FE5213: Quantitative Macroeconomics and Finance with Python

Project on Linear Dynamics in a Monetary Model

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Overview: Central banks aim to stabilize inflation and output fluctuations using monetary policy instruments such as the interest rate, typically via a Taylor rule. The analysis of monetary policy is an important issue in New Keynesian monetary economics. In this project, we will analyze the linear dynamics in a monetary model using Python. It involves formulating and setting up the state-space representation, solving impulse responses to economic and monetary shocks, and conducting comparative statics as the model parameters change.

Contents

1 Problem Setup

1.1 Economic Dynamics

We consider a simple New Keynesian model, where the economy is described by a Phillips curve and IS curve. In particular,

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + u_t, \tag{1}$$

$$x_t = \mathbb{E}_t x_{t+1} - \sigma \left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n \right), \tag{2}$$

where (1) is the New Keynesian Phillips curve and (2) is the IS curve. In these equations, π_t denotes inflation, x_t output gap, i_t nominal interest rate (policy instrument), u_t cost-push shock, r_t^n demand shock, β discount factor, κ slope of the Phillips curve, σ sensitivity to the real interest rate.

1.2 Central Bank's Policy Rule

The central bank sets the interest rate via the Taylor rule, which takes the following form:

$$i_t = \phi_\pi \pi_t + \phi_x x_t + \nu_t, \tag{3}$$

where ν_t is the monetary policy surprise/shock.

1.3 Shocks Assumption

Assume that u_t, r_t^n, and ν_t follow the following AR(1) process:

$$u_t = \rho_u u_{t-1} + \sigma_u \varepsilon_{ut}, \tag{4}$$

$$r_t^n = \rho_r r_{t-1}^n + \sigma_r \varepsilon_{rt}, \tag{5}$$

$$\nu_t = \rho_\nu \nu_{t-1} + \sigma_\nu \varepsilon_{\nu t}, \tag{6}$$

where $\varepsilon_{ut} \sim N(0,1)$, $\varepsilon_{rt} \sim N(0,1)$, and $\varepsilon_{\nu t} \sim N(0,1)$.

1.4 Model Summary

Variables	Meaning	Setting
π_t	inflation	New Keynesian Phillips curve (1)
x_t	output gap	IS curve (2)
i_t	nominal interest rate (policy instrument)	Taylor rule (3)
u_t	cost-push shock	AR(1) process (4)
r_t^n	demand shock	AR(1) process (5)
ν_t	monetary policy shock	AR(1) process (6)
ε_{ut}	unit white noise for cost-push shock	$N(0,1)$
ε_{rt}	unit white noise for demand shock	$N(0,1)$
$\varepsilon_{\nu t}$	unit white noise for monetary policy shock	$N(0,1)$

Table 1: Variables in the model: the output (π_t, x_t, i_t), three shocks (u_t, r_t^n, ν_t), with unit white noise $(\varepsilon_{ut}, \varepsilon_{rt}, \varepsilon_{\nu t})$.

Parameter	Meaning	Benchmark Value
β	discount factor	0.99
σ	sensitivity to the real interest rate	1/6
κ	slope of the Phillips curve	0.024
ϕ_π	sensitivity to the inflation	1.5
ϕ_x	sensitivity to the output gap	0.5
ρ_r	AR(1) coefficient for demand shock	0.35
ρ_u	AR(1) coefficient for cost-push shock	0.35
ρ_ν	AR(1) coefficient for monetary policy shock	0.35
σ_r	std for demand shock white noise	3.7
σ_u	std for cost-push shock white noise	0.4
σ_ν	std for monetary policy shock white noise	1

Table 2: Parameters in the model. In the benchmark analysis, use the calibrated parameters taken from the literature.

2 Model Solution

2.1 Rational Expectations Equilibrium

The rational expectations equilibrium for this model can be solved as follows. Conjecture a model solution where the output is linear in the state comprised of three shocks (u_t, r_t^n, ν_t),

$$\begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = \mathbf{P} \begin{bmatrix} u_t \\ r_t^n \\ \nu_t \end{bmatrix} := \begin{bmatrix} \gamma_\pi^u & \gamma_\pi^r & \gamma_\pi^\nu \\ \gamma_x^u & \gamma_x^r & \gamma_x^\nu \\ \gamma_i^u & \gamma_i^r & \gamma_i^\nu \end{bmatrix} \begin{bmatrix} u_t \\ r_t^n \\ \nu_t \end{bmatrix}. \tag{7}$$

With (7), we can write the one-period-ahead expectations accordingly,

$$\begin{aligned} \mathbb{E}_t \pi_{t+1} &= \gamma_\pi^u \rho_u u_t + \gamma_\pi^r \rho_r r_t^n + \gamma_\pi^\nu \rho_\nu \nu_t, \\ \mathbb{E}_t x_{t+1} &= \gamma_x^u \rho_u u_t + \gamma_x^r \rho_r r_t^n + \gamma_x^\nu \rho_\nu \nu_t, \\ \mathbb{E}_t i_{t+1} &= \gamma_i^u \rho_u u_t + \gamma_i^r \rho_r r_t^n + \gamma_i^\nu \rho_\nu \nu_t. \end{aligned} \tag{8}$$

To solve for $\mathbf{P}$, we can substitute items in (1)(2)(3) with (7) and (8) and solve $\mathbf{P}$ using the method of undetermined coefficients. In particular, for (1),

$$\gamma_\pi^u u_t + \gamma_\pi^r r_t^n + \gamma_\pi^\nu \nu_t = \beta\left(\gamma_\pi^u \rho_u u_t + \gamma_\pi^r \rho_r r_t^n + \gamma_\pi^\nu \rho_\nu \nu_t\right) + \kappa\left(\gamma_x^u u_t + \gamma_x^r r_t^n + \gamma_x^\nu \nu_t\right) + u_t,$$

collecting items and re-arranging,

$$\left(\gamma_\pi^u - \beta\gamma_\pi^u \rho_u - \kappa\gamma_x^u - 1\right) u_t + \left(\gamma_\pi^r - \beta\gamma_\pi^r \rho_r - \kappa\gamma_x^r\right) r_t^n + \left(\gamma_\pi^\nu - \beta\gamma_\pi^\nu \rho_\nu - \kappa\gamma_x^\nu\right) \nu_t = 0,$$

This implies that

$$\gamma_\pi^u = \beta\gamma_\pi^u \rho_u + \kappa\gamma_x^u + 1, \tag{9}$$

$$\gamma_\pi^r = \beta\gamma_\pi^r \rho_r + \kappa\gamma_x^r, \tag{10}$$

$$\gamma_\pi^\nu = \beta\gamma_\pi^\nu \rho_\nu + \kappa\gamma_x^\nu. \tag{11}$$

Similarly, for (2)

$$\gamma_x^u u_t + \gamma_x^r r_t^n + \gamma_x^\nu \nu_t = (\gamma_x^u \rho_u u_t + \gamma_x^r \rho_r r_t^n + \gamma_x^\nu \rho_\nu \nu_t) - \sigma \left[(\gamma_i^u u_t + \gamma_i^r r_t^n + \gamma_i^\nu \nu_t) - (\gamma_\pi^u \rho_u u_t + \gamma_\pi^r \rho_r r_t^n + \gamma_\pi^\nu \rho_\nu \nu_t) - r_t^n \right],$$

collecting items, we have

$$\gamma_x^u = \gamma_x^u \rho_u - \sigma \left(\gamma_i^u - \gamma_\pi^u \rho_u \right), \tag{12}$$
$$\gamma_x^r = \gamma_x^r \rho_r - \sigma \left(\gamma_i^r - \gamma_\pi^r \rho_r - 1 \right), \tag{13}$$
$$\gamma_x^\nu = \gamma_x^\nu \rho_\nu - \sigma \left(\gamma_i^\nu - \gamma_\pi^\nu \rho_\nu \right). \tag{14}$$

For (3),

$$\gamma_i^u u_t + \gamma_i^r r_t^n + \gamma_i^\nu \nu_t = \phi_\pi \left(\gamma_\pi^u u_t + \gamma_\pi^r r_t^n + \gamma_\pi^\nu \nu_t \right) + \phi_x \left(\gamma_x^u u_t + \gamma_x^r r_t^n + \gamma_x^\nu \nu_t \right) + \nu_t,$$

so we have

$$\gamma_i^u = \phi_\pi \gamma_\pi^u + \phi_x \gamma_x^u, \tag{15}$$
$$\gamma_i^r = \phi_\pi \gamma_\pi^r + \phi_x \gamma_x^r, \tag{16}$$
$$\gamma_i^\nu = \phi_\pi \gamma_\pi^\nu + \phi_x \gamma_x^\nu + 1. \tag{17}$$

Note that in the system of equations (9) through (17), there are 9 equations with 9 unknowns, which are entries in the matrix $\mathbf{P}$. We denote the vectorization of the matrix $\mathbf{P}$ as vector $\mathbf{x}$,

$$\mathbf{P} = \begin{bmatrix} \gamma_\pi^u & \gamma_\pi^r & \gamma_\pi^\nu \\ \gamma_x^u & \gamma_x^r & \gamma_x^\nu \\ \gamma_i^u & \gamma_i^r & \gamma_i^\nu \end{bmatrix} \implies \mathbf{x} := \mathrm{vec}(\mathbf{P}) = \left[\gamma_\pi^u,\ \gamma_x^u,\ \gamma_i^u,\ \gamma_\pi^r,\ \gamma_x^r,\ \gamma_i^r,\ \gamma_\pi^\nu,\ \gamma_x^\nu,\ \gamma_i^\nu \right]^\top.$$

The system of linear equations (9) through (17) can be packed into matrix form

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{18}$$

where

$$\mathbf{A} = \begin{bmatrix} 1-\beta\rho_u & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\beta\rho_r & -\kappa & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-\beta\rho_\nu & -\kappa & 0 \\ \sigma\rho_u & \rho_u - 1 & -\sigma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma\rho_r & \rho_r - 1 & -\sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma\rho_\nu & \rho_\nu - 1 & -\sigma \\ \phi_\pi & \phi_x & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_\pi & \phi_x & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \phi_\pi & \phi_x & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -\sigma \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

(18) can be easily solved using matrix inversion, so that we get $\mathrm{vec}(\mathbf{P}) = \mathbf{A}^{-1}\mathbf{b}$. In this way, $\mathbf{P}$ is solved and known.

2.2 State-Space Form

With $\mathbf{P}$, the model is characterized by a linear state-space system with the measurement equation

$$\begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = \mathbf{P} \begin{bmatrix} u_t \\ r_t^n \\ \nu_t \end{bmatrix}, \tag{19}$$

and the transition equation

$$\begin{bmatrix} u_t \\ r_t^n \\ \nu_t \end{bmatrix} = \begin{bmatrix} \rho_u & 0 & 0 \\ 0 & \rho_r & 0 \\ 0 & 0 & \rho_\nu \end{bmatrix} \begin{bmatrix} u_{t-1} \\ r_{t-1}^n \\ \nu_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma_u & 0 & 0 \\ 0 & \sigma_r & 0 \\ 0 & 0 & \sigma_\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{ut} \\ \varepsilon_{rt} \\ \varepsilon_{\nu t} \end{bmatrix}. \tag{20}$$

2.3 Main Problems to Address

- Solve $\mathbf{P}$ and formulate the model solutions as a linear state space system.
- Plot and discuss the dynamic responses of $\pi_{t+i}, x_{t+i}, i_{t+i}$ to a one standard deviation change in economic shocks $\varepsilon_{rt}, \varepsilon_{ut}$ and the monetary policy shock $\varepsilon_{\nu t}$.
- Investigate how the results would change if we change $\kappa, \rho_u, \rho_r, \phi_\pi, \phi_x$. Discuss your results.

3 Results