since portfolios below this line are dominated by those on the red line
- The **global minimum variance portfolio** is the portfolio with the smallest σ_p – note that this may not always be the "best" portfolio for an investor

How do we find the efficient frontier?

- A constraint optimization problem
- Recall that the portfolio variance is

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} w_i w_j \sigma_{ij}$$

- At each $\overline{r_p}$, minimize σ_p subject to the following constraints

$$\overline{r_p} = \sum_{i=1}^{N} w_i \overline{r_i}, \ \sum_{i=1}^{N} w_i = 1$$

- We could then solve for w_i
- Repeat this procedure for different $\overline{r_p}$ which will give the minimum variance frontier, from which the efficient frontier can be obtained

Lagrange multipliers

- A useful technique for solving optimization problems
- Let's say we want to maximize the function f of N variables x_i subject to 2 constraints $g(x)$ and $h(x)$, i.e. Maximize $f(x_1, x_2, \ldots, x_N)$ where $g(x_1, x_2, \ldots, x_N)=0$ and $h(x_1, x_2, \ldots, x_N)=0$
- Introduce the Lagrangian $L = f(\mathbf{x}) - \lambda_1 g(\mathbf{x}) - \lambda_2 h(\mathbf{x})$
- Set the partial derivative of L w.r.t. each x_i to 0 (N equations)
- Together with the original 2 constraints, we have $N+2$ equations and $N+2$ unknowns (N weights, λ_1, λ_2)
- λ_1 and λ_2 are known as the Lagrange multipliers

Solution of the optimization problem

- Instead of minimizing σ_P^2, we minimize $\frac{1}{2}\sigma_P^2$

- The Lagrangian is formed by

$$L = \frac{1}{2}\left(\sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} w_i w_j \sigma_{ij} \right) - \lambda_1 \left(\sum_{i=1}^{N} w_i \overline{r_i} - \overline{r_P} \right) - \lambda_2 \left(\sum_{i=1}^{N} w_i - 1 \right)$$

- Set $\frac{\partial L}{\partial w_i} = 0, \ i = 1, \cdots, N$

  and together with the original two constraints to form N+2 equations

Solution of the optimization problem

- The Lagrange multipliers and the portfolio weights would satisfy the following N+2 equations

$$\text{Condition 1}: w_i\sigma_i^2 + \sum_{j=1}^{N}\sigma_{ij}w_j - \lambda_1\overline{r_i} - \lambda_2 = 0,\ i = 1,\cdots,N$$

$$\text{Condition 2}: \sum_{i=1}^{N} w_i\overline{r_i} = \overline{r_P}$$

$$\text{Condition 3}: \sum_{i=1}^{N} w_i = 1$$

An example with 3 assets

- Assume the following parameters

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1;\ \overline{r_1} = 1, \overline{r_2} = 2, \overline{r_3} = 3$$

$$\rho_{12} = \rho_{23} = \rho_{13} = 0, \text{or } \sigma_{12} = \sigma_{23} = \sigma_{13} = 0$$

- The set of equations is

$$\begin{aligned} w_1 - \lambda_1 - \lambda_2 &= 0 \\ w_2 - 2\lambda_1 - \lambda_2 &= 0 \\ w_3 - 3\lambda_1 - \lambda_2 &= 0 \\ w_1 + 2w_2 + 3w_3 &= \overline{r_P} \\ w_1 + w_2 + w_3 &= 1 \end{aligned}$$

- The solution is:

$$\lambda_1 = \left(\overline{r_P}/2\right) - 1,\ \lambda_2 = 2\frac{1}{3} - \overline{r_P}$$

$$w_1 = \frac{4}{3} - \left(\overline{r_P}/2\right),\ w_2 = \frac{1}{3},\ w_3 = \left(\overline{r_P}/2\right) - \frac{2}{3}$$

$$\sigma_P = \sqrt{\frac{7}{3} - 2\overline{r_P} + \frac{\overline{r_P}^2}{2}}$$

When short sale is not allowed

- If short position is not allowed (i.e. cannot have a weight which is greater than 1 or smaller than 0), add the constraint

$$w_i \geq 0 \text{ for all } i = 1 \cdots N$$

- This problem cannot be reduced to a set of linear equations, given that the objective function includes quadratic terms and the constraints are linear equalities and inequalities

- Note that when short sale is allowed, most of the w_i will have non-zero values, whereas when short sale is not allowed, many weights are equal to 0

An example with 3 assets

- Using the previous parameters, the solution has the following form

$1 \le \bar{r} \le \frac{4}{3}$	$\frac{4}{3} \le \bar{r} \le \frac{8}{3}$	$\frac{8}{3} \le \bar{r} \le 3$
$w_1 = 2 - \bar{r}$	$\frac{4}{3} - \frac{\bar{r}}{2}$	0
$w_2 = \bar{r} - 1$	$\frac{1}{3}$	$3 - \bar{r}$
$w_3 = 0$	$\frac{\bar{r}}{2} - \frac{2}{3}$	$\bar{r} - 2$
$\sigma = \sqrt{2\bar{r}^2 - 6\bar{r} + 5}$	$\sqrt{\frac{2}{3} - 2\bar{r}_1 + \frac{\bar{r}^2}{2}}$	$\sqrt{2\bar{r}^2 - 10\bar{r} + 13}$.

An example with 3 assets

- If short position is not allowed (i.e. each weight is 0 or above), $1 \le r_P \le 3$ where $r_P = w_1 + 2w_2 + 3w_3$
- From the previous solution, $w_3 = \left(\overline{r_P} / 2\right) - 2/3$
- When $1 \le \overline{r_P} \le 4/3$, w_3 becomes negative; if a short position is not allowed, it would be set to 0
  - We can then solve w_1 and w_2 by $w_1 + 2w_2 = \overline{r_P}$, $w_1 + w_2 = 1$
- A similar argument applies for w_1, which would be set to 0 when $8/3 \le r_P \le 3$

Expected return maximization

- The common way of stating problem is the risk minimization formulation (as shown in the previous slides)
- Fund managers often find that they are limited to certain risk, and want to find an optimum portfolio to generate the highest return
- The formulation is:

$$\text{Maximize } \overline{r_P} = \sum_{i=1}^{N} w_i \overline{r_i}$$

$$\text{Subject to } \sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} w_i w_j \sigma_{ij}$$

$$\sum_{i=1}^{N} w_i = 1$$

Two-fund theorem

- In the computation of the efficient frontier, we are supposed to find the solution from the optimization process by setting different values of the expected return
- However, the mean-variance efficient set has an important property, known as the two-fund theorem:
  - *Two efficient funds (portfolios) can be established so that any efficient portfolio can be duplicated, in terms of mean and variance, as a combination of these two. In other words, all investors seeking efficient portfolios need only invest in combinations of these two funds.* (Luenberger 1998, p.163)
- Implication: only two mutual funds are needed to provide investment service for everyone!

Proof of the two-fund theorem

- Assume that there are two known solutions,

$$w^1 = \left(w_1^1, w_2^1, \cdots, w_n^1\right), \lambda_1^1, \lambda_2^1$$
$$w^2 = \left(w_1^2, w_2^2, \cdots, w_n^2\right), \lambda_1^2, \lambda_2^2$$

  which corresponds to expected rates of return $\overline{r_P^1}, \overline{r_P^2}$
- We form a portfolio by assigning a weight of α to w^1 and $(1-\alpha)$ to w^2 to form a new weight vector, i.e. $\alpha w^1 + (1-\alpha)w^2$
- By varying α, any portfolio on the efficient frontier can be obtained
- The two-fund theorem is proved if it can be shown that this new portfolio also satisfies the conditions given on slide 33

Proof of the two-fund theorem

- Condition 1 is satisfied because if w^1 and w^2 both make the left hand side of that equation equal to zero, it also applies to any linear combination of w^1 and w^2
- Condition 2 is satisfied by noting that

$$\sum_{i=1}^{N}\left[\alpha w_i^1+(1-\alpha)w_i^2\right]\overline{r_i}=\alpha\sum_{i=1}^{N}w_i^1\overline{r_i}+(1-\alpha)\sum_{i=1}^{N}w_i^2\overline{r_i}$$
$$=\alpha\overline{r_P^1}+(1-\alpha)\overline{r_P^2}$$

- Condition 3 is satisfied because

$$\sum_{i=1}^{N}\left[\alpha w_i^1+(1-\alpha)w_i^2\right]=\alpha\sum_{i=1}^{N}w_i^1+(1-\alpha)\sum_{i=1}^{N}w_i^2=1$$

Two-fund theorem: an example

Security	covariance, σ_{ij}					mean, $\bar{r}_i$
1	2.30	0.93	0.62	0.74	−0.23	15.1
2	0.93	1.40	0.22	0.56	0.26	12.5
3	0.62	0.22	1.80	0.78	−0.27	14.7
4	0.74	0.56	0.78	3.40	−0.56	9.02
5	−0.23	0.26	−0.27	−0.56	2.60	17.68

- With this set of input parameters, two efficient portfolios can be found by setting:

$$\lambda_1 = 0, \lambda_2 = 1: \quad \sum_{j=1}^{5} \sigma_{ij} v_j^1 = 1$$

$$\lambda_1 = 1, \lambda_2 = 0: \quad \sum_{j=1}^{5} \sigma_{ij} v_j^2 = \bar{r}_i$$

Two-fund theorem: an example

security	v^1	v^2	w_g	w_d
1	0.141	3.652	0.088	0.158
2	0.401	3.583	0.251	0.155
3	0.452	7.284	0.282	0.314
4	0.166	0.874	0.104	0.038
5	0.440	7.706	0.275	0.334
mean			14.413	15.202
variance			0.625	0.659
standard deviation			0.791	0.812

- In the solution obtained above, v^1 and v^2 do not have weights that sum to 1, therefore the final solution w_g and w_d are obtained by normalizing the weights to 1

$$w_i^g = v_i^1 / \sum_{j=1}^{5} v_j^1,\ w_i^d = v_i^2 / \sum_{j=1}^{5} v_j^2,\ i = 1 \text{ to } 5$$

Step 2: Allocation between risky and risk free assets

- We assume that we have already found the efficient frontier of risky assets
- There are many portfolios on the efficient frontier; which portfolio should we choose to be combined with a risk free asset, in order to find a portfolio with the "best" risk-return characteristics?

Characteristics of a portfolio with risk free and risky assets

- Say A is the risky portfolio and B is a risk free asset
- By definition σ_B=0, and a portfolio P formed from A and B would have the following characteristics:

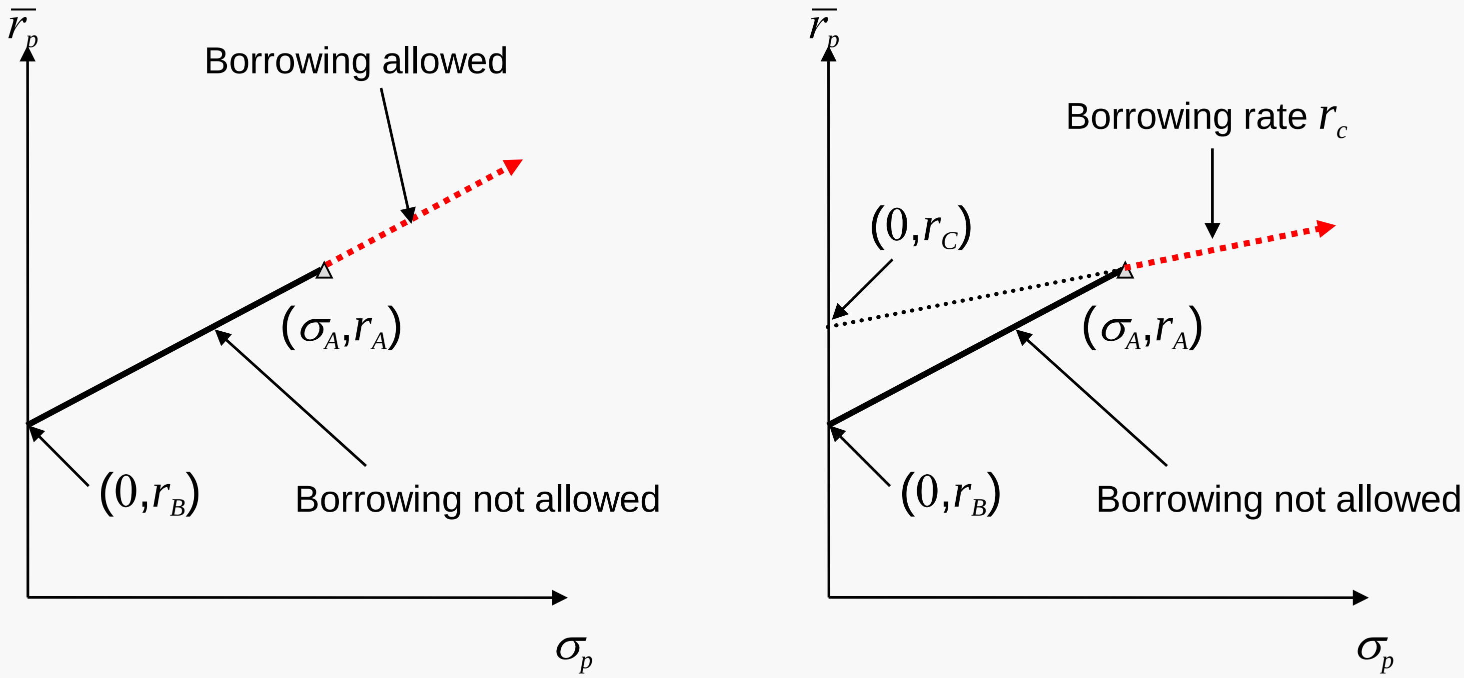
$$\sigma_p = w_A \sigma_A$$

$$\overline{r_p} = w_A \overline{r_A} + (1 - w_A)\overline{r_B}$$

$$= \overline{r_B} + \frac{(\overline{r_A} - \overline{r_B})}{\sigma_A}\sigma_p$$

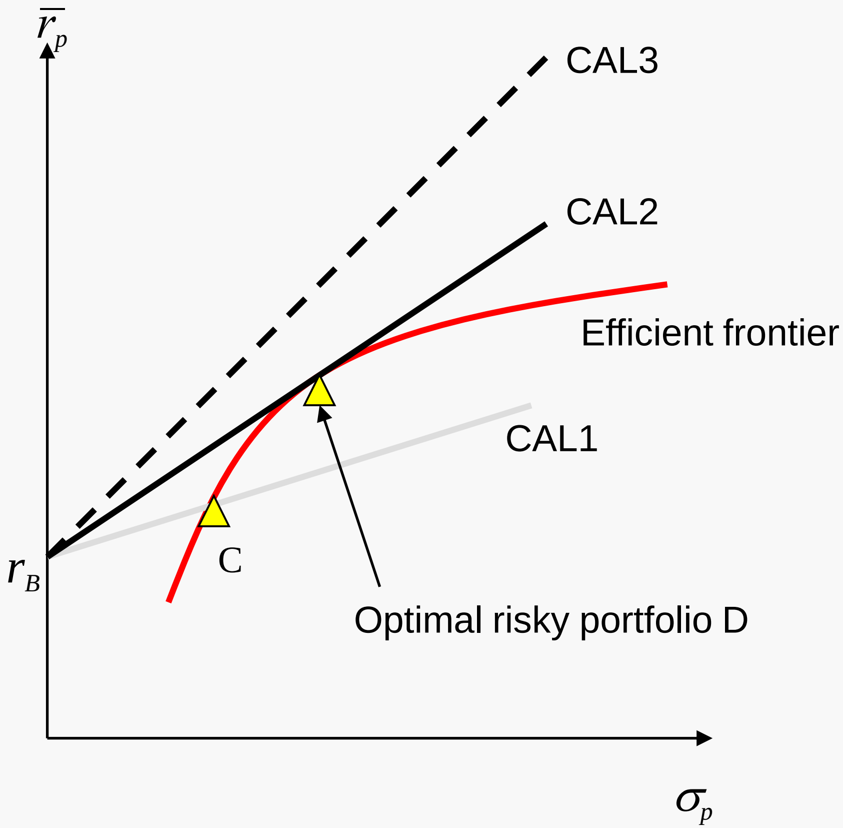
- This is a straight line, known as the **Capital Allocation Line (CAL)**
- The slope is called the ***reward-to-variability ratio***
  - For each unit increase of σ_P, this ratio shows the extra increase in return
- The line passes through the points $(0, r_B)$ and (σ_A, r_A)

Capital allocation lines



- An investor may have a different borrowing/lending rate
- If borrowing is allowed (i.e. weight of an asset can be negative), it is known as a leverage position

Which risky portfolio should we choose?



- All risky portfolios on the efficient frontier are optimal combinations; e.g. we could choose portfolio C or D
- However, portfolio D, when combined with the risk free asset B, would produce the steepest CAL; CAL2 is a tangent to the efficient frontier
- CAL 3 is not a valid CAL because no risky portfolio could achieve such a risk/return profile

How do we find the optimal CAL?

- Another optimization problem
- The CAL is $$\overline{r_p} = \overline{r_B} + \frac{(\overline{r_A} - \overline{r_B})}{\sigma_A}\sigma_p = \overline{r_B} + \theta\sigma_p$$
- We want to optimize the slope of the CAL, i.e. maximize $$\theta = \frac{\overline{r_A} - \overline{r_B}}{\sigma_A}$$

  subject to the constraints

$$\overline{r_A} = \sum_{i=1}^{N} w_i \overline{r_i}, \ \sum_{i=1}^{N} w_i = 1$$

Short sales allowed with riskless borrowing and lending

- No further constraint is added to the two conditions stated on the previous slide
- Instead of using Lagrangian multipliers, this can be turned into a simple maximization problem, noting that

$$\overline{r_B} = \sum_{i=1}^{N} w_i \overline{r_B}$$

$$\theta = \frac{\sum_{i=1}^{N} w_i \left(\overline{r_i} - \overline{r_B} \right)}{\left[\sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} w_i w_j \sigma_{ij} \right]^{1/2}}$$

Short sales allowed with riskless borrowing and lending

- The solution can be found by setting the partial derivative w.r.t. each w_i to be zero, and each of the derivative can be written as

$$\frac{\partial\theta}{\partial w_1} = \frac{\partial\theta}{\partial w_2} = \cdots = \frac{\partial\theta}{\partial w_N} = 0$$

$$\frac{\partial\theta}{\partial w_i} = -(\lambda w_1\sigma_{1i} + \lambda w_2\sigma_{2i} + \cdots + \lambda w_i\sigma_i^2 + \cdots \lambda w_N\sigma_{Ni}) + \overline{r_A} - \overline{r_B}$$

$$= 0$$

$$\lambda = \left(\overline{r_A} - \overline{r_B}\right)/\,\sigma_P^2$$

- where w_i can then be obtained from the solution of a set of simultaneous equations

Short sales not allowed with riskless borrowing and lending

- Compared to the basic case, the optimization problem involves one extra constraint, i.e.

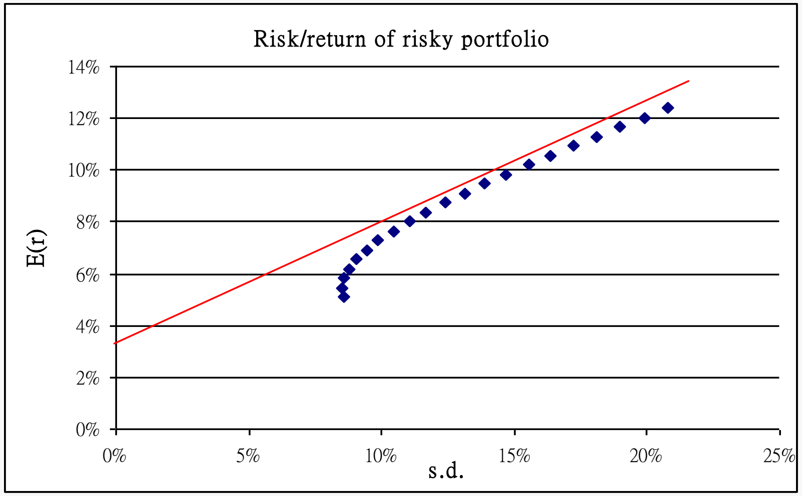
$$w_i \geq 0 \text{ for all } i$$

- This becomes a quadratic programming problem as the objective function to be optimization involves quadratic terms

Efficient frontier example

sigma 1	20.8%	return 1	12.4%
sigma 2	8.6%	return 2	5.1%
rho	0.30	rate	3.0%

	w1	w2	sigma	E(r)	slope
1	0	1	8.60%	5.10%	0.2442
2	0.05	0.95	8.54%	5.47%	0.2886
3	0.1	0.9	8.60%	5.83%	0.3292
4	0.15	0.85	8.77%	6.20%	0.3644
5	0.2	0.8	9.05%	6.56%	0.3936
6	0.25	0.75	9.42%	6.93%	0.4166
7	0.3	0.7	9.89%	7.29%	0.4340
8	0.35	0.65	10.42%	7.66%	0.4466
9	0.4	0.6	11.03%	8.02%	0.4552
10	0.45	0.55	11.69%	8.39%	0.4608
11	0.5	0.5	12.39%	8.75%	0.4641
12	0.55	0.45	13.13%	9.12%	0.4657
13	0.6	0.4	13.90%	9.48%	0.4660
14	0.65	0.35	14.71%	9.85%	0.4655
15	0.7	0.3	15.53%	10.21%	0.4643
16	0.75	0.25	16.37%	10.58%	0.4626
17	0.8	0.2	17.23%	10.94%	0.4607
18	0.85	0.15	18.11%	11.31%	0.4586
19	0.9	0.1	19.00%	11.67%	0.4564
20	0.95	0.05	19.89%	12.04%	0.4542
21	1	0	20.80%	12.40%	0.4519



Finding the CAL

Use Solver, maximize cell M19 (slope of the CAL) by changing cell I19

w1	w2	sigma	E(r)	slope
0.59042	0.40958	13.75%	9.41%	0.466

Minimum variance portfolio

Use Solver, minimize cell K24 (sigma) by changing cell I24

w1	w2	sigma	E(r)
0.05083	0.94917	8.54%	5.47%

Choice of a risky portfolio

- Depending on the risky assets available, the optimal portfolio on the efficient frontier can be chosen via the methods described earlier

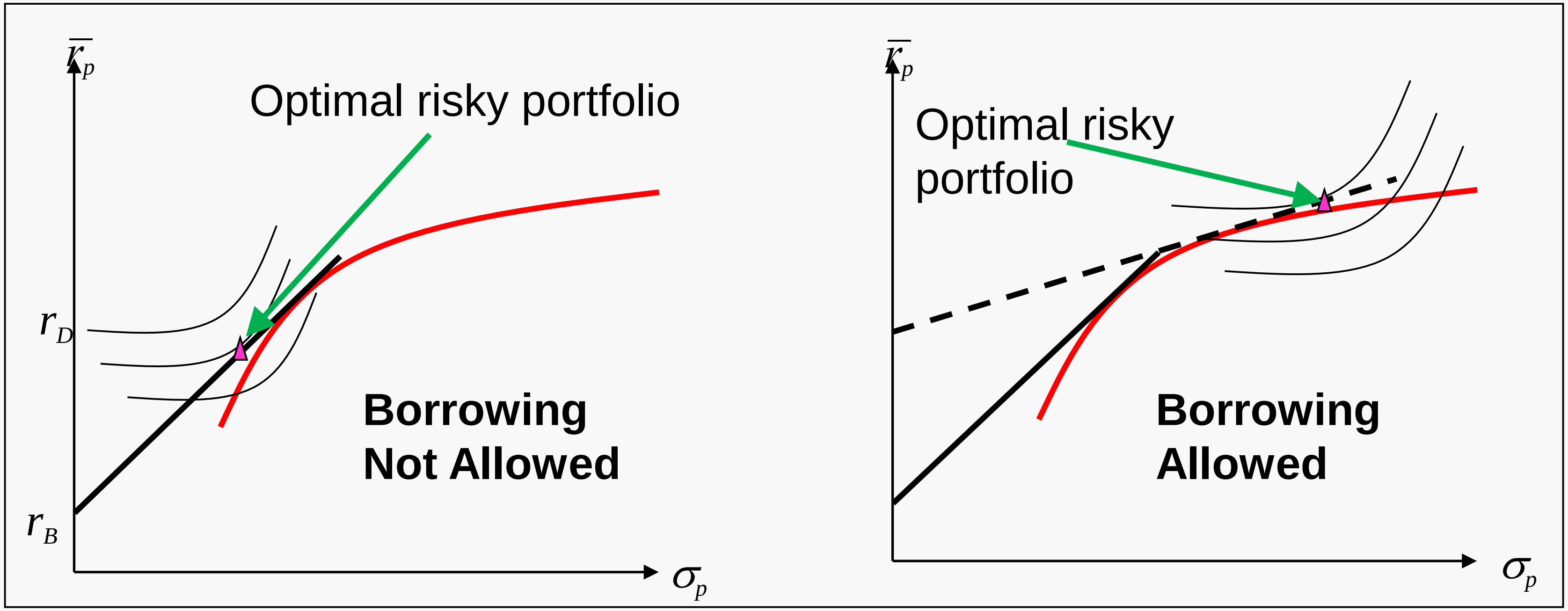
- One may pursue a *passive* strategy, where there is no need to engage in any security analysis
  - The portfolio is a well-diversified portfolio, e.g. a portfolio that mirrors the S&P 500 index – this is sometimes known as the "market portfolio"

- The CAL that combines a risk-free asset (e.g. US 1-month T-bills) and this passive portfolio is called the Capital Market Line (CML)

Step 3: combining with the investor’s preference

- From steps 1 and 2, we have found the optimal CAL, which is the set of the most efficient portfolios that combine a risky portfolio and a risk free asset
- Note that this CAL is applicable to any investor (“the separation property”)
- We could then combine the results with the preference of the investor, by plotting the results above with the set of indifference curves

Finding the optimal combination



- We find the portfolio that is tangential to the highest utility curve
- The choice of optimal portfolio depends on whether borrowing is allowed
  - Identify the portfolio with the highest utility

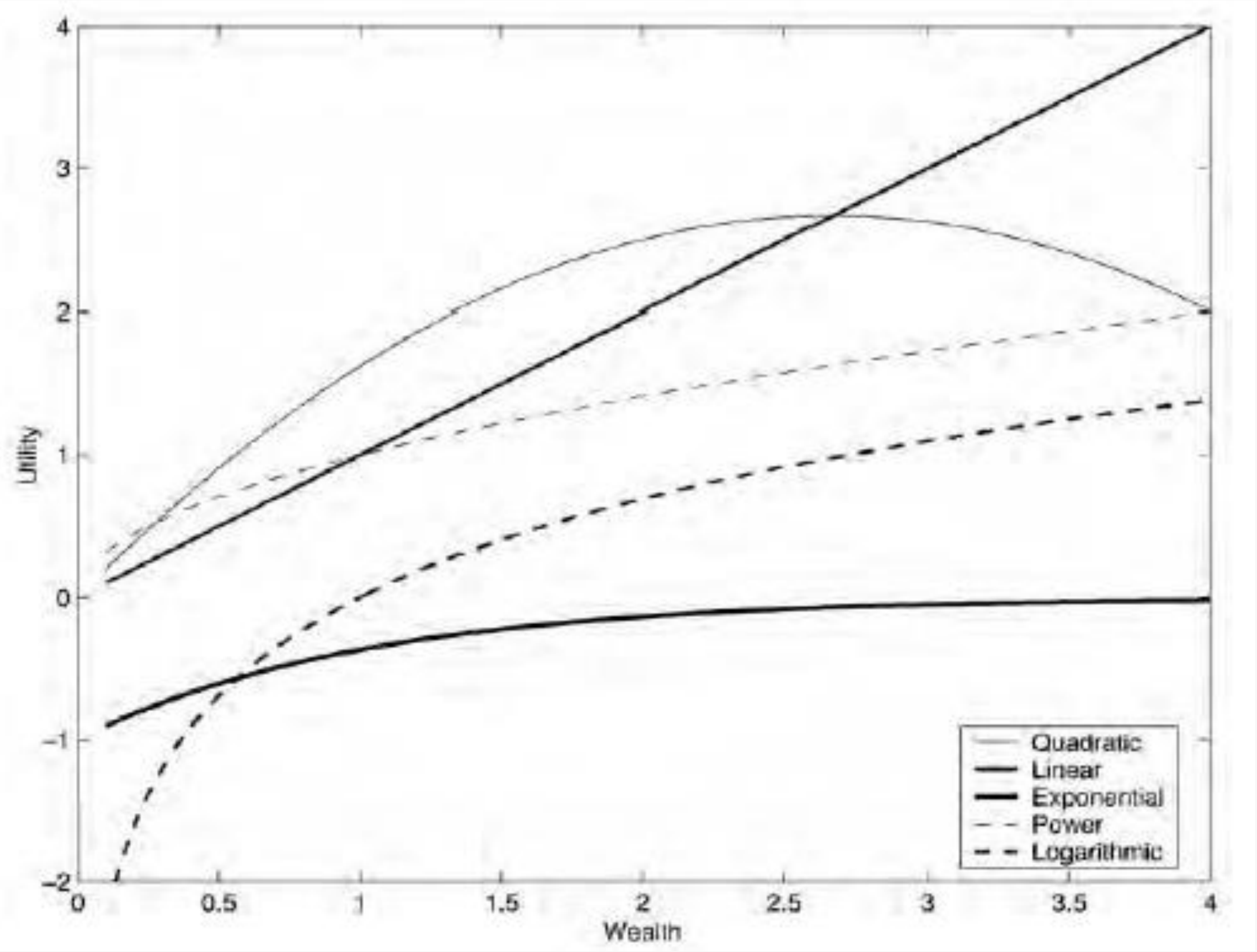
Utility functions

- Describe how entities make decisions when faced with a set of choices

- Assign a (numeric) value to all possible choices faced by the entity

- Maximize the utility based on some given constraints
  - Formalized by von Neumann and Morgenstern in 1944

Examples of utility functions



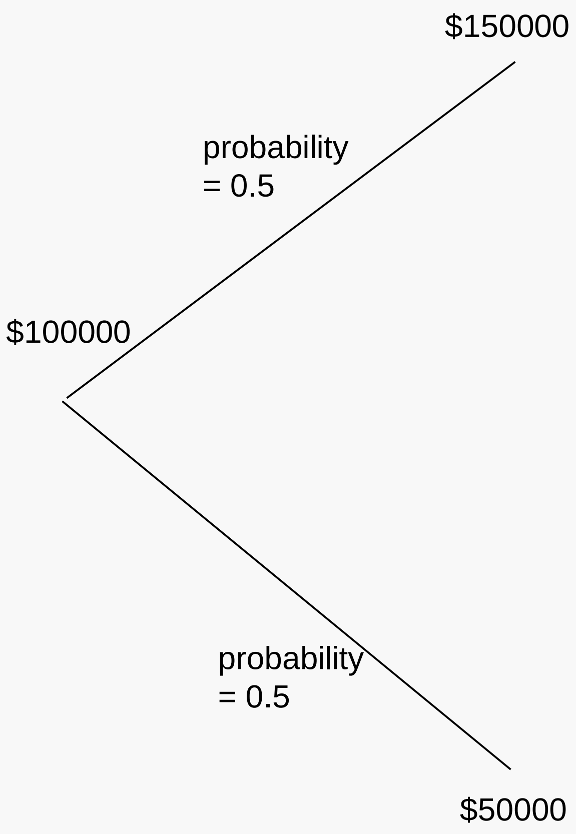
Examples of utility functions

- Linear utility $$u(x) = a + bx$$
- Exponential utility $$u(x) = 1 - e^{-ax}, x > 0$$
- Power utility $$u(x) = x^{\alpha}, 0 < \alpha < 1$$
- Logarithmic utility $$u(x) = a \ln x + b,\ a > 0$$

Utility function and certainty equivalent

- Investors do not assign the same value per dollar to all payoffs
  - The greater their wealth, the less their “appreciation” for each extra dollar
- One way to express this “risk aversion” behavior is through a logarithmic utility function: $U[R(n)] = ln[R(n)]$
- The expected utility value of the game given in the St Petersburg Paradox (lecture 1, slide 38) if we use a logarithmic utility function is:
  - Expected utility = Sum(probability x $ln[R(n)]$) = 0.693
- The certain wealth level necessary to yield this utility value is $2, as ln(2)=0.693 – this is known as the certainty equivalent
  - Therefore, one should play this game if it costs less than $2 to enter the game

Utility function and certainty equivalent



- Expected value
  = 0.5x150000+0.5x50000
  = 100000
- Expected utility
  =0.5xln(150000)+0.5xln(50000)
  = 11.37
- Certainty equivalent:
  ln(CE)=11.37=> CE = 86681.87
  - i.e. this is the amount which has the same utility as if the game is played
- If this game is rejected, the utility is ln(100000)=11.51

Quadratic utility and mean-variance criteria

- The quadratic utility function can be defined as

$$u(x) = ax - \frac{b}{2}x^2$$

- This is only applicable to the range $x \leq a/b$, as the function is only increasing when this relationship is satisfied
  - A good approximation to other types of utility functions within this range
- For b>0, the function is strictly concave, which represents a risk-averse investor
- Mean-variance analysis = maximize expected utility based on the quadratic utility function

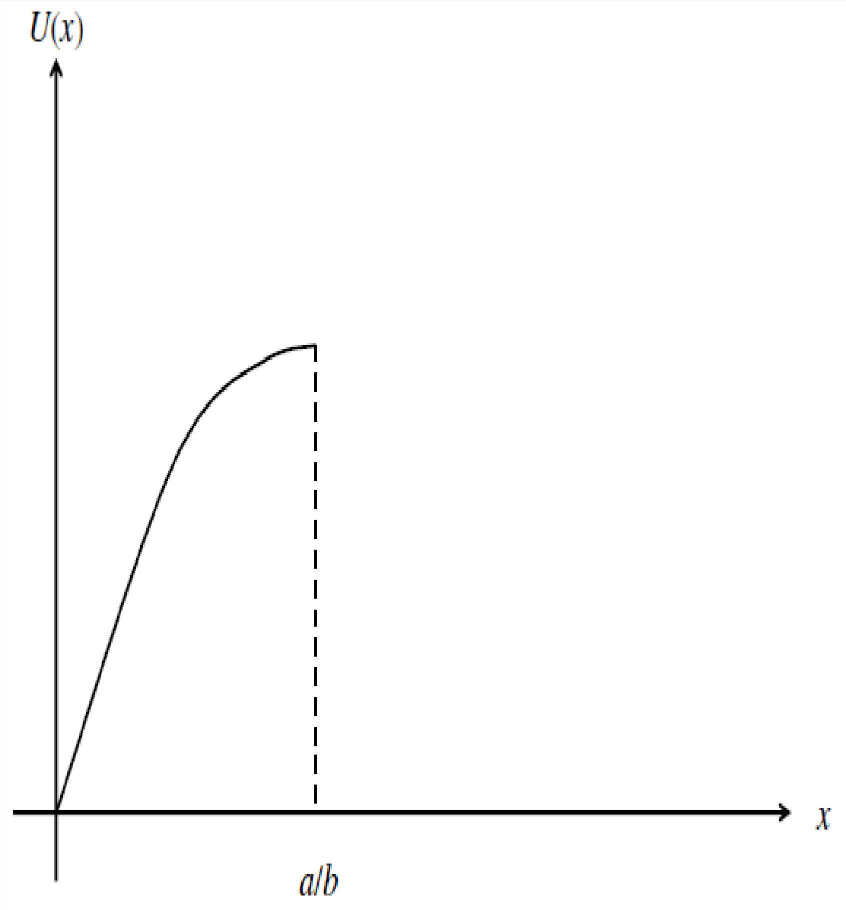
Optimization with expected utility

- Assume the random wealth value of a portfolio is y
- Under the expected utility criterion, we calculate

$$
\begin{aligned}
E[u(y)] &= E\left[ay - \frac{b}{2}y^2\right] \\
&= aE[y] - \frac{b}{2}E[y^2] \\
&= aE[y] - \frac{b}{2}(E[y])^2 - \frac{b}{2}\mathrm{var}(y)
\end{aligned}
$$

- Thus the expected utility is dependent only on the mean and variance of y

Optimization with expected utility



- For a given value of $E[y]$, maximizing $E[u(y)]$ = minimizing $\mathrm{var}(y)$
- For a given $\mathrm{var}(y)$, maximizing $E[u(y)]$ = maximizing $E[y]$
- These are only applicable in the range

$$0 \le x \le a / b$$

What have we achieved so far?

- **Step 1:** Find the efficient frontier of risky assets (e.g. combining stocks and bonds)
- **Step 2:** Find the CAL which has the highest slope tangential to the efficient frontier; hence we find the optimal risky portfolio (i.e. we know the weightings of the assets of the optimal risky portfolio)
- ***Steps 1 and 2 should be applicable to EVERYONE, i.e. everyone should have the same risky portfolio***
- **Step 3:** Find the highest utility curve that is tangential to the CAL; hence we find the optimal weightings of the risk free asset and the risky portfolio

References

- Zvi Bodie, Alex Kane and Alan Marcus, *Investments*, 9th edition, McGraw Hill, 2010
- Edwin Elton, Martin Gruber, Stephen Brown and William Goetzmann, *Modern Portfolio Theory and Investment Analysis*, 7th edition, Wiley, 2007
- David Luenberger, *Investment Science*, Oxford University Press, 1998
- Frank Fabozzi, Petter Kolm, Dessislava Pachamanova, Sergio Focardi, *Robust Portfolio Optimization and Management*, Wiley, 2007

Implementation issues

- How do we reduce the complexity of the problem?
  - index models

- How do we choose assets in the portfolio?
  - Constraints in the selection of assets
  - Selecting a risk free asset
  - International diversification
  - Other practical problems

Why do we need factor models?

- Size of correlation matrix
  - Number of terms = N(N-1)/2 where N is the number of assets
  - E.g. when N = 100, number of terms = 4950, which is difficult to maintain
- Easy to lead to inconsistent result
  - If correlation matrix is not positive definite, negative portfolio variance could be obtained
  - For example

			Correlation matrix		
Asset	s.d.	Weight	A	B	C
A	20%	-1	1.00	0.90	0.90
B	20%	1	0.90	1.00	*0.00*
C	20%	1	0.90	*0.00*	1.00

- Portfolio variance is -0.024
- **This is not possible as variance must be positive**

- To simplify the problem, we assume that the co-movements between stocks are due to one or more common influences or indices (known as the **single-** or **multi-index model**)

Single factor model

- It is observed that covariance between stocks tend to be positive
  - Intuitively this could be explained because stocks follow the same economic factors
- Assume that we combine all influences to a single factor F, so that

$$r_i = E(r_i) + \beta_i F + e_i$$

Return of asset i

Expected return of asset i

Sensitivity of asset i to factor F

Macro factor F

Unanticipated firm specific events

Single index model

- We could use a broad based market capitalization weighted stock index, e.g. the S&P 500, as a proxy for the common macro factor
- The return of the stock is written as

$$R_i = \alpha_i + \beta_i R_M + e_i$$

where R_M is the return of the market index, α_i is the stock specific expected return, e_i is the stock specific unexpected return, β_i is the sensitivity factor to the market index

Properties of the single index model

- By construction, we have the following properties
  - Mean of e_i = $E(e_i)$ = 0
  - Assets are unrelated to each other: $E(e_i e_j)$ = 0
  - Stock specific event unrelated to the market factor:

$$\text{cov}(e_i R_M) = 0 \Rightarrow E\left[(e_i - 0)(R_M - \overline{R_M})\right] = 0$$

- The mean return

$$\begin{aligned} E(R_i) &= E[\alpha_i + \beta_i R_M + e_i] \\ &= E(\alpha_i) + E(\beta_i R_M) + E(e_i) \\ &= \alpha_i + \beta_i \overline{R_M} \end{aligned}$$

Properties of the single index model

- The variance of an asset's return

$$
\begin{aligned}
\sigma_i^2 &= E\left(R_i - \overline{R_i}\right)^2 \\
&= E\left[\beta_i(R_M - \overline{R_M}) + e_i\right]^2 \\
&= \beta_i^2 E(R_M - \overline{R_M})^2 + E(e_i)^2 \\
&= \beta_i^2 \sigma_M^2 + \sigma_{e_i}^2
\end{aligned}
$$

- The variance has two parts: the market-related risk (through $\beta_i \sigma_M$), and the unique risk of the stock (through σ_{e_i})

Properties of the single index model

- The covariance between two assets is

$$\sigma_{ij} = E\left[(R_i - \overline{R_i})(R_j - \overline{R_j})\right]$$
$$= E\left[\left(\beta_i(R_M - \overline{R_M}) + e_i\right)\left(\beta_j(R_M - \overline{R_M}) + e_j\right)\right]$$
$$= \beta_i \beta_j \sigma_M^2$$

- i.e. we only need to know the relationship between each asset and the index in order to calculate the covariance between any two assets
  - Assets move as a common response to market movements

Portfolio characteristics

- With the single index model, the portfolio mean and variance are given as:

$$\overline{R_P} = \sum_{i=1}^{N} X_i \overline{R_i} = \sum_{i=1}^{N} X_i \alpha_i + \sum_{i=1}^{N} X_i \beta_i \overline{R_M}$$

$$\sigma_P^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_M^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} X_i X_j \beta_i \beta_j \sigma_M^2 + \sum_{i=1}^{N} X_i^2 \sigma_{e_i}^2$$

- X_i is the weight of asset i
- The parameters required in their estimation are: α_i, β_i, $(\sigma_{e_i})^2$ for each stock, $\overline{R_M}$ and $(\sigma_M)^2$ for the market
  - No need to maintain the covariance matrix for all stocks

Index model and diversification

- We use a portfolio of N stocks, each with a weight of $1/N$
- Portfolio return is

$$
\begin{aligned}
R_P &= \frac{1}{N}\sum_{i=1}^{N}\left(\alpha_i + \beta_i R_M + e_i\right) \\
&= \frac{1}{N}\sum_{i=1}^{N}\alpha_i + \left(\frac{1}{N}\sum_{i=1}^{N}\beta_i\right)R_M + \frac{1}{N}\sum_{i=1}^{N}e_i \\
&= \alpha_P + \beta_P R_M + e_P \\
\overline{R_P} &= \alpha_P + \beta_P \overline{R_M}
\end{aligned}
$$

- α_P and β_P are the portfolio's α and β respectively

Index model and diversification

- The portfolio variance is

$$\sigma_P^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} X_i X_j \beta_i \beta_j \sigma_M^2 + \sum_{i=1}^{N} X_i^2 \sigma_{e_i}^2$$

$$= \beta_P^2 \sigma_M^2 + \sum_{i=1}^{N} X_i^2 \sigma_{e_i}^2$$

$$= \beta_P^2 \sigma_M^2 + \frac{1}{N}\sum_{i=1}^{N} \frac{1}{N} \sigma_{e_i}^2$$

- As N becomes larger the second term disappears

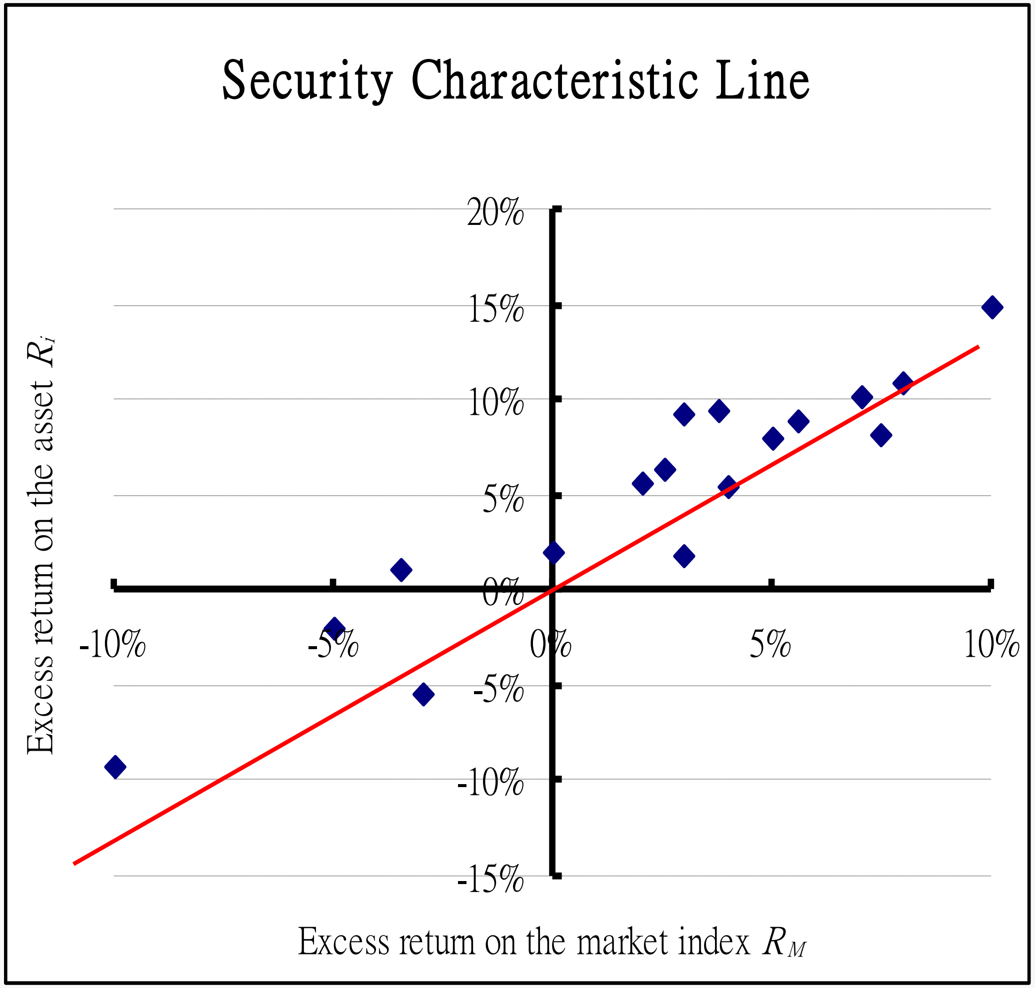
$$\sigma_P \approx \beta_P \sigma_M = \sigma_M \sum_{i=1}^{N} \frac{1}{N} \beta_i$$

- **The contribution of an asset to the risk of the portfolio is through β_i**

Advantages and disadvantages

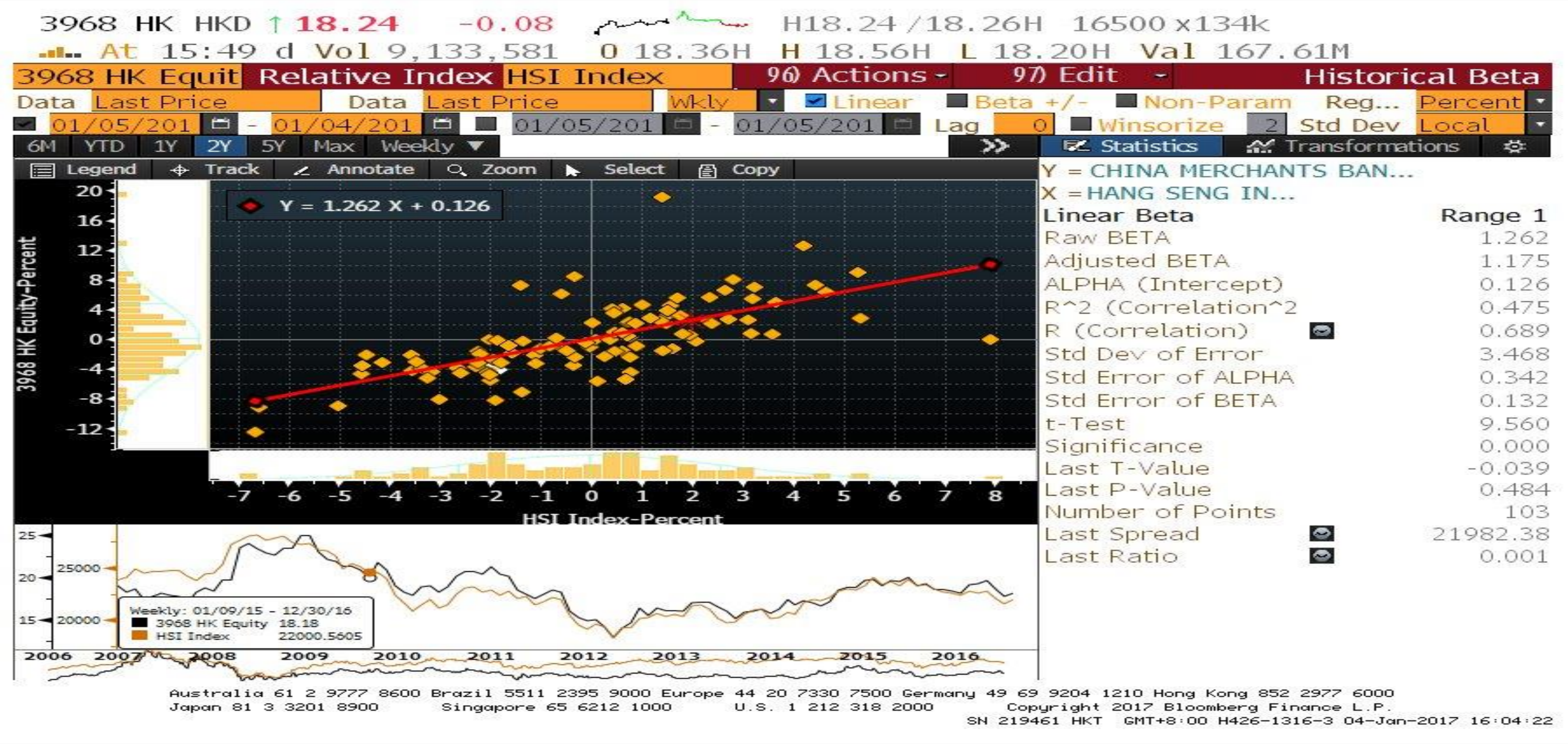
- For large universes of securities, much less data need to be maintained
- Affect who is going to perform security analysis
  - Analysts only need to specialize on one sector and work out a stock's correlation with a market index
  - E.g. the banking analyst only needs to concentrate on analyzing the performance of banks, not the relationship between banks and all other sectors
- But be aware of the much "simplified" world
  - E.g. there may exist industry specific events (not security specific or market specific, therefore not captured by either the macro factor F or the micro factor e)

Estimating the index model



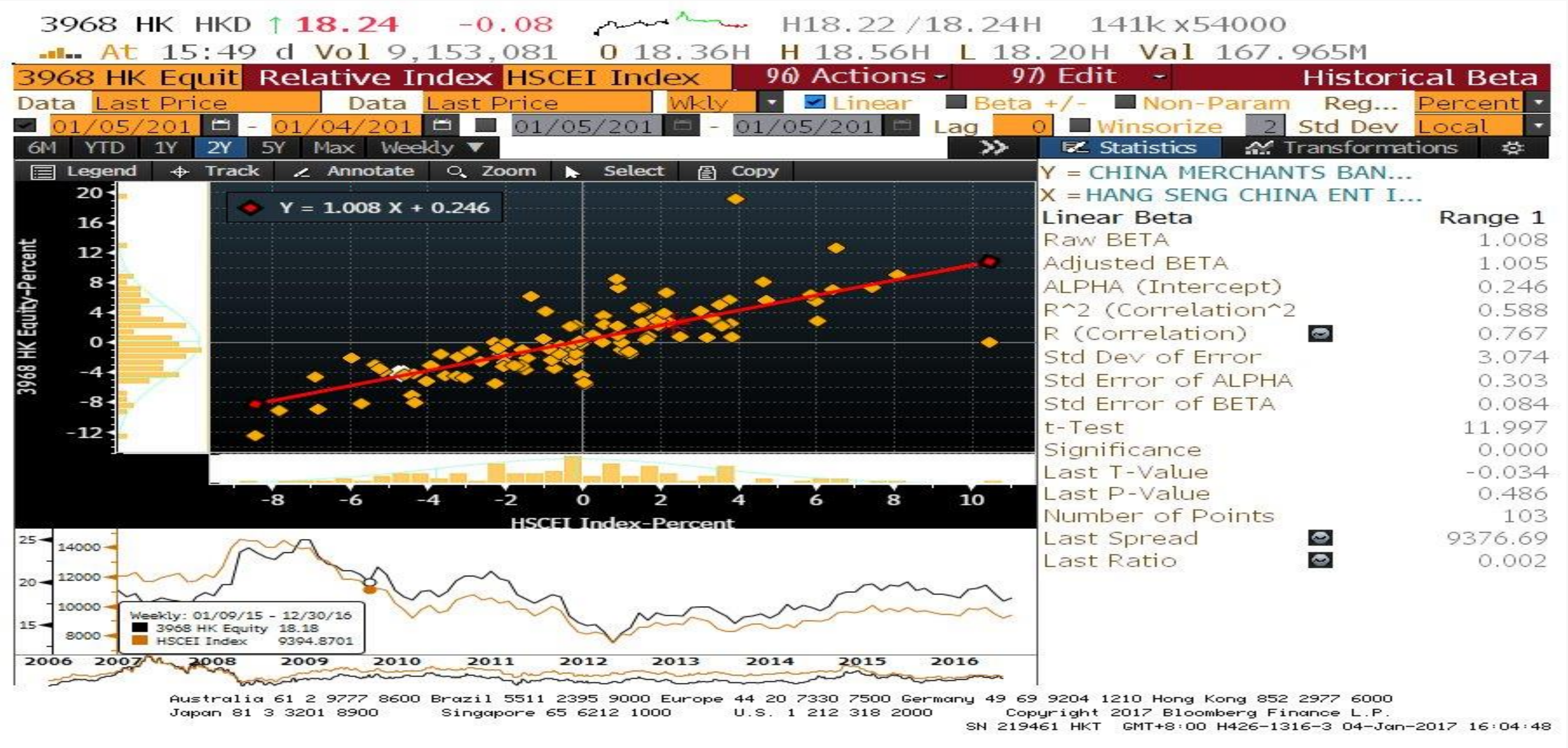
- Use data from different periods and plot a scattered diagram
- Find the "best-fit" straight line via linear regression
- This line is known as the **Security Characteristic Line (SCL)**, and the slope of this line is the best estimate for β.

Selecting the appropriate index



- Different results can be obtained if an irrelevant benchmark is chosen
- Source: Bloomberg

Selecting the appropriate index



- Source: Bloomberg

Multifactor model

- Instead of limiting to a single market factor, we could extend the model to include multiple factors
- One form is

$$R_i = \alpha_i + \beta_{iM} R_{iM} + \beta_{i1} R_{i1} + \beta_{i2} R_{i2} + \dots + e_i$$

- Examples of factors include economic growth, business cycle, long and short term interest rates, inflation, industrial production, strength of US dollar etc.
- For the model to work well, the factors should be independent from each other

Multifactor model

- Empirical studies were carried out to test the performance of these models
- Numerous possibilities exist, so we could not draw a definitive conclusion
- Some models give reasonable results, but many multifactor models do not perform as well as a single index model; this illustrates the difficulty in choosing appropriate factors (e.g. factors give rise to high explanatory power in one period may become unsuitable in the future)

Determining the opportunity set

- The importance of security analysis
  - Select individual stocks/bonds which have the best risk/return characteristics
  - Accurate forecasts are essential

- Other selecting criteria may prevail
  - Dividend assumption
  - Decision on the available set, e.g. ethical consideration, choice of international stocks or sectors

Portfolio selection in practice

- The original constraints in Markowitz's formulation serve as the starting point

- Complex practical issues such as the inclusion of transaction costs, optimization across multiple client accounts, and the need to take tax consequences into considerations

- A number of different constraints can be added to the optimization problem

Linear and quadratic constraints

- Long-only constraint

$$w_i \geq 0 \text{ for all } i = 1 \cdots N$$

- Turnover constraint
  - High turnover can result in large transaction costs
  - Liquidity may be an issue
  - Can have constraint of the form $|x_i| \leq \alpha U_i$
    where U_i is the average daily volume of a stock, x_i is the amount to be traded, and α is a percentage, e.g. 5%

Linear and quadratic constraints

- Holding constraint
  - A well-diversified portfolio should not have large concentration in any single asset, industry, sector, or country
  - Sometimes this constraint is imposed by the regulator
  - The constraint will take the form

$$L_i \leq w_i \leq U_i$$

    where L_i and U_i are the lower and upper bounds of the holding
  - The following constraint can also be introduced

$$L_i \leq \sum_{j \in I_i} w_j \leq U_i$$

    which limits the exposure to a particular sector or industry I_i

Linear and quadratic constraints

- Benchmark exposure and tracking error constraint
  - The fund manager may need to measure their performance against a benchmark, e.g. a stock index
  - A common constraint is to limit the deviations of the portfolio weights from the benchmark weights

$$\left\| w - w_{benchmark} \right\| \leq M$$

  - Another constraint is to limit the tracking error against the benchmark, e.g. by limiting the variance of the difference between the return of the portfolio and the benchmark

$$\mathrm{var}(R_p - R_{benchmark}) \leq \sigma^2$$

Combinatorial constraint

- Introduce the notation

$$\delta_i = \begin{cases} 1, \text{ if } w_i \neq 0 \\ 0, \text{ if } w_i = 0 \end{cases}$$

where w_i denotes the portfolio weight of the i^{th} asset

- Cardinality constraint
  - A portfolio manager may want to restrict the number of assets in the portfolio, e.g. use a smaller number of assets to replicate a benchmark

$$\sum_{i=1}^{N} \delta_i = K, \; K << N$$

- In order to solve the optimization problem with combinatorial constraints, more sophisticated algorithms are required

Availability of a risk free asset

- For the US market, investors usually consider the short dated US Treasury bills as risk free assets
  - Short term instruments, therefore assumed to be held to maturity and no price risk
- Other short term money market instruments also used

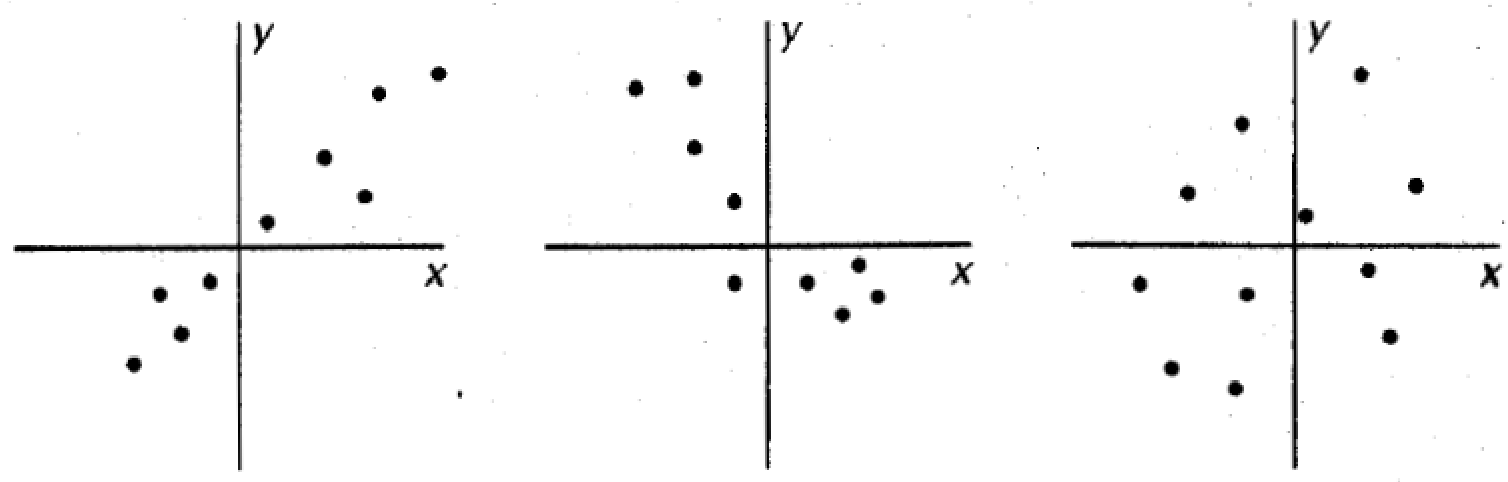
Other practical considerations

- Assumptions behind the theory
  - The mean-variance criterion
    - Samuelson (1970) showed that, although the distribution of returns shows higher moments, disregarding these would not affect portfolio choices
  - Quadratic utility functions
    - In Markowitz original work in 1959, it was shown that quadratic utility functions are good local approximates of more complicated forms
  - Normal distribution of returns
    - Even though individual asset returns may not be normal, portfolio returns will resemble a normal distribution

Other practical considerations

- Difficulty in obtaining reliable estimates for variances/covariances
  - Changing variance due to change of circumstances ("heteroscedasticity")
  - Changing correlation between assets
  - Increasing correlation in turbulent periods
  - Using history to forecast the future
    - The forecast is usually more better for variances than for the return; but is it accurate enough?
- Use more sophisticated econometric models, e.g. GARCH, for better predicting the future variances

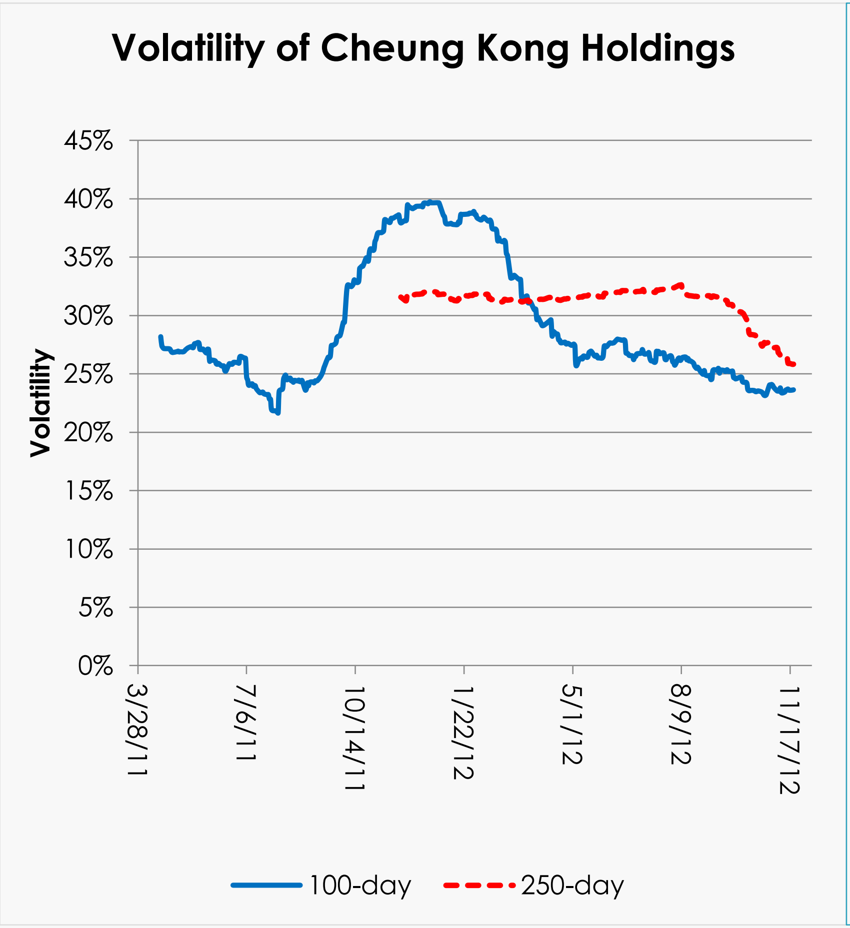
Positive and negative correlation

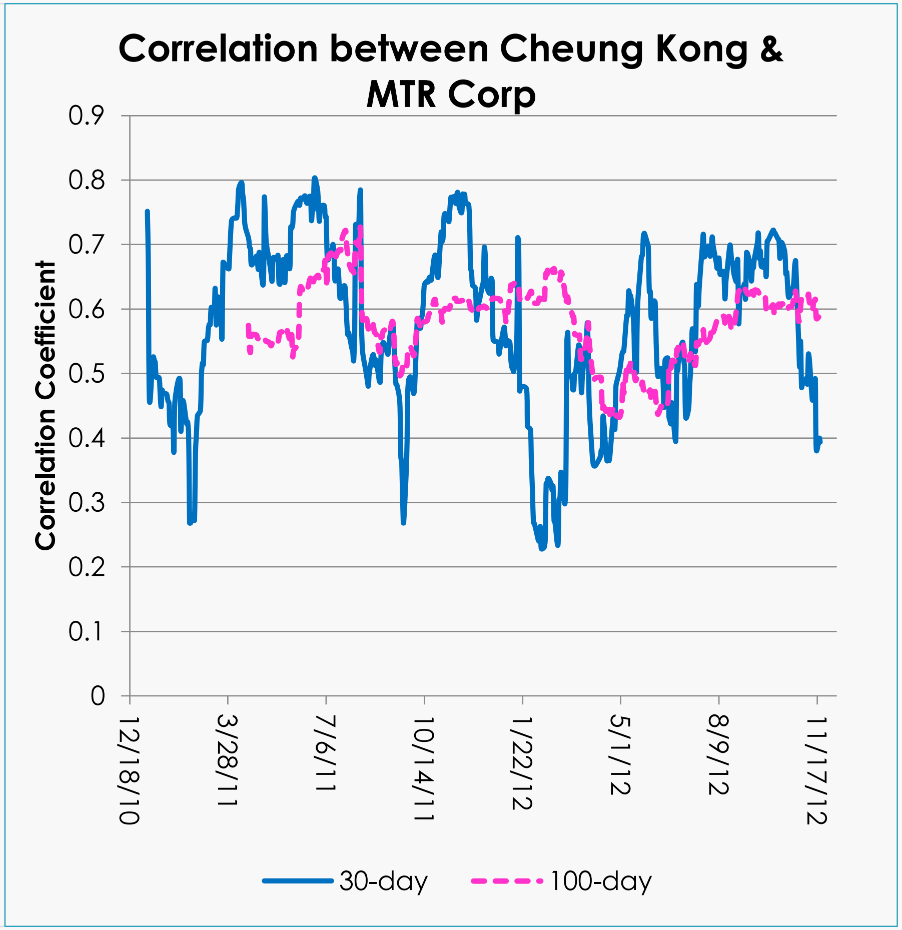


(a) Positively correlated (b) Negatively correlated (c) Uncorrelated

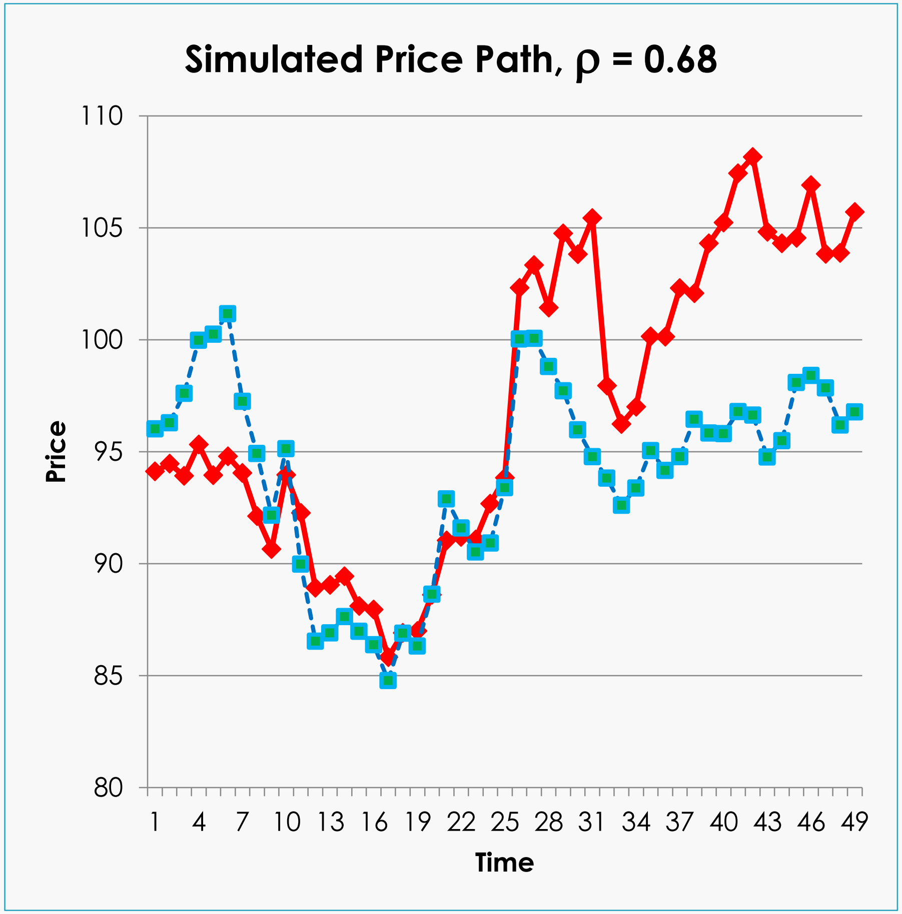
- If two variables are positively correlated, positive deviation from the mean of one variable would give rise to a <u>higher tendency</u> of a positive deviation from mean of the other variable

Variance/covariance graphs





Understanding correlation



- While it is easy to interpret a correlation of +1 or -1, intermediate values are hard to interpret in practice
- It is also complicated by the fact that the variance of the assets may not be the same, and could also be different at different times
- Data: Volatilities in the first half = 30%/40% for the two variables, volatilities = 60%/30% in the second half

Rationality arguments

- Investor's could be inconsistent, i.e. they make choices in an irrational (or even random) manner
- It is sometimes impossible to quantify their preferences in terms of a utility function
- Behavioral finance topics
  - Overconfidence, framing, mental accounting, regret avoidance etc.
  - Daniel Kahneman shared the Nobel prize in Economics in 2002 (joint work with Amos Tversky)

How could we use the MPT?

- According to theory, everyone should hold the same risky portfolio (in the same proportion)
  - *if this is the case, many people would lose their jobs!!!*
- Because of various reasons, many investors may be reluctant to invest in some asset classes
  - E.g. Hong Kong investors may find it inconvenient to invest in stocks in other countries
- It is often impractical to include all kinds of assets as inputs to the optimization process; we could still use Modern Portfolio Theory to optimize within each asset class

Problems with MPT

- Mean-variance optimization using historical data often yields unrealistic results
  - Unintuitive weights of assets in the portfolio
  - Concentrated in a few representative stocks
  - Small changes in the input parameters lead to drastic changes in portfolio allocations

- Note that the optimization results are less sensitive to errors in estimating variance, and that the population covariance is more stable over time than estimating returns from historical data