Three-Player Partisan Games

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Introduction

Goal

We are trying to build on work done to extend two-player game theory using Conway's construction of surreal numbers to three-player partizan games [Con76].

Currently, most research that has been done on games focused on two-player games. We attempted to answer some conjectures on three-player partizan games.

Set Up

Our research mostly focused on games of 3-player Hackenbush and Rhombination as most real life applications would have more than 2 players. In real world settings, it is indispensable to understand the winning strategies for each individual players when different players starts first.

- The rules for for 3-player Hackenbush are based on A. Cincotti's "Three-player partizan games" [Cin05], while the rules for Rhombination is built upon Katherine A. Greene's thesis [Gre17].

Number games: Similar to how Conway expresses two-player games, we can express three-player games through the possible game states after every player moves. If every player has no move we represent this game as $\{\ |\ |\ \}$. Since adding this game will not change the outcome, we call it 0. If the Left player has only one move which results in the 0 game and Center and Right players have no move, it would be expressed $\{0|\ |\ \}$, and we call this game 1_L. x^L is any possible game after Left moves. A game is a number game if $x^L \leq_L x^C, x^R$ and $x^C \leq_C x^L, x^R$ and $x^R \leq_R x^L, x^C$ where order is defined as follows:

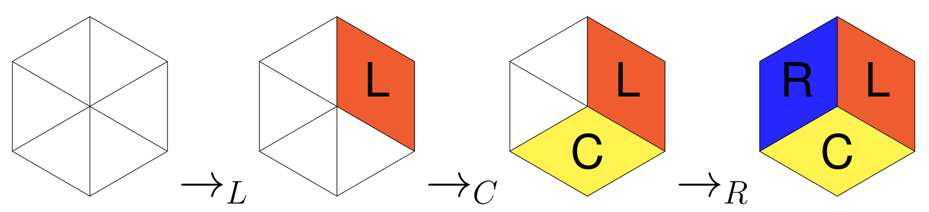
$$x \geq_L y \iff y \geq_L \text{ no } x^C \text{ and } y \geq_L \text{ no } x^R \text{ and no } y^L \geq_L x$$

$$x \geq_C y \iff y \geq_C \text{ no } x^L \text{ and } y \geq_C \text{ no } x^R \text{ and no } y^C \geq_C x$$

$$x \geq_R y \iff y \geq_R \text{ no } x^L \text{ and } y \geq_R \text{ no } x^C \text{ and no } y^R \geq_R x$$

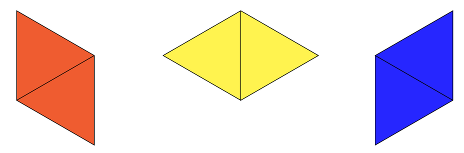
Summing Games: For the game $G+H$, a player chooses which game to act in. If Left plays, the resulting game will either be G^L+H or $G+H^L$.

Rhombination: In Rhombination each player has a different orientation of a tile they play on a grid of triangles where no tile can overlap. Player order is set, so Center plays after Left and Right plays after Center and so on. The first player who can not place a tile on their turn gets last place. The player before them gets first place. Assuming each player makes the move that gives them the best ranking, the outcome of any game can be determined. The outcome is expressed $\alpha\beta\gamma$ where α is the player who wins when Left plays first, β the the player who wins when Center plays first, and γ is the player who wins when Right plays first. In the game 0, the first player automatically loses since there are no moves and the previous player wins, so $0 \in \mathrm{RLC}$.



Another game in RLC

Since Left now has no moves, Right wins. Since each player plays once and then the first player loses, this is another game in RLC.



$1_L + 1_C + 1_R$

Note since each independent element of this game is a game itself with one move for a single player, the game that Left can move in is 1_L, the game Center can move in is 1_C, and the game Right can move in is 1_R. Since Center and Right lose if they have to move, $1_L \in \mathrm{LLC}$. Similarly $1_C \in \mathrm{RCC}$ and $1_R \in \mathrm{RLR}$. The resulting game, $1_L+1_C+1_R$, is in the outcome class RLC.

Journey of finding "zero"

Zero game: A game G is a 'zero game' if $G+H$ has the same outcome class as H for all game H. The simplest zero game is 0 where no player has any moves.

Zero games other than 0 are very important to the analysis of games. If we have zero games other than 0, we can simplify games, calculate additive inverses, and find how much benefit one player possesses in a game compared to the other two. We can even determine the winner without actually playing the game.

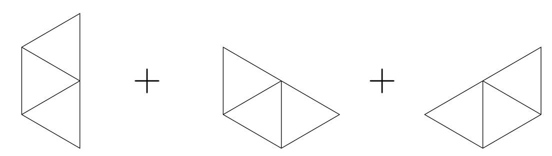
Rotation: In a general game G, rotation σ renames the players, so we define $\sigma(G) = \sigma(\{G^L|G^C|G^R\}) = \{\sigma(G^R)|\sigma(G^L)|\sigma(G^C)\}$ and $\sigma(0)=0$. All moves of Left become that of Center's now.

Seeing rotation, we may think that if we have a game G, and we create two games $\sigma(G)$ and $\sigma^2(G)$, their sum should be a zero game, and from our observation, a zero game must be in RLC, so here comes Greene's first conjecture.

Conjecture 1

For any game G, $G+G^{60}+G^{120} \in \mathrm{RLC}$.
Note: For Rhombination games, G^{60} is the same as $\sigma(G)$, and it renames players by rotating the game counterclockwise $60°$.

Unfortunately, we found a counterexample to this conjecture, so our dream of producing a stream of zero games through summing rotations is shattered.



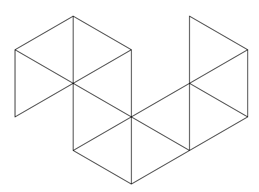
Counterexample to conjecture 1

Then we may think that although we didn't find a easy way to produce zero games, games in RLC may be zero games, so here comes Greene's second conjecture.

Conjecture 2

For any game G and any game $H \in \mathrm{RLC}$, $G+H$ and G have the same outcome class.

However, even adding the simplest RLC game $1_L+1_C+1_R$ to G may change its outcome class. And we generate a theorem which ends all possibilities of other zero games.



Counterexample to conjecture 2

Theorem. *There is no other "zero" game in the general game class.*

Proof. First we claim that for any game G, there exist numbers n and m such that $G+n(1_L)+m(1_C) \in \mathrm{CCC}$. Then suppose that x is a zero game that has some x^L. We can create a game $G = \{1_R+1_C|n(1_L)+m(1_C)|\ \}$ such that $G+x^L \in \mathrm{CCC}$. Then in game $G+x$, if Left goes to $G+x^L$, then Center can win by going to $n(1_L)+m(1_C)+x^L$. However, it is easy to see that $G \in \mathrm{RCC}$, so $G+x \neq G$. x^L must be empty for x to be a zero game.

By rotation property of three-player games, x is also not a zero game if x has a x^C or a x^R. Therefore, the only zero game is 0, and 0 is the unique additive inverse identity in general number games. □

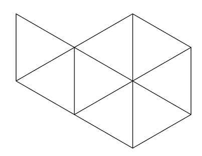
Corollary. *There is no additive inverse for a game $G \neq 0$ in general game class.*

Other Results

Conjecture 3

There are no Rhombination games in LRC, CLR, or RCL.

The right figure is a Rhombination game in LRC, and due to the rotational property of Rhombination games, $60°$ rotations will generate CLR and RCL. So conjecture 3 is false.



Counterexample to conjecture 3

Methods

Some counterexamples were found using a python program that generated Rhombination games and their sums and then determined their respective outcome classes.

Future Directions

Our research is mainly built upon Greene's thesis and explored some of her conjectures. We found some counterexamples to negate Greene's conjectures about possible methods to find other zero games, and we proved that there was no other zero game in a general game class.

However, we are still unsure whether there are other zero games in a restricted game class, such as Rhombination and Hackenbush, so testing other zero game's existence on these games might be interesting.

Games that have more than 3 players are also worth looking at. How will adding more players affect the outcome class and how players form a mutually constrained but dynamic balance with each other? These areas might be interesting for further exploration.

Another interesting future research area might be focused on the application of number games on situations and conflicts in real life. Can some of these surreal game models be applied into industry and economics? Will other factors affect the outcome class as well? How will the outcome change if some players cooperate together to compete against others? Situations might be more complicated in real world settings.

References

[Con76] J. H. Conway. *On Numbers and Games*. L.M.S. monographs. Academic Press, 1976.

[Cin05] A. Cincotti. "Three-player partizan games". In: *Theoret. Comput. Sci.* 332.1-3 (2005), pp. 367–389.

[Gre17] K. Greene. "Exploration of the Three-Player Partizan Game of Rhombination". MA thesis. Wake Forest University, 2017.

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