CMSC 5718 INTRODUCTION TO COMPUTATIONAL FINANCE

Lecture 4

Lecture outline

- Asset prices in equilibrium conditions
  - Capital Asset Pricing Model
  - Arbitrage Pricing Theory
- Testing the CAPM and APT

Equilibrium prices

- How would investors behave on aggregate?

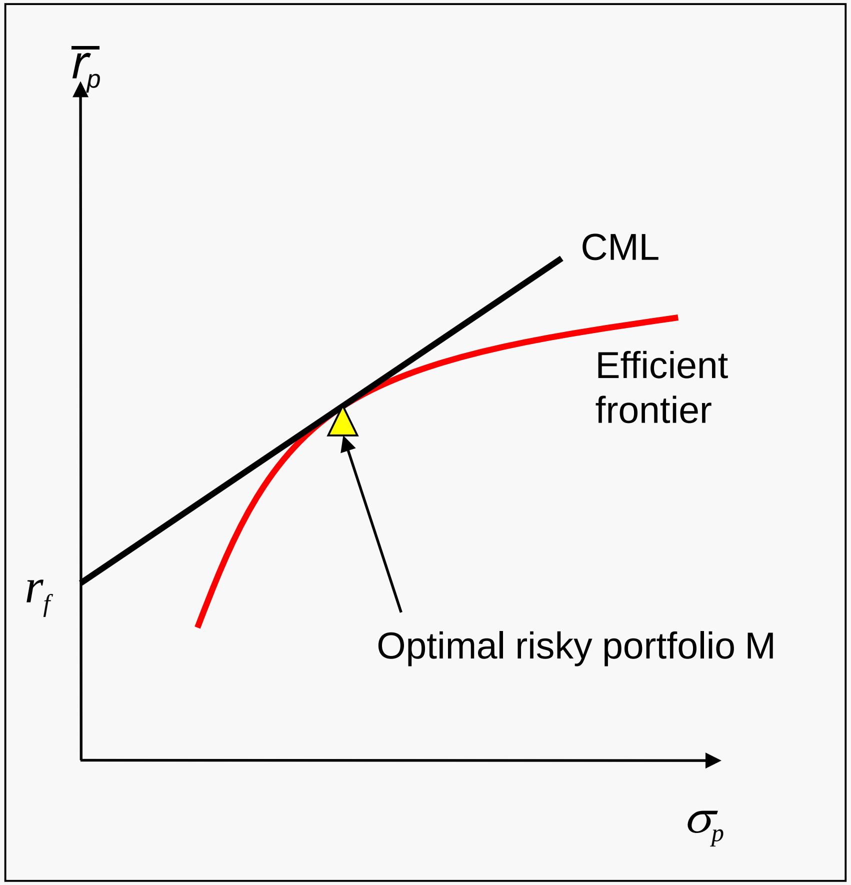
- We would like to answer the following question:
  - What is the relationship between the expected return and risk for any asset when markets are in equilibrium?

- One of the most celebrated financial theories is the **Capital Asset Pricing Model (CAPM)** (developed independently by Treynor, Sharpe, Lintner and Mossin)

Assumptions in CAPM

- The usual assumptions in portfolio theory
  - Mean variance optimization, non-satiation, risk averse investors, availability of risk free assets, zero taxes and transaction costs, availability of assets in small fractions, unrestricted riskless lending and borrowing, no impact costs in transactions
- All investors plan for one identical holding period, and they are price takers, i.e. their actions will not affect the traded prices of the securities
- All investors share the same economic view of the world; they have **homogeneous expectations** in regard to the expected returns, standard deviations, and covariances of securities

The Market Portfolio



- Because investors have the same expectations, they all face the same risky portfolio ***M***; this is known as the **Market Portfolio**
- The line combining the risk free asset with this portfolio M is known as the **Capital Market Line (CML)**

$$\overline{r_P} = r_f + \frac{\left(\overline{r_M} - r_f\right)}{\sigma_M}\sigma_P$$

Properties of the market portfolio

- Every security (stock) in the market would be included in the market portfolio
  - Since everyone is holding the same portfolio, if a security is left out, there would be absolutely no demand; stock price would fall and return increases, until it becomes attractive
  - Proportion of each security corresponds to the individual security's market value

- When all the price adjustment stops, the market will be brought into equilibrium

- Market portfolio is an inherently unobservable portfolio: as a proxy, a widely represented market index, e.g. the S&P 500, is commonly used

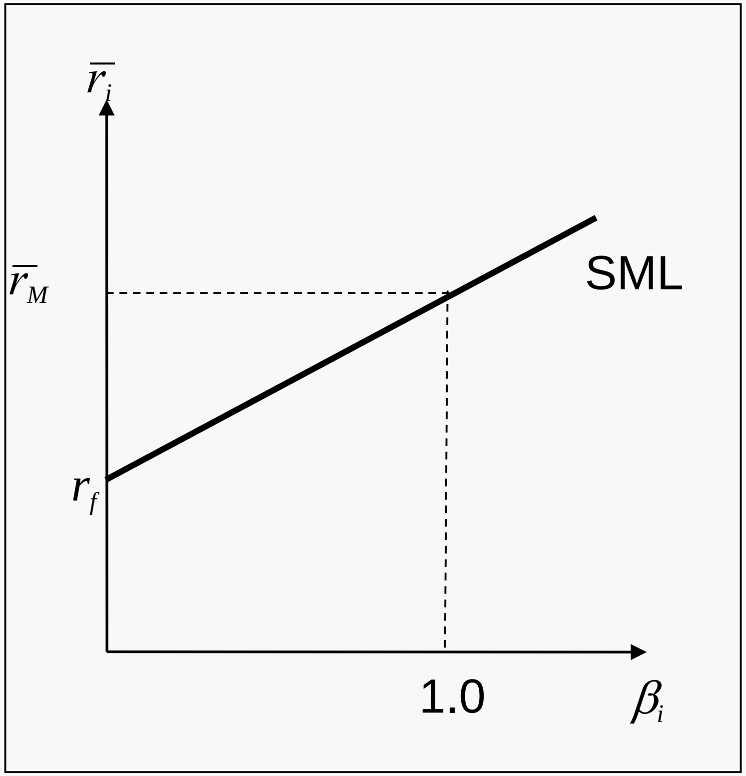
Implication for individual risky assets

- Remember for a large portfolio, the portfolio variance is

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} w_i w_j \sigma_{jk} \approx \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} w_i w_j \sigma_{jk}$$

- From the point of view of the portfolio, an asset's contribution to the variance of the market portfolio is determined by its covariance with the market portfolio = $w_A \sigma_{AM}$
- ***Accordingly, for an investor, the relevant measure of the asset's risk is its covariance with the market portfolio, not its variance***
- It follows that assets with larger σ_{AM} will have to provide proportionately larger returns, otherwise it would contribute to the risk of the market portfolio while not contributing to the return – it would be better to delete this asset from the market portfolio

CAPM and the Security Market Line (SML)



- The ***equilibrium relationship*** between risk and return of any asset is (by definition)

$$\overline{r_i} = r_f + \frac{\left(\overline{r_M} - r_f\right)}{\sigma_M^2}\sigma_{iM}$$
$$= r_f + \beta_i\left(\overline{r_M} - r_f\right)$$

Expected return = risk free rate + beta x Expected Market Risk Premium

- All assets should plot on this **Security Market Line** **(NOTE THE AXES)**
- For example, if the expected return of a market portfolio is 12%, risk free rate is 4%, and a stock has a beta of 1.2, its expected return is 4% + 1.2 x (12% – 4%) = 13.6%

A simple proof of CAPM

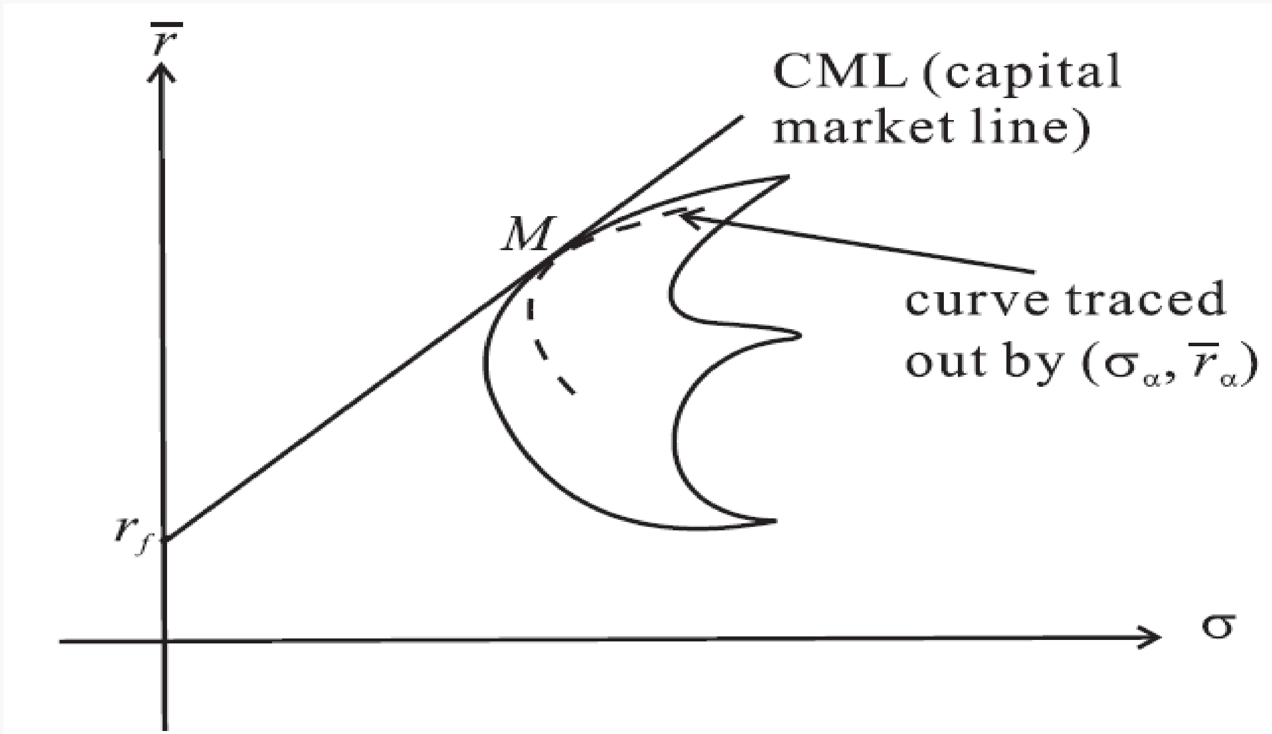
- Consider a portfolio that invests a portion α in asset i and a portion $(1-\alpha)$ in the market portfolio M

- Expected return: $\overline{r_\alpha} = \alpha \overline{r_i} + (1-\alpha)\overline{r_M}$

- Standard deviation:

$$\sigma_\alpha = \left[\alpha^2 \sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2 \sigma_M^2\right]^{1/2}$$

A simple proof of CAPM



- In the $\bar{r}-\sigma$ diagram, α=0 corresponds to the market portfolio M
- The curve obtained by varying α must be tangential to the CML at M

A simple proof of CAPM

$$\frac{d\overline{r_\alpha}}{d\alpha} = \overline{r_i} - \overline{r_M}$$

$$\frac{d\sigma_\alpha}{d\alpha} = \frac{\alpha\sigma_i^2 + (1-2\alpha)\sigma_{iM} + (\alpha-1)\sigma_M^2}{\sigma_\alpha}$$

$$= \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M} \text{ when } \alpha = 0$$

$$\left.\frac{d\overline{r_\alpha}}{d\sigma_\alpha}\right|_{\alpha=0} = \frac{d\overline{r_\alpha}/d\alpha}{d\sigma_\alpha/d\alpha} = \frac{(\overline{r_i} - \overline{r_M})\sigma_M}{\sigma_{iM} - \sigma_M^2} = \frac{\overline{r_M} - r_f}{\sigma_M} \text{ (slope of CML)}$$

$$\overline{r_i} = r_f + \left(\frac{\overline{r_M} - r_f}{\sigma_M^2}\right)\sigma_{iM} = r_f + \beta_i(\overline{r_M} - r_f)$$

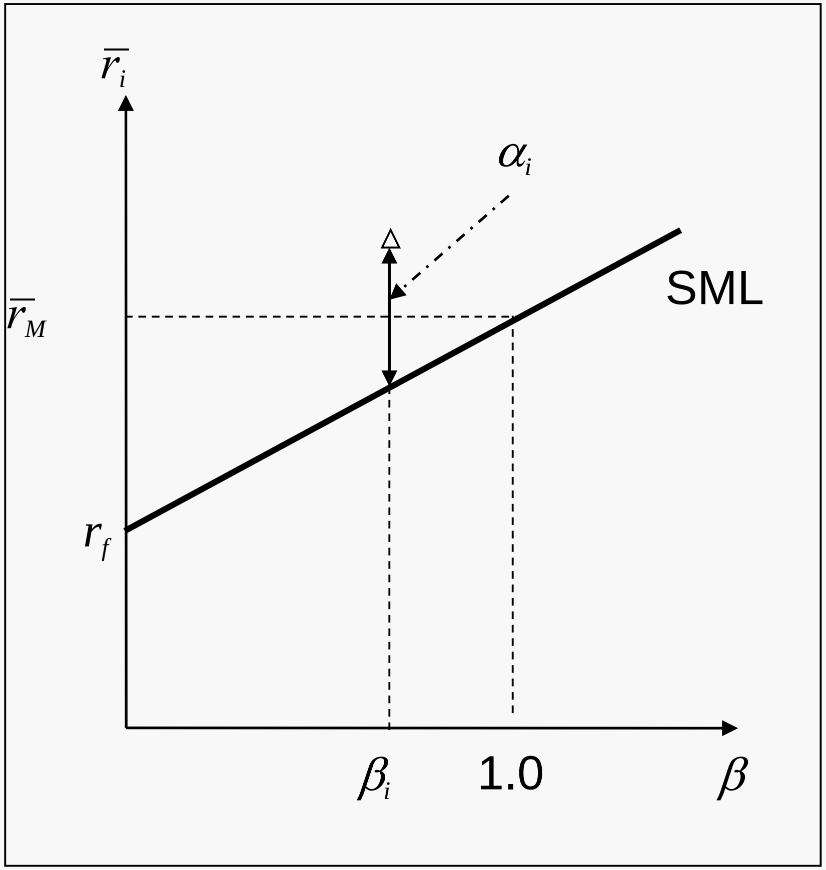
Using the CAPM: example

- $\bar{r}_M$=12%, r_f=7%, σ_M=32%
- i) what is the equation of the CML?
- ii) if $\bar{r}_P$ = 18%, what is the standard deviation of the portfolio?
- iii) how can you allocate $1000 to achieve the portfolio in (ii) above?
- iv) what is the beta of this portfolio?

Using the CAPM: example

- i) $$\overline{r_P} = r_f + \left(\frac{\overline{r_M} - r_f}{\sigma_M}\right)\sigma_P = 0.07 + 0.1562\sigma$$
- ii) $$\sigma_P = \frac{0.18 - 0.07}{0.1562} = 70.42\%$$
- iii) $$\overline{r_P} = \alpha \overline{r_M} + (1-\alpha) r_f$$
$$\Rightarrow \alpha = \frac{\overline{r_P} - r_f}{\overline{r_M} - r_f} = \frac{0.18 - 0.07}{0.12 - 0.07} = 2.2,\ 1 - \alpha = -1.2$$
  - i.e. the investor should short \$1200 of the risk free asset and long \$2200 of the market portfolio
- iv) since the beta of the risk free asset is 0, the beta of the portfolio equals the weight of the investment on the market portfolio, i.e. $\beta = \alpha = 2.2$

How do we use the SML?



- Given the beta of an asset, we could find the (equilibrium) expected return
- If the forecast expected return is higher than the equilibrium expected return, the security is mis-priced in the market (i.e. there is a disequilibrium)
- This extra return is usually known as **alpha**

$$\alpha_i = \bar{r}_i - \left[r_f + (\bar{r}_M - r_f)\beta_i \right]$$

- It is the job of the security analysts to search for stocks with excess alphas

How do we use the SML?

- (i) The SML is often used in capital budgeting decisions: for a firm considering a new project, the expected return represents a hurdle rate for the amount of risk taken (measured by beta)

- (ii) The expected return given by the SML could give guidance for assets not yet traded in the market e.g. IPO stocks

Using the SML: example

- Question:
  - The market expected return is 12% and the risk free rate is 4%
  - Stock A has a beta of 0.7 and an expected return of 10.5%
  - Stock B has a beta of 1.3 and an expected return of 13.7%
  - Which stock is a better buy?
- According to CAPM
  - r_A = 4% + 0.7 x (12% - 4%) = 9.6%
  - r_B = 4% + 1.3 x (12% - 4%) = 14.4%
- Expected return of A is above the SML and the expected return of B is below the SML; hence A is a better buy

Using the SML: example

- Risk free rate r_f = 8%, expected return of a market portfolio $E(r_M)$ = 16%; a firm considers a project that is expected to have a beta of 1.3. Should the project go ahead if the expected rate of return is 19%?
- The required rate of return is

$$r_f + \beta_i(E(r_M) - r_f) = 8\% + 1.3(16\% - 8\%) = 18.4\%$$

- If the expected rate of return of the project is 19%, the project should be accepted because it is higher than the required return (or known as the project's hurdle rate)
  - But since it is so close, other factors may be considered

CAPM and the index models

- If we have a single index model

$$R_i = \alpha_i + \beta_i R_M + e_i$$

- The covariance between stock i and the market index is

$$\sigma_{iM} = \beta_i \sigma_M^2 \Rightarrow \beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

- Referring to the equation of SML on slide 8, the same term for β is found
- Therefore we could estimate the $\boldsymbol{\beta}$ in CAPM by estimating $\boldsymbol{\beta}$ using a single index model, only assuming that the market index is used as a proxy for the market portfolio in CAPM

How do we estimate beta in CAPM?

- The actual excess return of stock i is

$$r_i - r_f = \alpha_i + \beta_i (r_M - r_f) + e_i$$

- This is re-written in the form

$$\begin{aligned} r_i &= \alpha_i + (1-\beta_i) r_f + \beta_i r_M + e_i \\ &= a + \beta_i r_M + e_i \end{aligned}$$

- We could perform a regression of r_i against r_M, and the slope gives β_i (a time-series regression)
- Note that the intercept of this line is an estimate of

$$\alpha_i + (1-\beta_i) r_f$$

Beta of a portfolio

- A portfolio consists of n risky assets with weights w_1, w_2, …, w_n:

$$r_P = \sum_{i=1}^{n} w_i r_i, \operatorname{cov}(r_P, r_M) = \sum_{i=1}^{n} w_i \operatorname{cov}(r_i, r_M)$$

$$\beta_P = \frac{\operatorname{cov}(r_P, r_M)}{\sigma_M^2} = \frac{\sum_{i=1}^{n} w_i \operatorname{cov}(r_i, r_M)}{\sigma_M^2} = \sum_{i=1}^{n} w_i \beta_i$$

- i.e. the portfolio beta β_P is the weighted average of the betas of the assets in the portfolio
- Relationship followed by the portfolio:
  - Given $\overline{r_P} = \sum_{i=1}^{n} w_i \overline{r_i}, \ \sum_{i=1}^{n} w_i = 1, \ \overline{r_i} - r_f = \beta_i(\overline{r_M} - r_f)$

$$\Rightarrow \overline{r_P} - r_f = \beta_P(\overline{r_M} - r_f)$$

Some results in CAPM

- From the SML, we can deduce that

$$\frac{\overline{r_i} - r_f}{\beta_i} = \frac{\overline{r_j} - r_f}{\beta_j} = \overline{r_M} - r_f$$

  i.e. the expected excess return above the risk free rate normalized by the beta of the asset is a constant for all assets

- Let P be an efficient portfolio on the CML, with α allocated to the market portfolio: $r_P = \alpha r_M + (1-\alpha) r_f$
- We have the following results

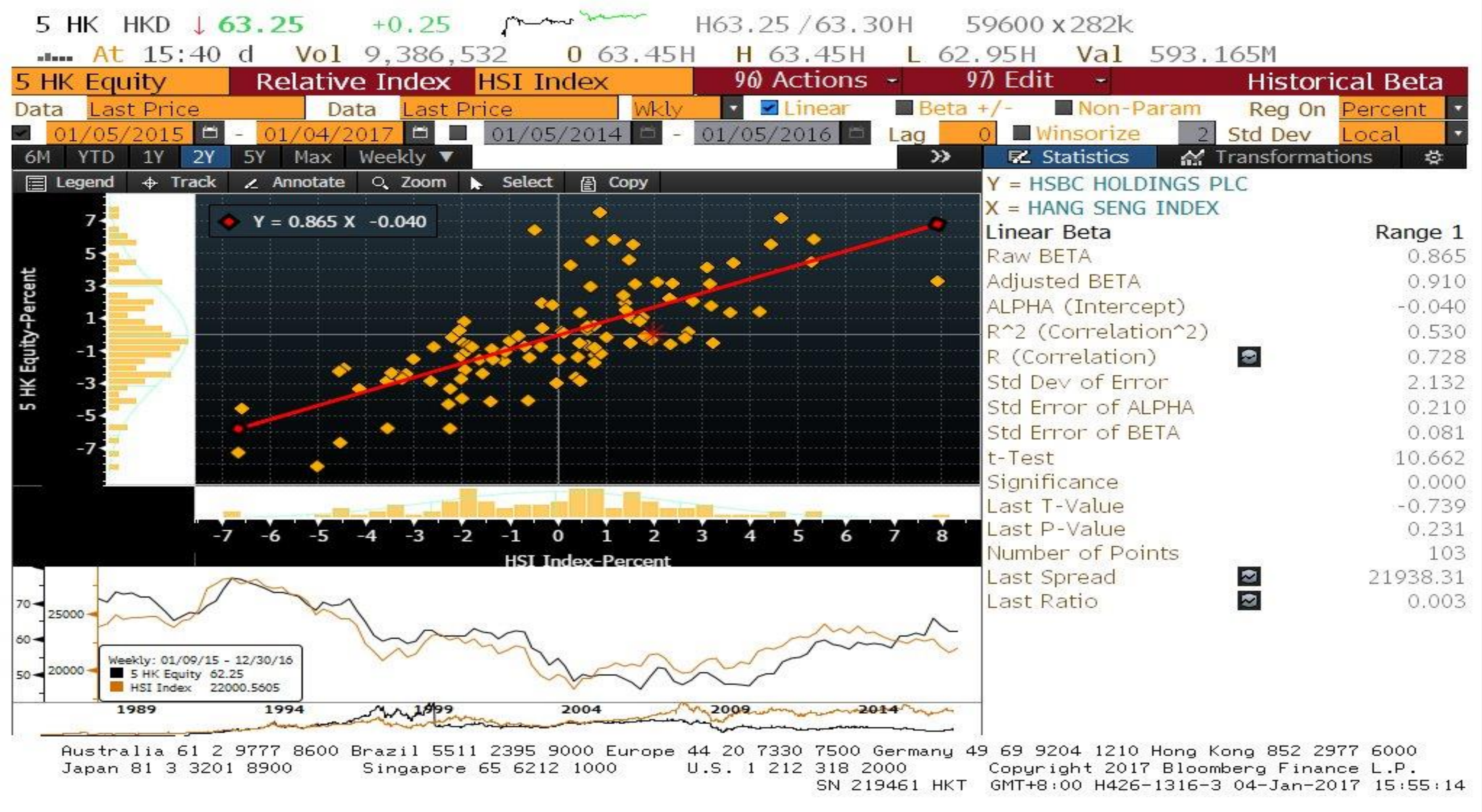
$$\sigma_P^2 = \alpha^2 \sigma_M^2, \ \operatorname{cov}(r_P, r_M) = \operatorname{cov}(\alpha r_M + (1-\alpha) r_f, r_M) = \alpha \sigma_M^2$$

$$\rho_{PM} = \frac{\operatorname{cov}(r_P, r_M)}{\sigma_P \sigma_M} = 1, \ \beta_P = \frac{\operatorname{cov}(r_P, r_M)}{\sigma_M^2} = \alpha$$

Index models in practice

- Published historical beta are available for many US stocks
  - But should we use historical beta to forecast future beta?
- Various adjustment methods are available, e.g.
  - Blume: beta would move towards 1, hence forecast beta = p + q x (historical beta), e.g. forecast beta = 0.33 + 0.67 x (historical beta)
  - Vasicek: adjust each stock's beta towards an "average beta"
  - Rosenberg: use firm-specific factors to forecast beta, e.g. firm size, dividend yield, earnings growth
- No conclusive evidence whether these adjustments are better than the historical beta or not
- HSI has been used in some of the following examples but it is not an ideal proxy representing the whole market

Historical Beta of HSBC
(source: Bloomberg LP)

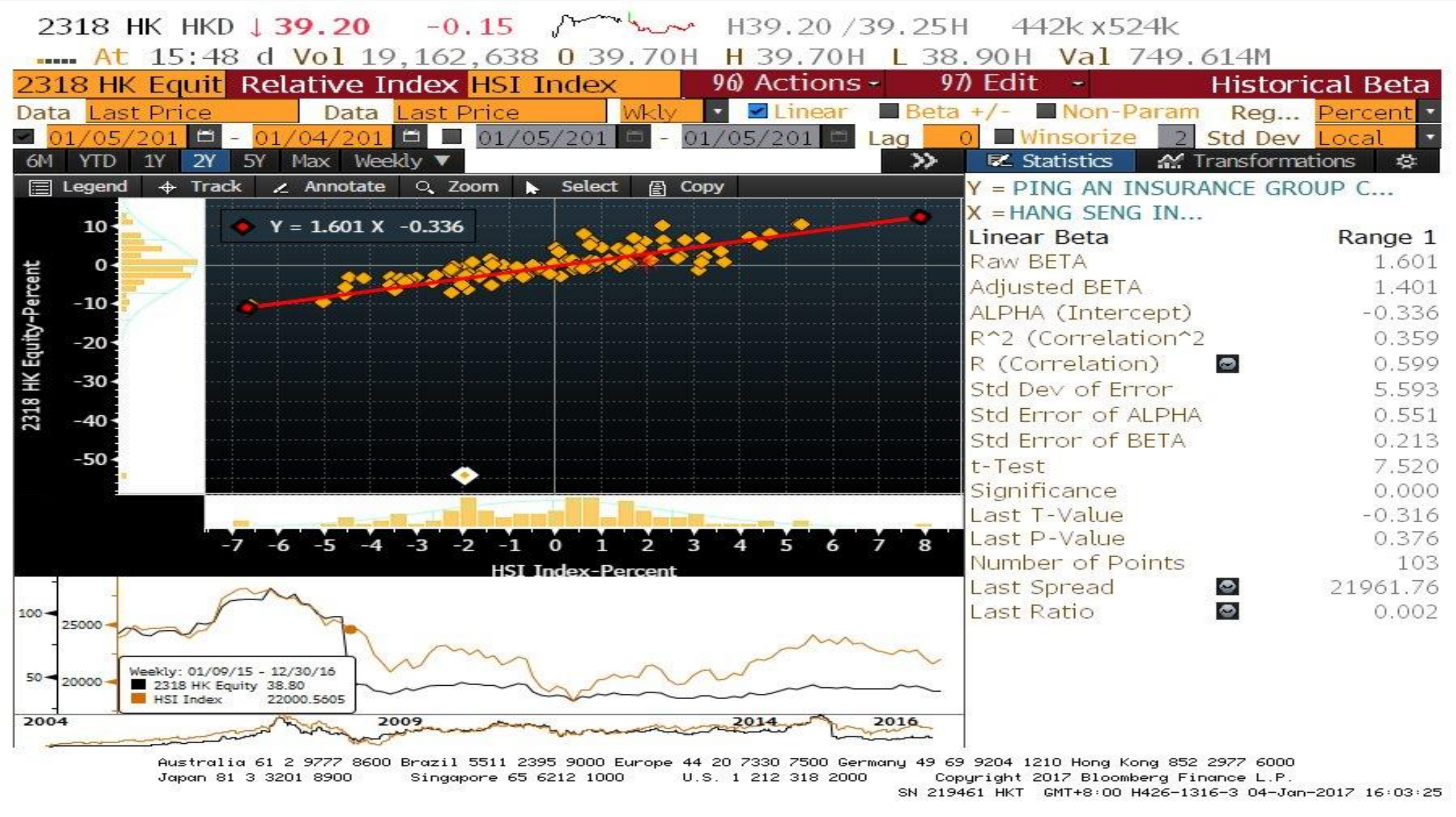


Historical Beta of AIA Group
(source: Bloomberg LP)

Historical Beta of CLP Holdings

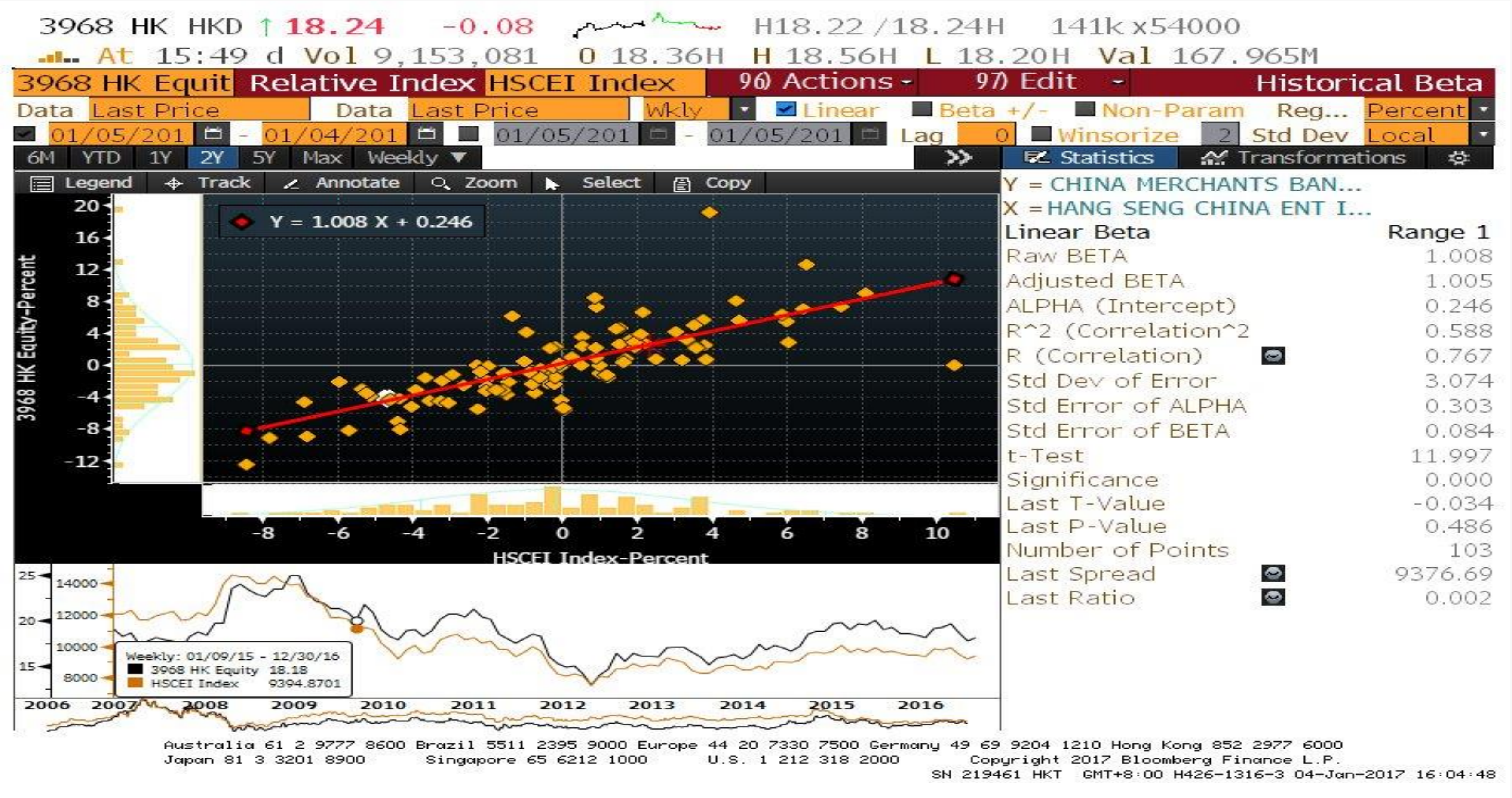
(source: Bloomberg LP)

Historical Beta of Ping An Insurance
(source: Bloomberg LP)



Historical Beta of China Merchants Bank (note the change in reference index)

(source: Bloomberg LP)



Industry Asset Betas

Industry	Beta	Industry	Beta
Airlines	1.80	Agriculture	1.00
Electronics	1.60	Food	1.00
Consumer Durables	1.45	Liquor	0.90
Producer Goods	1.30	Banks	0.85
Chemicals	1.25	International Oils	0.85
Shipping	1.20	Tobacco	0.80
Steel	1.05	Telephone Utilities	0.75
Containers	1.05	Energy Utilities	0.60
Nonferrous Metals	1.00	Gold	0.35

- Source: D. Mullins, "Does the Capital Asset Pricing Model Work?", Harvard Business Review, vol. 60, pp.105-114, 1982

CAPM as a pricing formula

- The standard CAPM is written in the form of return, not price
- Assume the purchase price of an asset is P (known), which is sold at Q (random) later
  - Rate of return = $(Q-P)/P$
  - From CAPM, fair price of P can be estimated

$$\frac{\overline{Q}-P}{P}=r_f+\beta_i(\overline{r_M}-r_f)\Rightarrow P=\frac{\overline{Q}}{1+r_f+\beta_i(\overline{r_M}-r_f)}=\overline{Q}\times DF$$

- DF is the risk adjusted discounting factor of a cashflow

CAPM as a pricing formula

- The price can also be written in an alternative form, so that the pricing formula becomes linear

$$\beta_i = \frac{\text{cov}[r_i, r_M]}{\sigma_M^2} = \frac{\text{cov}[(Q/P-1), r_M]}{\sigma_M^2} = \frac{\text{cov}(Q, r_M)}{P\sigma_M^2}$$

$$\therefore P = \frac{P\overline{Q}}{P(1+r_f) + \text{cov}(Q, r_M)(\overline{r_M} - r_f)/\sigma_M^2}$$

$$\Rightarrow P = \frac{1}{(1+r_f)}\left[\overline{Q} - \frac{\text{cov}(Q, r_M)(\overline{r_M} - r_f)}{\sigma_M^2}\right]$$

- The terms in the bracket are collectively known as the certainly equivalent of Q, which depend linearly on Q
  - Compensation for additional risk taken

Example

- The 1-year target price of the stock $\overline{Q}$ is \$100
- Other parameters:
  - β = 1.25
  - Expected return of the market portfolio $\overline{r_M}$ = 6%
  - Risk free rate r_f = 1%
- Fair price of the stock today

$$P = \frac{\overline{Q}}{1 + r_f + \beta_i(\overline{r_M} - r_f)} = \frac{100}{1 + 0.01 + 1.25(0.06 - 0.01)} = \$93.24$$

- If the current trading price is lower than P, then this stock is a good investment

Relaxing the constraints

- If investors have different risk free lending and borrowing rates, we could not find a unique market portfolio; the market portfolio may not even lie on the efficient frontier
- Fama (1970) showed that even if we extend our analysis to a multi-period setting, the single period CAPM may still be appropriate
- In the absence of risk free assets, Black (1972) formulated a model where a “zero-beta” portfolio could be created synthetically

Extension of CAPM: zero-beta model

- A fundamental assumption in CAPM is the availability of risk free assets, otherwise the choice of the risky portfolio on the efficient frontier will depend on the risk appetite of the investor, and the market portfolio may no longer be the common optimal portfolio

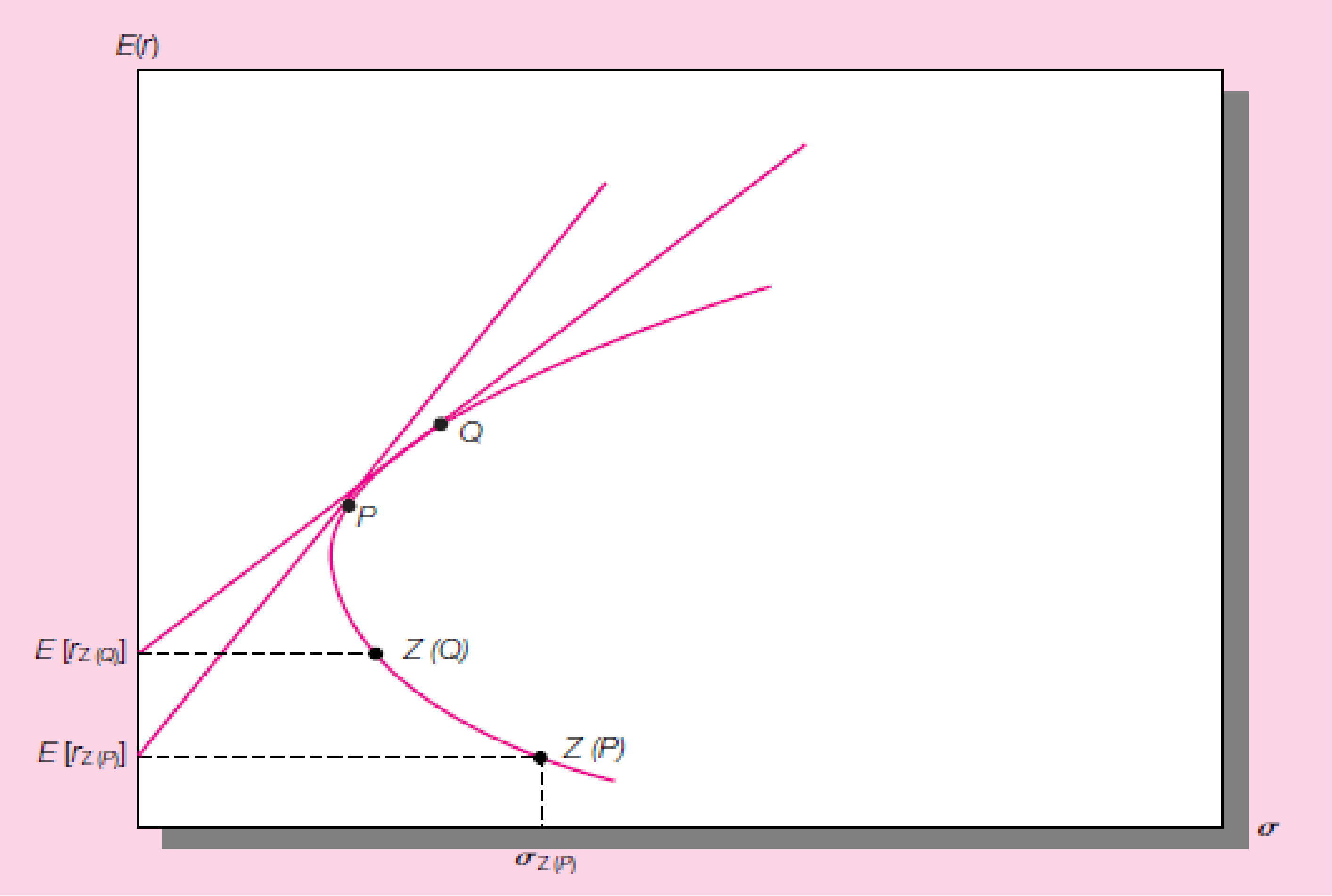
- Fischer Black has developed an extension of CAPM based on three properties of the mean-variance efficient portfolios
- (1) any portfolio constructed by combining efficient portfolios is itself on the efficient frontier ("Two-fund theorem")

Extension of CAPM: zero-beta model

- (2) Every portfolio on the efficient frontier has an inefficient counterpart, known as the zero-beta portfolio
  - Draw a tangent from a portfolio P to intersect with the y-axis, and find the expected return
  - Draw a horizontal line from this return to the minimum variance frontier, denoted as $Z(P)$

- The portfolio $Z(P)$ is uncorrelated with P

Extension of CAPM: zero-beta model



Extension of CAPM: zero-beta model

- (3) The expected return of any asset can be expressed as a linear function of two frontier portfolios

$$E(r_i) = E(r_Q) + [E(r_P) - E(r_Q)]\frac{\mathrm{cov}(r_i, r_P) - \mathrm{cov}(r_P, r_Q)}{\sigma_P^2 - \mathrm{cov}(r_P, r_Q)}$$

- Using M and $Z(M)$ and assuming $\mathrm{cov}[r_M, r_{Z(M)}]=0$, we have

$$E(r_i) = E(r_{Z(M)}) + [E(r_M) - E(r_{Z(M)})]\frac{\mathrm{cov}(r_i, r_M)}{\sigma_M^2}$$

- Basically, this is analogous to replacing the risk free asset with the zero-beta portfolio

Arbitrage Pricing Theory (APT)

- Strong assumptions used in deriving CAPM
  - Beta as the only source of risk?

- Could we relax some of these assumptions?

- APT uses arbitrage relationships
  - “Law of one price”: two items that are the same can’t sell at different prices
  - “same” means they have the same characteristics, i.e. same return and risk

Assumptions in APT

- Investors seek risk adjusted return, not absolute return (i.e. they are not risk neutral)
- Existence of a risk free rate for lending and borrowing
- Investors agree on the number and identity of the factors that are used in explaining asset returns
  - Homogeneous expectations
- Arbitrage opportunities do not exist

Arbitrage opportunities

- The table below shows the expected returns of portfolio A and stock B under different scenarios

	Scenario 1	Scenario 2	Scenario 3	Scenario 4
Portfolio A	23.33%	23.33%	20%	36.67%
Stock B	15%	23%	15%	36%

- Portfolio A outperforms stock B under any scenario
- A strategy is to short stock B and use the money obtained to buy the assets in portfolio A; a riskless profit could be earned no matter which scenario takes place
- When this happens, stock B's price would go down/portfolio A's price would go up; expected returns would change, and the arbitrage opportunity would disappear

Risk premiums under APT

- We start with a one factor model $r_P = E(r_P) + \beta_P F + e_P$
- If P is a well-diversified portfolio, its non-systematic risk would be 0

$$r_P = E(r_P) + \beta_P F$$

- We find another well-diversified portfolio Q, and form a new portfolio Z with weights in P and Q such that the overall β is 0; to prevent arbitrage this new portfolio must earn the risk free rate r_f

$$E(r_Z) = \frac{\beta_Q}{\beta_Q - \beta_P} E(r_P) + \frac{-\beta_P}{\beta_Q - \beta_P} E(r_Q) = r_f$$

- Therefore the risk premium must be proportional to its beta

$$\frac{E(r_P) - r_f}{\beta_P} = \frac{E(r_Q) - r_f}{\beta_Q}$$

The SML under APT

- Since the market portfolio M as a well-diversified portfolio; by definition, it has a β of 1

$$\frac{E(r_P)-r_f}{\beta_P}=\frac{E(r_M)-r_f}{1}$$

- Therefore we could write the relationship between the risk and return of a well diversified portfolio P

$$E(r_P)=r_f+\beta_P\left\lfloor E(r_M)-r_f\right\rfloor$$

- This has the same "form" as the SML depicted in CAPM
- It could also be shown that almost all individual assets k in the portfolio also follow the same relationship, i.e.

$$E(r_k)=r_f+\beta_k\left\lfloor E(r_M)-r_f\right\rfloor$$

Compare APT and CAPM

- APT relies on a few arbitrageurs to force the market back to equilibrium; CAPM relies on all investors to make small adjustments to their portfolios for the equilibrium to be established
- APT – no assumption of mean-variance behavior
- APT does not assume a complete “market” portfolio; any well diversified portfolio is sufficient
- APT by itself does not specify which factors are relevant; CAPM states that the only relevant factor is the security’s beta
- APT works for portfolios; the application to individual securities is not guaranteed
- Although APT and CAPM use different assumptions, the same SML is obtained

Empirical tests

- Should test the conclusions implied from the model, not whether assumptions are reasonable

- Testing CAPM requires a joint test between:
  - the theory of equilibrium itself
  - The stability of predictions

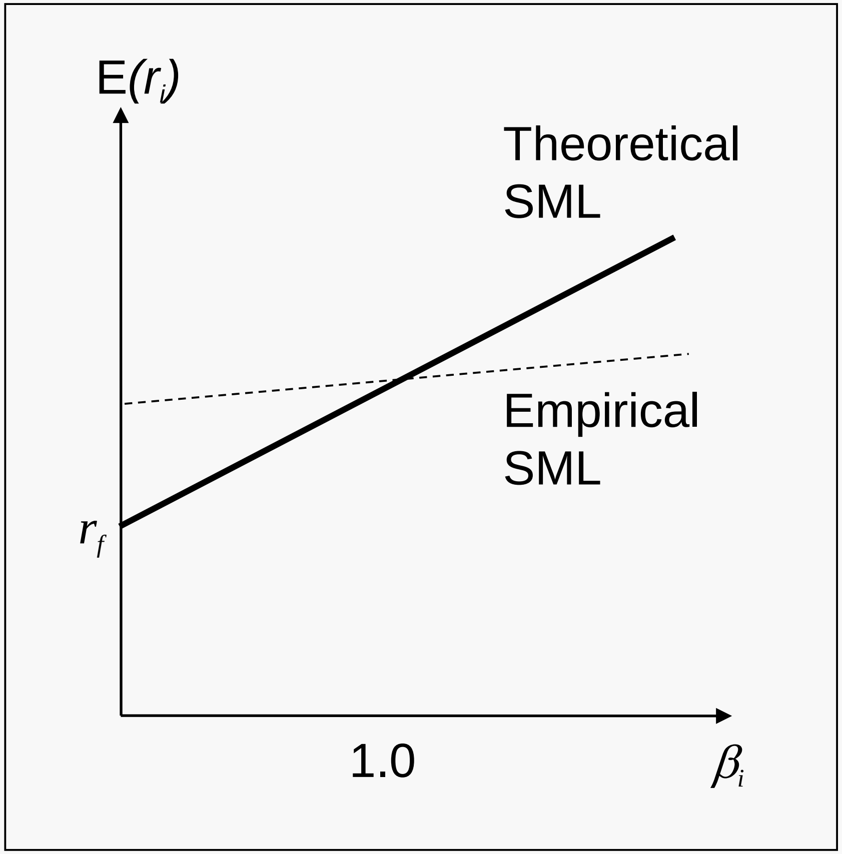
- Testing APT requires a joint test between:
  - The theory of equilibrium itself
  - The factor selection issue
    - need to select good factors in the model

Issues in empirical testing

- For CAPM
  - Assuming an index model, check that long term alpha = 0 and the expected return is related to beta linearly
    - use cross section data of many stocks
  - Are we using the correct proxy for the market portfolio? (*benchmark error* – part of “Roll’s critique” (1977))
    - a small change in the market proxy generates dramatically different expected returns

- For APT
  - Difficult to ascertain the theory because of the factor selection problem – even if the test results are not satisfactory, this may be due to a wrong choice of factors, not anything wrong in the theory

Some test results



- CAPM
  - The empirical line is too flat: the intercept is higher than the risk free rate, and the coefficient on beta is less than the average excess market return
  - beta does not seem to adequate explain the risk

Factors in the APT

- Identification of contributing factors in the APT is a major empirical research area

- Common macroeconomic factors include:
  - Inflation: impacts both the discount rate and the size of future cash flows
  - Risk premium: market's attitude to risk, measured by the difference between the return of (almost riskless) bonds and risky bonds
  - Industrial production: affect the real value of cash flow

Fama and French model

- Fama and French (1993, 1996) propose a 3-factor model for expected returns

$$\overline{r_i} = r_f + \beta_{iM}\left(\overline{r_M} - r_f\right) + \beta_{iS} R_{SMB} + \beta_{iH} R_{HML}$$

R_{SMB}: Expected difference in returns of small and big stocks

R_{HML}: Expected difference in returns of diversified portfolios of high and low B/M stocks

B/M: Book value of stock / Market value of stock

β_{iX}: Sensitivity to factor X

- Much better fit to empirical data
- However, there is no theoretical justification why these factors should be used

What have we learnt?

- The equilibrium expected return of an asset is linearly related to its risk, as measured by beta
  - *Having extra risk is the only way to achieve extra return*
  - the SML is obtained using both CAPM and APT
  - it is difficult to choose a correct “market portfolio”
  - the relationship is largely correct, but the slope and intercept may be different from that predicted by simple CAPM

What have we learnt?

- Note that while the CAPM/APT can use the concept of index models extensively, these models are very different in nature
  - CAPM/APT refer to the equilibrium price of the asset, whereas the factor model is only descriptive

- Studies applicable to Asian markets are difficult to carry out because of lack of information