ECE108 Assignment 1

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1 Set Operation

a) $R \subseteq S \iff R \subseteq ((S-T) \cup (R \cap T))$
This statement is false because the implication is only unidirectional.

Proving $R \subseteq S \rightarrow R \subseteq ((S-T) \cup (R \cap T))$
$R \subseteq ((S-T) \cup (R \cap T))$ can be simplified to using distributivity

$R \subseteq (((S-T) \cup R) \cap ((S-T) \cup T))$ where
$(S-T) \cup R$ gives you a set X such that $R \subseteq X$
$(S-T) \cup T$ gives you S ...

So... we get $R \subseteq (X \cap S)$ since $R \subseteq X$ and from our assumption $R \subseteq S$
The intersection gives at least R as an answer
Therefore $R \subseteq R$ is true

The opposite way cannot be proved because R does'nt have to be $\subseteq$ S $R \subseteq T$ will suffice the counter-example

Counter-example $S = \{4\}$ $R = \{1,3,5\}$ $T = R = \{1,3,5\}$
$S - T = \{4\}$ $R \cap T = \{1,3,5\}$
$R \subseteq (S-T) \cup (R \cap T) = \{1,3,4,5\}$ true in this case R is definitely not a subset of S

(b) $(A \cap C) \subseteq (B \cap C) \rightarrow A \subseteq B$
proof: let $x \in A \cap B$ we can conclude that...

by the definition of intersection, $\forall x \in A, x \in C$
we deduce that x must be in both A and C

by the definition of subset, $\forall x \in (A \cap C), x \in (B \cap C)$
we can say $\forall x \in A, x \in (B \cap C)$

which means by the definition of intersection that
$\forall x \in A, x \in B \wedge x \in C$
So... $\forall x \in A, x$ must be $\in B$
which is the definition of $A \subseteq B$

(c) $A \in B \wedge B \in C \rightarrow A \in C$
this statement is obviously false since
let $A = \{3\}$ let $B = \{\{3\}, 4\}$ let $C = \{\{\{3\}, 4\}, 5\}$

we can see that $A \in B \wedge B \in C$
but $A \notin C$

(d) $A \in B \wedge B \in C \rightarrow A \subseteq C$
this statement is obviously false since
let $A = \{3\}$ let $B = \{\{3\}, 4\}$ let $C = \{\{\{3\}, 4\}, 5\}$

we can see that $A \in B \wedge B \in C$
but $A \nsubseteq C$

(e) $A \in B \wedge B \subseteq C \rightarrow A \in C$
proof:
if $B \subseteq C$
that means $\forall x \in B, x \in C$
now $A \in B$ means that A is an element of B represented by $\forall x$
replacing $\forall x$ by A
we can then conclude $A \in B$ means $A \in C$

(f) $A \in B \wedge B \subseteq C \rightarrow A \subseteq C$
this is obviously false since we proved that $A \in C$ is true let $A = \{3\}$ let $B = \{\{3\}, 4\}$ let $C = \{\{3\}, 4, 5\}$

we can see that $A \in B \wedge B \subseteq C$
but $A \nsubseteq C$

2 Set operations

Given sets A and B under what condition does $A - B = B - A$
need to prove that $A = B \iff A - B = B - A$

Proof:
starting with $A = B \rightarrow A - B = B - A$
if $A = B$, then $A - B = B - A = \emptyset$

continuing with $A = B \leftarrow A - B = B - A$ Proof:

the definitions for differences are:
$A - B = \{x \mid x \in A \wedge x \notin B\}$
$B - A = \{x \mid x \in B \wedge x \notin A\}$
if $A - B = B - A$
then it means that $\exists x \in A \mid x \notin B$
because of the equality,
but the same x must exists in B not in A
we can see that no element
We can then conclude that
$A - B = \emptyset \wedge B - A = \emptyset$
using the definition of difference we can see that
in order for $A - B = \emptyset$, that means that $A \subseteq B$
in order for $B - A = \emptyset$, that means that $B \subseteq A$

therefore $A \subseteq B \wedge B \subseteq A$
this is the definition of equality $A = B$.

3 Functions

(a)
(i) if f is injective, we don't know anything about the relationship between co-domain and image ok i lied...
we only know about that image $\subseteq$ co-domain
(ii) image = co-domain when it is surjective
(iii) image = co-domain when it is bijective

(b)
(i) if f is injective, it means that the function is invertible and we can map the image ($\neq$ co-domain) back to it's domain
so f^{-1} has image the full codomain of f
image (f^{-1}) = dom (f)
(ii) turns out that if f is not injective, it cannot be inverted
since f^{-1} does not exist,
(iii) image (f^{-1}) = dom (f) when it is bijective

4 Functions

I assume that both X and Y are not empty sets....
(a) there exists an injection $f : X \to Y$

Proof:
if $X \subseteq Y$, this means
$\forall a(a \in X \to a \in Y)$
a "same" function can be applied that maps all the values in X to its same value, but in Y

$f : X \to Y$
$x \mapsto x$

Since all values in X is present in Y,
and every single value in X maps to one value inside of Y
and sets don't have duplicates
we've got an injective "equivalence" function

(b) there exists a surjection $g : Y \to X$
this is false if I didn't assume non-empty sets since if X = $\emptyset$, it is no longer a function

well we know that $X \subseteq Y$
so cardinality$(X) \leq$ cardinality(Y)
I can conclude that my domain will be either equal or bigger than my codomain
Therefore I can guarantee that my function will not be injective if my function is surjective

so I can have a function that

$$x \mapsto \begin{cases} x, & \text{if } x \in (X \cap Y) \\ \text{any Value in Y}, & \text{if } x \in (Y - X) \end{cases} \tag{1}$$

the mapping to any value will make sure that our function definition maps all

the domain to satisfy the definition of a function

5 Functions.. Even more

(a) $f : \mathbb{N} \to \mathbb{N}$ where $f : x \mapsto x$

Since the dom = codom, and the function maps the value to itself
we can conclude that:

function is injective because all values of the domain is mapped to a unique value in the codomain
function is surjective because all values of the codomain is being mapped to (range = codomain)
therefore, the function is bijective

(b) $g : \mathbb{N} \to \mathbb{N}$ where $g : x \mapsto x^2$
in this case, the function maps all the values in the domain to the square of itself.
since all values in the domain has a unique square value in the codomain, the function is injective
since range $\neq$ codomain because $3 \in \mathbb{N}$ but 3 is not mapped by any value in the domain, the function is **NOT** surjective
therefore not bijective

(c) $h : \mathbb{Q}^+ \to \mathbb{Q}^+$ where $h : x \mapsto 1/x$
I assume that the function maps x to the inverse of its unsimplified version
so the function h can be described as doing this:
$\forall_{xy} \in \mathbb{N}\frac{x}{y} \mapsto \frac{y}{x}$

we can then say that every value in the codomain has its correponding value in the domain
since we can map all $\frac{x}{y}$ to an unique $\frac{y}{x}$, it is injective

since we can represent all values of $\mathbb{Q}^+$ with y and $x \in \mathbb{N}$ and that there is no value that cannot be mapped by the domain, it is surjective

therefore h is bijective too

(d) possible to compose $f \circ g$ $f \circ h$ $g \circ h$
in order to have a correct composition $J \circ K$ also knowns as $K(J(x))$
We know that codom(J) $\subseteq$ dom(K) since the domain of K can be restricted to = codom(J)
therefore:
cod(f) = $\mathbb{N}$ and dom(g) = $\mathbb{N}$ therefore possible
$g(f(x)) : \mathbb{N} \to \mathbb{N}$ where $x \mapsto x^2$
cod(f) = $\mathbb{N}$ and dom(h) = $\mathbb{Q}^+$ therefore possible
$h(f(x)) : \mathbb{N} \to \mathbb{Q}^+$ where $x \mapsto \frac{1}{x}$
cod(g) = $\mathbb{N}$ and dom(h) = $\mathbb{Q}^+$ therefore possible
$h(g(x)) : \mathbb{N} \to \mathbb{Q}^+$ where $x \mapsto \frac{1}{x^2}$

6 Closure

a strict partial order is asymmetric and transitive
a partial order is reflective, antisymmetric and transitive

if we take the reflective closure of the relation $<$ (the new set is refered as R from now on)
then we have added all the xRx.
More precisely:
$\forall x < y, \exists (xRy \wedge xRx \wedge yRy)$
where $x \neq y$ since it is part of a strict poset
prove that transitivity is kept with the newly added elements:

as we see, for any arbitrary $xRy \in R$, we now have xRx and yRy:
using the definition of transitivity,
$xRy \wedge yRy \rightarrow xRy$
$xRx \wedge xRy \rightarrow xRy$
then we see that xRy is required to be in the set for it to be transitive
and indeed xRy is in the set from our assumption.

prove that the newly created set is antisymmetric:
the new set R now satisfies the new condition
$\forall x \forall y ((xRy \Rightarrow \neg yRx) \vee (x = y))$
this means that for an arbitrary xRy, there cannot be yRx unless $x = y$
this is the less formal definition of antisymmetric relations

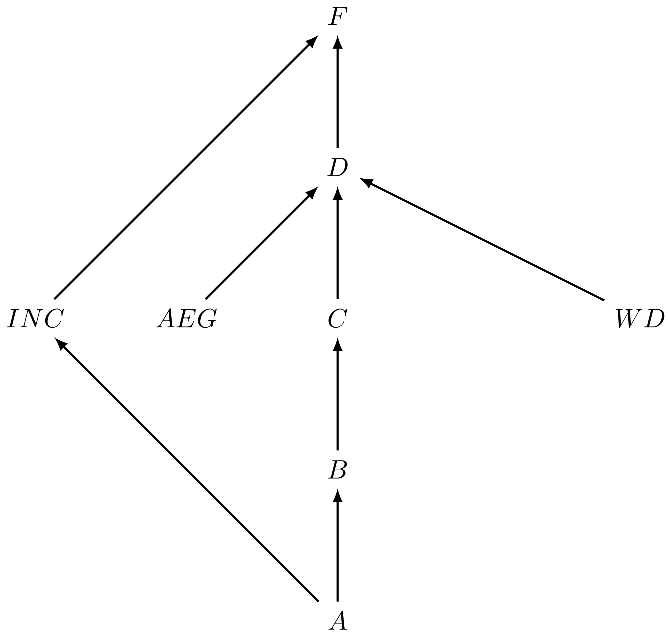
The proof for reflective is trivial since we had to take a reflective closure of the $<$ set.

Therefore, the new set R is a poset since it satisfies the 3 conditions

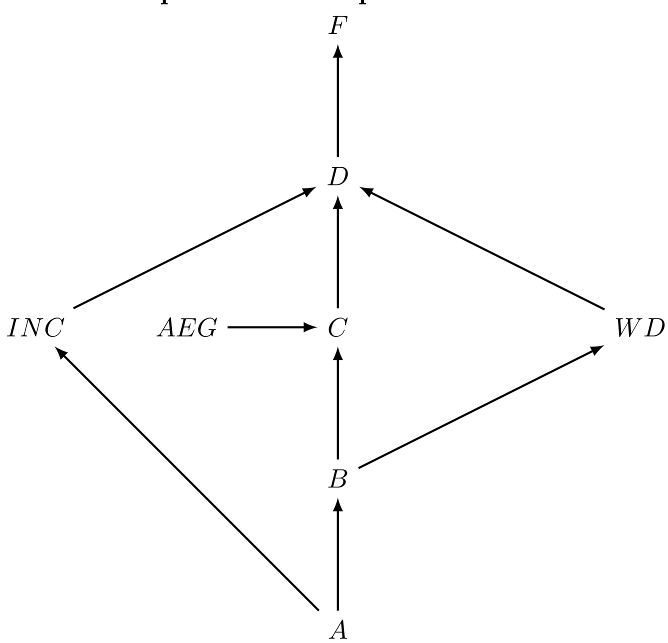
7 Hass Diagram

$INC \geq F$actually $INC > F$
$INC \geq D$
$A > INC$
$WD > F$
$WD > D$
$B \geq WD$
$AEG > F$
$AEG > D$
$AEG \geq C$

better relationship



better or equal relationship



8 Equivalent Relationship

need to show that T is reflective, symmetric and transitive

we first need to determine the set relationship between T and RS
Define $T \subseteq A^2$ such that $xTy \iff (xRy \wedge xSy)$
we see that for any arbitrary xTy , there exists xRy and xSy
we can conclude that for any element in T, the same element exists in S and R
mathematically, this is written as $T = R \subseteq T \wedge \subseteq S$
which implies $\subseteq (R \cap U)$

if $T = \emptyset$, then T would be an equivalent relationship since all the assumptions become false and implications become true
so for our proof, we are going to assume that there exists at least 1 element inside the relation T

proof for reflective:

if T is not reflective, then $\exists x(\neg xTx)$
so it means by double implication (iff) that $\exists x((\neg xRx) \vee (\neg xSx))$
but R and S are all equivalent relationships
Contraction occurs since both R and S are reflective and
$\forall x(xRx \wedge xSx)$
we conclude that T has to be reflective

Same logic follows for the 2 other conditions:

proof for symmetric:

if T is not symmetric, then $\exists x \exists y(xTy \wedge \neg yTx)$
so it means by double implication (iff) that $\exists x \exists y(xRy \wedge \neg yRx) \vee (xSy \wedge \neg ySx)$
but R and S are all equivalent relationships
Contraction occurs since both R and S are symmetric
we conclude that T has to be symmetric

proof for transitive:

if T is not transitive, then $\exists x \exists y \exists z(xTy \wedge yTz \wedge \neg xTz)$
so it means by double implication (iff) that the same xSz or xRz doesn't exist
$\exists x \exists y \exists z((xRy \wedge yRz \wedge \neg xRz) \vee (xSy \wedge ySz \wedge \neg xSz))$
but R and S are all equivalent relationships
Contraction occurs since both R and S are transitive, so both xRz and xSz exists
we conclude that T has to be transitive

We finally conclude that T is an equivalent relationship... $\square$

9 Posets

(a) $x \geq y \iff y \leq^{-1} x$
$x \geq y$ can be rewritten as $y \leq x$
since $y \leq x \neq y \leq^{-1} x$
for when $y \neq x$
false

(b) $x \geq y \iff y \leq^{'} x$
$x \geq y$ can be rewritten as $y \leq x$
since $y \leq x \neq y \leq^{'} x$
because $y \leq^{'} x = y > x$ by definition of complement
false

(c) $x < y \iff y \leq^{'} x$
prove that $x < y \Rightarrow y \leq^{'} x$
$x < y$ can be rewritten as $y > x$
and the complement of $y > x$ is $y \leq^{'} x$
therefore $y > x = y \leq^{'} x$

since I have proven they are equal, $\iff$ is proven
(d) $x > y \iff y(\leq^{-1})^{'} x$
$x > y$ can be rewritten as $y < x$
$y < x$'s inverse is $x <^{-1} y$
$x <^{-1} y$ can now be rewritten as $y >^{-1} x$
taking the complement might not be obvious, so let's split X into $\leq^{-1}, >^{-1}$ sets
taking the complement we get the set we don't have
$y(\leq^{-1})^{'} x$

we conclude that $x > y = y(\leq^{-1})^{'} x$
therefore the bidirection is proven since they are equal

(e) $x > y \iff y(\leq^{'})^{-1} x$
I doubt this is true, since **(d)** is true... turns out it is true
taking the complement we get $x > y = x \leq^{'} y$
if we then take the inverse we get $y(\leq^{'})^{-1} x$
indeed we just have to swap x and y to get the inverse

we conclude that $x > y = y(\leq^{'})^{-1} x$
therefore the bidirection is proven since they are equal

10 More Posets

$X \subseteq N$
$\forall x, y \in \mathbb{N}\, xRy \iff \exists_{z \in X}\, x + z = y$
(a) $0 \in X$
let us assume that set X has at least one element a
therefore, by the aRa must exist since R is a poset and has to be symmetric that implies that there must exist in X to replace z that will make the equation
$x + z = y$
with $x = a$ and $y = a$
$a + z = a$
we conclude that z must be $= 0$ and thus $0 \in X$
(b) I modify the question $a = x$ and $b = y$ so it becomes

$$\forall_{ab}(a \in X \wedge b \in X) \Rightarrow a + b \in X$$

we know from part **(a)** and assumption
$0, a, b \in X$
where a and b are just arbitrary values $\in X$
we then know that because R is a poset (symmetrical)
$0R0, aRa, bRb$ exists in the poset

we are trying to prove that $a + b \in X$
so when must $a + b$ be equal to z
in the equation $\exists_{z \in X} x + z = y$
let $x = y = (a + b)$
we know that $(x, y) \in \mathbb{N}$
so we need to show that $(a + b) \in \mathbb{N}$
since $(a \wedge b) \in X \wedge X \subseteq \mathbb{N}$
we can deduce that $(a \wedge b) \in \mathbb{N}$ by applying what we found in **question 1**
therefore since both a and b are $\in \mathbb{N}$
because addition is closed under $\mathbb{N}$
we conclude that the substituion is possible and $a + b \in \mathbb{N}$

therefore by setting $z = 0$ since $0 \in X$
we get $(a + b) + 0 = (a + b)$ which is true
this then implies by our assumption ($\iff$ statement) that $(a + b)R(a + b)$ exists, which implies that $a + b$ is part of the base set since the poset is reflective
(there must be $xRx \forall x \in X$)

11 posets until posets

(a) Proof that minimum element is unique
PBC let's assume a and b are the minimum elements of the subset Y

then by the $\iff$ statement,
$\forall y \in Y a \leq y$ (eq1)
$\forall y \in Y b \leq y$
and both $a \wedge b \in Y$
there is a contradiction here since we have $\forall y \in Y$
y can take the value of a and b
so it must be true that by manipulating (eq1) that
$b \in Y a \leq b$
which is saying that a is $\leq$ b but b cannot be smaller than any value because b is a minimum

therefore a minimum element is unique $\square$

(b)
Proof that minimum element $\Rightarrow$ minimal element

let a be the minimum element
then we know that $a \in Y \wedge \forall_{b \in Y}\, a \leq b$
PBC minimum element $\wedge \neg$ minimal element
so if a is not a minimal element, then $\exists_{z \in Y}\, z < a$
in plain english, there exists a value $z \in Y$ such that z is $<$ than a
but will know from the definition of minimum element that all values of set Y is $\geq a$
therefore contradiction occurs

minimum element $\Rightarrow$ minimal element $\square$

(c)
this is false,
consider the poset $R = \{(a, b), (c, b)(b, d)\}^{tran\ refl}$

$$\begin{tikzcd}
& d & \\
& b \arrow[u] & \\
a \arrow[ur] & & c \arrow[ul]
\end{tikzcd}$$

then let $Y = R$
the minimal element is a and c
the minimum element doesn't exist since a is not related to c
$\forall_{y \in Y}\, x \leq y$ is not satisfied

we see that minimal element doesn't have to be the minimum element

(d)
since totally ordered means connected partial order
$\forall_x \forall_y (xRy \vee yRx \vee x = y)$

We claim that there exists a minimum no matter which subset of the base set X we take.
This means every single element inside the base set will become at least once a minimum element
Lets start with a subset $Y = X$ with minimum value a then by the definition of minimum, a interacts with the entire set by being $\leq Y$ a is connected all the values in X

we then remove a from Y to create a set $Y^{'} \subseteq X$
let's say b is now the minimum, so b is connected to every element in X except for the element a , but a is already connected to b previously

we then remove b from $Y^{'}$ to create a new subset $Y^{''}$
we don't have to worry about a and b since both of them are connected

now let's say c is the new minimum, then c is not connected to every single elements in X including a and b done previously.
If we repeat this recursive process, we will find that all the elements are connected since every time we remove a minimum from the subset, the new minimum element is connected to X.

This is the proof that it is a total order

12 Functions, Relations and Cardinality

(a)
to be a function every single value inside of A needs to map to so some value of the cod(A) k let $A = \{a, b, c\}$ aa-ba-ca is an obvious one if a always maps to a, aa-ba-ca x3 aa-ba-cb aa-ba-cc
aa-bb-ca aa-bb-cb aa-bb-cc
aa-bc-ca aa-bc-cb aa-bc-cc
ab-ba-ca x9
ac-ba-ca x9
seems like the answer is N^N

(b)
Assuming A is not infinite,
in order to get a surjective mapping, all values in the domain must map to something different since Dom = Cod indeed $A = A$
this means that in order to get a surjective mapping, the function needs to be injective
by being both injective and surjective, the function is bijective
possible bijective functions for 3 elements
aa-bb-cc aa-bc-cb ab-ba-cc ab-bc-ca ac-ba-cb ac-bb-ca
by going through them one by one,

I conclude that the amount of bijective relationship is $N!$
therefore the amount of injective and surjective is also $N!$

(c)
well let's brute force through all the possible relations:
if 0 element $\to \{\}\binom{0}{0}$
if 1 element $\to \{\}, \{a, a\}\binom{1}{0to1}$ the size is 2
if 2 elements we get $\emptyset, aa, ab, ba, bb$, so $\binom{4}{0to4}$ the size is 16
if 3 elements we get $\emptyset, aa, ab, ac, ba, bb, bc, ca, cb, cc$ so $\binom{9}{0to9}$ the size is 512
so we can conclude in general we get

$$\sum_{k=0}^{N^2} \binom{N^2}{k}$$

is there a way to simplify?
ya...

$$2^{N^2}$$

(d)
to be both antisym and sym it means that the function needs to satisfy
$\forall x \forall y((xRy \Rightarrow yRx) \wedge ((xRy \wedge yRx) \Rightarrow x = y))$
this basically means that only xRx relationship can exist inside of the set;
attempt to prove:
since if xRy exists, then yRx exists by symmetric definition
but by the antisymmetric definition, $xRy \wedge yRx \Rightarrow x = y$
therefore $xRy = yRx$ therefore
$R = \{(x, x)\}$

by going through the similar process we realize its the sum of N choose K if 0 element $\to \{\}\binom{0}{0}$
if 1 element $\to \{\}, \{a, a\}\binom{1}{0to1}$ the size is 2
if 2 elements we get $\emptyset, aa, bb$, so $\binom{2}{0to2}$

$$\sum_{k=0}^{N} \binom{N}{k}$$

simplified to 2^N

(e)
now $R = \{\forall x \in A(xRx)\}$
if 0 element $\to \{\}$
if 1 element $\to \{\}, \{(a, a)\}$
if 2 elements we get $\emptyset, \{(a, a), (b, b)\}$
we see that since it's for all x, the amount of set is 2 for $N > 0$ and 1 for $N = 0$

(f)

it is now reflective, symmetric, transitive
if 0 element $\rightarrow \{\}$
if 1 element $\rightarrow \{\}, \{(a, a)\}$
if 2 elements we get $\emptyset, \{(a, a), (b, b)\}$ $aa - bb - ab - ba$
if 3 elements we get $\emptyset, aa - bb - cc$
also we get aa-bb-cc-ab-ba seems like 3 choose 2 aa-bb-cc-ac-ca aa-bb-cc-cb-bc
aa-bb-cc-ab-ba-ac-ca-cb-bc
if 4 elements we get $\emptyset, aa - bb - cc - dd$
aa-bb-cc-dd aa-bb-cc-dd-ab-ba with ab
aa-bb-cc-dd-ac-ca with ac
aa-bb-cc-dd-ad-da with ad
aa-bb-cc-dd-bc-cb with cb
aa-bb-cc-dd-bd-db with bd
aa-bb-cc-dd-cd-dc with cd
$\binom{4}{2}$
aa-bb-cc-dd-ab-ba-bc-cb
aa-bb-cc-dd-ac-ca-bd-db
aa-bb-cc-dd-ad-da-bc-cb
for num = 2 this step gives you 0
for num = 3 this step gives you 1
for num = 4 this step gives you 3
for num = 5 this step gives you 8
I ve come to a conclusion that it is $(N - 3) * (N - 1)$
aa-bb-cc-dd-ab-ba-bc-cb-ac-ca
aa-bb-cc-dd-bc-cb-cd-dc-bd-db with acd

it seems to me that it is 2 and $\binom{N}{1}$ then it looks like it is $\binom{N}{2}$ where the the 2 elements from the base set must not be adjacent to each other because if we take ab ba and