ECE108 Assignment3

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Contents

1 Rolling Dice **3**
- a Dicerolling1,2,3,4,5,6 . . . 3
- b DicerollingA,B,C . . . 4
- c DicerollingA,B,C Part2 . . . 4
- d Rolling 1s . . . 5
- e Rolling 1s generalized... . . . 5
- f Communication, actually even more generalization . . . 6

2 Binomial Theorem **7**
- a Base Case . . . 7
- b N+1 Case . . . 7

3 Graph Questions **10**
- a a graph with 5 vertices and 3 edges . . . 10
- b a graph with 5 odd-degree vertices with 3 edges . . . 10
- c a connected graph with 5 vertices and 3 edges? . . . 10
- d a directed graph with 5 vertices and 3 edges? . . . 11
- e a disconnected graph (a graph that is not connected) with 5 ices and 7 edges . . . 11
- f a disconnected, directed graph with 5 vertices and 7 edges . . . 11
- g a graph with 3 vertices and 5 edges . . . 11
- h a directed graph with 3 vertices and 5 edges . . . 12

4 Given a Graph **12**
- a Is this graph connected? . . . 12
- b Is this graph complete? Why/Why not? . . . 13
- c What is the minimum degree of the graph? . . . 13
- d What is the maximum degree of the graph? . . . 13

e What is the largest clique in the graph? 13
f Is the graph planar? Why/Why not? 13
g What is the shortest path from V1 to V17? How long is it? . . 13
h What is the longest path from V1 to V17? How long is it? . . 13

1 Rolling Dice

a Dicerolling1,2,3,4,5,6

How many times will you always get all six numbers at least one, assuming $n > 6A$? What is the probability of this occurring?

well let's see the probability of the complement happening, so how many times will I not get all six numbers:
Using the inclusion-exclusion principle:

1. No 1 :5^n
2. No 2 :5^n
3. No 3 :5^n
4. No 4 :5^n
5. No 5 :5^n
6. No 6 :5^n

the first set with no 1 seems to have 4^n in common with all 5 others
the second set with no 2 seems also to have the same thing
we can see a pattern here...
we remove $6C2(4^n) = 15(4^n)$ to remove all the duplications
then we add $6C3(3^n) = 20(3^n)$ because we removed too much
then we remove $6C4(2^n) = 15(2^n)$
then we add $6C5(1^n) = 6$
then we remove $6C6(0) = 0$

so we get total occurence of having all 6 numbers is:

$$(6^n) - (6(5^n) - 15(4^n) + 20(3^n) - 15(2^n) + 6)$$

the total probability is then:

$$\frac{(6^n) - (6(5^n) - 15(4^n) + 20(3^n) - 15(2^n) + 6)}{6^n}$$

b DicerollingA,B,C

How many times will the sum of the number of 1's and 2's equal the sum of the number of 3's, 4's, 5's and 6's? What is the probability of this occurring?
let's say

$$1\&2 \mapsto A$$

$$3\&4 \mapsto B$$

$$5\&6 \mapsto C$$

$A\&B\&C$ with all equal possibilities of showing up (1/3)
Then we can say we are only throwing a 3 sided dice, therefore the total amount of possibilities while throwing is

$$3^n$$

using the analogy of undistinguishable As and $B\&C$s our equation for $\|A\| = \|B\| + \|C\|$ is

$$\frac{n!}{\frac{n}{3}! * \frac{2n}{3}!}$$

the total probability is then:

$$\frac{\frac{n!}{\frac{n}{3}! * \frac{2n}{3}!}}{3^n}$$

c DicerollingA,B,C Part2

How many times will the sum of the number of 1's and 2's equal the sum of the number of 3's and 4's and equal the sum of the number of 5's and 6's? What is the probability of this occurring?
Same thing as part I, but now As are indisguishable from Bs and from Cs. Therefore,

$$\frac{n!}{\frac{n}{3}!\frac{n}{3}!\frac{n}{3}!}$$

with the total probability at :

$$\frac{\frac{n!}{\frac{n}{3}!\frac{n}{3}!\frac{n}{3}!}}{3^n}$$

d Rolling 1s

In how many ways will you roll a "1" at least wtice among all n dice roll s, assuming n¿2? What is the probability of this occurring?

Use the exclusion and inclusion principle

1. size of the set where there is absolutely zero 1: 5^n
2. size of the set where there is only one 1: $n(5^{n-1})$

since there is no overlapping,
the total occurence is:

$$6^n - 5^n - n(5^{n-1})$$

the probability is then

$$\frac{6^n - 5^n - n(5^{n-1})}{6^n}$$

e Rolling 1s generalized...

In how many ways will you roll a "1" at least k times, assuming $n > k$? What is the probability of this occurring?
Use the exclusion and inclusion principle

1. size of the set where there is absolutely zero 1: $\binom{n}{0}5^n$
2. size of the set where there is only one 1: $\binom{n}{1}(5^{n-1})$
3. size of the set where there are only 2 1: $\binom{n}{2}(5^{n-2})$
4. size of the set where there are only k 1: $\binom{n}{k}(5^{n-k})$

As we see, there is no overlap between the sets and we can just use sigma nnotation to give the answer.

$$6^n - \sum_{i=0}^{k} \binom{n}{i} 5^{n-i}$$

Probabilities:

$$\frac{6^n - \sum_{i=0}^{k} \binom{n}{i} 5^{n-i}}{6^n}$$

f Communication, actually even more generalization

You have a communications system that transmits bundles of data from a source to a destination. Most bundles are received correctly. However, there is a random chance of the data in a bundle being in error. Roughly every E^{th} bundle is in error. If you transmit n bundles, what is the probability of two bundles being in error? what is the probability of k bundles being in error?

Let's say the bundle is like a E-sided dice, if we roll a 1, we get error

1. size of the set where there is absolutely no error: $\binom{n}{0}(E-1)^n$
2. size of the set where there is only one 1: $\binom{n}{1}((E-1)^{n-1})$
3. size of the set where there are only 2 1: $\binom{n}{2}((E-1)^{n-2})$
4. ...
5. size of the set where there are only k 1: $\binom{n}{k}((E-1)^{n-k})$

Probabilities 2 errors:

$$\frac{E^n - \sum_{i=0}^{2}\binom{n}{i}(E-1)^{n-i}}{E^n}$$

$$\implies \frac{E^n - (E-1)^n - n(E-1)^{n-1}}{E^n}$$

Probabilities k error:

$$\frac{E^n - \sum_{i=0}^{k}\binom{n}{i}(E-1)^{n-i}}{E^n}$$

2 Binomial Theorem

Prove (using induction) the binomial theorem:

$$(x+a)^n = \sum_{r=0}^{n} C(nr)x^{n-r}a^r$$

a Base Case

n = 1 case:

$$(x+a) = \sum_{r=0}^{1} C(nr)x^{n-r}a^r$$

$$(x+a) = \binom{1}{0}x^1 + \binom{1}{1}a^1$$

$$(x+a) = x + a\blacksquare$$

b N+1 Case

Assuming:

$$(x+a)^n = \sum_{r=0}^{n} C(nr)x^{n-r}a^r$$

Show:

$$(x+a)^{n+1} = \sum_{r=0}^{n+1} C((n+1)r)x^{(n+1)-r}a^r$$

Proof:

$$(x+a)^{n+1} = (x+a)^n(x+a)$$

$$(x+a)^{n+1} = \sum_{r=0}^{n} C(nr)x^{n-r}a^r(x+a)$$

$$(x+a)^{n+1} = x\sum_{r=0}^{n} C(nr)x^{n-r}a^r + a\sum_{r=0}^{n} C(nr)x^{n-r}a^r$$

putting x and a inside of the sigma notation

$$(x+a)^{n+1} = \sum_{r=0}^{n} C(nr)x^{n-r+1}a^r + \sum_{r=0}^{n} C(nr)x^{n-r}a^{r+1}$$

pulling r = 0 out of the first sum

$$(x+a)^{n+1} = \binom{n}{0}x^{n+1} + \sum_{r=1}^{n} C(nr)x^{n-r+1}a^r + \sum_{r=0}^{n} C(nr)x^{n-r}a^{r+1}$$

pulling r = n out of the second sum

$$(x+a)^{n+1} = \binom{n}{0}x^{n+1} + \sum_{r=1}^{n} C(nr)x^{n-r+1}a^r + \binom{n}{n}a^{n+1} + \sum_{r=0}^{n-1} C(nr)x^{n-r}a^{r+1}$$

shifting the second sum forward

$$(x+a)^{n+1} = \binom{n}{0}x^{n+1} + \binom{n}{n}a^{n+1} + \sum_{r=1}^{n} C(nr)x^{n-r+1}a^r + \sum_{r=1}^{n} C(nr-1)x^{n-r+1}a^r$$

combining the sigmas

$$(x+a)^{n+1} = \binom{n}{0}x^{n+1} + \binom{n}{n}a^{n+1} + \sum_{r=1}^{n}((C(nr) + C(n,r-1))x^{n-r+1}a^r$$

$$(x+a)^{n+1} = \binom{n}{0}x^{n+1} + \binom{n}{n}a^{n+1} + \sum_{r=1}^{n}(\frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!})x^{n-r+1}a^r$$

trying to simply the choose

$$(x+a)^{n+1} = \binom{n}{0}x^{n+1} + \binom{n}{n}a^{n+1} + \sum_{r=1}^{n}(\frac{n!}{r(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)(n-r)!})x^{n-r+1}a^r$$

$$(x+a)^{n+1} = \binom{n}{0}x^{n+1} + \binom{n}{n}a^{n+1} + \sum_{r=1}^{n}(\frac{n!(r+n-r+1)}{r(r-1)!(n-r)!(n-r+1)})x^{n-r+1}a^r$$

$$(x+a)^{n+1} = \binom{n}{0}x^{n+1} + \binom{n}{n}a^{n+1} + \sum_{r=1}^{n}(\frac{n!(n+1)}{r!(n-r+1)!})x^{n-r+1}a^r$$

$$(x+a)^{n+1} = \binom{n}{0}x^{n+1} + \binom{n}{n}a^{n+1} + \sum_{r=1}^{n}\binom{n+1}{r}x^{n-r+1}a^r$$

$\binom{n}{0} = \binom{n+1}{0} = 1$ $\binom{n}{n} = \binom{n+1}{n+1} = 1$

$$(x+a)^{n+1} = \binom{n+1}{0}x^{n+1} + \binom{n+1}{n}a^{n+1} + \sum_{r=1}^{n}\binom{n+1}{r}x^{n-r+1}a^r$$

the two loose terms are the 0th and n+1th terms

$$(x+a)^{n+1} = \sum_{r=0}^{n+1} \binom{n+1}{r} x^{n+1-r} a^r$$

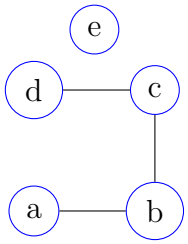
■

3 Graph Questions

a a graph with 5 vertices and 3 edges

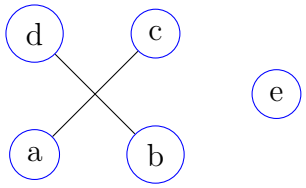
yes, any graph can exist with any amount of vertices and edges provided the number of edges are not bigger than the maximum possible amount.
for 5 vertices, the maximum amount of edges

$$\binom{5}{2} = 10$$



b a graph with 5 odd-degree vertices with 3 edges

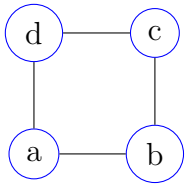
it is quite clear that with 3 edges, we are thinking of vertices with degree = 1.



it is clear that if we try to connect node e, with our remaining 1 edge, we will create a vertex of degree 2, therefore it is impossible.

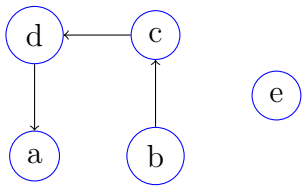
c a connected graph with 5 vertices and 3 edges?

a connected graph with 4 vertices requires at least 4 edges. therefore it is impossible to create a connected graph of 5 vertices with only 3 edges...



d a directed graph with 5 vertices and 3 edges?

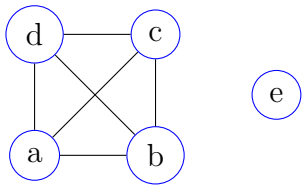
assuming you mean arcs by edges since in a digraph, there are no edges.



yes it it possible

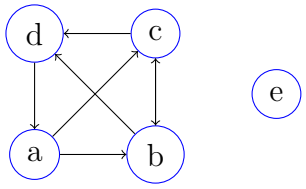
e a disconnected graph (a graph that is not connected) with 5 ices and 7 edges

no it is not possible, since K_4 has 6 edges (the maximum amount of edges we can connect with 4 vertices) so we have to connect the 5th node e using our last remaining edge.



f a disconnected, directed graph with 5 vertices and 7 edges

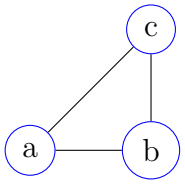
I assume edges mean arcs (directed edges). Then yes it is possible since the maximum amount of arcs for K_4 is 12.
So we can completely ignore the 5th node



g a graph with 3 vertices and 5 edges

No, since K_3 has 3 edges and the maximum amount of edges for 3 vertices is represented by

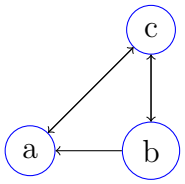
$$E(K_3) = 3$$



we can't fit 5 edges between 3 vertices.

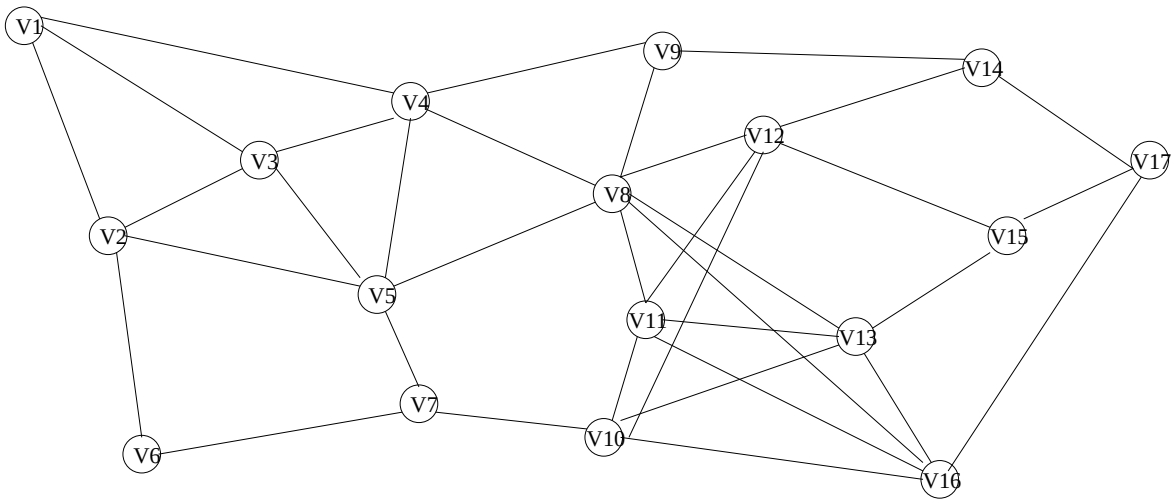
h a directed graph with 3 vertices and 5 edges

I assume that by edges, we are meaning arcs. then yes it is possible since the maximum amount of arcs for 3 vertices is 6



4 Given a Graph

Given the following graph:



a Is this graph connected?

Yes it is complete, since I can go from any node to any other node.

b Is this graph complete? Why/Why not?

No, there is no edge between V1 and V6.
However, there exists a subgraph K_4

c What is the minimum degree of the graph?

minimum degree is 2, on node V6

d What is the maximum degree of the graph?

maximum degree is 7, on node V8

e What is the largest clique in the graph?

It is a K_4
Here is one of the 2 K4

$$K_4$$
$$V8, V11, V16, V13$$

f Is the graph planar? Why/Why not?

No, there exists a subgraph $K_{3,3}$ $V10, V11, V8, V12, V13, V16$ inside.
We know that $K_{3,3}$ is non-planar.

g What is the shortest path from V1 to V17? How long is it?

$$V1 \to V4 \to V8 \to V16 \to V17$$

it is length 4

h What is the longest path from V1 to V17? How long is it?

$$V1 \to V4 \to V9 \to V14 \to V12 \to$$
$$V15 \to V13 \to V10 \to V7 \to$$
$$V6 \to V2 \to V3 \to V5 \to$$
$$V8 \to V11 \to V16 \to V17$$

length of 16. All the vertices are visited exactly once