ECE108 Assignment 1

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Feb 22, 2017

1 Set Operation

a) $R \subseteq S \iff R \subseteq ((S - T) \cup (R \cap T))$
This statement is false because the implication is only unidirectional.

Proving $R \subseteq S \rightarrow R \subseteq ((S - T) \cup (R \cap T))$
$R \subseteq ((S - T) \cup (R \cap T))$ can be simplified to using distributivity

$R \subseteq (((S - T) \cup R) \cap ((S - T) \cup T))$ where
$(S - T) \cup R$ gives you a set X such that $R \subseteq X$
$(S - T) \cup T$ gives you S ...

So... we get $R \subseteq (X \cap S)$ since $R \subseteq X$ and from our assumption $R \subseteq S$
The intersection gives at least R as an answer
Therefore $R \subseteq R$ is true

The opposite way cannot be proved because R does'nt have to be $\subseteq$ S $R \subseteq T$ will suffice the counter-example

Counter-example $S = \{4\}$ $R = \{1, 3, 5\}$ $T = R = \{1, 3, 5\}$
$S - T = \{4\}$ $R \cap T = \{1, 3, 5\}$
$R = (S - T) \cup (R \cap T) = \{1, 3, 4, 5\}$ in this case R is definitely not a subset of S

(b) $(A \cap C) \subseteq (B \cap C) \rightarrow A \subseteq B$ proof: let $x \in A \cap B$ we can conclude that...

by the definition of intersection, $\forall x \in A, x \in C$
we deduce that x must be in both A and C

by the definition of subset, $\forall x \in (A \cap C), x \in (B \cap C)$
we can say $\forall x \in A, x \in (B \cap C)$

which means by the definition of intersection that
$\forall x \in A, x \in B \wedge x \in C$
So... $\forall x \in A, x$ must be $\in B$
which is the definition of $A \subseteq B$

(c) $A \in B \wedge B \in C \rightarrow A \in C$
this statement is obviously false since
let $A = \{3\}$ let $B = \{\{3\}, 4\}$ let $C = \{\{\{3\}, 4\}, 5\}$

we can see that $A \in B \wedge B \in C$
but $A \notin C$

(d) $A \in B \wedge B \in C \rightarrow A \subseteq C$
this statement is obviously false since
let $A = \{3\}$ let $B = \{\{3\}, 4\}$ let $C = \{\{\{3\}, 4\}, 5\}$

we can see that $A \in B \wedge B \in C$
but $A \nsubseteq C$

(e) $A \in B \wedge B \subseteq C \rightarrow A \in C$
proof:
if $B \subseteq C$
that means $\forall x \in B, x \in C$
now $A \in B$ means that A is an element of B represented by $\forall x$
replacing $\forall x$ by A
we can then conclude $A \in B$ means $A \in C$

(f) $A \in B \wedge B \subseteq C \rightarrow A \subseteq C$
this is obviously false since we proved that $A \in C$ is true let $A = \{3\}$ let $B = \{\{3\}, 4\}$ let $C = \{\{3\}, 4, 5\}$

we can see that $A \in B \wedge B \subseteq C$
but $A \nsubseteq C$

2 Set operations

Given sets A and B under what condition does $A - B = B - A$
need to prove that $A = B \iff A - B = B - A$

Proof:
starting with $A = B \rightarrow A - B = B - A$
if $A = B$, then $A - B = B - A = \emptyset$

continuing with $A = B \leftarrow A - B = B - A$ Proof:

the definitions for differences are:
$A - B = \{x \mid x \in A \land x \notin B\}$
$B - A = \{x \mid x \in B \land x \notin A\}$
if $A - B = B - A$
then it means that $\exists x \in A \mid x \notin B$
because of the equality,
but the same x must exists in B not in A
we can see that no element
We can then conclude that
$A - B = \emptyset \land B - A = \emptyset$
using the definition of difference we can see that
in order for $A - B = \emptyset$, that means that $A \subseteq B$
in order for $B - A = \emptyset$, that means that $B \subseteq A$

therefore $A \subseteq B \land B \subseteq A$
this is the definition of equality $A = B$.

3 Functions

(a)
(i) if f is injective, we don't know anything about the relationship between co-domain and image we only know about that image $\subseteq$ co-domain
(ii) image = co-domain when it is surjective
(iii) image = co-domain when it is bijective

(b)
(i) if f is injective, it means that the function is invertible and we can map the image ($\neq$ co-domain) back to it's domain
so f^{-1} has image the full codomain of f
image (f^{-1}) = dom (f)
(ii) turns out that if f is not injective, it cannot be inverted
since f^{-1} does not exist,
(iii) image (f^{-1}) = dom (f) when it is bijective

4 Functions

I assume that both X and Y are not empty sets....
(a) there exists an injection $f : X \to Y$

Proof:
if $X \subseteq Y$, this means
$\forall a(a \in X \to a \in Y)$
a "same" function can be applied that maps all the values in X to its same value, but in Y

$f : X \to Y$
$x \mapsto x$

Since all values in X is present in Y,
and every single value in X maps to one value inside of Y
and sets don't have duplicates
we've got an injective "equivalence" function

(b) there exists a surjection $g : Y \to X$
well we know that $X \subseteq Y$
so cardinality$(X) \leq$ cardinality(Y)
I can conclude that my domain will be either equal or bigger than my codomain
Therefore I can guarantee that my function will not be injective if I need my function to be surjective

so a function that

$$x \mapsto \begin{cases} x, & \text{if } x \in (X \cap Y) \\ \text{any Value in Y}, & \text{if } x \in (Y - X) \end{cases} \tag{1}$$

the mapping to any value will make sure that our function definition maps all the domain to satisfy the definition of a function

5 Functions.. Even more

(a) $f : \mathbb{N} \to \mathbb{N}$ where $f : x \mapsto x$

Since the dom = codom, and the function maps the value to itself
we can conclude that:

function is injective because all values of the domain is mapped to a unique value in the codomain

function is surjective because all values of the codomain is being mapped to (range = codomain)
therefore, the function is bijective

(b) $g : \mathbb{N} \to \mathbb{N}$ where $g : x \mapsto x^2$
in this case, the function maps all the values in the domain to the square of itself.
since all values in the domain has a unique square value in the codomain, the function is injective
since range $\neq$ codomain because $3 \in \mathbb{N}$ but 3 is not mapped by any value in the domain, the function is **NOT** surjective
therefore not bijective

(c) $h : \mathbb{Q}^+ \to \mathbb{Q}^+$ where $h : x \mapsto 1/x$
I assume that the function maps x to the inverse of its most simplified version else, it is not even a function because $4 \mapsto \frac{1}{4}$ but also $\mapsto \frac{2}{8}$
right off the bat, we can see that the same number could be represented by 2 different values of the domain i.e $\frac{1}{2}$ $\frac{2}{4}$
since the inverse operation of these 2 fractions all map to 2, the function does not satisfy the injective definition.
similiarly, the surjective definition is not satisfied since $\frac{2}{8}$ will never be mapped to. refer to the assumption
obviously not bijective, so **None of the Above**

(d) possible to compose $f \circ g$ $f \circ h$ $g \circ h$
in order to have a correct composition $J \circ K$ also knowns as $K(J(x))$
We know that codom(J) $\subseteq$ dom(K) since the domain of K can be restricted to = codom(J)
therefore:
cod(f) = $\mathbb{N}$ and dom(g) = $\mathbb{N}$ therefore possible
$g(f(x)) : \mathbb{N} \to \mathbb{N}$ where $x \mapsto x^2$
cod(f) = $\mathbb{N}$ and dom(h) = $\mathbb{Q}^+$ therefore possible
$h(f(x)) : \mathbb{N} \to \mathbb{Q}^+$ where $x \mapsto \frac{1}{x}$
cod(g) = $\mathbb{N}$ and dom(h) = $\mathbb{Q}^+$ therefore possible
$h(g(x)) : \mathbb{N} \to \mathbb{Q}^+$ where $x \mapsto \frac{1}{x^2}$

6 Closure

a strict partial order is asymmetric, transitive
if we take the reflective closure of that then the relation will be asymmetric, transitive and reflective
then we only have to prove that reflective and asymmetric implies antisymmetric
prove:
refl $\wedge$ asym $\to$ antisym

i. Define $T \subseteq A2$ st $XTy \iff (xRyANDxSY)$ show T is refl sym and transitive to prove it

4. Given poset (x, smallerEq) prove or disprove (a) x ¿= y iff y ¡=-1 x

R-1 = (b,a) — (a,b) $\in$ R) (y,x) $\in$ ¡= so it mean (x,y) $\in$ ¡=-1

(b) x ¿= y $\iff$ y ¡='x (2,2) will prove it false;