3. Equilibrium in Capital Markets

3.1. Capital Asset Pricing Model

In the previous chapter, the main concern for the Modern Portfolio Theory is the selection of an optimal portfolio for an individual investor. In this chapter, we turn our attention to equilibrium prices, which is an indication of how investors would behave on aggregate. In particular, we would like to answer the following question:

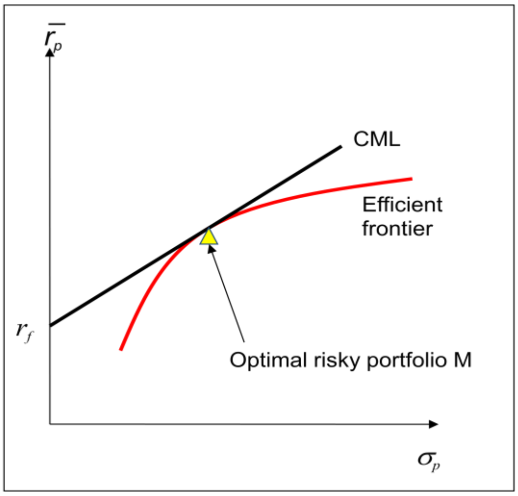
> *What is the relationship between the expected return and risk for any asset when markets are in equilibrium?*

One of the most celebrated financial theories is the **Capital Asset Pricing Model (CAPM)**, developed independently by Jack Treynor, William Sharpe, John Lintner and Jan Mossin in the early 1960s (Sharpe received the Nobel Prize in Economics in 1990 based on this work). While the model receives many criticisms because empirical results do not always support its conclusions (see section 3.3 below), it is still widely used in different applications because of its simple formulation.

3.1.1. Set-up

The starting point of CAPM takes the usual assumptions in portfolio theory: mean variance optimization, non-satiation, risk averse investors, availability of risk free assets, zero taxes and transaction costs, availability of assets in small fractions, unrestricted riskless lending and borrowing, and no impact costs in transactions. In addition, the following assumptions are made:

- All investors plan for one identical holding period.
- Investors are price takers, i.e. their actions will not affect the traded prices of the securities.
- All investors share the same economic view of the world; they have *homogeneous expectations* in regard to the expected returns, standard deviations, and covariances of securities. These information are available simultaneously to all investors.



Because investors have the same expectations, they all face the same risky portfolio M; this is known as the **Market Portfolio**. The line combining the risk free asset with this portfolio M is known as the **Capital Market Line (CML)**.

$$\bar{r}_P = r_f + \frac{\left(\bar{r}_M - r_f\right)}{\sigma_M}\sigma_P$$

$\bar{r}_M$ and σ_M are the expected return and risk of the Market Portfolio; r_f is the return of the risk free asset.

Properties of the Market Portfolio

One can immediately see that every security (stock) in the market must be included in the market portfolio. The argument is that since everyone is holding the same portfolio, if a security is left out there would be absolutely no demand. The price would fall until the anticipated return of this investment becomes attractive again. When all the price adjustment stops, the market will be brought into equilibrium. Since every security is included in the portfolio, the proportion of each security corresponds to the individual security's market value.

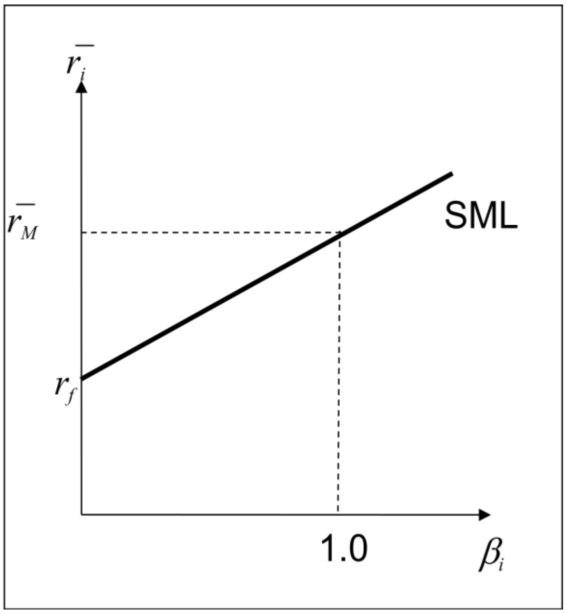
Since an equilibrium is only a theoretical construct and price adjustments never stop, the market portfolio is an inherently unobservable portfolio. A widely represented market index, e.g. the S&P 500 in the US, is commonly used as a proxy in constructing the model.

Recall that for a large portfolio, the portfolio variance is:

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} w_i w_j \sigma_{jk} \approx \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} w_i w_j \sigma_{jk}$$

When N is large, the terms in the double summation dominate the result as there will be many such terms (of the order of N^2), whereas there will only be N terms involving the variances. From the point of view of the portfolio, an asset's contribution to the variance of the market portfolio is determined by its covariance with the market portfolio = $w_A \sigma_{AM}$. ***Accordingly, for an investor, the relevant measure of the asset's risk is its covariance with the market portfolio, not its variance****.* This is different from the case of MPT, where the relevant measure of the asset's risk is the variance. In equilibrium, it follows that assets with larger σ_{AM} will have to provide proportionately larger returns, otherwise it would contribute to the risk of the market portfolio while not contributing to the return – it would be better to remove this asset from the market portfolio.

3.1.2. CAPM and the Security Market Line (SML)



The *equilibrium relationship* between risk and return of any asset is (by definition)

$$\begin{aligned} \bar{r}_i &= r_f + \frac{(\bar{r}_M - r_f)}{\sigma_M^2} \sigma_{iM} \\ &= r_f + \beta_i (\bar{r}_M - r_f) \end{aligned}$$

Expected return = risk free rate
+ beta x Expected Market Risk Premium

All assets should plot on this **Security Market Line** with the asset beta as an input. For example, if the expected return of a market portfolio is 12%, risk free rate is 4%, and a stock has a beta of 1.2, its expected return is 4% + 1.2 x (12% – 4%) = 13.6%.

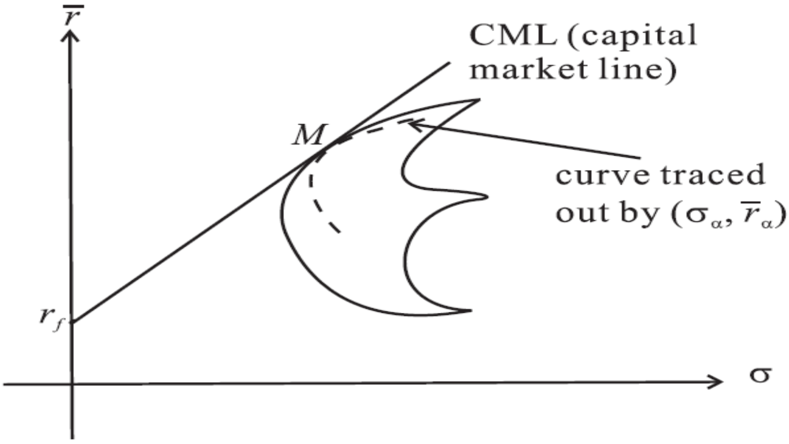
Two remarks about the SML:

- Note that this diagram is **a plot of the expected return against beta**. Contrast this with the diagrams that we see in MPT, in which expected return is plotted against the standard deviation of return. It indicates the change in the variable used in representing risk.
- The SML gives the equilibrium return, i.e. the theoretical return of the stock. In practice, stock price would seldom trade at this "fair value."

3.1.3. A simple proof of CAPM

Consider a portfolio that invests a portion α in asset i and a portion $(1-\alpha)$ in the market portfolio M. The Expected return is: $\overline{r_\alpha} = \alpha \overline{r_i} + (1-\alpha)\overline{r_M}$
and the standard deviation of return is:

$$\sigma_\alpha = \left[\alpha^2\sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2\sigma_M^2\right]^{1/2}$$



In the $\bar{r}$–σ diagram, α=0 corresponds to the market portfolio M. The curve obtained by varying α must be tangential to the CML at M

$$\frac{d\overline{r_\alpha}}{d\alpha} = \overline{r_i} - \overline{r_M}$$

$$\frac{d\sigma_\alpha}{d\alpha} = \frac{\alpha\sigma_i^2 + (1-2\alpha)\sigma_{iM} + (\alpha-1)\sigma_M^2}{\sigma_\alpha}$$

$$= \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M} \text{ when } \alpha = 0$$

$$\left.\frac{d\overline{r_\alpha}}{d\sigma_\alpha}\right|_{\alpha=0} = \frac{d\overline{r_\alpha}/d\alpha}{d\sigma_\alpha/d\alpha} = \frac{(\overline{r_i} - \overline{r_M})\sigma_M}{\sigma_{iM} - \sigma_M^2} = \frac{\overline{r_M} - r_f}{\sigma_M} \text{ (slope of CML)}$$

$$\overline{r_i} = r_f + \left(\frac{\overline{r_M} - r_f}{\sigma_M^2}\right)\sigma_{iM} = r_f + \beta_i(\overline{r_M} - r_f)$$

3.1.4. Using the CAPM

How do we use the CML?

Assume the following inputs: r_M=12%, r_f=7%, σ_M=32%

i) what is the equation of the CML?

ii) if $\overline{r}_P$ = 18%, what is the standard deviation of the portfolio?

iii) how can you allocate \$1000 to achieve the portfolio in (ii) above?

iv) what is the beta of this portfolio?

Answers

i) $$\overline{r}_P = r_f + \left(\frac{\overline{r}_M - r_f}{\sigma_M}\right)\sigma_P = 0.07 + 0.1562\sigma$$

ii) $$\sigma_P = \frac{0.18 - 0.07}{0.1562} = 70.42\%$$

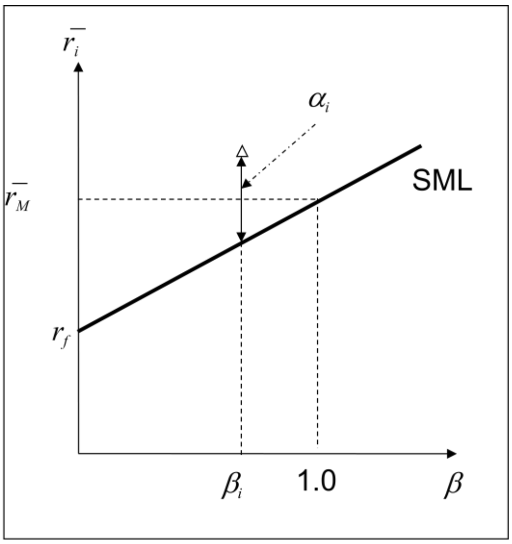
iii) $$\overline{r}_P = \alpha \overline{r}_M + (1-\alpha) r_f$$

$$\Rightarrow \alpha = \frac{\overline{r}_P - r_f}{\overline{r}_M - r_f} = \frac{0.18 - 0.07}{0.12 - 0.07} = 2.2,\ 1 - \alpha = -1.2$$

i.e. the investor should short \$1200 of the risk free asset and long \$2200 of the market portfolio.

iv) since the beta of the risk free asset is 0, the beta of the portfolio equals the weight of the investment on the market portfolio, i.e. β = α = 2.2.

How do we use the SML?



Given the beta of an asset, we could find the (equilibrium) expected return. If the forecast expected return is higher than the equilibrium expected return, the security is mis-priced in the market (i.e. there is a disequilibrium). This extra return is usually known as **alpha**

$$\alpha_i = \overline{r}_i - \left[r_f + (\overline{r}_M - r_f)\beta_i\right]$$

It is the job of the security analysts to search for stocks with excess alphas.

In another application, the SML is often used in capital budgeting decisions: for a firm considering a new project, the expected return represents a hurdle rate for the amount of risk taken (measured by beta). Finally, the expected return given by the SML could give guidance for assets not yet traded in the market e.g. Initial Public Offering (IPO) stocks.

Using the SML: example 1

The market expected return is 12% and the risk free rate is 4%. Stock A has a beta of 0.7 and an expected return of 10.5%. Stock B has a beta of 1.3 and an expected return of 13.7%. Which stock is a better buy?

Answer

According to CAPM

$$r_A = 4\% + 0.7 \text{ x } (12\% - 4\%) = 9.6\%$$
$$r_B = 4\% + 1.3 \text{ x } (12\% - 4\%) = 14.4\%$$

Expected return of A is above the SML and the expected return of B is below the SML; hence A is a better buy.

Using the SML: example 2

Risk free rate r_f = 8%, expected return of a market portfolio $\overline{r_M}$ = 16%; a firm considers a project that is expected to have a beta of 1.3. Should the project go ahead if the expected rate of return is 19%?

Answer

The required rate of return is

$$r_f + \beta_i(\overline{r_M} - r_f) = 8\% + 1.3(16\% - 8\%) = 18.4\%$$

If the expected rate of return of the project is 19%, the project should be accepted because it is higher than the required return (or known as the project's hurdle rate). But since it is so close, other factors need to be considered.

3.1.5. CAPM and the index models

If we have a single index model:

$$R_i = \alpha_i + \beta_i R_M + e_i$$

The covariance between stock i and the market index is

$$\sigma_{iM} = \beta_i \sigma_M^2 \Rightarrow \beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

Referring to the equation of SML in section 3.1.2., the same term for β is found. Therefore we could estimate the β in CAPM by estimating β using a single index model, only assuming that the market index is used as a proxy for the market portfolio in CAPM.

How do we estimate beta in CAPM?

The actual excess return of stock *i* is

$$r_i - r_f = \alpha_i + \beta_i (r_M - r_f) + e_i$$

This is re-written in the form

$$\begin{aligned} r_i &= \alpha_i + (1-\beta_i) r_f + \beta_i r_M + e_i \\ &= a + \beta_i r_M + e_i \end{aligned}$$

We could perform a regression of r_i against r_M, and the slope gives β_i (a time-series regression). Note that the intercept of this line is an estimate of

$$\alpha_i + (1-\beta_i) r_f$$

We can also work out a portfolio's beta from the components' values. If a portfolio consists of *n* risky assets with weights $w_1, w_2, \ldots, w_n$:

$$r_P = \sum_{i=1}^{n} w_i r_i, \operatorname{cov}(r_P, r_M) = \sum_{i=1}^{n} w_i \operatorname{cov}(r_i, r_M)$$

$$\beta_P = \frac{\operatorname{cov}(r_P, r_M)}{\sigma_M^2} = \frac{\sum_{i=1}^{n} w_i \operatorname{cov}(r_i, r_M)}{\sigma_M^2} = \sum_{i=1}^{n} w_i \beta_i$$

i.e. the portfolio beta β_P is the weighted average of the betas of the assets in the portfolio. We can deduce the SML followed by the portfolio:

Given $\overline{r_P} = \sum_{i=1}^{n} w_i \overline{r_i},\ \sum_{i=1}^{n} w_i = 1,\ \overline{r_i} - r_f = \beta_i (\overline{r_M} - r_f)$

$$\Rightarrow \overline{r_P} - r_f = \beta_P (\overline{r_M} - r_f)$$

Some results in CAPM

From the SML, we can deduce that:

$$\frac{\overline{r_i} - r_f}{\beta_i} = \frac{\overline{r_j} - r_f}{\beta_j} = \overline{r_M} - r_f$$

i.e. *the expected excess return above the risk free rate normalized by the beta of the asset is a constant for all assets*.

Let P be an efficient portfolio on the CML, with α allocated to the market portfolio:

$$r_P = \alpha r_M + (1-\alpha) r_f$$

We have the following results (note that σ_f for the risk free asset is 0 by definition):

$$\sigma_P^2 = \alpha^2 \sigma_M^2,\ \operatorname{cov}(r_P, r_M) = \operatorname{cov}(\alpha r_M + (1-\alpha) r_f, r_M) = \alpha \sigma_M^2$$

$$\rho_{PM} = \frac{\operatorname{cov}(r_P, r_M)}{\sigma_P \sigma_M} = 1,\ \beta_P = \frac{\operatorname{cov}(r_P, r_M)}{\sigma_M^2} = \alpha$$

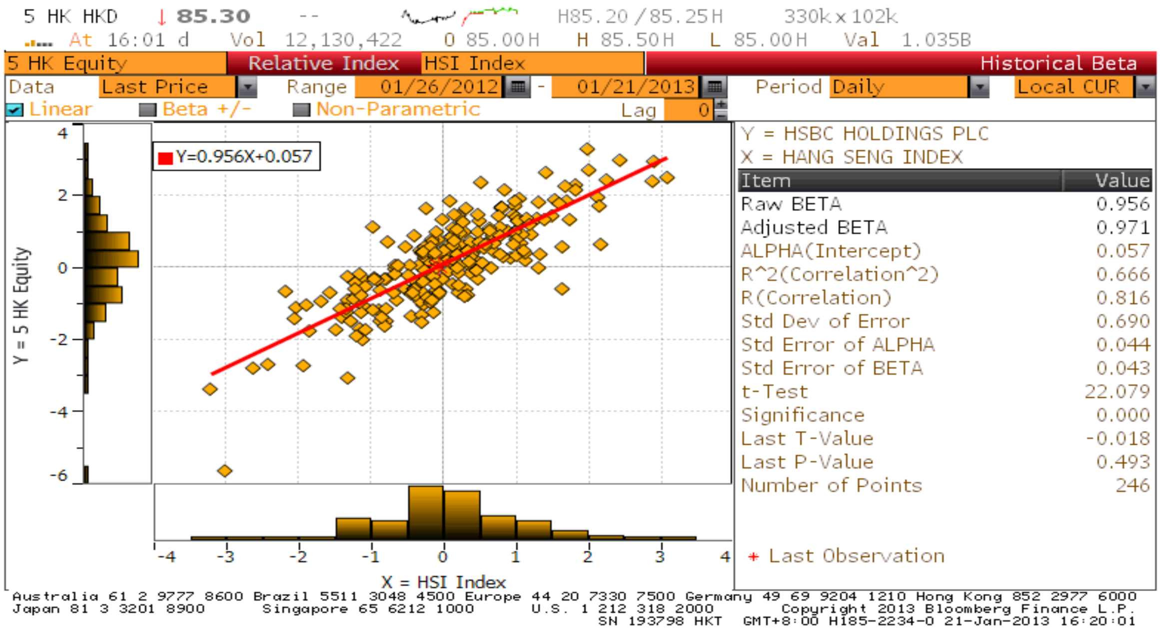
3.1.6. Index models in practice

Published historical beta is available for many US stocks. But since the future beta is what is needed in the model, various adjustment methods have been proposed, e.g.

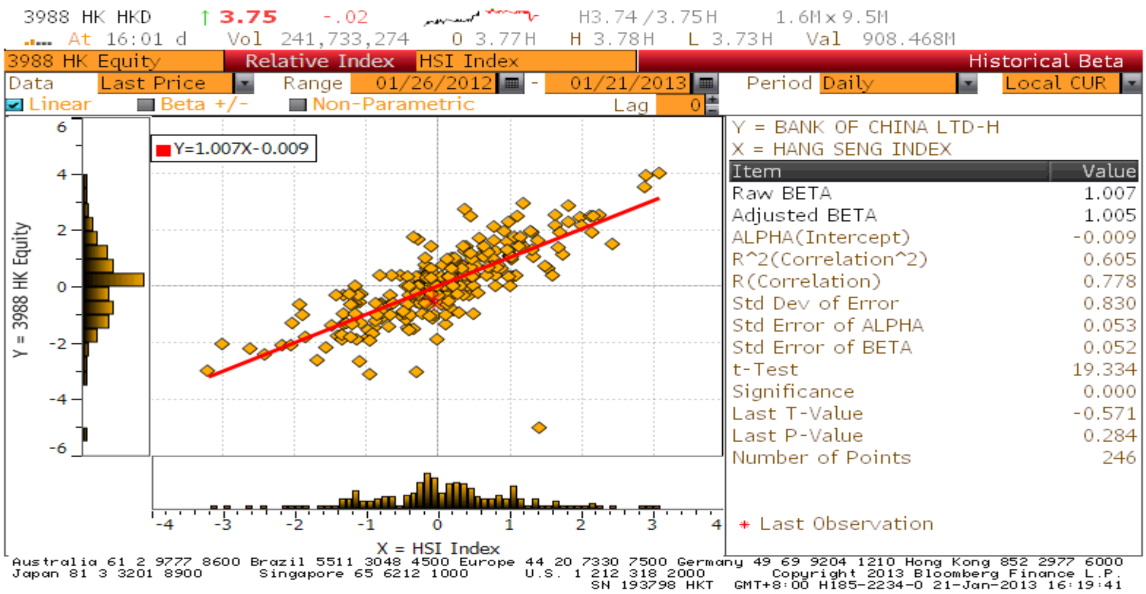
Blume: beta would move towards 1, hence forecast beta takes the form $p + q$ x (historical beta) where p and q are constants, e.g. forecast beta = 0.33 + 0.67 x historical beta.
Vasicek: adjust each stock's beta towards an "average beta".
Rosenberg: use firm-specific factors to forecast beta, e.g. firm size, dividend yield, earnings growth.

There is no conclusive evidence whether these adjustments are better than the historical beta or not. In the graphs below, we show some plots from the Hong Kong market. Hang Seng Index (HSI) has been used as the benchmark market portfolio in most of the examples. Although it is the most widely used index in Hong Kong, the HSI is not an ideal proxy representing the whole market because the index just comprises 50 of the biggest stocks.

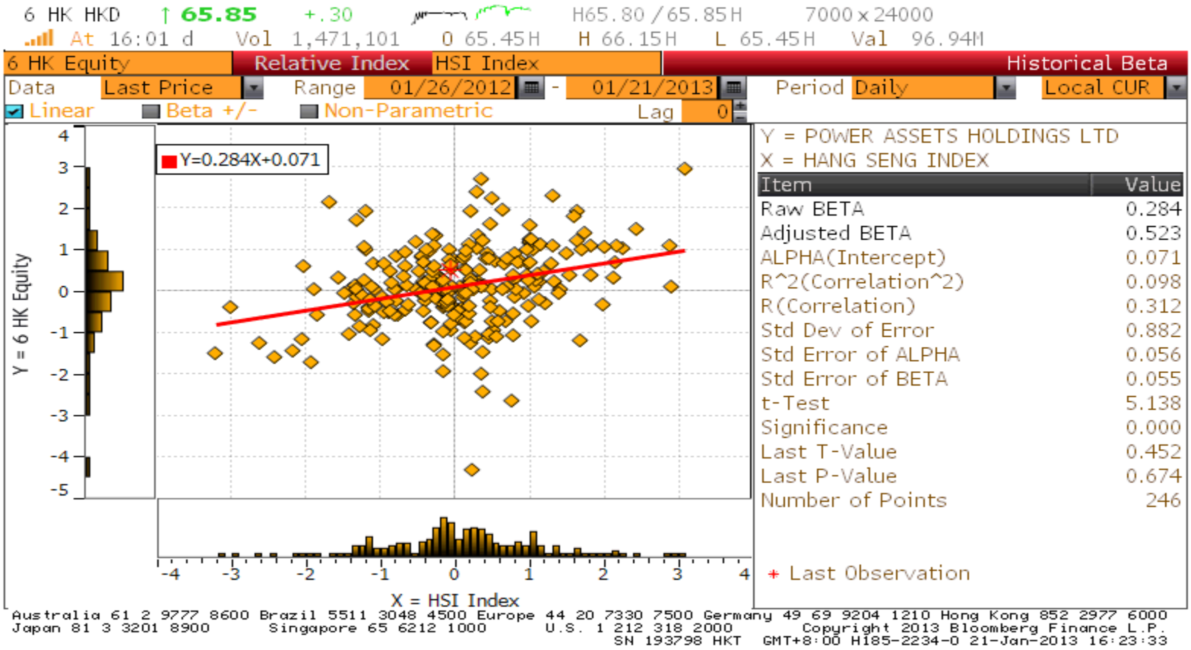
Historical Beta of HSBC (source: Bloomberg LP)



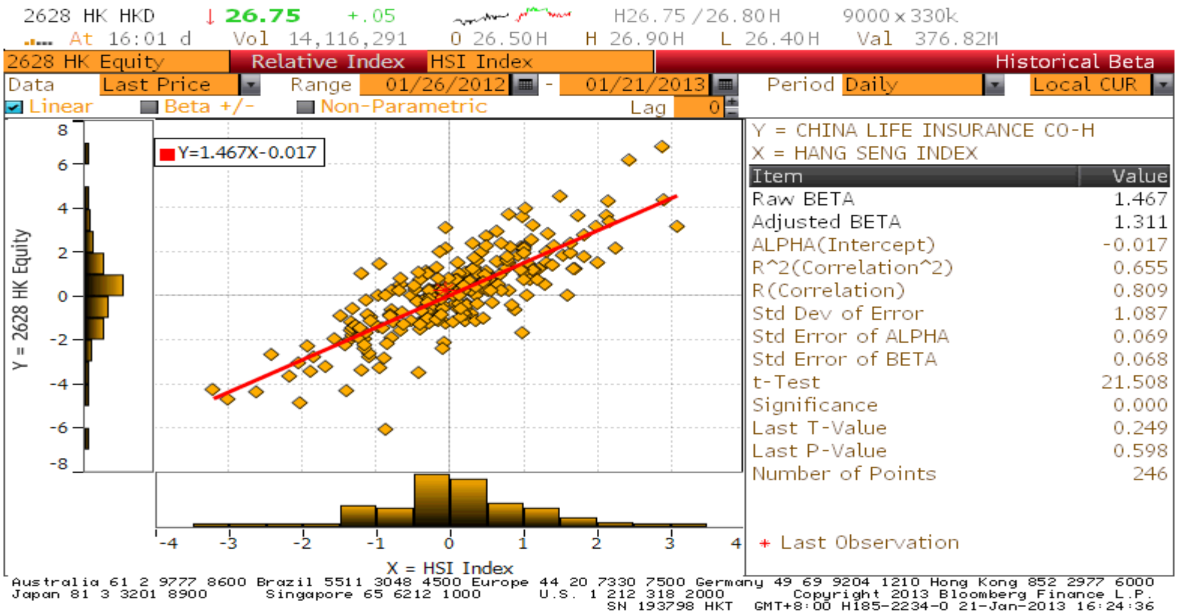
Historical Beta of Bank of China (source: Bloomberg LP)



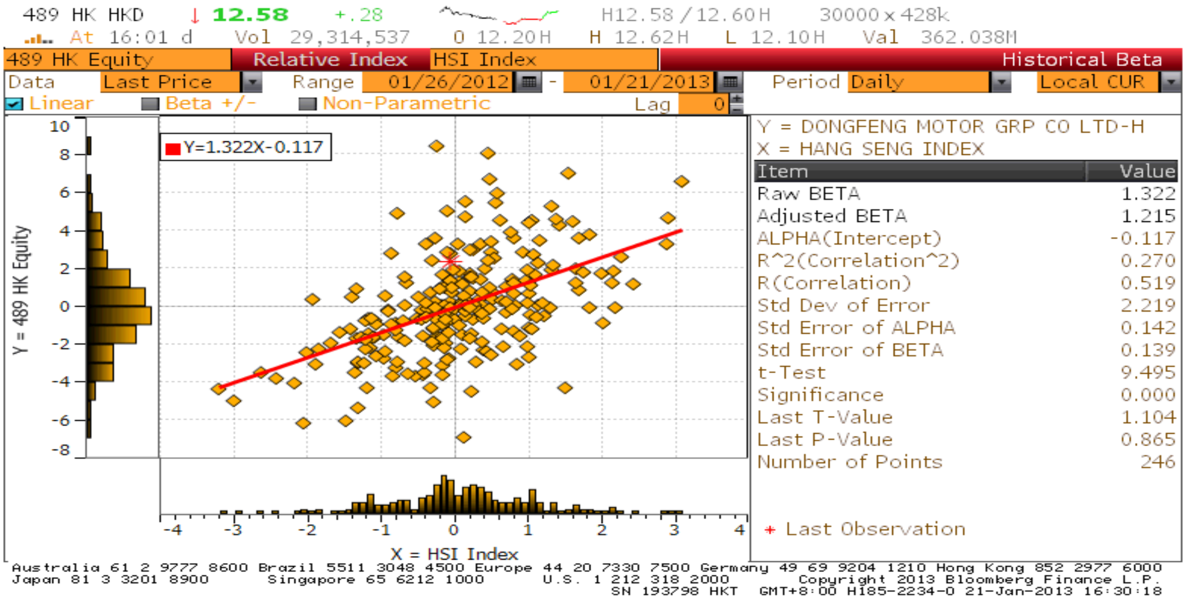
Historical Beta of Power Assets Holdings (source: Bloomberg LP)



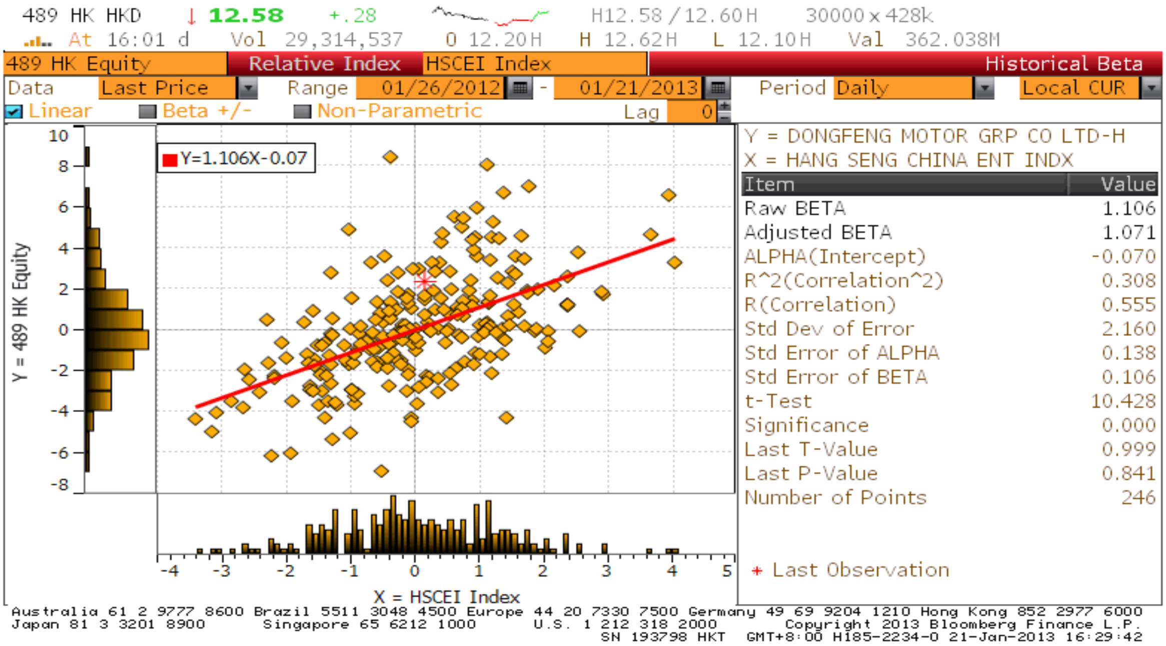
Historical Beta of China Life Insurance (source: Bloomberg LP)



Historical Beta of Dongfeng Motor (source: Bloomberg LP)



Dongfeng Motor is a Chinese company which is not part of the HSI. Hence we may obtain a different result if another index is used as the proxy, e.g. the Hang Seng China Enterprises Index. (Note that the beta changes from 1.215 for HSI to 1.071 for HSCEI).



Below is an example of Asset Betas for some industries in the United States. Note that the difference between industries can be quite substantial.

Industry	Beta	Industry	Beta
Airlines	1.80	Agriculture	1.00
Electronics	1.60	Food	1.00
Consumer Durables	1.45	Liquor	0.90
Producer Goods	1.30	Banks	0.85
Chemicals	1.25	International Oils	0.85
Shipping	1.20	Tobacco	0.80
Steel	1.05	Telephone Utilities	0.75
Containers	1.05	Energy Utilities	0.60
Nonferrous Metals	1.00	Gold	0.35

Source: D. Mullins, "Does the Capital Asset Pricing Model Work?", Harvard Business Review, vol. 60, pp.105-114, 1982

3.1.7. CAPM as a pricing formula

The standard CAPM is written in the form of return, not price. Assume the purchase price of an asset is P (known), which is sold at Q (random) later. Thus the rate of return = $(Q–P)/P$.

From CAPM, fair price of P can be estimated:

$$\frac{\overline{Q}-P}{P} = r_f + \beta_i(\overline{r_M} - r_f) \Rightarrow P = \frac{\overline{Q}}{1+r_f+\beta_i(\overline{r_M}-r_f)} = \overline{Q} \times DF$$

DF is the **risk adjusted discounting factor** of a cashflow.

The price can also be written in an alternative form, so that the pricing formula becomes linear:

$$\beta_i = \frac{\mathrm{cov}[r_i, r_M]}{\sigma_M^2} = \frac{\mathrm{cov}[(Q/P-1), r_M]}{\sigma_M^2} = \frac{\mathrm{cov}(Q, r_M)}{P\sigma_M^2}$$

$$\therefore P = \frac{P\overline{Q}}{P(1+r_f) + \mathrm{cov}(Q, r_M)(\overline{r_M} - r_f)/\sigma_M^2}$$

$$\Rightarrow P = \frac{1}{(1+r_f)}\left[\overline{Q} - \frac{\mathrm{cov}(Q, r_M)(\overline{r_M} - r_f)}{\sigma_M^2}\right]$$

The terms in the bracket are collectively known as the **certainly equivalent** of Q, which depend linearly on Q. This can be seen as a compensation for the additional risk taken.

Example

The 1-year target price of the stock $\overline{Q}$ is \$100. Other parameters are:

β	= 1.25
Expected return of the market portfolio $\overline{r_M}$	= 6%
Risk free rate r_f	= 1%

Fair price of the stock today

$$P = \frac{\overline{Q}}{1+r_f+\beta_i(\overline{r_M}-r_f)} = \frac{100}{1+0.01+1.25(0.06-0.01)} = \$93.24$$

If the current trading price is lower than P, then this stock is a good investment.

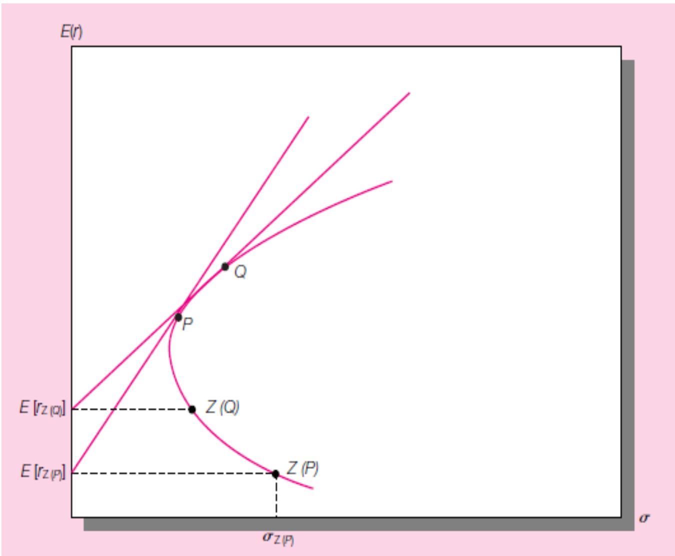
3.1.8. Relaxing the constraints

Research has shown that some of the straight constraints in the original formulation of CAPM can be relaxed. For example, Fama (1970) showed that even if we extend our analysis to a multi-period setting, the single period CAPM may still be appropriate.

One of the problems in the standard CAPM is that if investors have different risk free lending and borrowing rates, we could not find a unique market portfolio; the market portfolio may not even lie on the efficient frontier. Similarly, a fundamental assumption in CAPM is the availability of risk free assets, otherwise the choice of the risky portfolio on the efficient frontier will depend on the risk appetite of the investor, and the market portfolio may no longer be the common optimal portfolio.

In the absence of risk free assets, Fischer Black (in Black (1972)) formulated an extension to the model where a "zero-beta" portfolio could be created synthetically. This has been developed based on three properties of the mean-variance efficient portfolios:

1. Any portfolio constructed by combining efficient portfolios is itself on the efficient frontier ("Two-fund theorem")
2. Every portfolio on the efficient frontier has an inefficient counterpart, known as the **zero-beta portfolio.**



(source: Bodie, Kane and Marcus, Investments, 6th edition (2003))

A simple construction procedure can be used to derive this zero-beta portfolio. Firstly, draw a tangent from a portfolio P to intersect with the y-axis, and find the expected return (=$E(r_{Z(P)})$). Then draw a horizontal line from this return to the minimum variance frontier, denoted as $Z(P)$. The portfolio $Z(P)$ is uncorrelated with P.

3. The expected return of any asset can be expressed as a linear function of two frontier portfolios: [Notation: we use $E(r_i)$ to represent the expected return, same as $\bar{r}_i$ above]

$$E(r_i) = E(r_Q) + [E(r_P) - E(r_Q)]\frac{\text{cov}(r_i, r_P) - \text{cov}(r_P, r_Q)}{\sigma_P^2 - \text{cov}(r_P, r_Q)}$$

Using M and $Z(M)$ and assuming cov[r_M, $r_{Z(M)}$]=0, we have

$$E(r_i) = E(r_{Z(M)}) + [E(r_M) - E(r_{Z(M)})]\frac{\text{cov}(r_i, r_M)}{\sigma_M^2}$$

Basically, this is analogous to replacing the risk free asset with the zero-beta portfolio.

3.2. Arbitrage Pricing Theory (APT)

Strong assumptions have been made in deriving CAPM. For example, a particularly restrictive assumption is that all kinds of risk are captured by only one parameter, i.e. the beta. While the simplicity of this model is a feature that makes it useful, since it came out there has been much research effort in trying to relax some of the original assumptions. One of the most successful and elegant theories was first proposed by Stephen Ross in 1976, known as the Arbitrage Pricing Theory (APT). The starting point of this model is the use of *arbitrage relationships*, which can be known as the "law of one price": two items that are the same cannot sell at different prices. (Note that "same" here means that the items have the same characteristics, i.e. same expected return and risk.)

3.2.1. Assumptions in APT

There are less assumptions in APT than in CAPM; these assumptions are:

- Investors seek risk adjusted return, not absolute return (i.e. they are not risk neutral).
- Existence of a risk free rate for lending and borrowing.
- Investors agree on the number and identity of the factors that are used in explaining asset returns, as well as having homogeneous expectations in their characteristics.
- An important assumption that is specific to APT is that **arbitrage opportunities do not exist.**

3.2.2. Set up of the model

Arbitrage opportunities

The table below shows the expected returns of portfolio A and stock B under different scenarios:

	Scenario 1	Scenario 2	Scenario 3	Scenario 4
Portfolio A	23.33%	23.33%	20%	36.67%
Stock B	15%	23%	15%	36%

If these four scenarios contain all the possibilities, it means that portfolio A outperforms stock B under any scenario. A good investment strategy is to short stock B and use the money obtained to buy the assets in portfolio A; a riskless profit could be earned no matter which scenario takes place. When this happens, stock B's price would go down (because more people are selling) and portfolio A's price would go up (because more people are buying). This trading action would only stop when the expected returns change. For example, A becomes more expensive and thus the expected return decreases (if the future target price remains the same), and the arbitrage opportunity would disappear when A no longer dominates B.

Risk premiums under APT

We start with a one factor model $r_P = E(r_P) + \beta_P F + e_P$. If P is a well-diversified portfolio, its non-systematic risk would be 0, i.e. $r_P = E(r_P) + \beta_P F$.

We find another well-diversified portfolio Q, and form a new portfolio Z with weights in P and Q such that the overall β is 0; to prevent arbitrage this new portfolio must earn the risk free rate r_f:

$$E(r_Z) = \frac{\beta_Q}{\beta_Q - \beta_P} E(r_P) + \frac{-\beta_P}{\beta_Q - \beta_P} E(r_Q) = r_f$$

Re-arranging the terms, we find that the *risk premium* (defined in Topic 2, p.2) must be proportional to its beta:

$$\frac{E(r_P) - r_f}{\beta_P} = \frac{E(r_Q) - r_f}{\beta_Q}$$

The SML under APT

We note that the market portfolio M is a well-diversified portfolio, which has $\beta = 1$:

$$\frac{E(r_P) - r_f}{\beta_P} = \frac{E(r_M) - r_f}{1}$$

From this identity, we can write down the relationship between the risk and return of a well diversified portfolio P:

$$E(r_P) = r_f + \beta_P \left\lfloor E(r_M) - r_f \right\rfloor$$

We could immediately see that this relationship has the same "form" as the SML depicted in CAPM, which means that the SML applies to any well-diversified portfolio, irrespective of the correctness of CAPM.

Furthermore, it could be shown that almost all individual assets k in the portfolio also follow the same relationship, i.e.

$$E(r_k) = r_f + \beta_k \left\lfloor E(r_M) - r_f \right\rfloor$$

A heuristic argument goes as follows. Firstly, we assume that individual assets do not follow this relationship. However, we notice that when we form a well-diversified portfolio from these assets, the portfolio indeed follows the SML. This is possible by construction. Next, we form another well-diversified portfolio, and by definition it will also follow the SML. We can form any number of well-diversified portfolios and all of them will follow the SML. Hence it is highly unlikely that we can identify individual assets that do not follow the SML while all linear combinations of these assets follow the SML.

Multi-factor APT

In the case of a well-diversified portfolio, we assume a two factor model:

$$r_P = E(r_P) + \beta_{P1} F_1 + \beta_{P2} F_2 + e_P$$

where β_{Pi} is the sensitivity to factor F_i. It can be shown that, if we increase the number of assets in the portfolio, the non-systematic risk could be diversified, so that

$$r_P = E(r_P) + \beta_{P1}F_1 + \beta_{P2}F_2$$
$$\sigma_P^2 = \beta_{P1}^2\sigma_{F1}^2 + \beta_{P2}^2\sigma_{F2}^2$$

Here we need to introduce a concept known as a *pure factor portfolio*. Using many assets, we could create portfolios which are sensitive to only one of the factors, i.e.

$$r_{P1} = E(r_{P1}) + F_1$$
$$r_{P2} = E(r_{P2}) + F_2$$

These portfolios have a beta of 1 in one of the factors and have zero sensitivity to the other factors. The risk premium on each factor portfolio = $E(r_{Pi}) - r_f$. Note that the non-systematic risk component e_{P1} has been assumed to be diversified away.

If a stock k has the following characteristic:

$$r_k = E(r_k) + \beta_{k1}F_1 + \beta_{k2}F_2 + e_k$$

The return of this portfolio comes from a summation of the risk free rate plus the risk premiums given by each of the two factors. We could use pure factor portfolios to construct a portfolio A which has the same sensitivities as stock k, where the weights are β_{k1} of P_1, β_{k2} of P_2, and $(1 - \beta_{k1} - \beta_{k2})$ of risk free asset. Portfolio A would have an expected return:

$$E(r_A) = \beta_{k1}E(r_{k1}) + \beta_{k2}E(r_{k2}) + (1 - \beta_{k1} - \beta_{k2})r_f$$
$$= r_f + \beta_{k1}[(r_{k1}) - r_f] + \beta_{k2}[(r_{k2}) - r_f]$$

Note that this equation is similar to the form of SML with one factor.

Under equilibrium conditions, stock k's expected return must equal the expected return of portfolio A, otherwise arbitrage occurs. The reason is because stock k and portfolio A have the same sensitivity to the underlying factors. If expected return from stock k was less than that of portfolio A, portfolio A "dominates" stock k. Arbitrage trading would take place, i.e. an arbitrageur could buy A and sell k, and hope to make an arbitrage profit (given that both assets are sensitive to the same factors). This strategy may not be riskless because the actual return of k has a specific component e_k which can be random; however, on average, this risk has a mean of zero. On the other hand, if the expected return from stock k were more than that of portfolio A, there is no obvious dominance. A possible strategy is for an arbitrageur to buy k and sell A, and hope to realize the difference between the expected returns. However, since k contains unsystematic risk, there is no guarantee that k's return must be higher than A, so that the possible profit is not risk free.

3.2.3. Comparison between APT and CAPM

Both CAPM and APT aim to derive the behavior of any asset under equilibrium conditions. Although we have similar results in terms of the final relationships, their natures are very different.

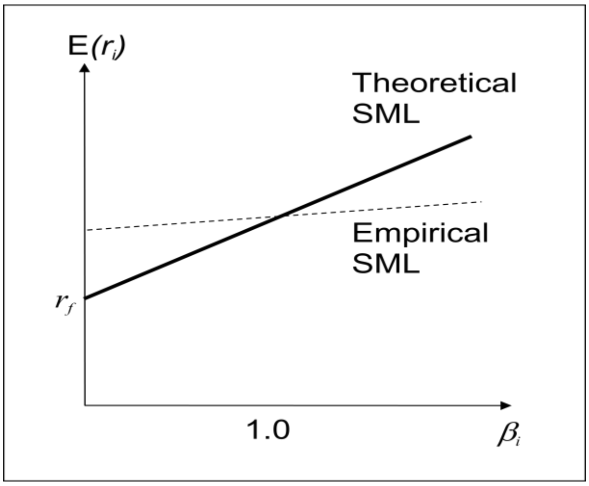
- APT relies on a few arbitrageurs to force the market back to equilibrium; CAPM relies on all investors to make small adjustments to their portfolios for the equilibrium to be established
- In the APT, there is no need to assume that everyone follows a mean-variance optimization.

- APT does not assume a complete "market" portfolio; any well diversified portfolio is sufficient (to be used in the factor model).
- APT by itself does not specify which factors are relevant; CAPM states that the only relevant factor is the security's beta.
- APT works for portfolios; the application to individual securities is not guaranteed.
- *Although APT and CAPM use different assumptions, the same SML is obtained.*

3.3. Empirical tests

Since CAPM and APT were developed, there have been much research effort in testing the validity of these theories. However, this is not a trivial task. We should note that in order to show that the models actually work, one should aim at testing the conclusions implied from the model. As long as the results are sound, it is not necessary to demonstrate that the assumptions are reasonable – they can be seen as simplifications of the real world.

Testing the CAPM requires a joint test between the theory of equilibrium itself and the stability of predictions. One way of testing is to start by assuming an index model, and then one can try to demonstrate that the long term alpha = 0 (because it is supposed that all stocks cannot acquire returns which are different from those predicted by the SML) and the expected return is related to beta linearly. It is necessary to make use of cross section data of many stocks. In addition, one of the famous testing results is known as a *benchmark error* (part of "Roll's critique, " from Roll(1977)) Since the market portfolio is not observable, one has to use a proxy. However, it is noted that a small change in the market proxy generates dramatically different expected returns.



A typical test using historical data would show that the empirical line is too flat: the intercept is higher than the risk free rate, and the coefficient on beta is less than the average excess market return. A conclusion from this diagram indicates that beta does not seem to be adequate in explaining the risk.

Testing the APT requires a different procedure. It is a joint test between the theory of equilibrium itself and the factor selection issue. In order for the model to work, good factors need to be identified. However, a main difficulty arises: it is hard to ascertain the theory because of the factor selection problem – even if the test results are not satisfactory, it may be due to a wrong choice of factors, while there may not be anything wrong in the theory. Indeed, identification of contributing factors in the APT is a major empirical research area. Some common macroeconomic factors include:

Inflation: impacts both the discount rate and the size of future cash flows
Risk premium: market's attitude to risk, measured by the difference between the return of (almost riskless) bonds and risky bonds
Industrial production: affect the real value of cash flow

Fama and French (1993, 1996) have proposed a 3-factor model for expected returns:

$$\overline{r_i} = r_f + \beta_{iM}\left(\overline{r_M} - r_f\right) + \beta_{iS} R_{SMB} + \beta_{iH} R_{HML}$$

R_{SMB}: Expected difference in returns of small and big stocks
R_{HML}: Expected difference in returns of diversified portfolios of high and low B/M stocks
B/M: Book value of stock / Market value of stock
β_{iX}: Sensitivity to factor X

This is a popular model which has a much better fit to empirical data (in the US) compared to other models. However, as with factor models in general, there is no theoretical justification why these factors should be used.

3.4. Final Remarks

CAPM is one of the most important finance models, and its implications are wide-ranging. The most significant characteristic of this model is that the equilibrium expected return of an asset is linearly related to its risk, as measured by beta. Consistent with the Modern Portfolio Theory, CAPM postulates that *having extra risk is the only way to achieve extra return.* Furthermore, it was found that the same SML is obtained using both CAPM and APT. On the other hand, some assumptions in CAPM seem to be restrictive which affects its usefulness in practice. For example, it is difficult to choose a correct "market portfolio". Finally, it is noted that while the CAPM relationship is largely correct, the slope and intercept may be different from that predicted by the standard formulation.

We should understand that while the CAPM/APT can use the concept of index models extensively, these models are very different in nature. The index/factor models are purely descriptive, i.e. it aims to give an empirical relationship between the returns and the input factors, without explaining why there is a relationship in the first place. On the other hand, CAPM/APT are prescriptive models which give fair values of the equilibrium prices of the assets (with different dependent variables). The model formulation may be simple, but empirical results are encouraging. However, we also note that studies applicable to many Asian markets are more difficult to conduct because of lack of information, and further research needs to be carried out to demonstrate the validity of CAPM/APT in these markets.