Root Dynamics of Random Polynomials Under Differentiation

IGL Scholars: Parth Deshmukh, Dillon Max, Sally Rong, Yifan Zhang
Graduate Mentors: Kohei Noda, Qiang Wu   Faculty Mentors: Yuliy Baryshnikov, Tomoyuki Shirai

Introduction

We study the root dynamics of random polynomials under differentiation, i.e., how the roots of a polynomial approach the roots of its differentials. We mainly focus on the modulus and velocity of roots under differentiation.

Example 1: Roots of a Kostlan Polynomial

A Kostlan polynomial is a polynomial with complex Gaussian coefficients scaled by the square root of binomial coefficients.

$$p(z) = \sum_{k=0}^{n} C_k \cdot \sqrt{\binom{n}{k}} \cdot z^k,\ C_k \sim N(0,1) + iN(0,1).$$

Below is a plot of roots of a Kostlan polynomial and its differentials. Color denotes the order of the differential. Note that the trajectories form almost straight lines.

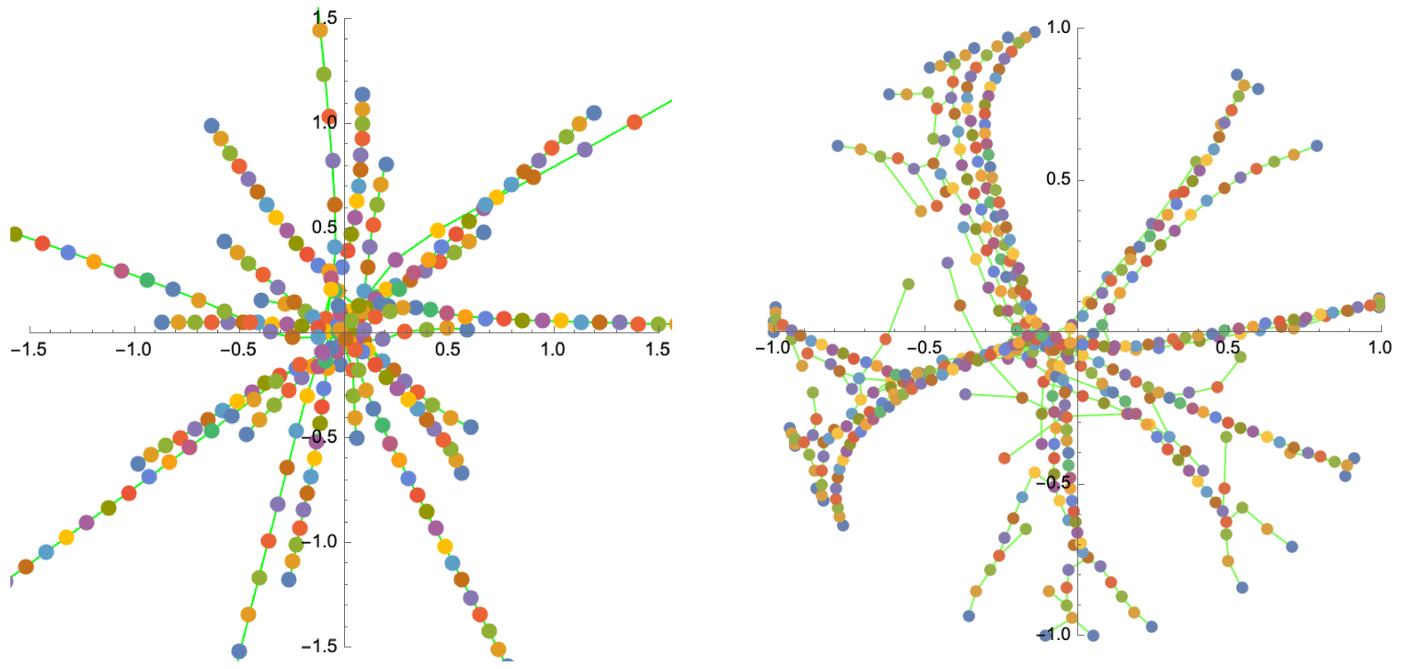


Figure 1: *Root dynamics of a Kostlan polynomial and an uniform polynomial under differentiation.*

Example 2: Roots of Unity

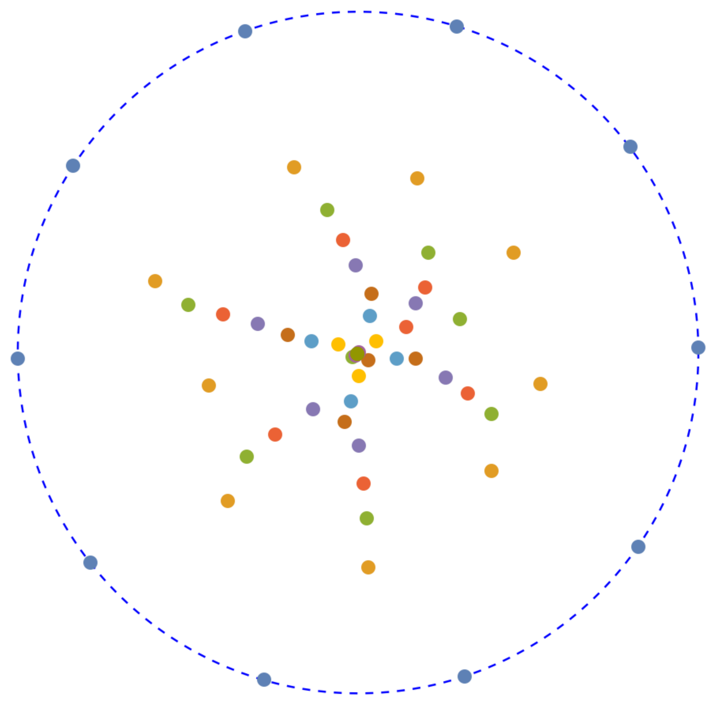


Figure 2: *Root dynamics of perturbed roots of unity.*

From the roots of unity, we can construct a polynomial

$$p(z) = \prod_{k=1}^{n} (z - e^{i(2\pi k/n)}) = z^n - 1.$$

Roots of unity collapse to the origin right after the first differentiation. However, if we add some perturbation to the roots, the pattern is less predictable.

Main Result 1: Perturbation

We introduce small, controllable "randomness" to the roots of unity case $z^n - 1$ by rotating a single root at $z = 1$ by an angle $2\pi t$ about the origin.

$$p(z) = z^n - 1 + (1 - e^{2\pi i t}) \sum_{k=0}^{n-1} z^k.$$

As the perturbed root approaches the other roots, it exhibits an attractive behavior:

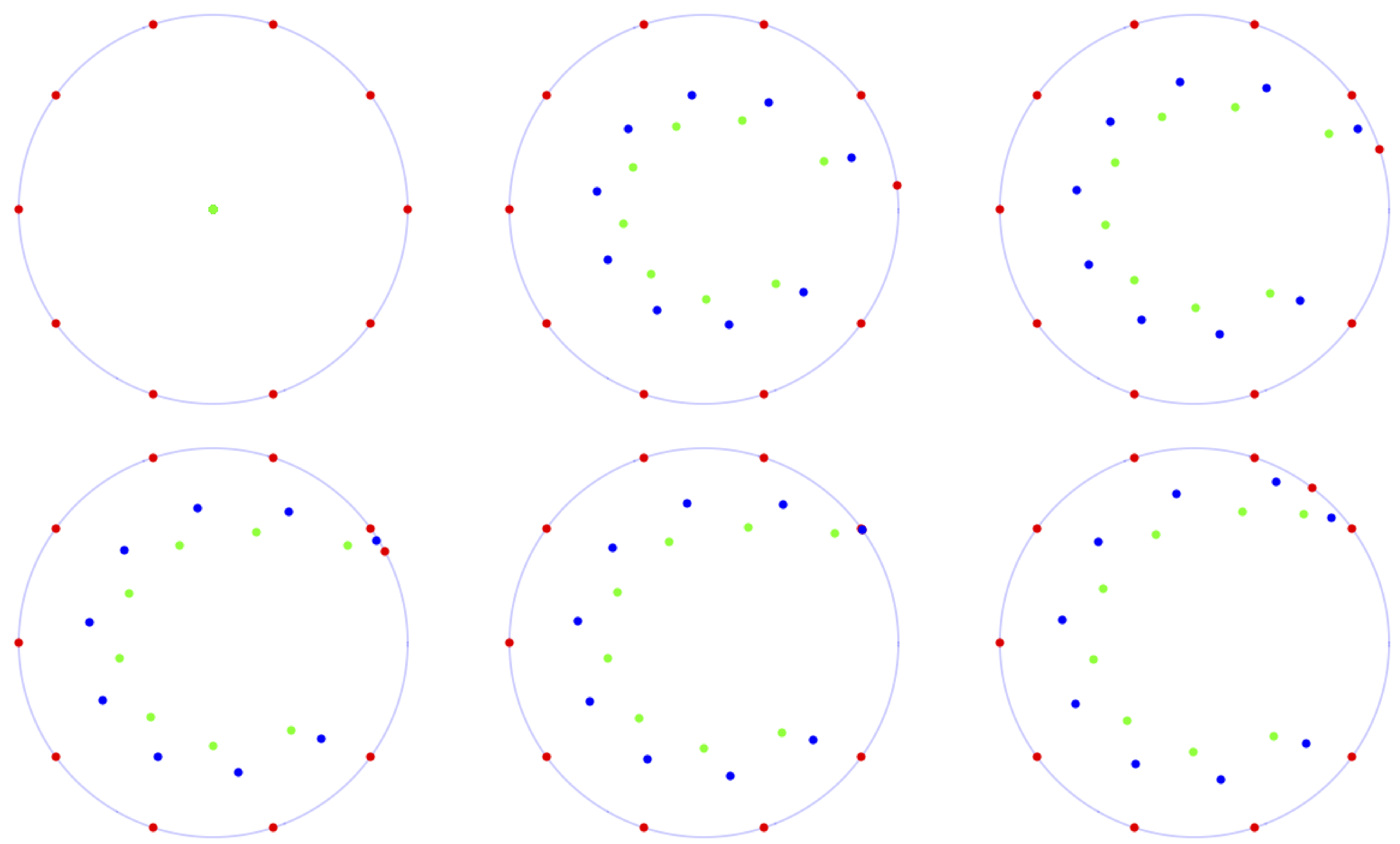
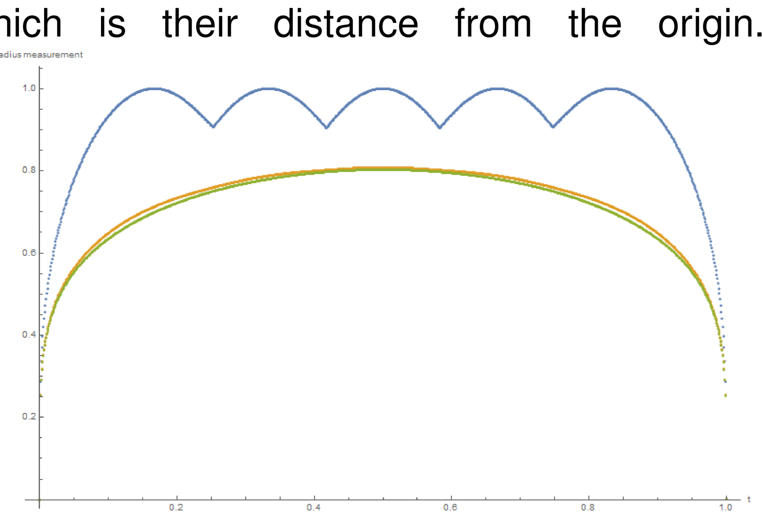


Figure 3: *10-degree polynomial with $t \in [0, 0.15]$. Note the attractive behavior in fourth/fifth picture near t = 0.1. Blue points represent first order derivative, green points represent second order derivative.*

To understand the distance the roots travel when differentiated once (critical points), we measure their modulus which is their distance from the origin.

Modulus of critical points versus t for a 6-degree polynomial:

- Blue: maximum modulus of critical points
- Orange: average modulus of critical points
- Green: $d(t)$ as shown below



Theorem 1

$$d(t) = \left(\frac{|1 - e^{2\pi i t}|}{n} \right)^{1/(n-1)} + o(1)$$

closely models the average modulus at higher values of n. The number of peaks in the maximum modulus is $n - 1$ and each peak is at a multiplicity.

The peaks behavior arises when roots overlap. We explore this with multiplicities.

Main Result 2: Multiplicity

Instead of perturbing the roots of unity, we increase the multiplicity at each roots to m.

$$p(z) = (z^d - 1)^m.$$

Moduli of Roots Under Differentiation

Definition
The thresholds of collapse are the moduli of roots before the innermost roots collapse to the origin. Note that for $m > 2$, thresholds of collapse occur every d times differentiation, and each threshold contains multiple radii.

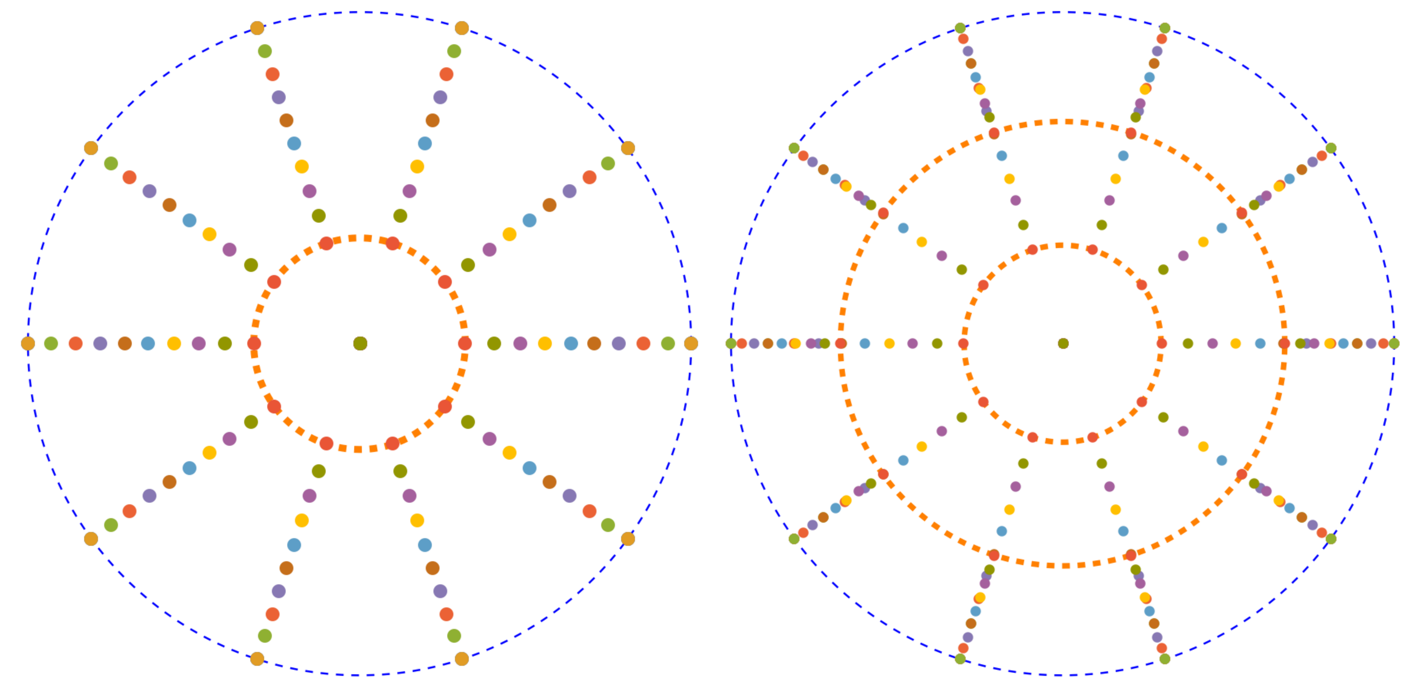


Figure 4: *Root dynamics of 10th roots of unity with multiplicity 2 and 3 up to 10th differentiation.*

Theorem 2
For $p(z) = (z^d - 1)^m, m \geq 2$, the kth $(k = 1, \cdots, m-1)$ thresholds of collapse occur at radii

$$\frac{k^k}{(k+1)^{k+1}}, \frac{2^2(k+1)^{k+1}}{(k+2)^{k+2}}, \frac{3^3(k+2)^{k+2}}{2^2(k+3)^{k+3}}, \cdots, \frac{(m-k)^{m-k}(m-1)^{m-1}}{(m-k-1)^{m-k-1} m^m},$$

as $d \to \infty$.

Examples

- When $m = 2$, the threshold of collapse approaches $1/4$ as $d \to \infty$
- When $m = 3$, the first thresholds $\to 1/4, 16/27$ and the second threshold $\to 4/27$ as $d \to \infty$.

Velocity of Roots Under Differentiation

Theorem 3
The roots approach the thresholds of collapse at asymptotically constant velocity under differentiation.

Future Work

- Investigate the direction of roots movement.
- Investigate the relation between root density and root velocity.

References

[1] Boris Hanin. Pairings of Zeros and Critical Points for Random Polynomials. arXiv:1601.06417v1

Support for this project was provided by the Illinois Geometry Lab and the Department of Mathematics at the University of Illinois at Urbana-Champaign.