ECE108 Assignment 1

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1 Set Operation

a) $R \subseteq S \iff R \subseteq ((S-T) \cup (R \cap T))$
This statement is false because the implication is only unidirectional.

Proving $R \subseteq S \rightarrow R \subseteq ((S-T) \cup (R \cap T))$
$R \subseteq ((S-T) \cup (R \cap T))$ can be simplified to using distributivity

$R \subseteq (((S-T) \cup R) \cap ((S-T) \cup T))$ where
$(S-T) \cup R$ gives you a set X such that $R \subseteq X$
$(S-T) \cup T$ gives you S ...

So... we get $R \subseteq (X \cap S)$ since $R \subseteq X$ and from our assumption $R \subseteq S$
The intersection gives at least R as an answer
Therefore $R \subseteq R$ is true

The opposite way cannot be proved because R does'nt have to be $\subseteq$ S $R \subseteq T$ will suffice the counter-example

Counter-example $S = \{4\}$ $R = \{1,3,5\}$ $T = R = \{1,3,5\}$
$S - T = \{4\}$ $R \cap T = \{1,3,5\}$
$R = (S-T) \cup (R \cap T) = \{1,3,4,5\}$ in this case R is definitely not a subset of S

(b) $(A \cap C) \subseteq (B \cap C) \rightarrow A \subseteq B$ proof: let $x \in A \cap B$ we can conclude that...

by the definition of intersection, $\forall x \in A, x \in C$
we deduce that x must be in both A and C

by the definition of subset, $\forall x \in (A \cap C), x \in (B \cap C)$
we can say $\forall x \in A, x \in (B \cap C)$

which means by the definition of intersection that
$\forall x \in A, x \in B \wedge x \in C$
So... $\forall x \in A, x$ must be $\in B$
which is the definition of $A \subseteq B$

(c) $A \in B \wedge B \in C \rightarrow A \in C$
this statement is obviously false since
let $A = \{3\}$ let $B = \{\{3\}, 4\}$ let $C = \{\{\{3\}, 4\}, 5\}$

we can see that $A \in B \wedge B \in C$
but $A \notin C$

(d) $A \in B \wedge B \in C \rightarrow A \subseteq C$
this statement is obviously false since
let $A = \{3\}$ let $B = \{\{3\}, 4\}$ let $C = \{\{\{3\}, 4\}, 5\}$

we can see that $A \in B \wedge B \in C$
but $A \not\subseteq C$

(e) $A \in B \wedge B \subseteq C \rightarrow A \in C$
proof:
if $B \subseteq C$
that means $\forall x \in B, x \in C$
now $A \in B$ means that A is an element of B represented by $\forall x$
replacing $\forall x$ by A
we can then conclude $A \in B$ means $A \in C$

(f) $A \in B \wedge B \subseteq C \rightarrow A \subseteq C$
this is obviously false since we proved that $A \in C$ is true let $A = \{3\}$ let $B = \{\{3\}, 4\}$ let $C = \{\{3\}, 4, 5\}$

we can see that $A \in B \wedge B \subseteq C$
but $A \not\subseteq C$

2 Set operations

Given sets A and B under what condition does $A - B = B - A$
need to prove that $A = B \iff A - B = B - A$

Proof:
starting with $A = B \rightarrow A - B = B - A$
if $A = B$, then $A - B = B - A = \emptyset$

continuing with $A = B \leftarrow A - B = B - A$ Proof:

the definitions for differences are:
$A - B = \{x \mid x \in A \wedge x \notin B\}$
$B - A = \{x \mid x \in B \wedge x \notin A\}$
if $A - B = B - A$
then it means that $\exists x \in A \mid x \notin B$
because of the equality,
but the same x must exists in B not in A
we can see that no element
We can then conclude that
$A - B = \emptyset \wedge B - A = \emptyset$
using the definition of difference we can see that
in order for $A - B = \emptyset$, that means that $A \subseteq B$
in order for $B - A = \emptyset$, that means that $B \subseteq A$

therefore $A \subseteq B \wedge B \subseteq A$
this is the definition of equality $A = B$.

3 Functions

(a)
(i) if f is injective, we don't know anything about the relationship between co-domain and image we only know about that image $\subseteq$ co-domain
(ii) image = co-domain when it is surjective
(iii) image = co-domain when it is bijective

(b)
(i) if f is injective, it means that the function is invertible and we can map the image ($\neq$ co-domain) back to it's domain
so f^{-1} has image the full codomain of f
image (f^{-1}) = dom (f)
(ii) turns out that if f is not injective, it cannot be inverted
since f^{-1} does not exist,
(iii) image (f^{-1}) = dom (f) when it is bijective

4 Functions

I assume that both X and Y are not empty sets....
(a) there exists an injection $f : X \to Y$

Proof:
if $X \subseteq Y$, this means
$\forall a(a \in X \to a \in Y)$
a "same" function can be applied that maps all the values in X to its same value, but in Y

$f : X \to Y$
$x \mapsto x$

Since all values in X is present in Y,
and every single value in X maps to one value inside of Y
and sets don't have duplicates
we've got an injective "equivalence" function

(b) there exists a surjection $g : Y \to X$
well we know that $X \subseteq Y$
so cardinality$(X) \leq$ cardinality(Y)
I can conclude that my domain will be either equal or bigger than my codomain
Therefore I can guarantee that my function will not be injective if I need my function to be surjective

so a function that

$$x \mapsto \begin{cases} x, & \text{if } x \in (X \cap Y) \\ \text{any Value in Y}, & \text{if } x \in (Y - X) \end{cases} \tag{1}$$

the mapping to any value will make sure that our function definition maps all the domain to satisfy the definition of a function

5 Functions.. Even more

(a) $f : \mathbb{N} \to \mathbb{N}$ where $f : x \mapsto x$

Since the dom = codom, and the function maps the value to itself
we can conclude that:

function is injective because all values of the domain is mapped to a unique value in the codomain

function is surjective because all values of the codomain is being mapped to (range = codomain)
therefore, the function is bijective

(b) $g : \mathbb{N} \to \mathbb{N}$ where $g : x \mapsto x^2$
in this case, the function maps all the values in the domain to the square of itself.
since all values in the domain has a unique square value in the codomain, the function is injective
since range $\neq$ codomain because $3 \in \mathbb{N}$ but 3 is not mapped by any value in the domain, the function is **NOT** surjective
therefore not bijective

(c) $h : \mathbb{Q}^+ \to \mathbb{Q}^+$ where $h : x \mapsto 1/x$
I assume that the function maps x to the inverse of its most simplified version else, it is not even a function because $4 \mapsto \frac{1}{4}$ but also $\mapsto \frac{2}{8}$
right off the bat, we can see that the same number could be represented by 2 different values of the domain i.e $\frac{1}{2}$ $\frac{2}{4}$
since the inverse operation of these 2 fractions all map to 2, the function does not satisfy the injective definition.
similiarly, the surjective definition is not satisfied since $\frac{2}{8}$ will never be mapped to. refer to the assumption
obviously not bijective, so **None of the Above**

(d) possible to compose $f \circ g$ $f \circ h$ $g \circ h$
in order to have a correct composition $J \circ K$ also knowns as $K(J(x))$
We know that codom(J) $\subseteq$ dom(K) since the domain of K can be restricted to = codom(J)
therefore:
cod(f) = $\mathbb{N}$ and dom(g) = $\mathbb{N}$ therefore possible
$g(f(x)) : \mathbb{N} \to \mathbb{N}$ where $x \mapsto x^2$
cod(f) = $\mathbb{N}$ and dom(h) = $\mathbb{Q}^+$ therefore possible
$h(f(x)) : \mathbb{N} \to \mathbb{Q}^+$ where $x \mapsto \frac{1}{x}$
cod(g) = $\mathbb{N}$ and dom(h) = $\mathbb{Q}^+$ therefore possible
$h(g(x)) : \mathbb{N} \to \mathbb{Q}^+$ where $x \mapsto \frac{1}{x^2}$

6 Closure

a strict partial order is asymmetric and transitive
a partial order is reflective, antisymmetric and transitive

if we take the reflective closure of the relation $<$ (the new set is refered as R from now on)
then we have added all the $x < x$.
More precisely:

$\forall x < y, \exists(xRy \wedge xRx \wedge yRy)$
where $x \neq y$ since it is part of a strict poset
prove that transitivity is kept with the newly added elements:

as we see, for any arbitrary $xRy \in R$, we now have xRx and yRy:
using the definition of transitivity,
$xRy \wedge yRy \rightarrow xRy$
$xRx \wedge xRy \rightarrow xRy$
then we see that xRy is required to be in the set for it to be transitive
and indeed xRy is in the set from our assumption.

prove that the newly created set is antisymmetric:
the new set R now satisfies the new condition
$\forall x \forall y((xRy \Rightarrow \neg yRx) \vee (x = y))$
this means that for an arbitrary xRy, there cannot be yRx unless $x = y$
this is the less formal definition of antisymmetric relations

The proof for reflective is trivial since we had to take a reflective closure of the $<$ set.

Therefore, the new set R is a poset since it satisfies the 3 conditions

7 Hass Diagram

8 Equivalent Relationship

need to show that T is reflective, symmetric and transitive

we first need to determine the set relationship between T and RS
Define $T \subseteq A^2$ such that $xTy \iff (xRy \wedge xSy)$
we see that for any arbitrary xTy , there exists xRy and xSy
we can conclude that for any element in T, the same element exists in S and R
mathematically, this is written as $T = R \subseteq T \wedge \subseteq S$
which implies $\subseteq (R \cap U)$

if $T = \emptyset$, then T would be an equivalent relationship since all the assumptions become false and implications become true
so for our proof, we are going to assume that there exists at least 1 element inside the relation T

proof for reflective:

if T is not reflective, then $\exists x(\neg xTx)$
so it means by double implication (iff) that $\exists x((\neg xRx) \vee (\neg xSx))$

but R and S are all equivalent relationships
Contraction occurs since both R and S are reflective and
$\forall x(xRx \land xSx)$
we conclude that T has to be reflective

Same logic follows for the 2 other conditions:

proof for symmetric:

if T is not symmetric, then $\exists x \exists y(xTy \land \neg yTx)$
so it means by double implication (iff) that $\exists x \exists y(xRy \land \neg yRx) \lor (xSy \land \neg ySx)$
but R and S are all equivalent relationships
Contraction occurs since both R and S are symmetric
we conclude that T has to be symmetric

proof for transitive:

if T is not transitive, then $\exists x \exists y \exists z(xTy \land yTz \land \neg xTz)$
so it means by double implication (iff) that the same xSz or xRz doesn't exist
$\exists x \exists y \exists z((xRy \land yRz \land \neg xRz) \lor (xSy \land ySz \land \neg xSz))$
but R and S are all equivalent relationships
Contraction occurs since both R and S are transitive, so both xRz and xSz exists
we conclude that T has to be transitive

We finally conclude that T is an equivalent relationship... $\square$

9 Posets

(a) $x \geq y \iff y \leq^{-1} x$
$x \geq y$ can be rewritten as $y \leq x$
since $y \leq x \neq y \leq^{-1} x$
false

(b) $x \geq y \iff y \leq^{'} x$
$x \geq y$ can be rewritten as $y \leq x$
since $y \leq x \neq y \leq^{'} x$
false

(c) $x < y \iff y \leq^{'} x$
prove that $x < y \Rightarrow y \leq^{'} x$
$x < y$ can be rewritten as $y > x$
and the complement of $y > x$ is $y \leq^{'} x$
therefore $y > x = y \leq^{'} x$

since I have proven they are equal, $\iff$ is proven

(d) $x > y \iff y(\leq^{-1})' x$
$x > y$ can be rewritten as $y < x$
$y < x$'s inverse is $x <^{-1} y$
$x <^{-1} y$ can now be rewritten as $y >^{-1} x$
taking the complement might not be obvious, so let's split X into $\leq^{-1}, >^{-1}$ sets
taking the complement we get the set we don't have
$y(\leq^{-1})' x$

we conclude that $x > y = y(\leq^{-1})' x$
therefore the bidirection is proven since they are equal

(e) $x > y \iff y(\leq')^{-1} x$
I doubt this is true, since **(d)** is true... turns out it is true
$x > y$ can be rewritten as $y < x$
taking the complement we get $y < x = y \geq' x$
if we then take the inverse we get $x(\geq')^{-1} y$
indeed we just have to swap x and y to get the inverse
this could be rewritten as $y(\leq')^{-1} x$

we conclude that $x > y = y(\leq')^{-1} x$
therefore the bidirection is proven since they are equal

10 More Posets

11 posets until posets

12 Functions, Relations and Cardinality

(a)
to be a function every single value inside of A needs to map to so some value of the cod(A)
let $A = \{a, b, c\}$ aa-ba-ca is an obvious one if a always maps to a, aa-ba-ca x3
aa-ba-cb aa-ba-cc
aa-bb-ca aa-bb-cb aa-bb-cc
aa-bc-ca aa-bc-cb aa-bc-cc
ab-ba-ca x9
ac-ba-ca x9
seems like the answer is N^N

(b)
Assuming A is not infinite,
in order to get a surjective mapping, all values in the domain must map to something different since Dom = Cod indeed $A = A$

this means that in order to get a surjective mapping, the function needs to be injective
by being both injective and surjective, the function is bijective
possible bijective functions for 3 elements
aa-bb-cc aa-bc-cb ab-ba-cc ab-bc-ca ac-ba-cb ac-bb-ca
by going through them one by one,
I conclude that the amount of bijective relationship is $N!$
therefore the amount of injective and surjective is also $N!$

i. Define $T \subseteq A2$ st $XTy \iff (xRyANDxSY)$ show T is refl sym and transitive to prove it

4. Given poset (x, smallerEq) prove or disprove (a) x ¿= y iff y ¡=-1 x

R-1 = (b,a) — (a,b) $\in$ R) (y,x) $\in$ ¡= so it mean (x,y) $\in$ ¡=-1

(b) x ¿= y $\iff$ y ¡=’x (2,2) will prove it false;