6. Derivative Instruments

6.1. Introduction

Financial derivatives, as its name suggests, are contracts that are "derived" from some underlying asset prices. For example, there are derivatives on stocks, exchange rates, interest rates, and even derivatives on derivatives.

The payoff pattern for basic assets is linear: the change in the asset value is constant when the underlying asset price moves by one unit. On the other hand, the derivative payoff could be linear or non-linear. For some derivatives such as swaps and futures, if the underlying asset price moves by one unit, the derivative instrument's value would move linearly. Another major category of derivatives, known as options, would have non-linear payoff. If the underlying asset price moves by one unit, the derivative instrument's value could sometimes move dramatically. While the basic derivatives are dependent on one underlying asset, some complex derivatives can involve multiple assets and currencies.

Derivative instruments are effective means of *risk transfer*, allowing entities with different risk profiles to manage their risks actively, thus giving rise to a *complete market*. Furthermore, many derivative contracts have a built-in *leverage* effect, hence any gains or losses from holding these positions will be amplified. Given that some of the listed instruments (those that are traded in organized exchanges) could be very liquid, these instruments are very efficient in acting as a speculative tool. For example, the typical daily trading volume of Hang Seng Index futures is about 120,000 contracts, which is roughly equivalent to HKD 130 billion (assuming HSI is at 22,000 points). This represents more than 100% of cash market turnover per day.

Before we describe the instruments in more detail, it is important to have a basic understanding of the concepts of *fair value* and *no arbitrage*, which play an important part in pricing. Given that the derivative instrument is based on the prices of underlying assets, we may apply some kind of mathematical model and calculate its theoretical value. However, in the market, these instruments may not always trade at their fair theoretical values, thus giving rise to *arbitrage* trading opportunities. By an arbitrage, it is possible to set up a trading strategy such that a riskless profit could be earned (the risks here normally only refer to the risks coming from changes in market prices).

We use a simple example from trading stocks to illustrate this concept. In the summer months, there will be some overlap between the trading hours in Hong Kong and London. At 3:10pm Hong Kong time, let's say HSBC is trading at HKD 82.0 in HK and GBP 6.67 in London. The GBP/HKD exchange rate is 12.50. A possible arbitrage strategy is to sell N shares in London, convert the money into HKD, and buy N shares in HK. The net profit (before transaction cost) is HKD 1.375 per share (= 6.67 x 12.50 – 82.0). Of course, in real life we need to consider the transaction cost (such as trading fee and stamp duty) as well as convertibility issues. In this example, the opportunity arises because an instrument is not trading at its fair value because of supply and demand conditions in different markets; otherwise arbitrage should not exist.

6.2. Basic derivative contracts
6.2.1. Forward contracts

A forward contract is a simple derivative contract which is traded between two parties directly and not through an exchange. A delivery price would be negotiated today that would be applicable to the contract when the trade is completed at a time in the future. Normally this contact has a price of zero at trade date, i.e. both parties do not need to make initial payments when the contact is traded. In many markets, contracts of different maturities based on the same underlying exist, with different delivery prices depending on the maturities. Examples from the currency and interest rate markets include:

- FX forward

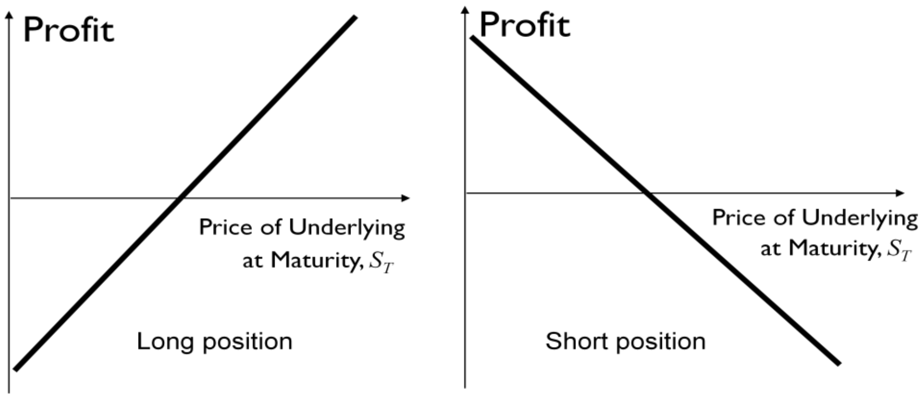
A agrees to buy GBP 1 Million from B in 3-months, in exchange for USD 1.62 Million.

- Forward Rate Agreement (FRA)

A agrees to borrow USD 1 Million from B for 6-months at 0.75% p.a., and the loan would start 1-year from now.

At the maturity of the contract, the contracts are settled when both parties fulfill their obligations. This is known as *physical settlement*. However, trades related to interest rates are usually *cash-settled.* For example, for the FRA above, if 6-month interest rate resets at 1.19% in 1 year's time, then A would earn a profit of USD 1000000 x (1.19% – 0.75%) x 0.5 = USD 2,200 payable at time 1.5 yr, or 2200 / (1+0.0119 x 0.5) = USD 2,186.99 at time 1 year (0.5 is the accrual period, because the borrowing period is only 6 months, which is assumed to be equal to 0.5 year here. Interest rate is always expressed as a % per annum, and thus it has to be multiplied by the correct accrual period).

The P&L profiles of typical forward positions look like these:



Note that the payoff is linear to the price movement.

Theoretical pricing of forward contracts

A simple model, often known as the cost-of-carry model, can be applied. There are a few different forms for this model. In the case of stock forward, we often use:

$$F = Se^{(r_c - d_c)t} \text{ or } F = S[1 + (r_a - d_a)t]$$

F is the fair forward price, S is the current stock price, t is the time to maturity (in years), r_c is the interest rate and d_c is the dividend yield, given in the continuous compounding convention; r_a and d_a are the corresponding rates given in annual compounding convention; the conversion is given by:

$$e^{r_c t} = (1+r_a)^t, e^{d_c t} = (1+d_a)^t$$

For a currency forward, an alternative form often found in textbooks is (also see below):

$$F = Se^{(r_2 - r_1)t}$$

Commodity forward example

Spot Gold Price is USD 1240, 1-year USD interest rate is 1.5%. Therefore S = 1240, t = 1, r_a = 0.015, d_a = 0. Today's forward price of gold in 1 year is calculated as:

F = 1240 x (1+0.015 x 1) = 1258.60

Note that this price is the theoretical price, which is often different from the market price.

Interest rate parity

One of the fundamental relationships in economics is known as the interest rate parity, which can also be illustrated as an example in currency forward. Assume that the interest rate of currency 1 = r_1, interest rate of currency 2 = r_2. Spot FX rate is expressed in the convention: 1 unit of currency 1 = S units of currency 2. At time t,

- 1 unit of currency 1 becomes $(1+r_1 t)$
- S units of currency 2 become $S(1+r_2 t)$

Initially, one could choose to hold 1 unit of currency 1 or S units of currency 2. As one should have equal benefit in holding either of these positions, their values at time t should be the same. Therefore the forward FX rate F as seen today is thus $S(1+r_2 t)/(1+r_1 t)$, i.e. $F = S(1+r_2 t)/(1+r_1 t)$. Note that this is the form that is often used in real markets instead of the textbook form using exponential functions.

A simple arbitrage trade

We can make use of the interest rate parity relationship to construct an arbitrage trade in real life. Given any 3 of the 4 variables from F, S, r_1, r_2, the 4th variable could be obtained (the time t is assumed to be fixed and given). If the parity equation does not hold, riskless profit could be made.

Assume the following market rates: Spot FX EUR 1 = USD 1.3585; 12-month forward FX is 1.3677; USD interest rate 3.10%; EUR interest rate 2.30%.

Theoretical 1-year forward FX rate is: F_{theo} = 1.3585 x (1+0.031)/(1+0.023)= 1.3691. As this theoretical rate is not equal to the market rate of 1.3677, an arbitrage opportunity exists. One way of constructing the strategy is by making four simultaneous trades on the current date:

1. Borrow EUR 1@2.3% for 12 months;
2. Sell EUR 1 to buy USD 1.3585 (spot FX transaction);
3. Deposit USD 1.3585@3.1% interest for 12 months;
4. Enter into forward FX, buy EUR/sell USD 1.4006@ forward rate 1.3677.

Note that the last trade is made at the market price of 1.3677 and not the theoretical price of 1.3691. *The theoretical price is not tradable*; trades can only be conducted at the market prices.

The arbitrage profit would be made when all the contracts mature after 12 months, irrespective of the market rates at that time. Only trades 1, 3 and 4 are outstanding as trade 2 is a spot FX trade and will be settled at the start. In trade 3 above, the USD amount becomes 1.3585 x (1+3.1%) = 1.4006. This amount will be used to settle the forward FX trade 4, by selling USD 1.4006 and receiving EUR 1.4006/1.3677 = 1.0241. Finally, the amount needed to settle trade 1 is EUR 1 x (1+2.3%) = 1.0230. The net profit of these trades is thus: EUR 1.0241 – 1.0230 = EUR 0.0011.

We note two characteristics in the above calculations. Firstly, as the market prices at maturity are not used (e.g. we don't need to know what the EUR/USD exchange rate is), the profit is fixed once all four trades are conducted on the first day. Secondly, there is **no cost involved** when entering these trades, so theoretically we can trade these in big amounts. Of course, in real markets, big trades can cause the market prices to move, and thus arbitrage conditions will no longer be satisfied.

6.2.2. Futures contracts

Forward contracts are privately negotiated between two parties, and the contract terms are defined in order to suit the needs of both. Another type of derivative instruments, known as a futures contract, is very similar to the forward. These contracts are traded in organized exchanges and have standardized terms, i.e. they are not negotiable. Contracts exist for different asset classes; examples include:

- Equity: S&P 500, Dow Jones, Nikkei 225, Hang Seng Index (HSI)
- Interest rates: US Treasury bonds, Eurodollar (LIBOR)
- FX: USD/JPY
- Commodity: Corn, wheat, gold, coffee, electricity, pork belly

There are contracts for other (and more complex) types of underlyings, e.g. futures on a volatility index, such as VIX in the US; futures on the forecast dividend of index stocks, e.g. for HSI and HSCEI in Hong Kong.

One type of popular futures contract is based on the equity index, e.g. Hang Seng Index futures. As with other futures contracts, the HSI futures has standardized terms, e.g. each point is worth HKD 50, and there are fixed rules to determine the number of different maturities available for trading on a particular trading day. Trading activities are carried out via an electronic trading platform, and they are highly transparent, i.e. everyone would have access to the current trading information. As noted before, there is much liquidity in these contracts and the value traded everyday is much higher than the cash market.

Compared to forward contracts, the counterparty in trading in a futures contract is the exchange. Most of the big exchanges are considered to be very reliable and thus these contracts are deemed to have very little counterparty risk (i.e. the risk of the counterparty not being able to fulfill the contract obligations). In terms of calculating the actual profit and loss of a contract, there is a big difference between forwards and futures. Recall our discussion about forward contracts above, that there is usually no initial cost in entering these contracts and the P&L will only be settled at maturity. For futures, there is a *daily mark-to-market* process, where the P&L will be allocated to a settlement account everyday. As long as the minimum amount in the account satisfies some kind of *margin requirement*, any trading profit can be withdrawn before contract maturity.

The calculation of fair value of index futures is very similar to that of forward contracts, but usually we make use of a slightly different form:[1]

$$F = (S - D)(1 + r_t t)$$

F: fair value of the futures
S: today's index level
r_t: interest rate from today to futures expiry
t: time from today to futures expiry
D: present value of all dividends paid before futures expiry (equivalent index points)

Assume today is January 22, 2014, spot HSI level S = 22,750, interest rate r = 1%, February futures expires on February 27, 2014, i.e. t = 36/365 = 0.09863 years (there are 36 days between Jan 22 and Feb 27), *estimated dividends* before futures expiry = 117.5 index points. Fair value F = (22750 – 117.5) x (1+0.01x0.9863) = 22654.82.

As discussed above, it is very likely that the market price of February futures would not be trading at this fair value. If the market price is M, and if $M > F$, futures is trading at a *premium*. Alternatively, if $M < F$, futures is trading at a *discount*. Note that whether the futures is trading at a premium or discount, the benchmark for comparison is the fair value of the futures, not the spot price. In this example, if M = 22680, it should be considered as trading at a premium although it is below the spot HSI level of 22750.

6.3. Options
6.3.1. Introduction

An option is a common type of derivative instrument which possesses a number of unique characteristics. Firstly, the holder of the option **has the right, but not the obligation**, to enforce the contract. This is different from the case of forwards, when both parties are obliged to fulfill their contracts at maturity. Because of this privilege, the option buyer usually needs to pay a fee (known as the option price or option premium). This is also different from the forwards (often traded at the market price), where there is no initial cost to either party of the transaction. Secondly, an option is a basic kind of derivatives which has a **non-linear payoff** pattern. This gives rise to some interesting features which would be discussed in later sections.

Some option terminology

Call option: the holder has the right to buy a pre-determined amount of the underlying asset at the strike (or exercise) price by a certain date.
Put option: the holder has the right to sell a pre-determined amount at the strike price by a certain date.
Option premium: the price paid by the option buyer to the option seller to purchase the right. This price is often paid at the beginning of the trade.
Option maturity date: last date at which the contract must terminate.

[1] Technically there is a small difference between the fair forward price and the futures price because of the daily mark-to-market process, especially when there is some kind of correlation between the interest rate and the underlying asset price. However, for short maturities, this adjustment can be ignored.

European option: the option holder could only exercise his/her right at maturity.
American option: the option holder could exercise the right at any time up to maturity.
Since the holder of an American option has more choices, therefore this option could NEVER be cheaper than the corresponding European option; usually it is more expensive (but note the properties discussed in the next section).
Exotic option: There are many types of options with more exotic (non-standard) payoff formulas. Typical ones include Digital (Bet), Barrier, Asian (Average), and quanto options. Some of these options require sophisticated modeling techniques for pricing and risk management, and they are often embedded in structured products sold by investment banks.

Option examples

We have already introduced the basic call and put options in section 4.1.4. These are often known as vanilla options. A similar stock option position would have the following terms: long 10,000 European call options on Cheung Kong, strike = HKD 130, maturity January 29, 2015. Options exist in all kinds of asset classes. A typical currency option would be: long USD 5 Million notional American call USD/put JPY option, strike 109.00, maturity June 30, 2015. In this case, since two currencies are involved, we need to specify clearly which currency would be bought and which currency would be sold under the option ("call USD/put JPY" here). One type of options on interest rate, known as an interest rate cap, would look like this: for every quarter in the next 3 years, if 3-month LIBOR (a reference interest rate) is higher than 1%, the holder gets (3-month LIBOR– 1)% on a notional of USD 10 Million. In this case, 1% is the strike of the option.

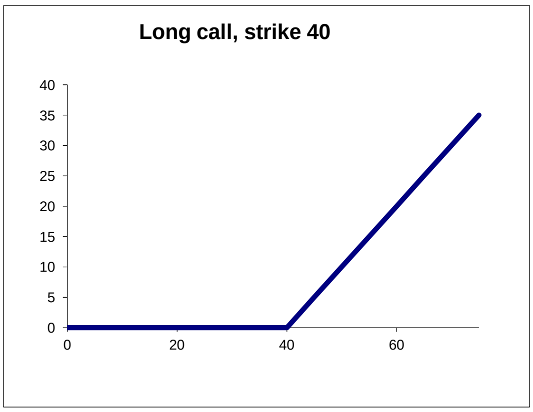
Payoff profiles at maturity

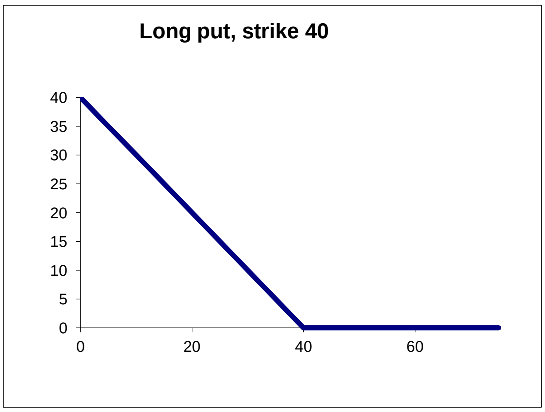
The payoff formulas at maturity for European calls and puts are:

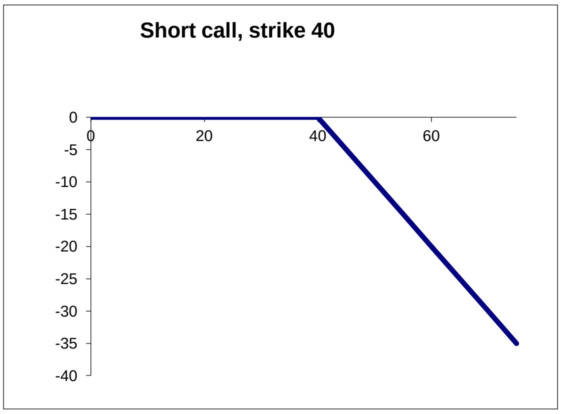
European call payoff = max (S_T – strike, 0)

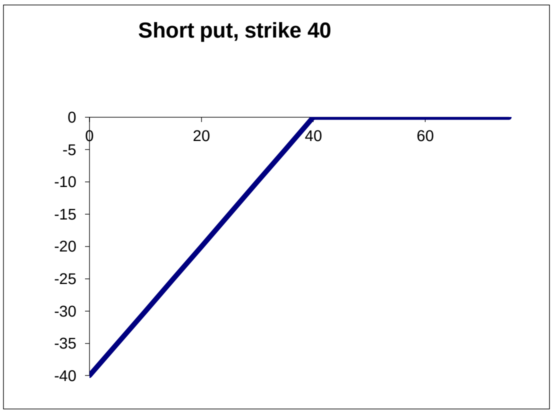
European put payoff = max (strike – S_T, 0)

where S_T is the price of the underlying as seen at maturity. The characteristics of basic options can be seen in the following diagrams. These diagrams plot the payoff of the option at maturity against the stock price. The four standard positions are: 1) long call; 2) long put; 3) short call; and 4) short put. Mastering these basic diagrams will be very useful in analyzing the option properties and strategies. (Note that an alternative way of presenting the option characteristics is mentioned in section 6.3.3 below).





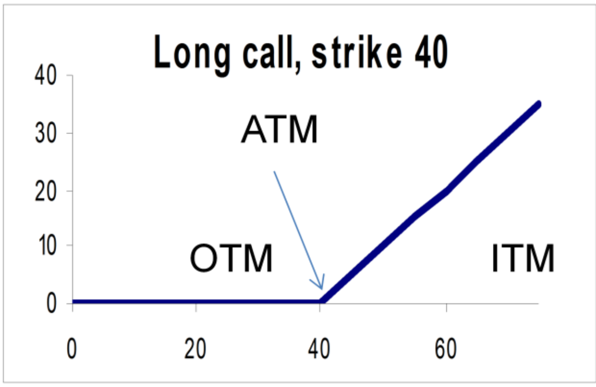


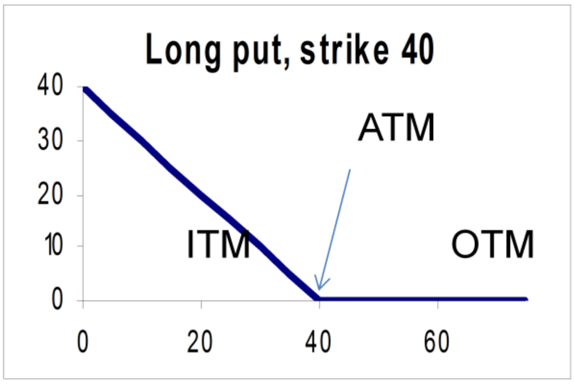


Intrinsic value and moneyness

The intrinsic value of an option is its exercise value assuming that the option can be exercised immediately. Of course, this is not feasible for European options, but the intrinsic value can still be calculated as such. For example, if the call option strike is $40 and the current spot price is $50, the intrinsic value is 50–40= $10. However, if stock price is $37, the intrinsic value of this call option is 0. Later we will see that the current option price would be different from this value, and requires a pricing model for correct valuation.

A related concept is moneyness. It is defined as follows: *at-the-money* (or ATM, asset price = strike), *in-the-money* (or ITM, payout of option if exercised immediately is positive, i.e. the intrinsic value is greater than 0), *out-of-the-money* (or OTM, payout of option is 0). For example, if call option strike = $40; spot price = $37 (out-of-the-money), $40 (at-the-money), $50 (in-the-money). For a put option with strike = $40; spot price = $35 (in-the-money), $40 (at-the-money); $45 (out-of-the-money). The following diagrams represent a schematic demonstration. However, note that this concept does not only apply to describe the situation at maturity.





Option settlement

When an option reaches the maturity date, there are two ways for settling the contract. For example, an investor holds 2,000 European call options on a stock with strike $75. Let's say the share price at maturity is $80 and the option buyer decides to exercise. In the case of *physical settlement*, the option buyer would pay 2000x75 = $150,000 to the option seller and obtain 2,000 shares. Then the option buyer could decide to hold on to these

2,000 shares or sell them in the market (@$80) for a profit. For this method, it means that the option seller must hold these shares at maturity so that they can be delivered to the buyer. In the case of *cash settlement*, the option seller pays 2000 x (80 – 75) = $10,000 to the option buyer, representing the net profit that the buyer would get if the shares are sold at the current price. Supposedly the option seller does not need to own any share at maturity. The method of settlement is specified as part of the contract terms.

Listed and OTC options

Many options are traded in the over-the-counter (OTC) market, e.g. between a company and an investment bank. Terms are tailor-made and agreed between the two parties. In addition, there is a very liquid listed market in many countries, with standardized terms. For example, for listed HSI options in Hong Kong, there are European calls and puts, where 1 index point represents HKD 50, and they have similar maturities as HSI futures (with more maturity months). The strikes are fixed by the exchange at 200 points intervals. A snapshot is given in the table below:

Contract	Bid	Ask	Last Traded	High	Low	Volume	Prev. Day Settlement Price	Net Change	Prev. Day Open Interest
P Jan-14 - 22000	11	13	11	33	11	1,105	22	-11	2,657
C Jan-14 - 22200	-	-	732	732	720	35	819	-87	167
P Jan-14 - 22200	17	18	18	52	18	1,584	37	-19	2,951
C Jan-14 - 22400	779	805	795	835	530	10	635	160	75
P Jan-14 - 22400	28	29	28	83	26	2,263	59	-31	2,869
C Jan-14 - 22600	598	623	610	654	390	147	470	140	772
P Jan-14 - 22600	46	47	46	132	40	2,938	96	-50	3,235
C Jan-14 - 22800	436	453	440	535	250	316	324	116	977
P Jan-14 - 22800	76	80	80	200	63	3,062	152	-72	2,881
C Jan-14 - 23000	289	299	295	388	149	1,006	211	84	1,953
P Jan-14 - 23000	127	134	134	308	104	2,211	238	-104	1,132
C Jan-14 - 23200	180	184	183	257	85	907	127	56	3,221
P Jan-14 - 23200	211	220	219	415	170	1,352	350	-131	1,560
C Jan-14 - 23400	104	105	105	157	44	1,768	71	34	3,274
P Jan-14 - 23400	333	351	348	570	267	92	486	-138	804
C Jan-14 - 23600	53	59	55	90	23	2,130	36	19	2,637
P Jan-14 - 23600	-	1,401	490	730	460	34	684	-194	396
C Jan-14 - 23800	26	27	26	48	12	1,213	18	8	2,985
P Jan-14 - 23800	-	-	635	661	630	5	843	-208	848
C Jan-14 - 24000	10	15	12	24	6	1,016	8	4	2,315

Data as of Jan 17, 2014 (HSI closed at 23133.35, up 146.94 points from the previous day; HSI January futures closed at 23166, up 193 points from the previous day); source: www.hkex.com.hk.

6.3.2. Some properties of options

The fair pricing of options require the specification of the statistical distribution of the underlying price, which is the subject of the next topic. However, it is possible to derive some properties of options which are independent of the distributional assumptions. The classical exposition of the put-call parity was given in Stoll (1969) and many properties of options were described by Merton (1973). Note that the results in Stoll's paper are still valid today, while it was published before the appearance of the popular Black-Scholes model (published in 1973).

Notations

c :	European call option price;	C :	American Call option price
p :	European put option price;	P :	American Put option price
S_0 :	Stock price today;	S_T :	Stock price at maturity date T
K :	Strike price		
T :	maturity of option		
σ:	Volatility of stock price		
D :	Present value of dividends during option's life		
r :	Risk-free rate for maturity T with continuous compounding		

If not specified explicitly, it is assumed that the other parameters which can affect the pricing would remain the same. For example, if we are comparing $C(K_1)$ and $C(K_2)$, the other inputs such as maturity, volatility, interest rate and initial stock price are assumed to be equal.

Non-negativity of option prices

$$C \geq 0, P \geq 0, c \geq 0, p \geq 0$$

Reason: payoffs of options are non-negative. Given that there is a positive probability of receiving some kind of payoff, one should not receive money to hold this position, otherwise arbitrage exists.

Basic properties for American and European options

$$C \geq c, P \geq p$$

Reason: An American option has all the rights of the European option, plus the privilege of early exercise – this right has a non-negative value. However, later we will see that sometimes the American and European option prices can be the same under some conditions.

Option prices with different initial stock price levels

If $S_2 > S_1$, then:

$$C(S_2) > C(S_1)$$
$$c(S_2) > c(S_1)$$
$$P(S_2) < P(S_1)$$
$$p(S_2) < p(S_1)$$

Reason: payoff of a call option increases with increase stock price, and it also has a strictly higher chance to be exercised. Therefore higher stock price would lead to higher fair price. A similar but reverse argument holds for put options.

Option prices with different strike levels

If $K_2 > K_1$, then:

$$C(K_2) < C(K_1)$$
$$c(K_2) < c(K_1)$$
$$P(K_2) > P(K_1)$$
$$p(K_2) > p(K_1)$$

Reason: payoff of a call option decreases with increasing strike price, and it also has a strictly less opportunity to be exercised. Therefore higher strike price would lead to lower fair price. A similar but reverse argument holds for put options.

Upper bounds on calls and puts

$$S_0 \geq C \geq c$$
$$K \geq P \geq p$$

Reason: if strike price is not 0, the holder of the call option has to pay a certain amount to buy the stock, thus the option value must be less than the stock price itself.

$$Ke^{-rT} \geq p$$

Reason: at maturity, the put option cannot be worth more than the strike, thus it cannot be worth more than the present value of the strike today.

Lower Bound for European Call Prices (no dividend case)

$$c \geq \max(S_0 - Ke^{-rT}, 0)$$

Reason: consider the following two portfolios

- Portfolio A: one European call option c + cash Ke^{-rT}
- Portfolio B: one share S_0

There are only two possible scenarios at maturity. We can work out the payoffs for both portfolios A and B under these scenarios. For example, if $S_T < K$, the call option c will expire worthless and has a value of 0. Cash Ke^{-rT} will earn interest and the final amount becomes K. All the results are summarized below:

Stock price at maturity	$S_T < K$	$S_T \geq K$
Portfolio A	$0 + K = K$	$(S_T - K) + K = S_T$
Portfolio B	S_T	S_T
Result of comparison	$V_A > V_B$	$V_A = V_B$

It is evident that in both cases, the value of portfolio A (=V_A) is greater than or equal to the value of portfolio B (=V_B). Therefore the expression is verified.

We can illustrate this relationship with a numerical example. Assume that we have the following market prices: S_0=20, K=18, r=10%, T=1, c=3.00. Since $S_0 - Ke^{-rT} = 20 - 18e^{-0.1} = 3.71 > c$, we can come up with a strategy to make a guaranteed profit. The correct strategy is to sell the expensive leg and buy the cheap leg. Therefore:

- Buy call, short stock; cash inflow = 20 – 3 = $17
- Invest this amount for 1 year, to get $17e^{0.1} = 18.79$
- If stock price at maturity < 18, say at 17, buy stock at this price and close the short stock position.
- If stock price at maturity ≥ 18, exercise the option and buy the stock at 18 to close the short stock position. In order words, the maximum that one needs to pay in order to buy stock is $18.
- Thus there will be a guaranteed profit of at least 18.79 – 18 = $0.79.

Lower Bound for European Put Prices (no dividend case)

A similar expression can be derived in the case of a European put.

$$p \geq \max(Ke^{-rT} - S_0, 0)$$

Reason: consider the following two portfolios

- Portfolio C: one put option p + one share S_0
- Portfolio D: cash Ke^{-rT}

There are only two possible scenarios at maturity. We can proceed similar to the analysis in the section above. The results are summarized below:

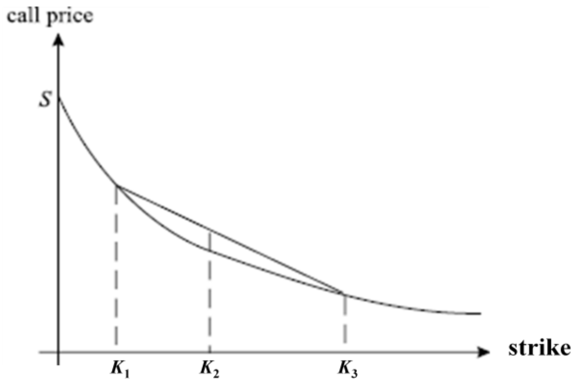
Stock price at maturity	$S_T < K$	$S_T \geq K$
Portfolio C	$(K - S_T) + S_T = K$	$0 + S_T = S_T$
Portfolio D	K	K
Result of comparison	$V_C = V_D$	$V_C \geq V_D$

With the same kind of reasoning, the expression is verified.

Convexity properties

Option prices are convex functions of the strike price. We use call options to illustrate the relationships. Let $K_2 = \lambda K_3 + (1 - \lambda) K_1$ where $0 < \lambda < 1$. A convex function means the following:

$$c(K_2) \leq \lambda\, c(K_3) + (1 - \lambda)\, c(K_1)$$
$$C(K_2) \leq \lambda\, c(K_3) + (1 - \lambda)\, C(K_1)$$



We can see that the call price equals S when strike $K = 0$, and tends to 0 when the strike is large. Consider the following two portfolios:

- Portfolio E: λ call option with strike K_3 + $(1 - \lambda)$ call option with strike K_1
- Portfolio F: one call option with strike K_2

The payoff patterns at maturity are given by:

Stock price at expiry	$S_T \le K_1$	$K_1 \le S_T \le K_2$	$K_2 \le S_T \le K_3$	$K_3 \le S_T$
Portfolio E	0	$(1 - \lambda)(S_T - K_1)$	$(1 - \lambda)(S_T - K_1)$	$\lambda(S_T - K_3) + (1 - \lambda)(S_T - K_1)$
Portfolio F	0	0	$S_T - K_2$	$S_T - K_2$
Result of comparison	$V_E = V_F$	$V_E \ge V_F$	$V_E \ge V_F$	$V_E = V_F$

In all these cases, the value of portfolio A (=V_E) is greater than or equal to the value of portfolio F (=V_F). Therefore the expression is verified.

Put-Call Parity for European options (no dividend case)

All of the above properties are inequalities, but an important equality known as the **put-call parity** gives the relationship between the call and put prices. Consider the following two portfolios:

- Portfolio A: one European call option c + cash Ke^{-rT}
- Portfolio C: one European put option p + one share S_0

There are only two possible scenarios at maturity. We can work out the payoffs for both portfolios A and C under these scenarios. For example, if $S_T < K$, the call option c will expire worthless and has a value of 0. Cash Ke^{-rT} will earn interest and the final amount becomes K. The results are summarized below:

Stock price at maturity	$S_T < K$	$S_T \ge K$
Portfolio A	$0 + K = K$	$(S_T - K) + K = S_T$
Portfolio C	$(K - S_T) + S_T = K$	$0 + S_T = S_T$
Result of comparison	$V_A = V_C$	$V_A = V_C$

It is evident that in both cases, the value of portfolio A (=V_A) is equal to the value of portfolio C (=V_C). Both are worth $\max(S_T, K)$ at maturity. If they have the same value at maturity, they must be worth the same today (to prevent arbitrage). This means that:

$$c + Ke^{-rT} = p + S_0$$

If this relationship does not hold, arbitrage opportunity exists. Assume we can find the following market prices: S_0=31, K=30, r=10%, T=0.25, c=3.00, p=2.25

$$\begin{array}{lll} c + Ke^{-rT} & = 3 + 30e^{-0.1\text{x}0.25} & = 32.26 \\ p + S_0 & = 2.25 + 31 & = 33.25 \end{array}$$

The values are not equal, and the correct strategy to take advantage of the situation is to buy the cheaper combination and simultaneously sell the more expensive combination at the correct proportions. There is more than one way to construct this strategy. Let's say we sell put and sell stock, and receive $33.25. We use part of the proceeds to buy a call, and will still be left with an upfront cash = –3+2.25+31 = $30.25. Next, we invest this amount for 3 months to get 30.25$e^{0.1\text{x}0.25}$ = 31.02.

At maturity, if S_T>30, the call is exercised (since we have a long position), and we will receive the shares and pay $30 per share. If S_T<30, the put will be exercised by the buyer (since we have a short position), and we will pay the strike price of $30 and buy the shares. In either case, the result is that 1 share would be purchased at a price of $30, which can be used to close out the existing short stock position. Since we only need $30 for these positions, but would receive $31.02 from the deposit, the guaranteed net profit is thus 31.02 – 30 = $1.02.

Just to summarize, for this strategy to succeed we need to arrange four trades: 1) a position in the call; 2) the put; 3) the shares; and 4) a cash amount (either borrow or deposit). If all these are traded at the initial prices, the profit is guaranteed at maturity, irrespective of the market conditions at that time.

Early exercise opportunity

American options give the holder an opportunity to exercise the options before maturity. It is sometimes worth paying extra premium for this additional flexibility. However, in the case of an American call option on a non-dividend paying stock, it can be shown mathematically that it is never optimal to exercise early. In practice, it means that the option should be sold rather than exercised if the investor wants to get out of the position before maturity. Of course, other practical factors may go into the decision, e.g. whether it is possible to close out the option at the theoretical price or not. If this is not feasible, one may need to exercise the option instead.

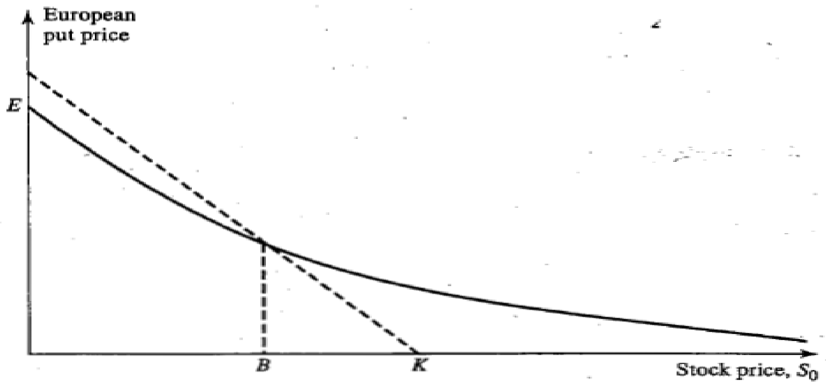
To facilitate discussion, let's say there is a deep in-the-money American call option, where S_0 = 100; T = 0.25; K = 60; D = 0. The decision to be made is: should you exercise immediately? Would it make any difference if:

(i) you want to hold the stock for the next 3 months?

(ii) you do not feel that the stock is worth holding for the next 3 months?

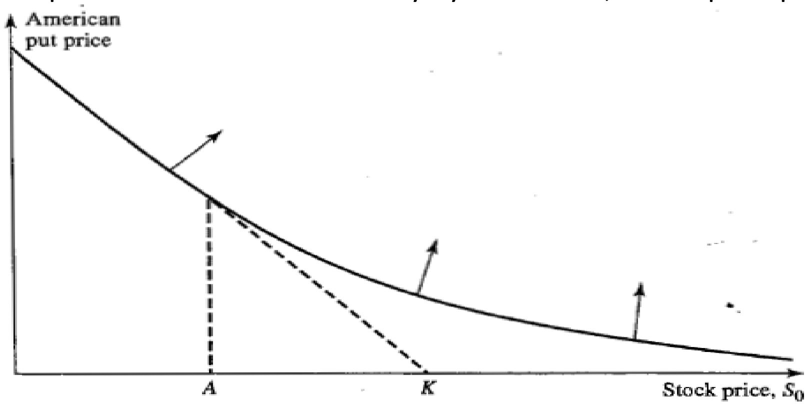
In the first case, since there is no dividend before maturity, no income is sacrificed if one withholds the exercise decision. If the option is exercised immediately, the strike price has to be paid; by delaying the exercise, payment of the strike price is made at maturity. Holding the call provides insurance against stock price falling below strike price, and hence there is no reason why the option should be exercised now. For the second case, the correct thing to do is to sell the option rather than getting the intrinsic value, because the option price is higher than the intrinsic value (see the detail formulas in the next topic).

The situation is more complicated for a put. The diagram below shows the variation in the theoretical option price of a European put when underlying spot price changes.



The dotted line shows the intrinsic value and the black line is the theoretical price. It can be seen that the European put option price can be below the intrinsic value once the stock price drops below B. Of course, for European options, one is not allowed to exercise early. [As a thinking exercise and without a theoretical pricing model, what do you think would be the fair price of a European put at the following market conditions: S_0 = 40; T = 0.25; r=10%, K = 100; D = 0?]

American options can be exercised early by the holder, so the price profile becomes:



The notation is reversed here: the dotted line is the intrinsic value, and the black line is the theoretical price. It is seen that when stock price is below A, the theoretical price of the option is equal to the intrinsic value. This means that when stock price is low enough, it is optimal to exercise the option. In fact, if r > 0, this condition is satisfied at a sufficiently low stock price, i.e. $P > p$. As a heuristic argument, consider the extreme case when stock price drops to 0. If the put option is exercised immediately, one can get \$$K$ now. However, since a stock price of 0 is the lowest possible value it can go, one can still only get \$$K$ if one exercises the option at maturity. Because of the positive interest rate, getting \$$K$

now is worth more than getting \$$K$ at maturity. Hence it is evident that at a low enough stock price, one should exercise the put immediately.

Put-Call Parity for American options (no dividend case)

The put-call parity relationship does not hold for American options, and we will not have an expression with equality signs. Nevertheless we can establish upper and lower bounds for the difference $C - P$:

$$S_0 - K \le C - P \le S_0 - Ke^{-rT}$$

Reason: upper bound

- since $C = c$ and $P > p$ (see an earlier section)
- $c - p = S_0 - Ke^{-rT}$ => $P > C - S_0 + Ke^{-rT}$ or $C - P \le S_0 - Ke^{-rT}$

Reason: lower bound

Consider the following two portfolios:

- Portfolio G: one European call c + cash K
- Portfolio H: one American put P + one share S_0

If the American put has not been exercised early, we have the following results:

Stock price at maturity	$S_T < K$	$S_T \ge K$
Portfolio G	$0 + Ke^{rT} = Ke^{rT}$	$(S_T - K) + Ke^{rT}$
Portfolio H	$(K - S_T) + S_T = K$	$0 + S_T = S_T$
Result of comparison	$V_G > V_H$	$V_G > V_H$

If the put option is exercised early, at the time of exercise (=t) we will have $S_T < K$, where portfolio H's value is $(K - S_t) + S_t = K$. Portfolio G's value is $c(t)+Ke^{rt}$ which must be greater than K given that call option price and the interest rate must be positive. A portfolio with an American call C would have a value no less than the European call, and thus $V_G \ge V_H$. In summary, portfolio G is always greater than or equal to Portfolio H, and thus the expression is verified.

Relationships when there are stock dividends

The above analysis can also be carried out in the presence of dividends. The lower bounds for European calls and puts are:

$$c \ge \max(S_0 - D - Ke^{-rT}, 0)$$
$$p \ge \max(D + Ke^{-rT} - S_0, 0)$$

D is the present value of all dividends received by the stock. The results can be derived using a similar reasoning to the cases of no dividend. Using the above notations, we adjust the portfolios in the previous examples as follows:

- Portfolio A': one call option c + cash $D+Ke^{-rT}$
- Portfolio D': cash $D+Ke^{-rT}$

By comparing the portfolios A' and B for the call and C and D' for the put, the above expressions can be verified.

The put-call parity for European options with stock dividends D is given by:

$$c + D + Ke^{-rT} = p + S_0$$

This expression can be verified by comparing portfolios A' and C. For American options, again we can only establish bounds for the difference between C and P:

$$S_0 - K \leq C - P \leq S_0 - Ke^{-rT}$$

Reason (upper bound)

- there is no change compared to the case of no dividend, as the presence of dividends will decrease C and increase P, making the spread even smaller.

Reason (lower bound)

Consider the two portfolios G' and H where

- Portfolio G': one European call c + cash K + cash D
- Portfolio H: one American put P + one share S_0

If the American put has not been exercised early, we have the following results:

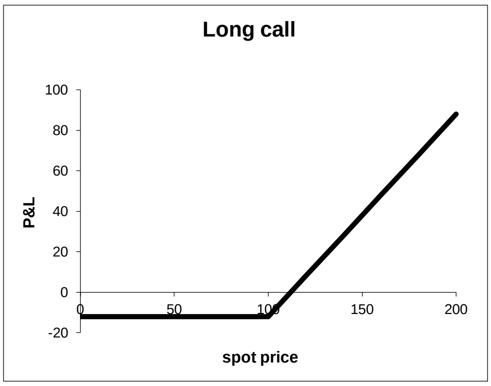
Stock price at maturity	$S_T < K$	$S_T \geq K$
Portfolio G'	$0+Ke^{rT}+De^{rT}$	$(S_T - K) + Ke^{rT} + De^{rT}$
Portfolio H	$K+De^{rT}$	$0 + S_T + De^{rT}$
Result of comparison	$V_{G'} > V_H$	$V_{G'} > V_H$

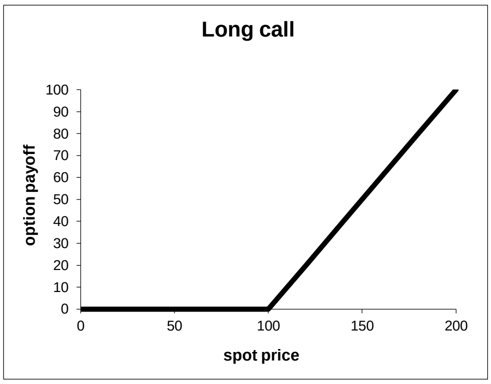
- If the put option is exercised early, at the time of exercise (=t) we will have $S_T < K$, where portfolio H's value is $(K - S_t) + S_t + De^{rt} = K+De^{rt}$ (assume that the dividend has already been paid). Portfolio G's value is $c(t)+Ke^{rt} + De^{rt}$ which must be greater than $K+De^{rt}$ given that call option price and the interest rate must be positive. A portfolio G'' with an American call C would have a value no less than the European call, and thus $V_{G''} \geq V_H$. In summary, portfolio G'' is always greater than or equal to Portfolio H, and thus the expression is verified.

6.3.3. Option strategies

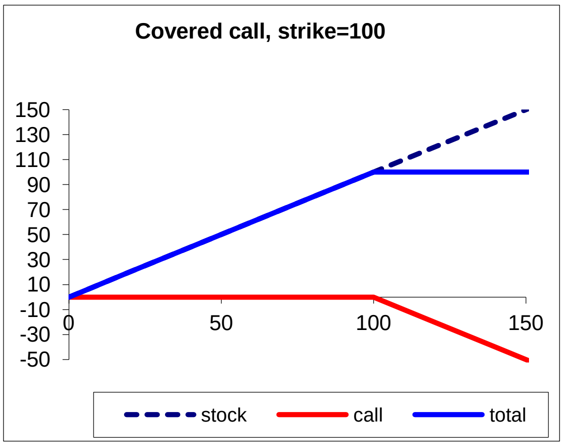
We can create combinations of a number of calls and/or puts and the stock in order to capture the specific views of investors. The reason why we want to do something like that is because we want to find the cheapest way that can express the views: we don't want to pay for something that is unlikely to happen. In the diagrams below, we concentrate on the payoff profiles at maturity; pricing of these options will be discussed in the next topic.

In some textbooks, payoff diagrams include the cost of the option, where the P&L of the position is plotted (as shown in the left diagram below). An alternative representation is shown on the right diagram, where only the option payoff is displayed. In the following sections, the notation in the right diagram is adopted.





Covered call

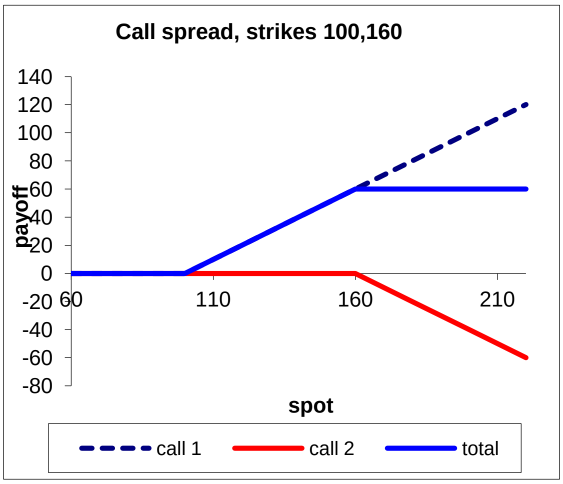


Strategy: Long 1 share, short 1 call.

View: anticipate that the stock price would not go very much higher.

Advantage: Collect option premium, but give up some upside potential.

Call spread / put spread



Strategy: call spread

- Long 1 call at strike K_1, short 1 call at strike K_2 where $K_2 > K_1$.
- earn a profit if stock price moves up.

Strategy: put spread

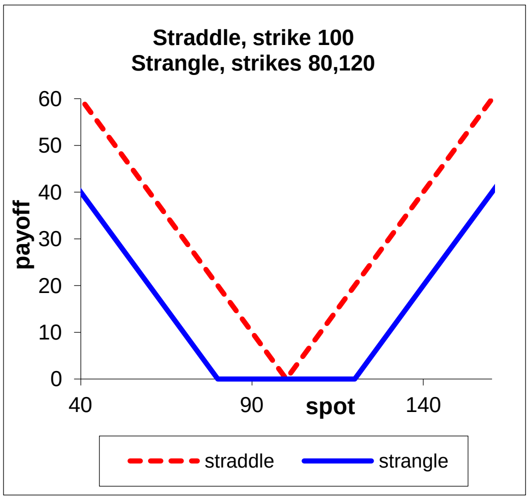
- Long 1 put at strike K_1, short 1 put at strike K_2 where $K_2 < K_1$ (compare with short 1 call at strike K_1, long 1 call at strike K_2 where $K_2 > K_1$) .
- earn a profit if stock price moves down.

Advantage: These strategies, also known as bull/bear spreads, limit the upside potential, hence they are cheaper than a single call or a single put.

The above diagrams are obtained by analyzing the payoffs of the components. Using the call spread as an example, let $K_2 > K_1$. Since $c(K_2) < c(K_1)$, a long call spread position will require an initial premium. At maturity, the various scenarios are:

Stock price at expiry	$S_T \le K_1$	$K_1 < S_T < K_2$	$K_2 \le S_T$
Payoff from long call position	0	$S_T - K_1$	$S_T - K_1$
Payoff from short call position	0	0	$K_2 - S_T$
Total payoff	0	$S_T - K_1$	$K_2 - K_1$

Straddle / strangle



Strategy: Straddle

- Long 1 call at strike K_1, long 1 put at strike K_1

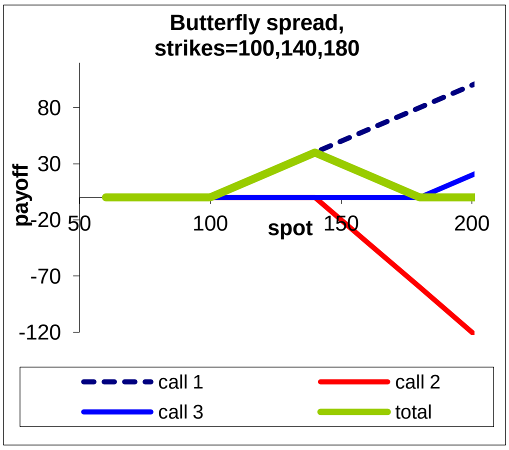
Strategy: Strangle

- Long 1 call at strike K_1, Long 1 put at strike K_2 where $K_1 > K_2$

View: Profitable if the stock price has moved away from the strike(s), but the investor is uncertain which direction it would move.

Disadvantage: Very expensive.

Butterfly spread



Strategy: Long 1 call at strike K_1, long 1 call a strike K_3, short 2 calls at strike K_2 where $K_2 = (K_1 + K_3)/2$.

View: Make a profit when final stock price is within a limited range. Maximum profit when spot price is at K_2.

Advantage: Compare with short straddle, the risk is lower, but is more costly to put on the trade.

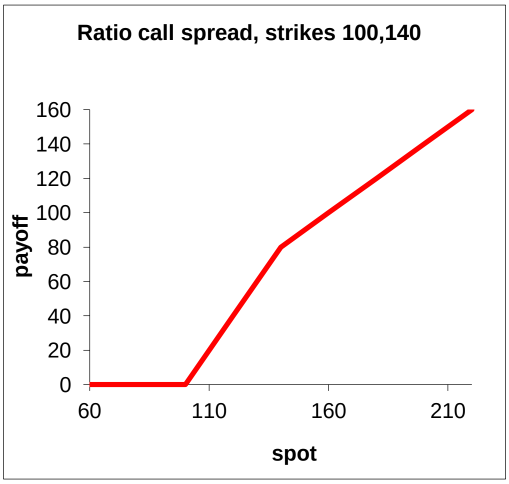
We can use the same method above to analyze its payoff. Let $K_3 > K_1$ and $K_2 = (K_1 + K_3)/2$. It can be proved from the convexity property of options that

$$c(K_2) \le 0.5c(K_3) + 0.5c(K_1) \Rightarrow 2c(K_2) \le c(K_3) + c(K_1)$$

i.e. long butterfly spread will require an initial premium. At maturity:

Stock price at expiry	$S_T \le K_1$	$K_1 \le S_T \le K_2$	$K_2 \le S_T \le K_3$	$K_3 \le S_T$
Payoff from $c(K_1)$	0	$S_T - K_1$	$S_T - K_1$	$S_T - K_1$
Payoff from $c(K_3)$	0	0	0	$S_T - K_3$
Payoff from $-2c(K_2)$	0	0	$-2(S_T - K_2)$	$-2(S_T - K_2)$
Total payoff	0	$S_T - K_1$	$K_3 - S_T$	0

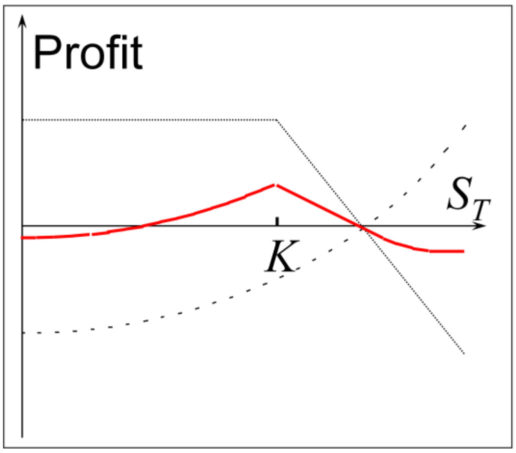
Ratio call spread



Strategy: e.g. Long 2 calls at strike K_1, short 1 call at strike K_2 where $K_1 < K_2$

View: Slightly cheaper than a straight call, but would not completely give up the upside as in a call spread

Calendar spread



Strategy: e.g. Long 1 call at strike K_1 with maturity t_2, short 1 call at strike K_1 with maturity t_1, where $t_1 < t_2$

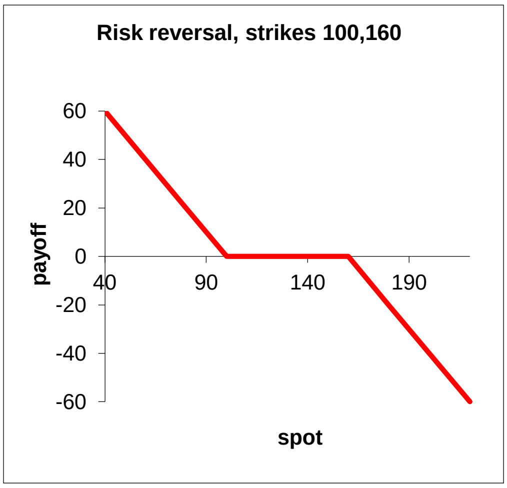
View: The diagram shows the profile when the shorter maturity option expires. The payoff pattern is similar to a butterfly spread

We can explain the shape of the payoff pattern as follows. The combination is a short position in $c(t_1)$ and a long position in $c(t_2)$. At time t_1,

- If stock price is very low (say close to 0), $c(t_1)$ is worth 0 and $c(t_2)$ will also have a value close to 0, which means that the difference is close to 0.
- If stock price is very high, $c(t_1)$ is exercised, and will have a value of $-(S_{t1} - K)$, whereas $c(t_2)$ is worth only slightly more than $(S_{t1} - K)$ (because the time value is small when it is deep in-the-money)

- If stock price is slightly less than strike K, $c(t_1)$ has zero value, whereas $c(t_2)$ is still very valuable because it is close to an at-the-money option. Therefore the value of the portfolio is highest when the spot price is close to K.

Zero-cost collar / risk reversal



Strategy: Long 1 put at strike K_1, short 1 call at strike K_2 where $K_1 < K_2$, such that the price of the call = the price of the put

View: This is a zero cost strategy where the long put position is being financed by the short call. It has some interesting characteristics, e.g. the second derivative (gamma) changes sign from positive to negative when spot price increases.

In some variations, the number of call and put in the combination may not be one to one.

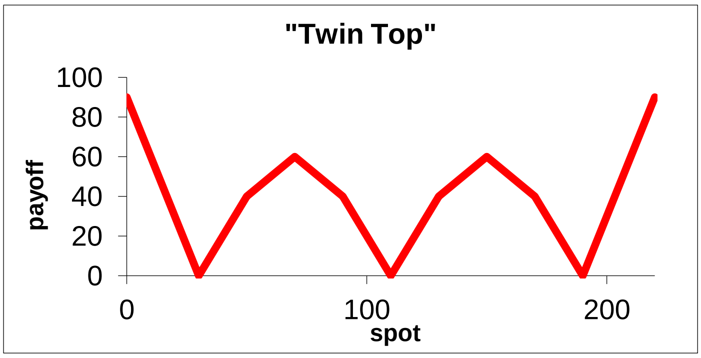
The strategy can sometimes be used to offer short term protection. For example, we have the following market parameters: current stock price = \$25, 1-month put option price (strike \$25) = \$4, 1-month call option price (strike \$40) = \$4. One portfolio strategy is to buy the shares at \$25, buy a \$25 strike put, sell a \$40 strike call, at a total cost of \$25. After one month, we may have the following scenarios:

- If stock price is below \$25, say \$20, the call is worthless but the put can be exercised, thus the investor has no loss.
- If stock price is between \$25 and \$40, both the call and the put would expire worthless; the investor would still hold the shares.
- If stock price is above \$40, the call would be exercised, where the investor has a gain of \$40-\$25 = \$15 but with no share position afterwards.

The net position is similar to a call spread.

A special payoff

Theoretically speaking, various combinations of calls and puts at different strikes can generate a wide variety of payoff patterns at maturity. For example, if we want to create something like this:



While it is feasible on paper, sometimes it may not be possible to execute all the various strikes and the cost could be high.
(One possible solution: Long 3 puts at strike 30; Long 2 calls at strike 30; Short 1 call at strike 50; Short 2 calls at strike 70; Short 1 call at strike 90; Long 4 calls at strike 110; Short 1 call at strike 130; Short 2 calls at strike 150; Short 1 call at strike 170; Long 5 calls at strike 190).

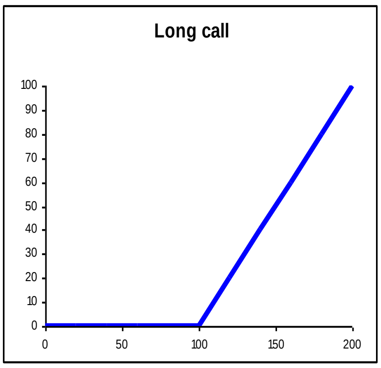
6.3.4. What is the option premium?

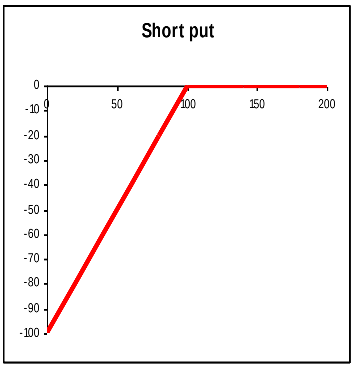
Later we would look at some mathematical models for option pricing. We note that the fair theoretical price equals the expected payoff of the option (under some technical conditions). One point that we need to emphasize is that, as a private investor, whether the trade should be entered or not should be based on the *subjective* probabilities of the expected payoff, which is not necessarily the fair price. For example, if the fair price of a butterfly strategy is $10, market price is $12, but if you think that there is a high probability of making more than $12, you may want to engage in this strategy. The fair price is only relevant if you need to **hedge** this position. This topic, known as expectation and arbitrage pricing, will be discussed again in more detail.

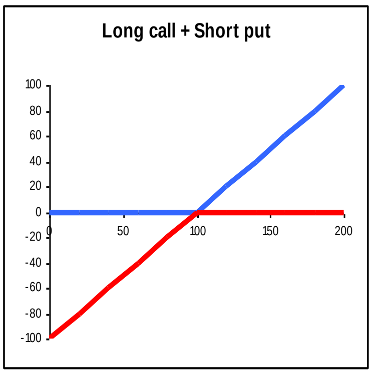
Before we use the formulas, we can also calculate the fair price of a derivative by calculating its expected hedging cost. In other words, if we could find a hedging strategy which could replicate the payoff of the option under any circumstances, then the cost of the strategy should be the same as the cost of the option (the principle of no arbitrage). For European options, one way of looking at this method of pricing is to make use of the payoff diagrams in the previous sections. Theoretically, any payoff pattern at maturity could be constructed by simply combining the basic patterns at different proportions. Of course, there is a tradeoff between cost and payoff, and certain options may not be available in the market. The reason is because some options could pose hedging difficulties for the strategy seller (hence higher cost), and thus sometimes we may not be able to find a seller of options at an attractive price.

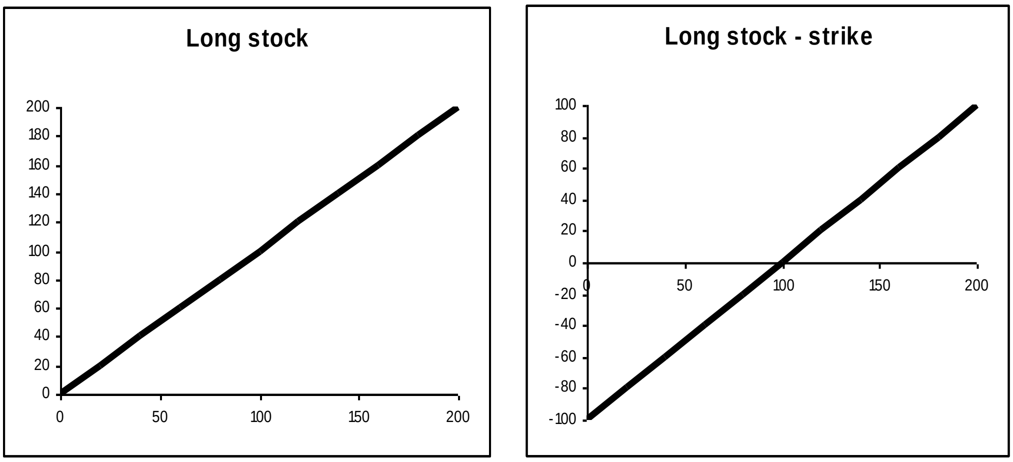
6.3.5. Synthetic positions

Some combinations of the underlying (e.g. stocks), bonds, calls and puts can be used to replicate the payoff patterns of other simple strategies. We can even derive the put-call parity with these combinations.









The combination using a long call and short put (both at strike K) is known as a *synthetic forward*. If we examine the diagrams above, we can find that the diagram for the synthetic forward is exactly the same as the diagram for the stock position, bought at the strike price (=100 in this example). In order words,

At maturity: $C - P = S_T - K$
Today: $C_0 - P_0 = S_0 - Ke^{-rT}$

This is exactly the put-call parity relationship. We can also make use of this equation to create other synthetic positions.

Synthetic call: Long 1 share, long 1 put, borrow cash Ke^{-rT}
Synthetic put: Long 1 call, short 1 share, deposit cash Ke^{-rT}

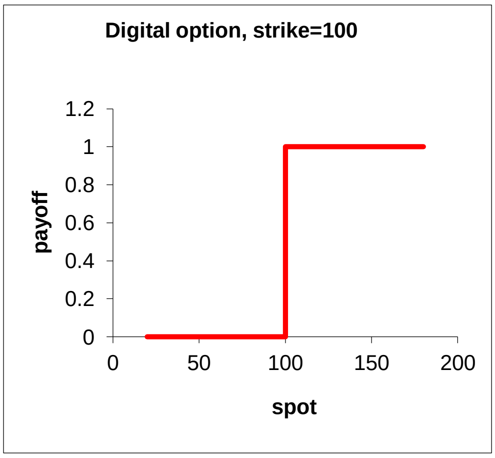
For example, we have the following market information: current share price = \$40, strike price = \$40, price of put option = \$4.233, time to maturity = 1 year, 1 year interest rate = 2.5% (simple compounding, instead of continuous compounding as given in the equation above). We want to create a short call option position. This is achieved by:

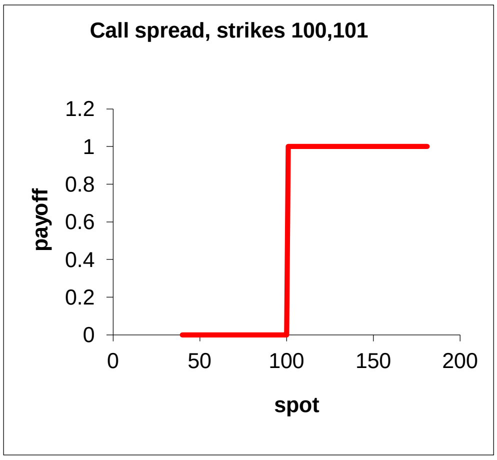
Short synthetic call: $-C_0 = -P_0 - S_0 + Ke^{-rT}$ or $-C_0 = -P_0 - S_0 + K/(1+rT)$
In other words, sell 1 put, sell 1 share, deposit cash $K/(1+rT)$

The call option price that would be received can be calculated as:
c = 4.233 + 40 – 40/(1+0.025x1) = \$5.209
(note the sign of the terms: short put means money is received, hence it has a positive sign when we want to calculate the total amount received).

Digital option

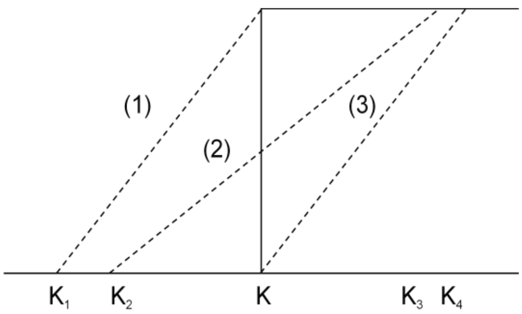
In this final example, we look at a basic “exotic” option called a digital option or a bet option. The option payoff is: if final spot price S_T is higher than K, then pay \$1, otherwise pay 0.





The payoff of the digital option is shown in the left diagram. Without using a correct theoretical model, we want to see whether this payoff can be approximated using vanilla options. The right diagram shows a call spread with strikes K_1 and K_2, with K_2 very close to K_1 (both are roughly equal to 100). We can see that the shapes of the payoffs are very similar. Therefore we can conclude that the theoretical price of the digital option should be very close to that obtained from the call spread, which can be priced with the standard option pricing model for European calls.

If we look into this pricing problem more closely, we can see that there are a number of different ways to come up with the approximate price. In practice, it depends on the position of the calculation agent, i.e. whether it wants to buy or sell the option.



The diagram shows a closer view to the payoff pattern of the digital option. Three different call spreads (shown by the dotted lines) can be used to approximate its price. The payoff of spread 1 (strikes K_1 and K) would always be higher than that of the digital, making it more expensive; conversely spread 3 should be cheaper than the digital. In terms of pricing, spread 2 would be closest to the digital option price, but its payoff is sometimes higher and sometimes lower than the digital (between spot price K_2 and K_3). In terms of risk management, this may not be the most desirable.

Finally, we need to scale the notional amount of the options so that the difference in strikes would pay \$1. For example, if we use 1 call option with strike K_2 and 1 call option with strike K_3 (spread 2 above), we will end up with a strategy that pays $\$(K_3 - K_2)$ when spot price is above K_3. In order to give the correct price to 1 digital option, the correct proportion of the different calls to use should be $1/(K_3 - K_2)$.