Calculus, Geometry, and Probability in n dimensions

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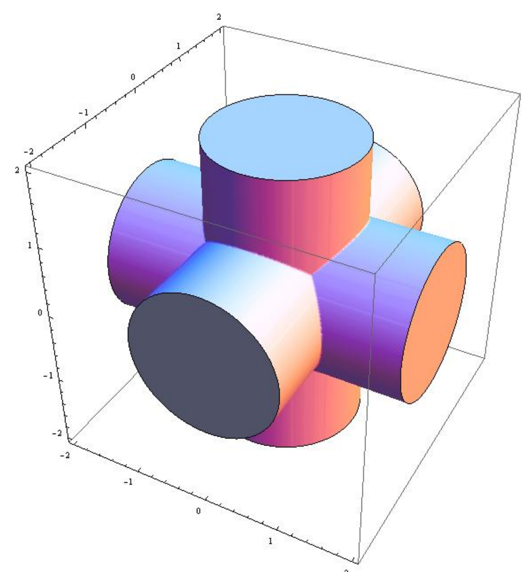
The Intersecting Cylinder Problem in 3D



Graphics created by 2013 IGL group [1]

Consider three cylinders of radius 1 each, centered at the coordinate axes. Let $C_{3,2}$ be the region of intersection of these cylinders, i.e., the region given by

$$x^2+y^2 \le 1,$$
$$x^2+z^2 \le 1,$$
$$y^2+z^2 \le 1.$$

The volume of $C_{3,2}$ is $16-8\sqrt{2}$. This result is due to **Charles Proteus Steinmetz** (1865 - 1923), a German-born American mathematician and electrical engineer. The region $C_{3,2}$ is called the **Steinmetz Solid** [2].

Intersecting Cylinders in n dimensions

The n-dimensional intersecting cylinder $C_{n,n-1}$ is defined as the intersection of the n cylinders

(1) $$X_1^2+\cdots+X_{n-1}^2 \le 1, \quad \ldots, \quad X_2^2+\cdots+X_n^2 \le 1.$$

The volume of $C_{n,n-1}$ for $n=4,5$ was determined in a 2013 IGL project [1] as, respectively,

$$48\left(\frac{\pi}{4}-\frac{1}{\sqrt{2}}\arctan\sqrt{2}\right), \quad 256\left(\frac{\pi}{12}-\frac{1}{\sqrt{2}}\arctan\frac{1}{2\sqrt{2}}\right).$$

A Simpler Problem

If we replace X_i^2 by u_i in (1), we get

(2) $$u_1+\cdots+u_{n-1} \le 1, \quad \ldots, \quad u_2+\cdots+u_n \le 1.$$

Let $T_{n,n-1}$ denote the region of n-tuples $(u_1,\ldots u_n) \in [0,1]^n$ satisfying (2).

Problem T

What is the volume of $T_{n,n-1}$ for general n?

Problem T is equivalent to the following problem.

Problem T*

Given n independent random numbers $X_1,\ldots,X_n$ in $[0,1]$, what is the probability, $P_{n,n-1}$, that **all** subsums of length $n-1$ are ≤ 1?

Sums of Random Numbers: Main Problem A

Problem A

Given n independent random numbers $X_1,\ldots,X_n$ in $[0,1]$, and $k \in \{1,\ldots,n\}$, what is the probability, $P_{n,k}$, that **all** subsums of length k are ≤ 1?

Theorem A

For general n and k, the probability $P_{n,k}$ is

$$P_{n,k}=\frac{1}{k^{n-k+1}(k-1)!}=\frac{1}{k^{n-k}k!}.$$

Special Cases

- $P_{n,2}=\frac{1}{2^{n-1}}$
- $P_{n,3}=\frac{1}{3^{n-3}3!}$
- $P_{n,n-1}=\frac{1}{(n-1)!(n-1)}$ (Solution to Problem T*)
- $P_{n,n}=\frac{1}{n!}$

Sums of Random Numbers: Main Problem B

Problem B

Given n independent random numbers $X_1,\ldots,X_n$ in $[0,1]$, and $k \in \{1,\ldots,n\}$, what is the probability, $P^*_{n,k}$, that **at least one** subsum of length k are ≤ 1?

Theorem B

For general n and k, the probability $P^*_{n,k}$ is

$$P^*_{n,k}=\frac{(k-1)!+\sum_{i=1}^{k-1}(-1)^{k-i}\cdot\binom{k-1}{i}\cdot i^n\cdot(i+1)^{-n+k-1}}{(k-1)!}.$$

Special Cases

- $P^*_{n,2}=1-1\cdot 2^{-n+1}$
- $P^*_{n,3}=\frac{1\cdot 2+2\cdot 2^{-n+2}-1\cdot 2^n 3^{-n+2}}{2}$
- $P^*_{n,4}=\frac{1\cdot 6-3\cdot 2^{-n+3}+3\cdot 2^n 3^{-n+3}-1\cdot 3^n 4^{-n+3}}{6}$
- $P^*_{n,n-1}=\frac{n!-|s(n,2)|}{((n-1)!)^2}$
  where $s(n,2)$ is the signed Stirling number of the first kind [3].
- $P^*_{n,n}=\frac{1}{n!}$

Numerical Values

k \ n	2	3	4	5	6
2	1/2	1/4	1/8	1/16	1/32
3	/	1/6	1/18	1/54	1/162
4	/	/	1/24	1/96	1/384
5	/	/	/	1/120	1/600
6	/	/	/	/	1/720

Table of values of $P_{n,k}$ for $k,n \le 6$

k \ n	2	3	4	5	6
2	1/2	3/4	7/8	15/16	31/32
3	/	1/6	13/36	115/216	865/1296
4	/	/	1/24	35/288	775/3456
5	/	/	/	1/120	223/7200
6	/	/	/	/	1/720

Table of values of $P^*_{n,k}$ for $k,n \le 6$

Future Directions

- Given n independent random numbers $X_1,\ldots,X_n$ in $[0,1]$, and $k \in \{1,\ldots,n\}$, what is the probability that **exactly m** subsums of length k are ≤ 1?
- Given n independent random numbers $X_1,\cdots,X_n$ in $[0,1]$, calculate the probability that all subsums (or at least one subsum) of length k are $\le \alpha$, **for general α**.
- Solve the original n-dimensional intersecting cylinder problem.
- Extend the results to subsums of X_i^p with $p \ge 2$.

References

[1] Kong, L., Lkhamsuren, L., Turner, A., Uppal, A., Hildebrand, A. J. (2013). *Intersecting Cylinders: From Archimedes and Zu Chongzhi to Steinmetz and Beyond.* IGL Project Report.

[2] Weisstein, Eric W. *Steinmetz Solid.* From MathWorld—A Wolfram Web Resource. Retrieved December 10, 2020, from `https://mathworld.wolfram.com/SteinmetzSolid.html`.

[3] Weisstein, Eric W. *Stirling Number of the First Kind.* From MathWorld—A Wolfram Web Resource. Retrieved December 10, 2020, from `https://mathworld.wolfram.com/StirlingNumberoftheFirstKind.html`.