Vectors

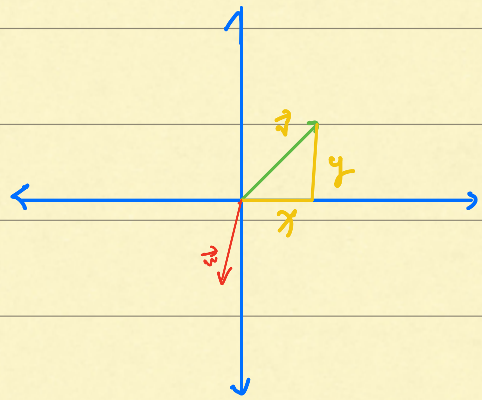
physics

CS

[2.5, 7, 4.7]

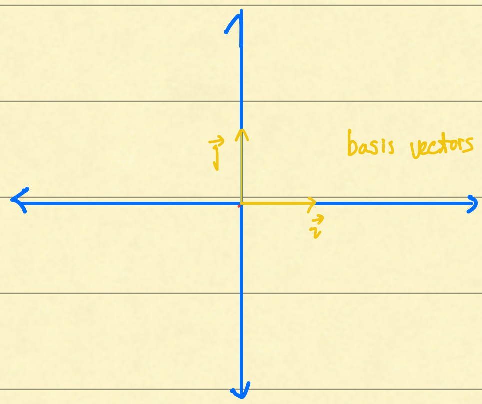
Math

$\vec{v} + \vec{w}$



1. Scaling → Scalar (Vector) $\vec{v} \cdot k$

Scalar multiplication



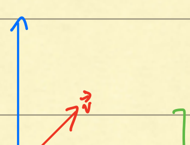
basis vectors (why this system?) → New Basis still works

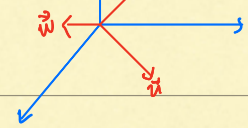
★Vectors depend on basis

linear Combination of vectors

$a\vec{v} + b\vec{w}$ → plane (unless $\vec{v}$ & $\vec{w}$ line up or are zero)
(span)

3D-Space





} linearly independent

Span → 3D

linearly dependent : when a vector adds no span to the linear combination, and when taking it away does not decrease the span.

$\vec{w}$ & $\vec{v}$ are linearly dependent

linear transformation

- no curves
- no change of origin
- keep grid parralled and equaly spaced

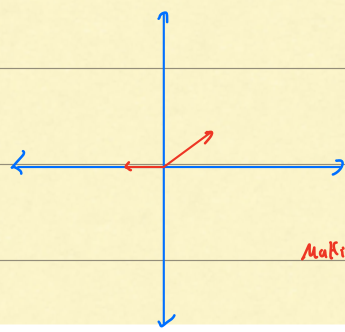
$\vec{v} = -1\vec{i} + 2\vec{j}$

↓

$$\begin{array}{ccccc} & & x & & y \\ \text{(transformed) } \vec{v} & = & -1\vec{i}\text{ (transformed)} & + & 2\vec{j}\text{ (transformed)} \\ \begin{bmatrix} 5 \\ 2 \end{bmatrix} & & \begin{bmatrix} 1 \\ -2 \end{bmatrix} & & \begin{bmatrix} 3 \\ 0 \end{bmatrix} \end{array}$$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1x + 3y \\ -2x + 0y \end{bmatrix}$ 2-D linear transformation described by <u>4 numbers</u>

Matrix Multiplication as Composition



Multiplication

Associativity : $(AB)C = A(BC)$

Sequence 2: $A(BC)\vec{v}$

1. **Combine B and C:**
   - Imagine combining the transformations of B and C into one.
   - This combined transformation, BC, is like applying both transformations simultaneously.
2. **Apply A Last:**
   - Transform the vector $\vec{v}$ using BC, resulting in $BC\vec{v}$.
   - Now, apply the transformation A to $BC\vec{v}$.

Key Insight

In both sequences, the same series of transformations are applied to the vector $\vec{v}$, just grouped differently. Whether we combine A and B first or B and C first, the final transformation applied to $\vec{v}$ is identical.

linear transformation for 3-D

$\vec{i}, \vec{j}, \vec{k}$ 3x3 matrix

$$\begin{bmatrix} & 3\times 3 & \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3\times 1 \end{bmatrix}$$

Determinant

★ the factor of a given region changes during linear transformation

- negative determinant means flipping orientation of space.

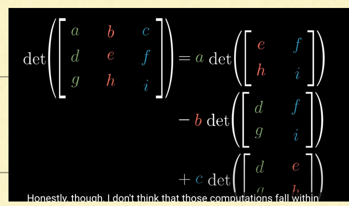
2 D   3 D

area scale   volume scale

"parallelepiped"

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$ad - bc$



$$Det(AB) = Det(A) \cdot Det(B)$$

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Key Insight

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Linear System of equations

$$2x + 5y + 3z = -3$$

$$4x + 0y + 8z = 0$$

$$1x + 3y + 0z = 2$$

$$A\vec{x} = \vec{B}$$

if $\det(A) = 0$

A has no inverse

$$A^{-1} \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\vec{x} = A\vec{b}$$

Rank: number of dimensions in the output of transformation

(full rank) → $\underbrace{\text{3-D : 3} \quad \text{2-D : 2} \quad \text{line : 1}}_{\text{Column Space of A}}$

fullrank only has one input that gives $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
if not, there will be many inputs that give $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
and thats not full rank Column space.

NullSpace "line of vectors squish onto orignin"

Space of all vectors
that land on origin

Non-Square Matrices

$$\begin{bmatrix} 2 \\ 7 \end{bmatrix} \xrightarrow{T(v)} \begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}$$

2d-input 3d-output

mapping a 2-D plane in a 3-D space

2 col → 2 basis vectors

$\vec{i}\ \vec{j}$ (no $\vec{k}$)

3 row → 3-D space

$$\begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 9 \end{bmatrix}$$

(from 2D → 3D)

3 1
4 1
5 9
$L(\hat{i})$ $L(\hat{j})$

3 col → 3 basis vectors

$$\text{2 row}\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}$$

↓

2-D space (from 3D → 2D)

3 1 4
1 5 9
$L(\hat{i})$ $L(\hat{j})$ $L(\hat{k})$

2-D → 1-D

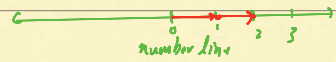
2 col → 2 basis

$$\text{1 row}\begin{bmatrix} 1 & 2 \end{bmatrix}$$

$L(\hat{i})$ $L(\hat{j})$

↓

1-D



Dot Product

(take vector → scalar)

→ a transformation like from 2D → 1D

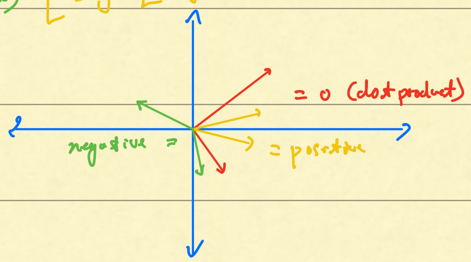
(Duality)

f_d (dot product)

$f_d(\vec{v}, \vec{u}) = A$ (distance)

$f_v(A) = \vec{v} \cdot \vec{u}$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 1 \cdot 2 + 2 \cdot 4 = 10$$



Wether vectors point in the same direction

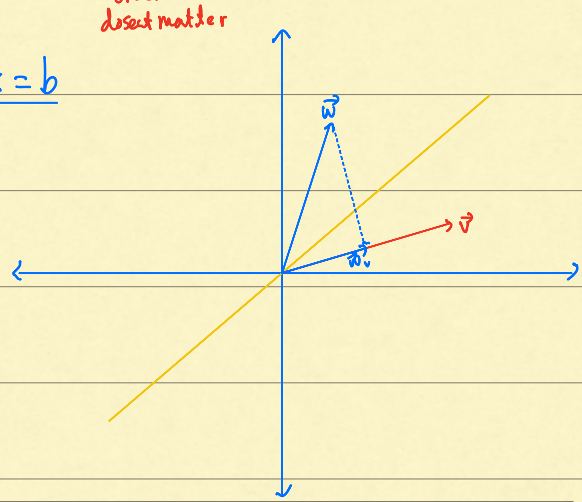
$\vec{v} \cdot \vec{w} = \underbrace{(\text{length of } \vec{v}) \cdot (\text{length of projected } \vec{w})}$

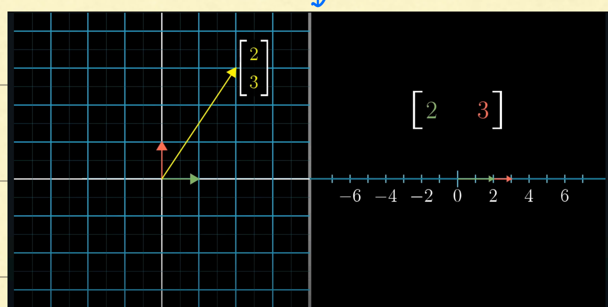
order doesn't matter

Dot product

$u \cdot v = \|u\| \|v\| \cdot \cos(\theta)$ ☆

$\underline{A \cdot x = b}$

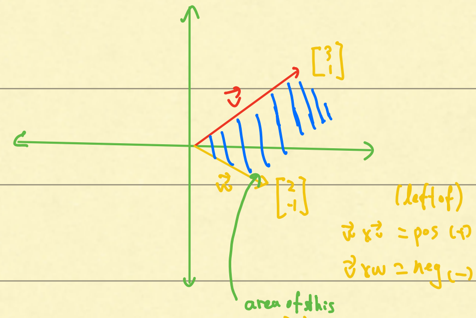




the dual of a 2D → 1D transformation is
a certain vector in the 2D space

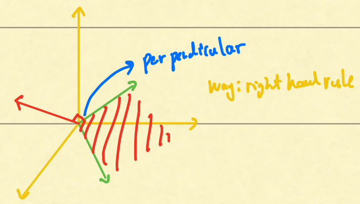
Useful to see if vectors are in
same, different, or perpendicular
direction.

Cross Product (standard)



$\det\left(\begin{bmatrix}3 & 2\\1 & -1\end{bmatrix}\right)$ Scaled up $3\vec{v}\times\vec{w} = 3(\vec{v}\times\vec{w})$

$\vec{v}\times\vec{w} = \underbrace{\vec{p}}_{\text{vector}}$ (area of parallelogram)



Duality of Cross product (3D cross-product)

$$\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\times\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} = \det\left(\begin{bmatrix}i & v_1 & w_1\\j & v_2 & w_2\\k & v_3 & w_3\end{bmatrix}\right)$$

if a linear transformation to the number line
↓
match it to the dual vector of transformation

Cramer's Rule (systems of unknowns)

Ex1 (2x2): $2x + 6y = -4$
$x + 8y = 2$

$D = \begin{vmatrix}2 & 6\\1 & 8\end{vmatrix}$ $D = 10$

replace xth column with answer

$D_x = \begin{vmatrix}-4 & 6\\2 & 8\end{vmatrix}$ $D_x = -20$

$D_y = \begin{vmatrix}2 & -4\\1 & 2\end{vmatrix}$ $D_y = 8$

$x = \frac{D_x}{D}$ $y = \frac{D_y}{D}$

$x = -2$ $y = \frac{1}{2}$

Ex (3x3) 2:

$3x + 3y + 5z = 1$
$3x + 5y + 9z = 0$
$5x + 9y + 17z = 0$

$D = \begin{vmatrix} 3 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{vmatrix} \begin{matrix} 3 & 3 \\ 3 & 5 \\ 5 & 9 \end{matrix}$

$-125 - 243 - 153 + 225 + 135 + 135$

$D = 4$

$D_x = \begin{vmatrix} 1 & 3 & 5 \\ 0 & 5 & 9 \\ 0 & 9 & 17 \end{vmatrix} \begin{matrix} 1 & 3 \\ 0 & 5 \\ 0 & 9 \end{matrix}$

$-0 - 81 - 0 + 85 \; 0 \; 0$

$D_x = 4$

$x = \frac{D_x}{D}$ $x = \frac{4}{4}$, $x = 1$

$D_y = \begin{vmatrix} 3 & 1 & 5 \\ 3 & 0 & 9 \\ 5 & 0 & 17 \end{vmatrix} \begin{matrix} 3 & 1 \\ 3 & 0 \\ 5 & 0 \end{matrix}$

$0 \; 0 - 51 \; 0 + 45 \; 0$

$D_y = -6$ $y = \frac{-3}{2}$

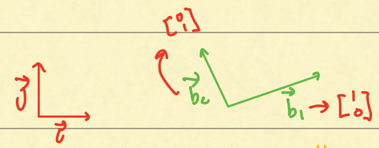
$D_z = \begin{vmatrix} 3 & 3 & 1 \\ 3 & 5 & 0 \\ 5 & 9 & 0 \end{vmatrix} \begin{matrix} 3 & 3 \\ 3 & 5 \\ 5 & 9 \end{matrix}$

$-25 \; 0 \; 0 \; 0 \; 0 + 27$

$D_z = 2$

$z = \frac{1}{2}$

Change of Basis



"different language"

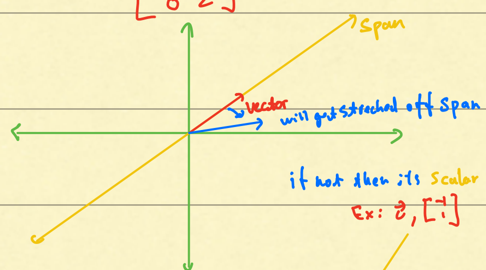
transformation of basis

$$\underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}}_{\text{back to } \vec{b}_1, \vec{b}_2} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{90^\circ \text{ rotation in } \vec{i}\,\vec{j}} \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}}_{\text{transform to } \vec{i}\,\vec{j} \text{ basis}} = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & 1/3 \end{bmatrix}$$

$$\boxed{A^{-1} \times \underbrace{M \times A}_{\text{"empathy" / "shift of perspective"}}} \star$$

Eigenvectors and Eigenvalues

$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \rightarrow$ transformation



if not then its scalar transformation

Ex: $\vec{i}$, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Eigen vectors & Eigenvalues → the value it gets stretched or squished
(negative means flipped)

Rotations: 3D space → Eigenvector is the axis of Rotation
↳ Eigenvalue 1 → no squish or strech

scalar

$$A\vec{v} = \lambda\vec{v}$$

matrix

Same as scaling
(eigen vector)

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$