4. Performance and Risk Measures

4.1. Application of asset pricing models in performance evaluation
4.1.1. Introduction

Given that there are many different kinds of trading strategies, how do we compare their performance? One result that we want to achieve is to identify a good manager from others who get good investment results because of market or other random factors, i.e. we want to distinguish between skill and luck. In particular, we want to apply what we have learnt in the past few chapters, and try to make use of the link between risk and return.

Examples of fund performance

	產品風險級別 Product Risk Ratings	基金成立日 Inception Date	YTD	3M	2013	2012	2011	2010	2009	3年波幅 Ann. Volatility 3Y	3年夏普比率 Ann. Sharpe Ratio 3Y
股票債券混合 BALANCED / MIXED ASSETS											
環球股票債券混合 Global Balanced											
貝萊德環球資產配置基金 A2 USD BGF Global Allocation Fund A2 USD	3	3/1/1997	2.99%	2.68%	13.98%	8.02%	-4.31%	8.51%	22.23%	10.05%	0.54
美國資產配置 Mixed Asset US											
德盛收益及增長基金 – AT – USD Allianz Income and Growth – AT – USD	3	18/11/2011	5.16%	3.43%	17.36%	11.19%	NA	NA	NA	NA	NA
股票：已發展市場 EQUITIES: DEVELOPED MARKETS											
環球股票 Global Equity											
安本環球 – 世界股票基金 A2 USD Aberdeen Global – World Equity A2 USD	5	1/2/1993	8.11%	5.45%	12.40%	14.45%	-2.37%	10.19%	36.17%	13.86%	0.62
北美股票 North American Equity											
富蘭克林美國機會基金 A (acc) USD Franklin US Opportunities A (acc) USD	5	3/4/2000	3.13%	3.99%	38.60%	9.33%	-3.88%	20.68%	41.65%	16.36%	0.73
摩根美國價值基金 A Dis USD JPM US Value A Dis USD	4	19/10/2000	7.45%	4.07%	30.65%	13.13%	1.41%	14.34%	20.88%	12.25%	1.17
聯博 – 精選美國股票基金 A USD AllianceBernstein – Select US Equity Pf A USD	4	28/10/2011	5.07%	4.60%	29.35%	14.32%	NA	NA	NA	NA	NA
美盛銳思美國小型資本機會基金 A Dis A USD Legg Mason Royce US Small Cap Opportunity Fund A Dis A USD	5	8/11/2002	2.83%	-0.29%	40.48%	21.47%	-12.39%	34.04%	59.13%	20.29%	0.65
歐洲股票 European Equity											
貝萊德歐洲基金 A2 EUR BGF European Fund A2 EUR	4	30/11/1993	1.43%	-0.83%	21.78%	20.45%	-10.41%	13.00%	38.64%	12.97%	0.39
富蘭克林互惠歐洲基金 A (acc) EUR Franklin Mutual European A (acc) EUR	4	31/12/2001	1.05%	1.18%	24.84%	17.97%	-8.22%	11.76%	21.02%	12.08%	0.43

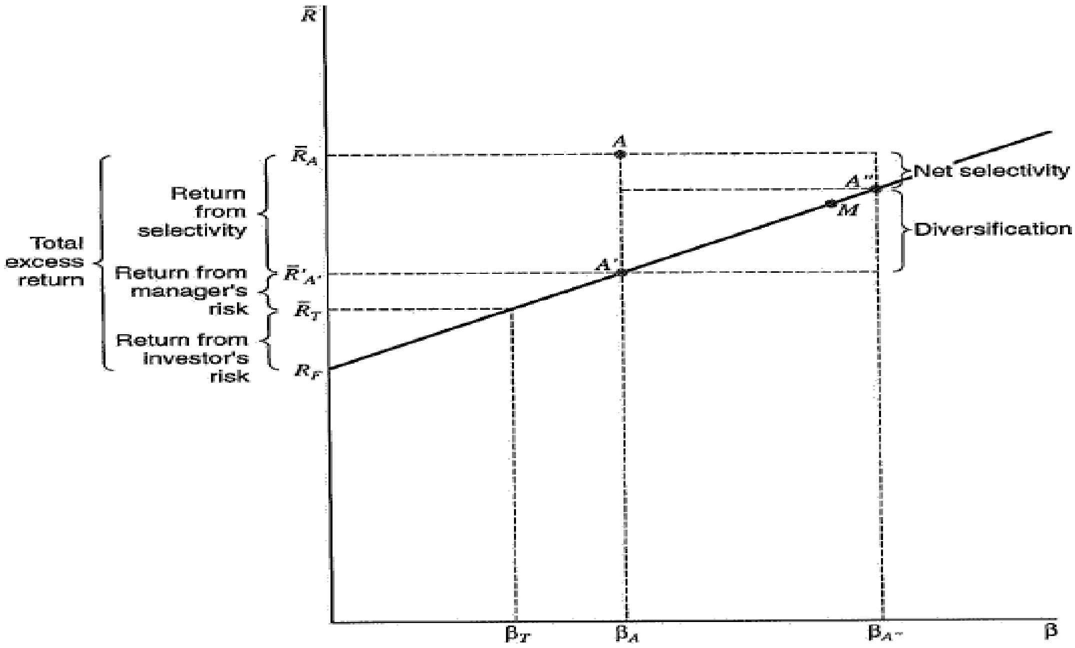
Source: Funds Select, Second Quarter 2014, Standard Chartered Bank, Hong Kong

The above table is a snapshot of a summary presentation of mutual fund performance. We can see that there are many different kinds of mutual funds, focusing on different markets and asset classes. Typical information for a fund will include the historical performance figures for the past few years, its risk (represented by the annualized volatility of the fund price), and some performance benchmarks. (The Sharpe ratio given here will be discussed in section 4.1.3.).

4.1.2. Performance attribution

In the table above, the 3-month performance of North American funds are much better than that of the European Equity funds. We want to answer a fundamental question: does a given fund manager provide an actual value-added service with respect to a given benchmark, or the performance can be attributed to luck alone?

Fama's attribution scheme



We can use the CAPM to calculate risk-adjusted performance. In an early work, Fama (1972) has developed a detailed scheme of analysis (the figure here is taken from Elton et al. (2007)). The diagram is a plot of return against beta, same as the one depicting the SML, which is denoted by line R_F–M , and plots the return of all possible combinations of the riskless asset and the market portfolio. A represents a portfolio with diversifiable risk.

The performance attribution procedure starts from the standard analysis, where we can identify the Jensen's alpha for A = A–A', and this is defined as the return from selectivity (see also the definition of Jensen's alpha in the next section). Next we form a portfolio A" with the same total risk as the portfolio A. Since total risk of a portfolio is represented by σ (instead of β), the condition is:

$$\sigma_A^2 = \sigma_{A''}^2 = \beta_{A''}\sigma_M^2 \Rightarrow \beta_{A''} = \sigma_A^2 / \sigma_M^2$$

The risk of A'' is proportional to σ_M as it is a portfolio on the SML, which means that it is a combination of the market portfolio M and the risk free asset (with σ=0). Extra return earn by portfolio A over the portfolio A" (with the same total risk) is called Net Selectivity, and the return from selectivity is the sum of net selectivity and the extra return from diversification (basically we are saying that A achieves extra return partly because the portfolio has some extra risk in the form of increase β).

Secondly, the extra return from $\overline{R_{A'}} - R_F$ is decomposed into two components. The investor may have a target risk level given by β_T, which will generate a return of $\overline{R_T}$. The extra return above the risk free rate = $\overline{R_T} - R_F$ is known as the "investor's risk" as it is the investor's view to choose an investment which has more risk, hence achieving a higher return. The remaining return ($\overline{R_{A'}} - \overline{R_T}$) is the return earned because the manager chose a different risk level than the target, known as the "manager's risk."

These three parts, i.e. return from selectivity, the investor's risk, and the manager's risk, combine together to form the total excess return of portfolio A.

Example

Risk free rate R_F = 4%, return of market portfolio $\overline{R_M}$ = 12%, s.d. of market portfolio σ_M = 11%, Beta of portfolio A β_A = 1.3, expected return $\overline{R_A}$ = 17.53%, s.d. of portfolio A σ_A = 14%

Return of A from SML	= 0.04 + 1.3x(0.12–0.04) = 14.4%
Beta of portfolio A'':	$\beta_{A''} = \sigma_A^2 / \sigma_M^2 = 0.14^2 / 0.11^2 = 1.62$
Return of A'' from SML	= 0.04 +1.62x(0.12–0.04) = 16.96%
Return from selectivity	= Jensen's alpha = 17.53–14.4 = 3.13%
Net selectivity	= 17.53-16.96 = 0.57%
Return from diversification	= 16.96–14.4 =3.13–0.57 = 2.56%

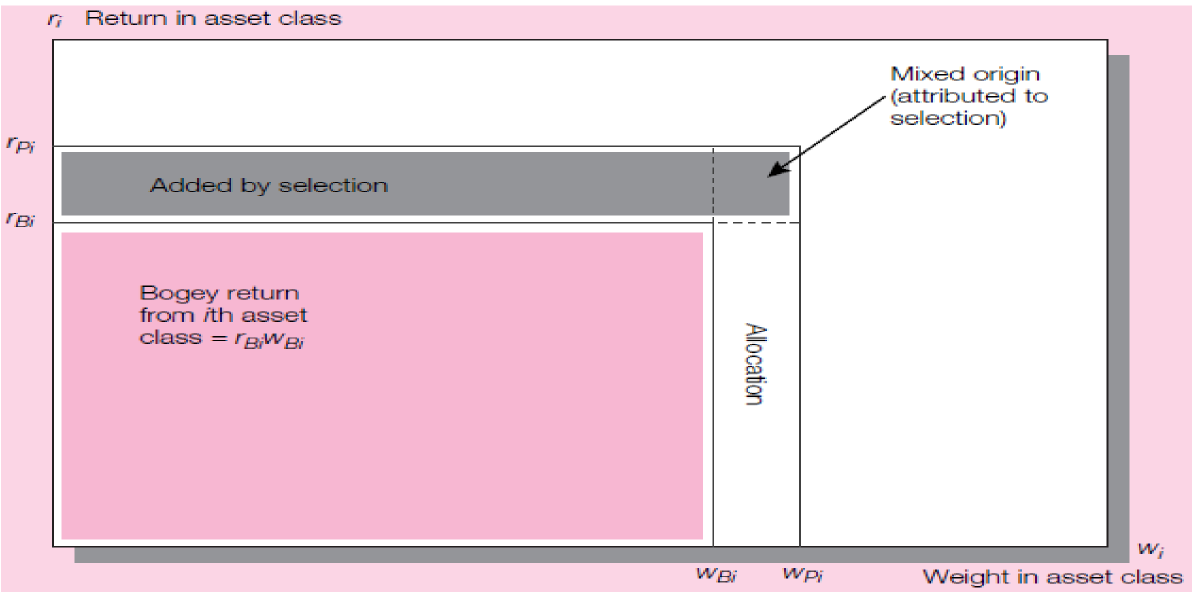
Assume investor's target portfolio C has a beta = 1.2

Return of C from SML	= 0.04 +1.2x(0.12–0.04)= 13.6%
Investor's risk	= 13.6 – 4 = 9.6%
Manager's risk	= 14.4 – 13.6 = 0.8%

An alternative performance attribution method

This method makes use of the fact that performance of a fund is often measured against a benchmark. In this case, a benchmark called the bogey portfolio is created, with fixed weights in each asset class. For example, we can use broad market indices in each asset class to represent the passive strategies (such as S&P 500 for stocks). Furthermore, one needs to decide on the "neutral" weights to be allocated to each asset class; this depends on the investor's risk tolerance. Once these have been fixed, the performance in each asset class is divided as:

Total return = return from asset allocation + return from security selection



[Figure from Bodie, Kane and Marcus, Investments, 6th edition, McGraw Hill (2003)]

The rates of return of the managed portfolio r_P and the bogey portfolio r_B are:

$$r_P = \sum_{i=1}^{n} w_{Pi} r_{Pi}, \;\; r_B = \sum_{i=1}^{n} w_{Bi} r_{Bi}$$

where w_{Pi} and w_{Bi} are the weights of individual assets in each of portfolio P and B respectively. The difference between the two rates of return is:

$$r_P - r_B = \sum_{i=1}^{n} (w_{Pi} r_{Pi} - w_{Bi} r_{Bi})$$

Total contribution from asset class i is decomposed as:

$$\underbrace{w_{Pi} r_{Pi} - w_{Bi} r_{Bi}}_{\text{total contribution}} = \underbrace{(w_{Pi} - w_{Bi}) r_{Bi}}_{\text{asset allocation}} + \underbrace{w_{Pi}(r_{Pi} - r_{Bi})}_{\text{security selection}}$$

It can be seen from the diagram above that there is a region (at the top right hand corner) that can be attributed to either asset allocation or security selection. In the current formula, this has been attributed to security selection (because the term in asset allocation is multiplied by r_{Bi} and not r_{Pi}).

An example is given in Bodie et al. (2003). The bogey portfolio has 60% equity, 30% fixed income, 10% cash, whereas the managed portfolio has a 70/7/23 distribution. The return of the bogey portfolio is 3.97%, managed portfolio is 5.34%, i.e. excess return is 5.34–3.97 = 1.37%.

Attribution of the returns is as follows:

A. Contribution of Asset Allocation to Performance

Market	(1) Actual Weight in Market	(2) Benchmark Weight in Market	(3) Excess Weight	(4) Market Return (%)	(5) = (3) × (4) Contribution to Performance (%)
Equity	.70	.60	.10	5.81	.5810
Fixed-income	.07	.30	−.23	1.45	−.3335
Cash	.23	.10	.13	.48	.0624
Contribution of asset allocation					.3099

B. Contribution of Selection to Total Performance

Market	(1) Portfolio Performance (%)	(2) Index Performance (%)	(3) Excess Performance (%)	(4) Portfolio Weight	(5) = (3) × (4) Contribution (%)
Equity	7.28	5.81	1.47	.70	1.03
Fixed-income	1.89	1.45	0.44	.07	0.03
Contribution of selection within markets					1.06

Performance from market timing

A good manager is supposed to be able to time the market, i.e. hold more stocks when the market is going up, but reduce the stock positions when the market is going down. In a successful market timing strategy, the portfolio should have more exposure to the market factor (i.e. beta is higher than 1) when the market is bullish, but have less exposure (beta smaller than 1) when the market is bearish. Instead of a linear model as in the standard CAPM, the Security Characteristic Line is modified as (Treynor and Mazuy (1966)):

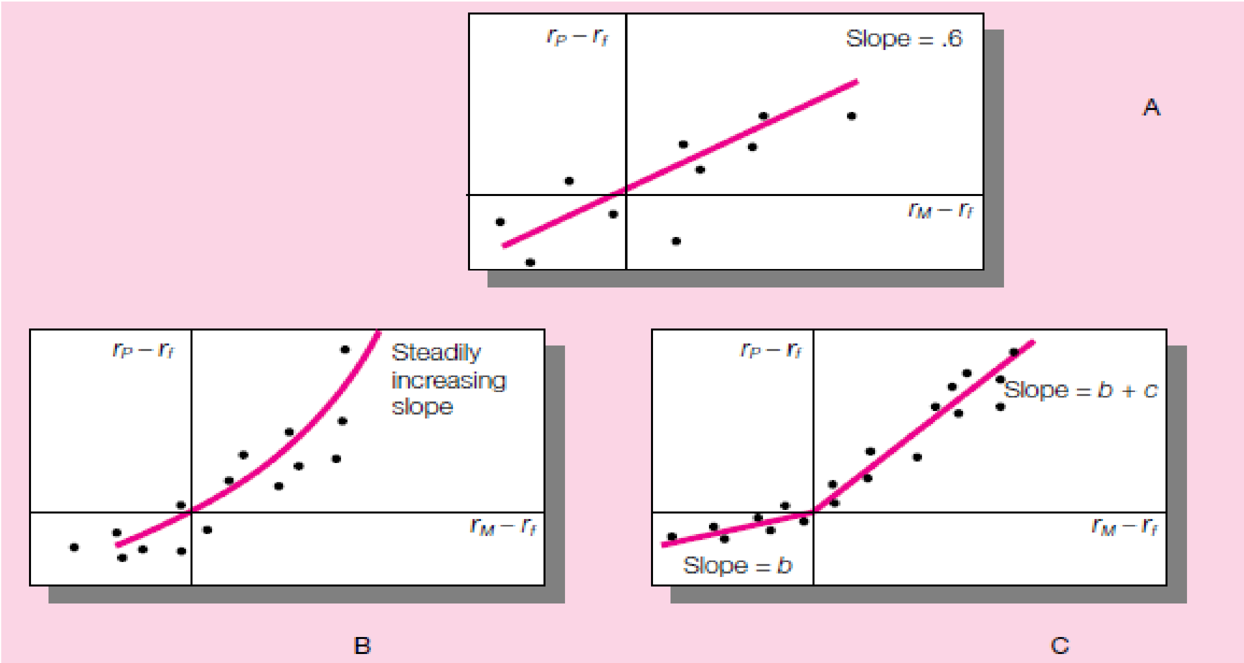
$$R_P - R_f = a + b(R_M - R_f) + c(R_M - R_f)^2 + e_P$$

where a, b, and c are constants estimated by regression. A positive c indicates evidence of timing ability.

In a similar manner, Henrickson and Merton (1981) suggest a two-beta model:

$$R_P - R_f = a + b(R_M - R_f) + c(R_M - R_f)D + e_P$$

D=1 if $R_M > R_f$, and D=0 if $R_M <= R_f$. Basically the SCL is represented by two straight lines.



A: No market timing, constant beta; B: market timing with increasing beta; C: market timing with two values of beta (Bodie et al., 2003)

In the examples given here, if plotting the historical data gives a diagram that looks like B or C, it is more likely that the fund manager has been successful in timing the market.

4.1.3. Standard performance measures

The most widely used benchmarks in measuring investment performance are the following:

- From the Capital Allocation Line (CAL):

$$\text{Sharpe ratio} = RS = \frac{(\overline{r_A} - r_f)}{\sigma_A}$$

- From the Security Market Line (SML):

$$\text{Treynor ratio} = RT = \frac{(\overline{r_A} - r_f)}{\beta_A}$$

$$\text{Jensen's alpha } \alpha_A = \overline{r_A} - \left[r_f + (\overline{r_M} - r_f)\beta_A\right]$$

$$\text{Information ratio} = RI = \frac{\alpha_A}{\sigma_{e_A}} \text{ where } e_A \text{ is a portfolio's tracking error}$$

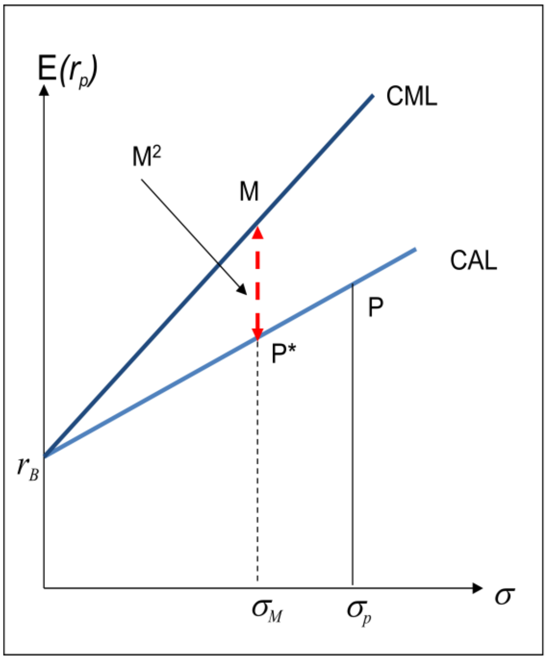
When should we use each benchmark?

Depending on the investment assumptions, one of the benchmarks can be more appropriate than the others. There are three different scenarios:

Case 1: If the portfolio represents the entire investment of an individual, then total volatility matters (measured by σ), and the Sharpe ratio is appropriate.
Case 2: If the portfolio is to be mixed with a passive market-index portfolio, the information ratio is more relevant as it gives the improvement in the Sharpe ratio of the overall portfolio.
Case 3: If a portfolio is just part of a diversified portfolio, then systematic risk matters (measured by β), and the Treynor ratio is more appropriate.

No matter which case is adopted, Jensen's alpha is widely used as it indicates a superior performance according to CAPM.

In each measure, a higher value is better. These benchmarks could be used to rank different portfolios. However, it is sometimes difficult to interpret the meaning of each ratio directly. For example, let's say the Sharpe ratio of a market index is 0.7 and a portfolio has a Sharpe ratio of 0.65 (i.e. it underperforms the market). It is difficult to assess the economic significance of a difference of 0.05 in the ratio.



An alternative measure known as M^2 was proposed and popularized by Modigliani and Modigliani (1997) (from the last names of the two authors). From portfolio P and the risk free asset, we can construct a portfolio P^* with the same σ as the market portfolio M. Next we define M^2 as:

$$M^2 = r_{P^*} - r_M = (RS_P - RS_M)\sigma_M$$

where RS_P and RS_M are the Sharpe ratios of portfolios P and M. Thus this ratio gives the excess return of the portfolio over the market portfolio. (In the example on the left, M^2 is negative.)

Performance measure example

	Port P	Port Q	Market
Excess return (%)	2.76	7.56	1.63
s.d.	6.17	14.89	8.48
alpha	1.63	5.28	
beta	0.69	1.40	1.00
r2	0.91	0.64	1.00
sigma(e)	1.95	8.98	
Sharpe Ratio	0.45	0.51	0.19
Treynor Ratio	4.00	5.40	1.63
Information Ratio	0.84	0.59	
M2	2.16	2.68	

Portfolio P has a lower β but smaller residual risk (σ_e) than portfolio Q. Both P and Q outperform the market (higher Sharpe ratios and positive alphas). Q has a higher Sharpe ratio, Treynor ratio and M^2 than P, therefore it is better in cases 1 and 3. P is better if it is to be mixed with an index portfolio as it has a higher Information Ratio (case 2 above).

Relationships between the ratios

$$RT_A = \frac{\overline{r_A} - r_f}{\beta_A} = \frac{\alpha_A}{\beta_A} + \left(\overline{r_M} - r_f\right) = \frac{\alpha_A}{\beta_A} + RT_M$$

RT_M is the Treynor ratio of the market, with $\beta_M = 1$.

$$RS_A = \frac{\overline{r_A} - r_f}{\sigma_A} = \frac{\alpha_A}{\sigma_A} + \frac{\beta_A\left(\overline{r_M} - r_f\right)}{\sigma_A} = \frac{\alpha_A}{\sigma_A} + \frac{\rho_{AM}\left(\overline{r_M} - r_f\right)}{\sigma_M} = \frac{\alpha_A}{\sigma_A} + \rho_{AM} RS_M$$

RS_M is the Sharpe ratio of the market and ρ_{AM} is the correlation coefficient between the portfolio and market, which is often smaller than 1. If $\rho_{AM} \approx 1$ and the portfolio is well diversified, then $RS \approx RT / \sigma_M$.

From these results, we can see that a higher α will lead to higher Sharpe and Treynor ratios. However, we should also note that a portfolio A with a positive α_A may not always result in a higher Sharpe ratio than the market portfolio. If we write out the expression:

$$RS_A - RS_M = \frac{\alpha_A}{\sigma_A} + (\rho_{AM} - 1) RS_M$$

the difference can be negative if ρ_{AM} is small. Secondly, Sharpe and Treynor ratios may rank portfolios differently as α_A is used differently. One condition that can ensure the same ranking by Sharpe and Treynor ratios is:

$$(RT_A > RT_B \Leftrightarrow RS_A > RS_B) \text{ if } \rho_{AM} = \rho_{BM}$$

Proof:

$$RT_A > RT_B \Leftrightarrow \frac{\overline{r_A} - r_f}{\beta_A} > \frac{\overline{r_B} - r_f}{\beta_B}$$

$$\Leftrightarrow \frac{\sigma_M}{\rho_{AM}} \frac{\overline{r_A} - r_f}{\sigma_A} > \frac{\sigma_M}{\rho_{BM}} \frac{\overline{r_B} - r_f}{\sigma_B} \Leftrightarrow \frac{RS_A}{\rho_{AM}} > \frac{RS_B}{\rho_{BM}}$$

Note that if $RT_A > RT_B$ and $\rho_{AM} > \rho_{BM}$, then $RS_A > RS_B$; otherwise no general comparison is possible.

To compare Sharpe and Jensen measures:

$$RS_A = \frac{\alpha_A}{\sigma_A} + \rho_{AM} RS_M \,,\, RS_B = \frac{\alpha_B}{\sigma_B} + \rho_{BM} RS_M$$

$$RS_A > RS_B \Leftrightarrow \alpha_A - \frac{\sigma_A}{\sigma_B} \alpha_B > RS_M (\rho_{BM} - \rho_{AM}) \sigma_A$$

The last expression indicates that the Sharpe and Jensen measures can often produce different rankings, given the various parameters that are involved in the expression.

4.1.4. Financial instruments with non-linear payoffs

The standard ratios are appropriate for used in measuring portfolios with assets that have linear payoffs (such as stocks and bonds). However, there exist a large class of instruments with non-linear payoffs, such as financial derivatives, which can give rise to different behavior for the ratios. An important example is an option, where the holder has the right, but not the obligation, to choose whether to exercise the contract or not.

Two basic types of options are the European calls and puts. In a European call, the final payoff formula is: payoff = max(S_{final} – strike, 0) where S_{final} is the stock price at option maturity. For example, a European call option on HSBC has a strike price of $65. At option maturity, if stock price = \$70, the payoff per option is 70–65 = \$5. However, if stock price = \$60, the option is worthless. For a European put option, the payoff = max(strike – S_{final} , 0). If a put option has strike = \$65, and if stock price at maturity = \$70, the option is worthless. Otherwise if stock price = \$60, payoff per option is 65–60 = \$5.

Manipulating the Sharpe ratio

Goetzmann et al. (2002) show some examples where the Sharpe ratio may become misleading. In one such example, a great trading strategy can turn out to have a low Sharpe ratio. Assume a certain analyst can pick stocks that can always outperform the other companies within an industry. A natural strategy is to long the outperforming stocks, and short the underperforming stocks in each sector. This strategy will always provide a positive return; however, because there are variations in returns (i.e. high variance), the Sharpe ratio can be low because the variance appears in the denominator.

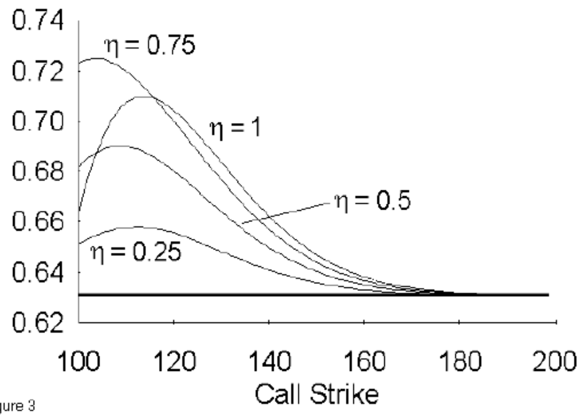


Figure 3

In another example, it is shown that the Sharpe ratio is especially misleading when applied to portfolios with options. The Sharpe ratio of the stock is 0.631 and a portfolio consists of long 1 share and short η calls. There is no guarantee that this strategy must work, but a Sharpe ratio above 0.73 can be achieved by varying η.

Some new benchmarks have been developed to alleviate the problems mentioned above. Here we would like to introduce the concept of dispersion measures, which are measures of uncertainty. The most widely known and typical one, standard deviation of return, would consider both positive and negative deviations from the mean and treat them equally. Other measures include:

Mean-absolute deviation (MAD): $E\left(\left|\sum_{j=1}^{n} w_j r_j - E\left[\sum_{j=1}^{n} w_j r_j\right]\right|\right)$

More generally, Mean-absolute moment: $\left\{E\left(\left|\sum_{j=1}^{n} w_j r_j - E\left[\sum_{j=1}^{n} w_j r_j\right]\right|^q\right)\right\}^{1/q}$

Alternative definitions of alpha and beta

CAPM assumes that asset returns are normal and investors have mean-variance preferences (i.e. ignore the third moment or skewness). For portfolios with arbitrary return distributions, the following modifications are used (Leland (1999)) :

$$CAPM\ \beta_P = \frac{\operatorname{cov}(r_P, r_M)}{\sigma_M^2},\ \text{Adjusted } B_P = \frac{\operatorname{cov}[r_P, -(1+r_M)^{-b}]}{\operatorname{cov}[r_M, -(1+r_M)^{-b}]}$$

If excess return of the market is normal, b=1. If excess return of the market is lognormal,

$$b = \frac{\ln[E(1+r_M)] - \ln(1+r_f)}{\mathrm{var}[\ln(1+r_M)]}$$

Adjusted Alpha is calculated using the CAPM formula, but with the adjusted B_p.

Long the market portfolio, short one call

Strike	E(r)	β	α	adj B	adj A
90	5.51%	0.038	0.24%	0.073	0.00%
100	6.76%	0.163	0.62%	0.251	0.00%
110	8.61%	0.394	0.85%	0.515	0.00%
120	10.27%	0.650	0.72%	0.753	0.00%
130	11.30%	0.838	0.57%	0.900	0.00%
140	11.77%	0.939	0.20%	0.967	0.00%

This example is computed assuming a lognormal portfolio with expected return 12% and s.d. 15%. α is computed with a risk free rate of 5%. The factor b is 3.63. From Leland (1999).

Long the market portfolio, long one put

Strike	E(r)	β	α	adj B	adj A
90	11.49%	0.962	-0.24%	0.927	0.00%
100	10.24%	0.837	-0.62%	0.749	0.00%
110	8.40%	0.606	-0.84%	0.485	0.00%
120	6.73%	0.351	-0.72%	0.247	0.00%
130	5.70%	0.163	-0.44%	0.101	0.00%
140	5.24%	0.062	-0.19%	0.034	0.00%

Downside measures

Instead of allocating risks equally between upside and downside returns, downside risk measures only take into account of the losses with respect to a certain per-defined level. An example is the Roy's safety-first criterion: portfolio optimization is based on minimizing the probability that the portfolio return will be lower than a threshold return R_0. By Tchebycheff's inequality, it can be shown that

$$P(R_P < R_0) \le \frac{\sigma_P^2}{\overline{R_P} - R_0}$$

where $P()$ denotes a probability function.

We can see that the Sharpe ratio is based on a dispersion risk measure around the mean, thus variations above or below the mean have equal contributions. Following the ideas above, we define two measures that only take into account of the asymmetry and the downside risks:

- Semi-variance $SV(R) = E\left[\left[(E[R]-R)^+\right]^2\right]$
- Lower partial moment $SV(R) = E\left[\left[(E[R]-R)^+\right]^p\right]$
  where p is an integer > 2

Sortino ratio

This is a very popular ratio used for comparing hedge fund performances. Conceptually it is similar to the Sharpe ratio, but it only takes into account of the loss expectations:

$$Sor(R) = \frac{E(R) - L}{\sqrt{E\left[\left[(L-R)^+\right]^2\right]}}$$

where L is a minimal acceptable return level. Depending on the application, different L can be chosen. For example, in order to control the loss risk, L could be set to 0. Alternatively, the riskless rate r_f can be used as the benchmark, i.e. $L = r_f$. For comparison purposes L can also be chosen as the mean expected returns of other funds.

The Omega performance measure

Another performance measure is introduced by Keating and Shadwick (2002). It takes into account of the whole probability distribution, considering both the gain and loss probabilities:

$$\Omega_F(L) = \frac{E_P\left[(X-L)^+\right]}{E_P\left[(L-X)^+\right]}$$

where L is a reference level of wealth chosen by the investor. This is the ratio of the expectation of gains above L on the expectation of losses below L, or the ratio of a call on a put with the same strike L, where the options are computed with the historical probability P rather than the risk-neutral probability.

Example of hedge fund performance

Illustration 5: Hedge fund strategies' risks for the period from 1999 to 2008

	Risk Dimension				Risk-Adjusted Performance	
Reference period: January 1999-December 2008	Maximum Drawdown (in %)	Volatility (in %)*	Downside Risk (in %)*	Modified Value-at-Risk (in %)***	Sharp Ratio*/**	Sortino Ratio*/**
Convertible Arbitrage	29.27%	6.74%	8.69%	3.55%	0.24	0.26
CTA Global	11.68%	8.80%	4.85%	3.52%	0.48	1.01
Distressed Securities	22.60%	5.88%	5.56%	2.50%	1.04	1.22
Emerging Markets	34.54%	11.60%	9.04%	5.05%	0.65	0.91
Equity Market Neutral	11.08%	3.17%	4.34%	1.28%	1.05	0.91
Event Driven	20.07%	5.83%	5.34%	2.56%	0.84	1.04
Fixed Income Arbitrage	17.60%	4.21%	6.44%	2.03%	0.36	0.33
Global Macro	7.92%	5.27%	2.62%	1.56%	0.95	2.16
Long/Short Equity	21.04%	7.60%	5.21%	3.14%	0.51	0.86
Merger Arbitrage	5.65%	3.58%	2.95%	1.31%	1.21	1.68
Relative Value	15.94%	4.52%	5.27%	2.08%	0.81	0.82
Short Selling	36.30%	17.74%	11.31%	7.91%	-0.01	0.05
Funds of Funds	20.22%	6.18%	4.78%	2.43%	0.50	0.78

Source: Veronique Le Sourd, "Hedge Fund Performance in 2008," *EDHEC Risk and Asset Management Research Centre*, Feb 2009.

4.2. Risk measures

4.2.1. Value-at-Risk (VaR)

Defining risk is a difficult task, especially in the case of complex operations. Different measures have been developed in order to manage risk effectively. One of the most often used concepts was first proposed in the mid-1990's, known as **Value-at-Risk (VaR)**.

According to Hull (2003), it is: "*An attempt to provide a single number summarizing the total risk in a portfolio of financial assets for senior management*". A typical statement is:

"We are X % certain that we will not lose more than V dollars in the next N days."

- X is the confidence interval (usually X =95 or 99)
- N is the holding period (N=1 day or 10 days are common)
- V is the VaR

In computing the VaR, a normal distribution is commonly assumed for the asset returns because of the Central Limit Theorem and its analytical tractability. The *holding period* reflects the liquidity risk, which should roughly equal to the time needed to liquidate the portfolio. Finally, the *confidence level* is kind of arbitrary; there is no absolute justification as to why 95% or 99% is a correct standard. However, a consistent measure should be used to compare the change in VaR on a daily basis.

4.2.2. Computing the VaR

The Standard Case

For a portfolio with no option, VaR is given by the following formula:

$$\text{VaR} = a \times \text{expected \% move in portfolio value}$$

There are two parts in this definition. In the first part, if we assume a normal distribution, at 99% confidence interval, a = 2.33; at 95% confidence interval, a = 1.96 (values obtained from a standardized normal distribution table).

For the second part,

$$\text{expected \% move in portfolio value} = N \times \sigma_{\text{p}} \sqrt{t}$$

where N is the portfolio value, σ_{p} is the portfolio s.d. and t is the holding period (in years). This move is scaled by the nominal amount of the portfolio. Furthermore, if we assume assets prices follow normal distributions, the portfolio variance is given by:

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} w_j w_k \sigma_{jk}$$

As an example, assume that the mark-to-market value of the current portfolio N = \$100 Million; the variability of the risk factors σ_{p} = 15% per annum; the time horizon or the holding period t = 10 business days (or 10/252 year, assuming 252 business days in a year); and set the confidence interval a to 99% (i.e. 2.33σ for a normal distribution). The VaR V is given by

$$\begin{aligned} V &= N \times \sigma_p \times \sqrt{t} \times a \\ &= 100 \times 0.15 \times \sqrt{\frac{10}{252}} \times 2.33 \\ &= \$6.96 \text{ Million} \end{aligned}$$

Portfolios with derivatives (especially options)

If the portfolio has options, how to represent the portfolio value becomes a complex (but realistic) topic. Because of the non-linear payoff, the risk profile of such a portfolio could be highly irregular. For a portfolio that contains a large number of option positions, the only sensible method is to perform a full scale scenario analysis, which can be computationally intensive. In order to simplify the problem, a number of models have been proposed, but we need to understand the limitations of each of these methods.

Delta-normal method

Each option position is replaced by its delta equivalent:

$$\Delta = \frac{\delta P}{\delta S} \Rightarrow \delta P = \Delta \delta S = \Delta S \delta x$$

where $\delta P/\delta S$ is the change in the portfolio value w.r.t. a unit change in the underlying asset price. δx is the % change in the asset price in one day. For a portfolio with n assets,

$$\delta P = \sum_{i=1}^{n} \Delta_i S_i \delta x_i$$

We could then use the equation above to calculate the portfolio variance, using δP to replace the term $N\sigma_p$.

As an example, we have the following inputs:

Delta of Hang Seng Bank (HSB) options = 800 shares
Delta of Sun Hung Kai (SHK) options = 1200 shares
HSB share price = \$130, SHK share price = \$72
Volatility of HSB = 25%, SHK = 30%,
correlation = 0.60
Portfolio standard deviation =

$$\sigma_{HSB} = 800 \times 130 \times 0.25 = 26000$$

$$\sigma_{SHK} = 1200 \times 72 \times 0.30 = 25920$$

$$\sigma_P = \sqrt{26000^2 + 25920^2 + 2 \times 26000 \times 25920 \times 0.6} = 46439$$

VaR at 99% confidence interval and 1-day holding period is $2.33 \times 46439 \times \sqrt{1/252} = \$6816$.

Historical simulation method

Instead of using a model to represent the distribution of the asset prices, it is assumed that the daily historical return can represent future movements, and these movements are applied to the current market prices in order to generate possible future scenarios. The advantages of this method are:

- Simple to implement if data are readily available.
- Allows non-linearities and non-normal distributions and correlation behavior.

On the other hand, there may be some difficulty in getting sufficient data, especially for new companies or new markets. Furthermore, as we only have one sample path (from history), one can argue that it may not be a reliable guide of the future. Hence it is often used as a reference, and should be supplemented with other measures.

One example from the FX market is shown below.

Reference date	2/15/2013		
spot fx	1.6850	Position	
R(USD)	2.700%	USD fwd	16,760,000
R(GBP)	4.200%	GBP fwd	-10,000,000
days to maturity	90	Current MTM (USD)	-29,659

	days to				Absolute daily change			Simulated movements				
	maturity	spot FX	R(USD)	R(GBP)	FX % retur	R(USD)	R(GBP)	FX	R(USD)	R(GBP)	MTM	P&L
7/29/2010	90	1.5425	4.8750%	5.6250%				1.6850	2.7000%	4.2000%	-29,659	
8/1/2010	87	1.5360	4.8125%	5.6875%	-0.00421	-0.0625%	0.0625%	1.6779	2.6375%	4.2625%	43,613	73,272
8/2/2010	86	1.5355	4.8125%	5.6250%	-0.00033	0.0000%	-0.0625%	1.6774	2.6375%	4.2000%	45,891	2,278
8/3/2010	85	1.5573	4.8125%	5.5000%	0.01420	0.0000%	-0.1250%	1.7012	2.6375%	4.0750%	-195,480	-241,371
8/4/2010	84	1.5357	4.7500%	5.4375%	-0.01387	-0.0625%	-0.0625%	1.6776	2.5750%	4.0125%	37,677	233,157
8/5/2010	83	1.5436	4.7500%	5.5625%	0.00514	0.0000%	0.1250%	1.6862	2.5750%	4.1375%	-43,761	-81,439
8/8/2010	80	1.5394	4.8750%	5.6250%	-0.00272	0.1250%	0.0625%	1.6816	2.7000%	4.2000%	-2,708	41,053
8/9/2010	79	1.5280	4.8750%	5.5000%	-0.00741	0.0000%	-0.1250%	1.6692	2.7000%	4.0750%	115,608	118,316
8/10/2010	78	1.5370	4.8750%	5.5000%	0.00589	0.0000%	0.0000%	1.6790	2.7000%	4.0750%	17,553	-98,055
8/11/2010	77	1.5580	5.0000%	5.5625%	0.01366	0.1250%	0.0625%	1.7019	2.8250%	4.1375%	-212,715	-230,268

Assume that a position in 90-day FX forward was held on Feb 15, 2013, which was long USD 16.76 Million against short GBP 10 million (the historical prices are not accurate here). Current interest rates for USD and GBP were 2.70% and 4.20% respectively. Starting from July 29, 2010, the daily movements in the FX rate and the interest rates are used to generate the movements in each day. These are then applied to the current rates. For example, between 8/2/2010 and 8/3/2010, the GBP interest rate has dropped by 0.125%. This is applied to the simulated rate, so that the movement is from 4.20% to 4.075%. For FX rate, the percentage movements are used instead of absolute movements. From these simulated rates, the theoretical mark-to-market (MTM) value is calculated and the daily P&L can be obtained. After calculating the daily P&L, they are ranked and the 95% worst case is selected as the VaR (assuming a 95% confidence interval).

Monte Carlo simulations

This method uses a mathematical model to represent asset price movements. The simplest model which is used widely in derivative pricing is the Geometric Brownian Motion (more details are given in Topic 7). Using the model, paths of asset prices are generated. For each asset path, the mark-to-market P&L of the portfolio is calculated. Similar to the historical simulation method above, the P&L of each path is ranked, and the 99 (or 95) percentile return is the VaR. Typically 100,000 paths or more will be used.

One advantage of this method is that it can accommodate portfolios with very complex positions. However, if the calculation of portfolio value is time consuming, the VaR calculation can become very computational intensive. On the other hand, these calculations are now often implemented with parallel computing algorithms as each path is independent to each other. Furthermore, the accuracy of the method relies on strong assumptions in the underlying model (i.e. model risk is high).

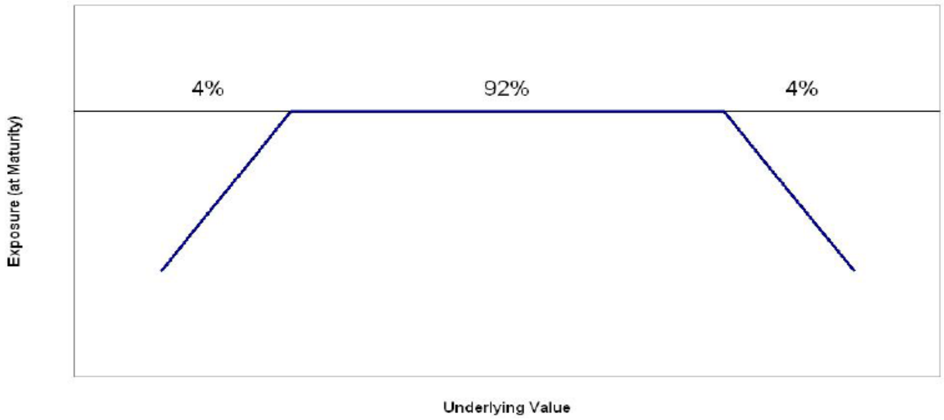
4.2.3. Problems with the VaR models

The pioneer work from Artzner et al. (1999) has defined some important properties which should be satisfied by good risk measures. Mathematically, if X and Y are two portfolios and $\rho()$ is a risk measure, the conditions of coherence are given by:

Monotonicity:	if $X \le Y, \rho(X) \ge \rho(Y)$
Translation invariance:	$\rho(X+k) = \rho(X) - k$
Homogeneity:	$\rho(b \circ X) = b \circ \rho(X)$
Sub-additivity:	$\rho(X+Y) \le \rho(X) + \rho(Y)$

It can be shown that VaR is not a good risk measure as it does not satisfy the last property!

We need to examine the last property in more detail. The condition of sub-additivity comes from the effect of diversification. Supposedly, given the effect of diversification, a portfolio with assets X and Y should have risks that are lower than the plain sum of the risks of these assets. However, we can see that sometimes the VaR of a portfolio can be higher than the sum of the VaRs of the individual assets of the portfolio, especially when the portfolios contain positions with non-linear payoffs. From Artzner et al. (1999), assumes that Trader A sells an out-of-the-money call option, with a probability of 4% that the call be in-the-money. Trader B sells an out-of-the-money put option, with a probability of 4% that the put be in-the-money. The payoffs and the probabilities are shown below:



The 95% VaR of each individual position is 0 since there is only a 4% probability that there will be a loss (VaR(A)+VaR(B)=0). However, the 95% VaR of the portfolio is higher than 0, since the portfolio only has a probability of 92% that it will not end up with a loss.

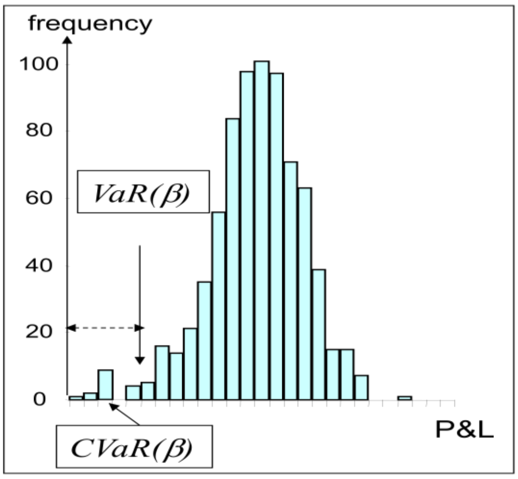
In another similar example, assume that each of two independent projects has a probability 0.98 of a loss of $1 million and 0.02 probability of a loss of $10 million. Obviously the 97.5% VaR for each project is $1 million and the simple sum of the two VaRs is $2 million. When combining the two projects into a single portfolio, the loss distribution can be calculated as follows:

- Probability = 0.02 x 0.02 = 0.04% of a loss of $20 million
- Probability = 2 x 0.02 x 0.98 = 3.92% of a loss of $11 million
- Probability = 0.98 x 0.98 = 96.04% of a loss of $2 million

The 97.5% lies in the segment (3.92%), where the loss is $11 million. VaR of the portfolio is thus $11 million, which is much higher than the sum of the individual VaRs.

4.2.4. Conditional value-at-risk (CVaR)

Another method which can overcome the theoretical inconsistency faced by the VaR is the Conditional Value-at-Risk (CVaR), which is also known as the Expected shortfall. If VaR is the best outcome of a set of bad outcomes on a bad day, the CVaR is the *average bad outcome on a bad day*. In particular, it measures the expected losses in the tail of the distribution of possible portfolio losses. As such, calculating CVaR can be computationally more intensive and requires a more *detailed* description of the loss distribution.



The diagram shows a typical example of the frequency distribution of portfolio P&L. VaR represents a quantile, e.g. 95% or 99%.

$$\Pr(\text{loss} <= \text{VaR}(\beta)) = \beta$$

e.g. $\beta = 99\%$

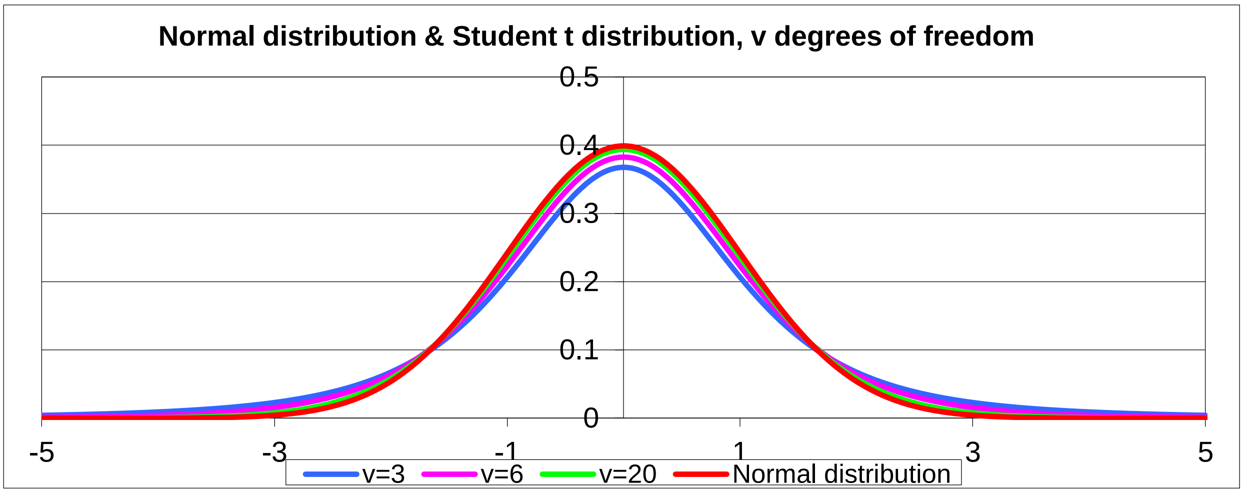
On the other hand, CVaR measures the expected portfolio loss rather than the quantile.

$$\text{CVaR}(\beta) = E(\text{loss} \mid \text{loss} > \text{VaR}(\beta))$$

It is obvious that CVaR is greater than VaR.

4.2.5. Other distributions used in risk management

With a normal distribution, there is a very small probability for extreme moves (e.g. 3 standard deviations). However, it is well known that extreme moves (“Black Swan” events) do occur much more frequently (e.g. stock market crash in 1987, Asian crises in 1997, 9/11 event in 2001, financial crisis of 2007/2008, etc.). In order to represent these moves more accurately, other fat-tailed distributions have been adopted, e.g. Student-t distribution and the use of Extreme Value Theory. While they can better describe the extreme movements, these models are not so analytically tractable.



4.2.6. Some comments on how to use VaR

Despite the known pitfalls, VaR is still the most widely used measure for risk management. However, one must understand thoroughly the assumptions behind the VaR number. For example, one should never see VaR as a kind of "worst" loss. Even if we are calculating a 99% confidence interval, it still means that for 1 % of the time, or about 3 days per year, the loss could be greater than the amount as indicated by the VaR. This is predicted by the model, and it is normal. In addition, it may be difficult to compare the VaR calculated by different institutions, given that there is not a set of universally accepted assumptions and standard parameters. Therefore a proper way to use VaR is to monitor its trend for a portfolio, and note any unusual change. Ultimately, it should be recognized that he calculated VaR is not a magic number – it should always be used as one of the measures in managing risks, together with sound risk management procedures.

Additional References

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