CMSC 5718 INTRODUCTION TO COMPUTATIONAL FINANCE

Lecture 8

Outline

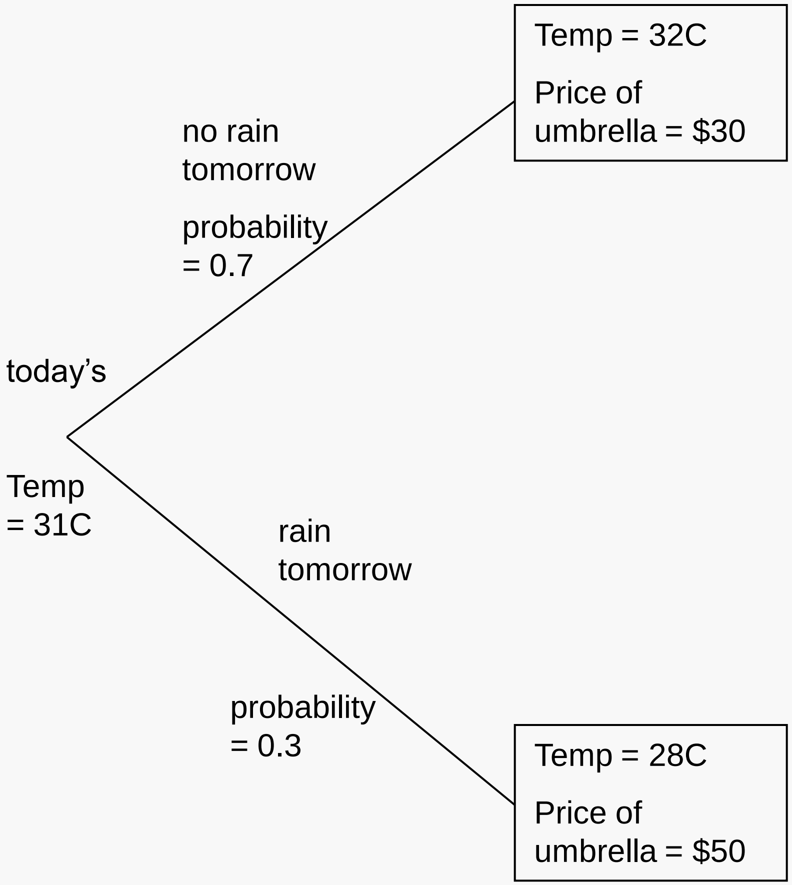
- An informal introduction to the mathematical formulation of option pricing
  - Expectation and arbitrage
  - Model of stock price behavior
  - Black-Scholes arguments
  - Issues about volatility and alternative models

- Reference: John Hull, 7th edition (2009), chapters 12, 13, 18; or 8th edition (2011), chapters 13, 14, 19

Pricing by replication

- Let's suppose there is a new kind of drink, "Fresh", selling for \$50 a glass, and people ***love*** it
- Only the seller of the drink knows that the drink could be mixed by adding 50% orange juice, 30% apple juice, and 20% mango juice
- If the market for orange, apple and mango juice is already saturated (i.e. not easy to make money), the "Fresh" drink is a perfect way to make a ***huge*** amount of money initially
- The trick would work until people discover the ingredients and the proportion; the price would then be pushed down

Pricing by expectation



- Assume that the price of an umbrella is a function of temperature, say

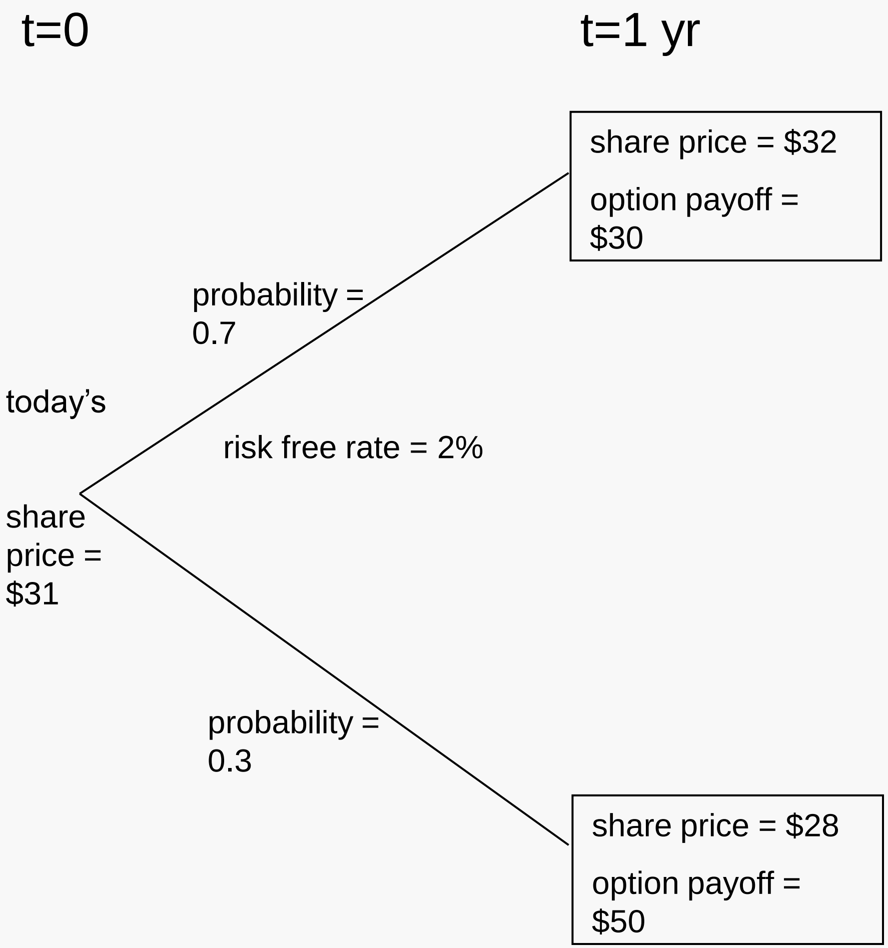
$$P = 190 - 5 \times temperature$$

- What is the expected price of an umbrella tomorrow?

$$\begin{aligned} E(P) &= \sum probability \times price \\ &= 0.7 \times 30 + 0.3 \times 50 \\ &= \$36 \end{aligned}$$

- **This is correct!!!**

Pricing by arbitrage



- Assume that the payoff of a derivative contract is a function of share price, say

$$P = 190 - 5 \times \text{share price}$$

- What is the expected price of the contract after 1 year?

$$E(P) = \sum probability \times payoff$$
$$= 0.7 \times 30 + 0.3 \times 50$$
$$= \$36$$

- Expected price today = present value of $36 = exp(-0.02) x 36 = $35.287
- **This is wrong!!!**

Pricing by arbitrage

- We hold a portfolio of Δ shares and \$$B$ bond
  - portfolio's worth today is $\Delta S_0 + B$
- After 1 year, the portfolio is worth $\Delta S + B\exp(rt)$
- We choose Δ and B so that the portfolio would always have the same value as the derivative contract, i.e.

$$\Delta \times 32 + B\exp(0.02) = 30$$

$$\Delta \times 28 + B\exp(0.02) = 50$$

- Solving the above, we obtain $\Delta = -5$, $B = 186.2377$
- **Since the portfolio is always worth the same as the derivative contract, by the principle of no arbitrage, they could not have a different price today**
- Derivative contract price today is thus –5 x 31 + 186.2377 = \$31.2377

Pricing by arbitrage

- The subjective probabilities (p = 0.7, 1$-p$ = 0.3) do not enter into the pricing equation; *only the risk free rate is relevant*
  - this is known as pricing in the Risk Neutral world
  - Note the difference between the expected share price (=0.7 x 32 + 0.3 x 28 = $30.80) and the fair forward value = ($31 x exp(0.02) = $31.626)
  - We could solve for a value of p^* so that (p^* x 32 + (1$-p^*$) x 28 = $31.626); this distribution is known as the risk neutral probability distribution
- The strategy would only work if we could buy/sell the underlying instrument without restrictions
  - this is known as a “complete market”
  - in the earlier example of umbrella pricing, we could not buy/sell a contract called “Temperature”

Types of Stochastic Processes

- Discrete time; discrete variable
- Discrete time; continuous variable
- Continuous time; discrete variable
- Continuous time; continuous variable

- We can use any of the four types of stochastic processes to model stock prices

- The continuous time, continuous variable process proves to be the most useful for the purposes of valuing derivatives

- Additional reference: Kerry Back, *A Course in Derivative Securities: Introduction to Theory and Computation*, Springer (2005), Chapter 2.

Markov Processes

- In a Markov process future movements in a variable depend only on where we are, not the history of how we got there; in other words,
  - given the current value $X(s)$, the value of $X(t)$, $t>s$, depends only on $X(s)$ but not on any value $X(u)$ where $u<s$

- We assume that stock prices follow Markov processes
  - If this is true, then technical analysis (i.e. the study of charts to predict further stock market movements) would be useless

Wiener Process

- Sometimes also known as a Brownian process
- We consider a variable $Z(t)$ whose value changes continuously
- Define $\phi(\mu,v)$ as a normal distribution with mean μ and variance v
- The change in a small interval of time Δt is ΔZ
- The variable follows a Wiener process if

$$\Delta Z = \varepsilon\sqrt{\Delta t} \text{ where } \varepsilon \text{ is } \phi(0,1)$$

- The values of ΔZ for any two different (non-overlapping) periods of time are independent

Properties of a Wiener Process

- Mean and variance of $[Z(t) - Z(t_0)]$ are 0 and $T\,(= t - t_0)$
- Probability distribution

$$
\begin{aligned}
P[Z(t) \le z | Z(t_0) = z_0] &= P[Z(t) - Z(t_0) \le z - z_0] \\
&= \frac{1}{\sqrt{2\pi(t - t_0)}} \int_{-\infty}^{z-z_0} \exp\left(-\frac{x^2}{2(t - t_0)}\right) dx \\
&= N\left(\frac{z - z_0}{\sqrt{t - t_0}}\right).
\end{aligned}
$$

Generalized Wiener Processes

- A Wiener process has a drift rate of 0 (i.e. average change per unit time) and a variance rate of 1

- In a generalized Wiener process the drift rate and the variance rate can be set equal to any chosen constants

- The variable x follows a generalized Wiener process with a drift rate of a and a variance rate of b^2 if

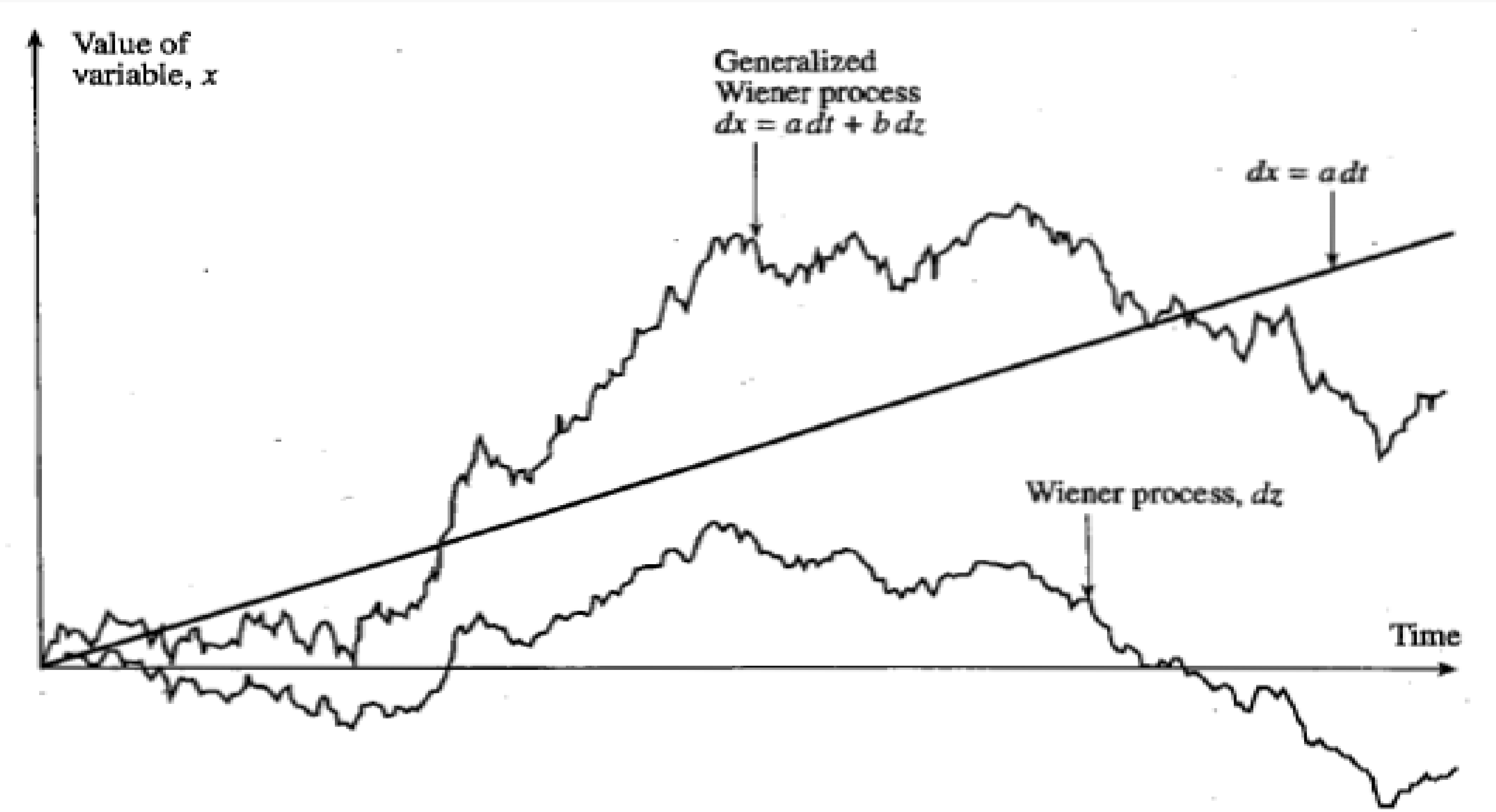
$$dx = a\,dt + b\,dz$$

Generalized Wiener Processes

$$\Delta x = a\,\Delta t + b\,\varepsilon\sqrt{\Delta t}$$

- Mean change in x in time T is aT
- Variance of change in x in time T is b^2T
- Standard deviation of change in x in time T is $b\sqrt{T}$

Generalized Wiener process



Itô Process

- In an Itô process the drift rate and the variance rate are functions of time

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

- The discrete time equivalent

$$\Delta x = a(x,t)\Delta t + b(x,t)\varepsilon\sqrt{\Delta t}$$

  is only true in the limit as Δt tends to zero

- For stock prices, assume

$$dS = \mu S\,dt + \sigma S\,dz$$

  where μ is the expected return and σ is the volatility

- The discrete time equivalent is $\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$

Itô's Lemma

- If we know the stochastic process followed by a random variable x, Itô's lemma tells us the stochastic process followed by some function $G(x, t)$

- Since a derivative contract is a function of the price of the underlying and time, Itô's lemma plays an important part in the analysis of derivative securities

Itô's Lemma: heuristic proof

- A Taylor's series expansion of $G(x, t)$ gives

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + ½\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x \partial t}\Delta x\,\Delta t + ½\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \ldots$$

- Even if we ignore the second order terms, the term involving Δx^2 cannot be dropped, because Δx is of order $\sqrt{\Delta t}$

Itô's Lemma: heuristic proof

Suppose

$$dx = a(x,t)dt + b(x,t)dz$$

so that

$$\Delta x = a\,\Delta t + b\,\varepsilon\sqrt{\Delta t}$$

Then ignoring terms of higher order than Δt

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + ½\,\frac{\partial^2 G}{\partial x^2}b^2\varepsilon^2\Delta t$$

Since $\varepsilon \approx \phi(0,1),\ E(\varepsilon) = 0, E(\varepsilon^2) = 1 + [E(\varepsilon)]^2 = 1$

It follows that $E(\varepsilon^2\Delta t) = \Delta t$, variance of Δt is $O(\Delta t^2)$

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\Delta t$$

Application of Ito's Lemma to a Stock Price Process

Taking limits: $dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + ½ \frac{\partial^2 G}{\partial x^2} b^2 dt$

Substituting: $dx = a\, dt + b\, dz$

We obtain: $dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + ½ \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b\, dz$

The stock price process is

$$dS = \mu S\, dt + \sigma S\, dz$$

For a function G of S and t

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + ½ \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S\, dz$$

Examples

1. The forward price of a stock for a contract maturing at time T

$$G = S\, e^{r(T-t)}, \frac{\partial G}{\partial S} = e^{r(T-t)} = \frac{G}{S}, \frac{\partial^2 G}{\partial S^2} = 0, \frac{\partial G}{\partial t} = -rG$$

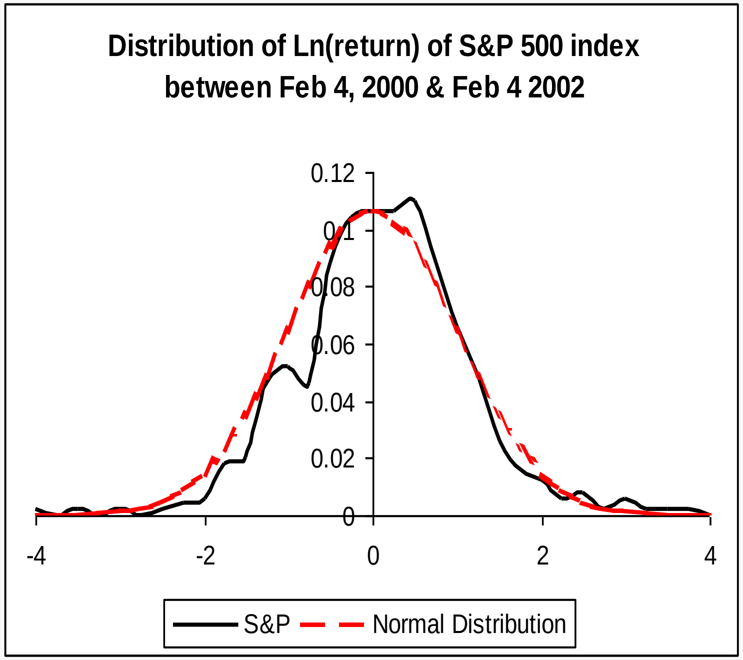
$$dG = (\mu - r)G\, dt + \sigma G\, dz$$

2. $G = \ln S, \frac{\partial G}{\partial S} = \frac{1}{S}, \frac{\partial^2 G}{\partial S^2} = \frac{-1}{S^2}, \frac{\partial G}{\partial t} = 0$

$$dG = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma\, dz$$

Stock price behaviour

- The real world stock process: Geometric Brownian Motion



$$\frac{dS}{S} = \mu dt + \sigma dz$$

$$d(\ln S) = (\mu - \frac{\sigma^2}{2})dt + \sigma dz$$

$$S_t = S_0 \exp\left[(\mu - \frac{\sigma^2}{2})T + \sigma\varepsilon\sqrt{T} \right]$$

- μ is the stock's growth rate
- ε is a random variable drawn from a standardized normal distribution (mean = 0, variance = 1)
- σ is the annualized volatility of S

The Lognormal Property

- It follows from this assumption that

$$\ln S_T - \ln S_0 \approx \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

or

$$\ln S_T \approx \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

- Since the logarithm of S_T is normal, S_T is lognormally distributed

Continuously Compounded Return

- If x is the continuously compounded return

$$S_T = S_0 \, e^{xT}$$

$$x = \frac{1}{T} \ln \frac{S_T}{S_0}$$

$$x \approx \phi\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T} \right)$$

The Expected Return

$$E(S_T) = S_0\, e^{\mu T}$$

$$\mathrm{var}(S_T) = S_0{}^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$$

- The expected value of the stock price is $S_0 e^{\mu T}$
- The expected return on the stock is $\mu - \sigma^2/2$ not μ
- This is because

$$\ln[E(S_T / S_0)] \quad \text{and} \quad E[\ln(S_T / S_0)]$$

are not the same

The Expected Return

- (Example taken from John Hull's book)
- A sequence of returns of a stock is as follows:
  - 15%, 20%, 30%, –20%, 25%
- Arithmetic mean = (15+20+30–20+25)/500 = 14%
  - This is the expected return of the stock μ, for each period
- Expected return of an investor
  = (1.15 x 1.20 x 1.30 x 0.80 x 1.25)^(1/5) – 1 = 12.4%
  - This is the expected compound return of the stock

Historical development

- Fischer Black and Myron Scholes managed to publish the famous paper in 1973 (after being rejected by a few journals)
- Robert Merton published a paper in the same year, giving an alternative derivation which leads to the same formula
- The Black-Scholes formula is almost identical to a formula given by Paul Samuelson in 1965, only that the subjective growth rate μ is now replaced by the risk free rate r
- Scholes and Merton earned the Nobel prize in 1997 (Fischer Black died in 1995)

The Black-Scholes argument

- The option price and the stock price depend on the same underlying source of uncertainty
- A portfolio P of short 1 derivative contract (with value f), long Δ shares (with value ΔS)
- Choose Δ such that the portfolio is riskless (stochastic term = 0)
- **This portfolio must earn the risk free rate, otherwise arbitrage exists**
- Note that
  - this portfolio is only risk free instantaneously; Δ would change when S, t have changed
  - the growth rate μ does not enter the equation
- We could then set up a partial differential equation (PDE) satisfied by ANY derivative

Black-Scholes PDE

$$P = -f + \Delta S$$

$$dP = -df + \Delta dS$$

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)dt + \sigma S \frac{\partial f}{\partial S} dz$$

$$\Rightarrow dP = -\left[\left(\Delta - \frac{\partial f}{\partial S}\right)\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right]dt + \sigma S\left(\Delta - \frac{\partial f}{\partial S}\right)dz$$

- If we choose $\Delta = \frac{\partial f}{\partial S}$ the dz term is 0, which means the portfolio is riskless.

  P should thus earn the risk free rate r

$$P_{t+dt} = P_t \exp(rdt), dP = rPdt$$

$$dP = \left[-\frac{\partial f}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right]dt = rPdt = r\left(\frac{\partial f}{\partial S} S - f\right)dt$$

$$\Rightarrow \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \quad \text{(Black - Scholes PDE)}$$

Differential Equation approach

- Black-Scholes differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

- Specify the initial condition and boundary conditions in order to obtain a solution, usually from numerical schemes, e.g. (explicit or implicit) finite difference methods
  - E.g. For European calls,

$$\text{at } t = T,\ f = \max(S - K, 0)$$

$$f(t) = 0 \text{ for } S = 0, \frac{\partial f}{\partial S} = 1 \text{ for } S \to \infty \ \{\text{boundary conditions}\}$$

Risk-Neutral Valuation

- The variable μ does not appear in the Black-Scholes equation
- The equation is independent of all variables affected by risk preference
- The solution to the differential equation is therefore the same in a risk-free world as it is in the real world
- This leads to the principle of risk-neutral valuation
  - **Extremely important result**

Assumptions in the Black-Scholes formulation

- (i) Continuous trading, i.e. prices move in infinitesimal small increments
- (ii) constant riskless interest rate
- (iii) the asset pays no dividend
- (iv) there are no transaction costs and taxes
- (v) the assets are perfectly divisible (c.f. board lots in shares)
- (vi) short selling is allowed

The Black-Scholes Formulas

$$c = S_0 \, N(d_1) - K \, e^{-rT} N(d_2)$$

$$p = K \, e^{-rT} \, N(-d_2) - S_0 \, N(-d_1)$$

where $$d_1 = \frac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0 / K) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

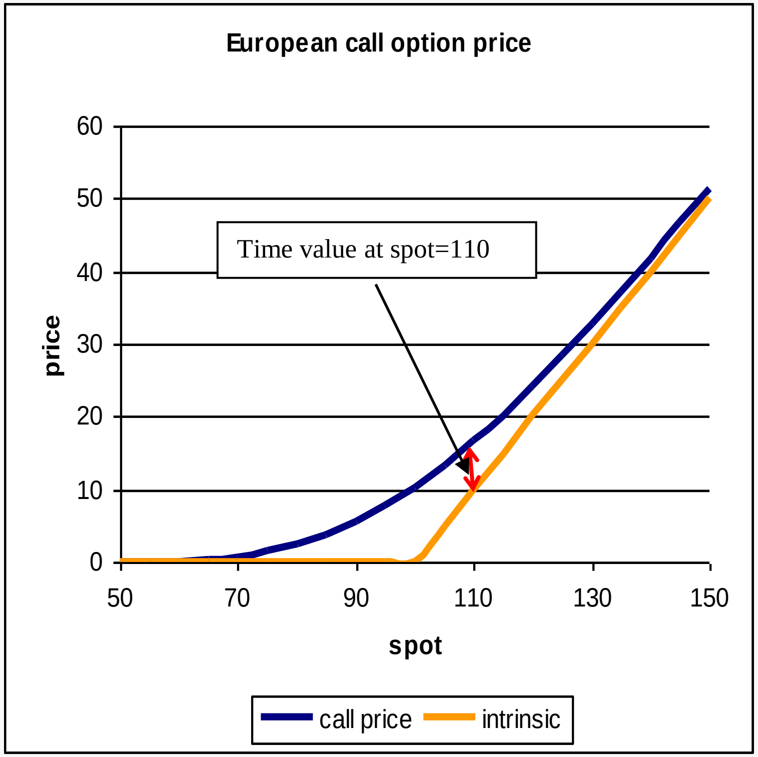
- c and p are the European call and put prices
- $N()$ is the cumulative normal distribution

Properties of Black-Scholes Formula

- As S_0 becomes very large c tends to $S_0 - Ke^{-rT}$ and p tends to zero

- As S_0 becomes very small c tends to zero and p tends to $Ke^{-rT} - S_0$

Time value and intrinsic value



Strike 100, maturity 3 months, r=2%, volatility 50%,

- Option price is made up of two parts
  = time value + intrinsic value
- **Intrinsic value**: the value if the option is exercised immediately
  - = maximum of (spot price – strike,0) for a call
  - = maximum of (strike – spot price,0) for a put
- **Time value**: the reward for holding the option
  - Could be very small if the option is deep in-the-money or deep out-of-the-money
- Example
  - Call option, spot price 110, strike price 100, option price 16.6
  - Intrinsic value = 110 – 100 = 10, time value = 16.6 – 10 = 6.6
  - If spot price is 90 and the option price is 5.4, intrinsic value = 0, time value = 5.4

Methods in option pricing

- Analytical solution
  - convolution of payoff and probability density function
- Monte Carlo simulation
  - generation of many random paths and obtain the price via averaging the result
- Numerical solution of PDE
  - Explicit finite difference methods – "Trees"
  - Implicit finite difference methods

The Feynman-Kac formulation

- A technique for solving the Black-Scholes PDE
- The Feynman-Kac result: a PDE with a form

$$\frac{\partial f}{\partial t} + \mu(S,t)\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2(S,t)\frac{\partial^2 f}{\partial S^2} - rf = 0$$

- with boundary condition $H(S,t)$ has solution

$$f(S,t) = \exp(-rt)\mathrm{E}\big(H(S,t)\big)$$

- where the expectation E is taken with respect to a process S defined by

$$dS = \mu(S,t)dt + \sigma(S,t)dz$$

- In the risk neutral world, if S has a constant volatility, we could write

$$dS = rSdt + \sigma Sdz$$

Analytical formula example

- Payoff function for a European call at maturity: $\max(S-K,0)$
- Expected value (option price) at maturity is

$$f = \int_0^\infty H(S) \bullet G(S)\,dS, \quad H(S) = \begin{cases} S-K, & S \ge K \\ 0, & S < K \end{cases}$$

$$G(S) = \frac{1}{S\sigma\sqrt{2\pi t}} \exp\left\{ -\frac{1}{2}\left(\frac{\ln(S)-\mu}{\sigma\sqrt{t}} \right)^2 \right\}$$

$$\mu = \ln(S_0) + \left(r - \frac{\sigma^2}{2} \right) t$$

$$\therefore f = \int_K^\infty (S-K) \bullet G(S)\,dS$$

- $G(S)$ is the probability density function if S follows a lognormal distribution
- Expected value today = f x $\exp(-rt)$ (the Black-Scholes formula)

Simulation approach

$$dS = rSdt + \sigma Sdz$$

- In discrete form, we can approximate this expression by:

$$\Delta S_i = rS_i \Delta t_i + \sigma S_i \varepsilon \sqrt{\Delta t_i}$$

$$S_{i+1} = S_i + \Delta S_i$$

- We generate many different paths of $S_0 \ldots S_n$ (typically at least 10,000 paths) using a random number generator. For each path, work out the option payoff given the payoff function
- The option price is just the present value of the average of all these payoffs

Simple extensions of the Black-Scholes framework

- Two of the assumptions in the standard Black-Scholes framework can be relaxed in a straightforward manner
  - If r is a function of time, replace r by $\frac{1}{t}\int_0^t r(u)\,du$
  - If σ is a function of time, obtain σ from the following:

$$\sigma^2 = \frac{1}{t}\int_0^t \sigma^2(u)\,du$$

Black's Model

- Instead of assuming the spot price follows a lognormal distribution, a more useful result is obtained when the return of the forward price is lognormal and has a constant volatility

$$c = P(0,T)[F_0 N(d_1) - KN(d_2)]$$

$$p = P(0,T)[KN(-d_2) - F_0 N(-d_1)]$$

$$d_1 = \frac{\ln(F_0/K) + \sigma_F^2 T}{\sigma_F \sqrt{T}} \qquad d_2 = \frac{\ln(F_0/K) - \sigma_F^2 T}{\sigma_F \sqrt{T}}$$

$P(0,T)$ is the price of a zero coupon bond with maturity T

Black's model

- Note the subtle difference between Black-Scholes model and Black's model
  - We can easily obtained the Black-Scholes formula from the Black's formula if interest rate is deterministic
  - F captures the dividend yield and stochastic interest rate

What is volatility?

- Formal definition: standard deviation of the daily log return, expressed in an annualized fashion
- *A statistical concept*
- Rule of thumb: Annual volatility = Y %, 68% chance that the stock will move +/- Y/16% daily
  - E.g. Stock price is 100 and volatility is 32%; it implies that tomorrow there is a 68% chance that the closing price of the stock would range between 98 and 102
- Note that, it says 68% chance that it would stay in the range – there is still a significant chance that it could trade outside of this range
  - Based on the movements of a few days, we could not say whether the underlying volatility has changed or not

How do we "calculate volatility"?

- From a historical data series

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2} \times \sqrt{252}$$

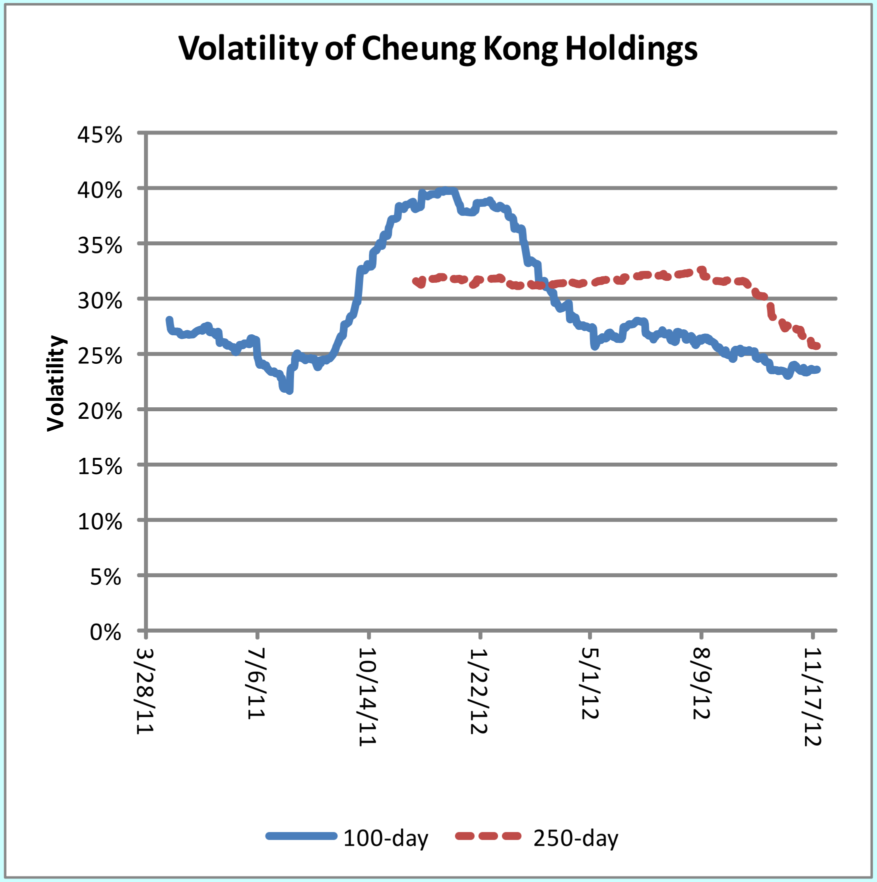
$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right), S_i \text{ is the stock price at time } i, \bar{u} \text{ is the mean of the } u_i\text{'s}$$

$$252 \text{ is assumed to be the number of trading days per year}$$

- Difficult to choose the number of data points
  - often choose n=100 or 250 of the most recent observations
  - Alternatively, plot a moving series of 100 or 250 observations and observe the trend
- Other more sophisticated econometric methods could be used

Example of historical volatility calculation (c.f. lecture 2&3, p.92)

	Cheung Kong closing price	ln(return)	Volatility 100-day	250-day
10/19/12	113.50	-0.0070	23.47%	28.07%
10/22/12	114.90	0.0123	23.36%	27.35%
10/24/12	119.90	0.0426	23.02%	27.64%
10/25/12	119.40	-0.0042	22.99%	27.56%
10/26/12	117.80	-0.0135	23.06%	27.60%
10/29/12	112.30	-0.0478	23.94%	27.68%
10/30/12	112.20	-0.0009	23.95%	27.68%
10/31/12	114.50	0.0203	23.99%	27.71%
11/01/12	115.00	0.0044	23.90%	27.37%
11/02/12	115.80	0.0069	23.73%	27.24%
11/05/12	115.60	-0.0017	23.44%	27.24%
11/06/12	115.50	-0.0009	23.44%	26.81%
11/07/12	116.10	0.0052	23.44%	26.71%
11/08/12	113.50	-0.0226	23.71%	26.54%
11/09/12	112.90	-0.0053	23.30%	26.44%
11/12/12	114.40	0.0132	23.36%	26.47%
11/13/12	112.50	-0.0167	23.54%	26.50%
11/14/12	113.70	0.0106	23.57%	26.49%
11/15/12	112.60	-0.0097	23.62%	25.86%
11/16/12	112.40	-0.0018	23.52%	25.86%
11/19/12	113.70	0.0115	23.53%	25.77%
11/20/12	113.20	-0.0044	23.56%	25.78%



Time Varying Volatility

- Total variance of a stock is $\sigma^2 t$
- Suppose the volatility is σ_1 for the first year and σ_2 for the second and third
- Total accumulated variance at the end of three years is $\sigma_1{}^2 + 2\sigma_2{}^2$
- The 3-year average volatility is

$$3\bar{\sigma}^2 = \sigma_1^2 + 2\sigma_2^2; \quad \bar{\sigma} = \sqrt{\frac{\sigma_1^2 + 2\sigma_2^2}{3}}$$

Volatility skew/smile

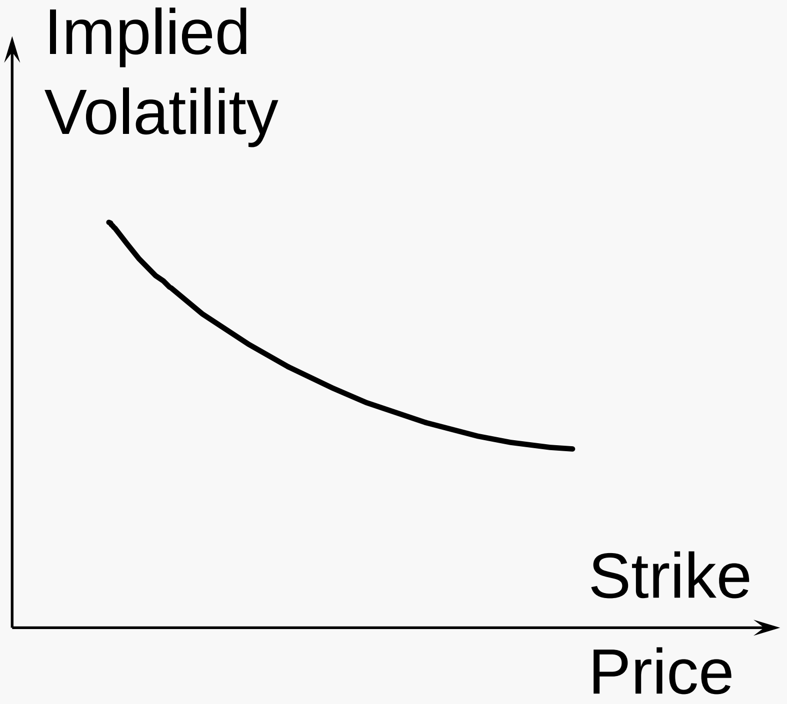
- Options at the same maturity but different strikes would be quoted at different volatilities
  - this is counter-intuitive, because the volatility should describe the movement of the underlying variable (e.g. a particular stock's price); it should have nothing to do with the characteristic of the option (i.e. strike)
- Smile implied volatility is
  - "A wrong number to put in the wrong formula to obtain the right price" - Rebonato (1999), p. 78
- The most common explanations of the smile phenomenon:
  - supply and demand of options at different strikes – more people demand downside puts than upside calls
  - the underlying distribution is not lognormal; it could have fat tails or jumps
  - the volatility of the underlying is stochastic

Term structure of volatility and volatility "smile"

	Strike		
Maturity	90	100	110
3Mth	47%	44%	41%
6Mth	44.5%	42%	38%
1 Year	39%	37%	35%
2 Years	35%	33%	32%
3 Years	32%	31%	30%
4 Years	30%	29.5%	29%
5 Years	29%	28.5%	28%

- The "volatility" is used as an alternative way to express the price of an option
- In many markets (especially for stock indices), we could see a different "volatility" being used to price options at different maturities and different strikes, an example is shown on the left
- *This is inconsistent with theoretical behaviour*: we cannot generate this kind of result from using the standard formulas with the historical time series

Typical Volatility Smile for Equity Options



Local volatility and stochastic volatility models

- Suppose European option prices at all strikes and maturities are available, we would like to find a state-time dependent volatility function which is consistent with these prices, as in the following formulation

$$\frac{dS}{S} = (r-q)dt + \sigma(S,t)dz$$

  - We can also recover the risk-neutral probability distribution of the asset price
- In a more sophisticated model, we can assume the volatility itself to be stochastic (c.f. Heston (1993) model)

$$\frac{dS}{S} = (r-q)dt + \sqrt{V}dz_S$$

$$dV = a(V_L - V)dt + \xi V^{\alpha} dz_V$$

Why do we need local volatility or stochastic volatility models?

- If our aim is to price vanilla European or American calls and puts, we don't need to use these volatility models

- The models would be important in describing the behavior of exotic products which may depend on the accurate (hedgeable) distribution of the underlying asset prices at different strike levels