FE5213: Quantitative Macroeconomics and Finance with Python

Project on Dynamics in Monetary Policy Models

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Overview: Central banks aim to stabilize inflation and output fluctuations using monetary policy instruments such as the interest rate, typically via a Taylor rule. The analysis of monetary policy is an important issue in New Keynesian monetary economics. In this project, we will carry out two analyses using Python: (1) the linear dynamics in a monetary model; (2) the solution to a central bank's optimal policy problem.

Contents

1 Linear Dynamics in a Monetary Model

In this section, we will analyze the linear dynamics in a monetary model. It involves formulating and setting up the state-space representation, solving impulse responses to economic and monetary shocks, and conducting comparative statics as the model parameters change.

1.1 Problem Setup

1.1.1 Economic Dynamics

We consider a simple New Keynesian model, where the economy is described by a Phillips curve and IS curve. In particular,

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + u_t, \tag{1}$$

$$x_t = \mathbb{E}_t x_{t+1} - \sigma \left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n \right), \tag{2}$$

where (1) is the New Keynesian Phillips curve and (2) is the IS curve. In these equations, π_t denotes inflation, x_t output gap, i_t nominal interest rate (policy instrument), u_t cost-push shock, r_t^n demand shock, β discount factor, κ slope of the Phillips curve, σ sensitivity to the real interest rate.

1.1.2 Central Bank's Policy Rule

The central bank sets the interest rate via the Taylor rule, which takes the following form:

$$i_t = \phi_\pi \pi_t + \phi_x x_t + \nu_t, \tag{3}$$

where ν_t is the monetary policy surprise/shock.

1.1.3 Shocks Assumption

Assume that u_t, r_t^n, and ν_t follow the following AR(1) process:

$$u_t = \rho_u u_{t-1} + \sigma_u \varepsilon_{ut}, \tag{4}$$

$$r_t^n = \rho_r r_{t-1}^n + \sigma_r \varepsilon_{rt}, \tag{5}$$

$$\nu_t = \rho_\nu \nu_{t-1} + \sigma_\nu \varepsilon_{\nu t}, \tag{6}$$

where $\varepsilon_{ut} \sim N(0,1)$, $\varepsilon_{rt} \sim N(0,1)$, and $\varepsilon_{\nu t} \sim N(0,1)$.

1.1.4 Model Summary

Variables	Meaning	Setting
π_t	inflation	New Keynesian Phillips curve (1)
x_t	output gap	IS curve (2)
i_t	nominal interest rate (policy instrument)	Taylor rule (3)
u_t	cost-push shock	AR(1) process (4)
r_t^n	demand shock	AR(1) process (5)
ν_t	monetary policy shock	AR(1) process (6)
ε_{ut}	unit white noise for cost-push shock	$N(0,1)$
ε_{rt}	unit white noise for demand shock	$N(0,1)$
$\varepsilon_{\nu t}$	unit white noise for monetary policy shock	$N(0,1)$

Table 1: Variables in the model: the output (π_t, x_t, i_t), three shocks (u_t, r_t^n, ν_t), with unit white noise $(\varepsilon_{ut}, \varepsilon_{rt}, \varepsilon_{\nu t})$.

Parameter	Meaning	Benchmark Value
β	discount factor	0.99
σ	sensitivity to the real interest rate	1/6
κ	slope of the Phillips curve	0.024
ϕ_π	sensitivity to the inflation	1.5
ϕ_x	sensitivity to the output gap	0.5
ρ_r	AR(1) coefficient for demand shock	0.35
ρ_u	AR(1) coefficient for cost-push shock	0.35
ρ_ν	AR(1) coefficient for monetary policy shock	0.35
σ_r	std for demand shock white noise	3.7
σ_u	std for cost-push shock white noise	0.4
σ_ν	std for monetary policy shock white noise	1

Table 2: Parameters in the model. In the benchmark analysis, use the calibrated parameters taken from the literature.

1.2 Problem Analysis

The rational expectations equilibrium for this model can be solved as follows.

1.2.1 Model Solution

Conjecture a model solution where the output is linear in the state comprised of three shocks (u_t, r_t^n, ν_t),

$$\begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = \mathbf{P} \begin{bmatrix} u_t \\ r_t^n \\ \nu_t \end{bmatrix} := \begin{bmatrix} \gamma_\pi^u & \gamma_\pi^r & \gamma_\pi^\nu \\ \gamma_x^u & \gamma_x^r & \gamma_x^\nu \\ \gamma_i^u & \gamma_i^r & \gamma_i^\nu \end{bmatrix} \begin{bmatrix} u_t \\ r_t^n \\ \nu_t \end{bmatrix}. \tag{7}$$

With (7), we can write the one-period-ahead expectations accordingly,

$$\begin{aligned} \mathbb{E}_t \pi_{t+1} &= \gamma_\pi^u \rho_u u_t + \gamma_\pi^r \rho_r r_t^n + \gamma_\pi^\nu \rho_\nu \nu_t, \\ \mathbb{E}_t x_{t+1} &= \gamma_x^u \rho_u u_t + \gamma_x^r \rho_r r_t^n + \gamma_x^\nu \rho_\nu \nu_t, \\ \mathbb{E}_t i_{t+1} &= \gamma_i^u \rho_u u_t + \gamma_i^r \rho_r r_t^n + \gamma_i^\nu \rho_\nu \nu_t. \end{aligned} \tag{8}$$

To solve for $\mathbf{P}$, we can substitute items in (1)(2)(3) with (7) and (8) and solve $\mathbf{P}$ using the method of undetermined coefficients. In particular, for (1),

$$\gamma_\pi^u u_t + \gamma_\pi^r r_t^n + \gamma_\pi^\nu \nu_t = \beta\left(\gamma_\pi^u \rho_u u_t + \gamma_\pi^r \rho_r r_t^n + \gamma_\pi^\nu \rho_\nu \nu_t\right) + \kappa\left(\gamma_x^u u_t + \gamma_x^r r_t^n + \gamma_x^\nu \nu_t\right) + u_t,$$

collecting items and re-arranging,

$$\left(\gamma_\pi^u - \beta\gamma_\pi^u \rho_u - \kappa\gamma_x^u - 1\right) u_t + \left(\gamma_\pi^r - \beta\gamma_\pi^r \rho_r - \kappa\gamma_x^r\right) r_t^n + \left(\gamma_\pi^\nu - \beta\gamma_\pi^\nu \rho_\nu - \kappa\gamma_x^\nu\right) \nu_t = 0,$$

This implies that

$$\begin{aligned}
\gamma_\pi^u &= \beta\gamma_\pi^u \rho_u + \kappa\gamma_x^u + 1, && (9)\\
\gamma_\pi^r &= \beta\gamma_\pi^r \rho_r + \kappa\gamma_x^r, && (10)\\
\gamma_\pi^\nu &= \beta\gamma_\pi^\nu \rho_\nu + \kappa\gamma_x^\nu. && (11)
\end{aligned}$$

Similarly, for (2)

$$\gamma_x^u u_t + \gamma_x^r r_t^n + \gamma_x^\nu \nu_t = \left(\gamma_x^u \rho_u u_t + \gamma_x^r \rho_r r_t^n + \gamma_x^\nu \rho_\nu \nu_t\right) - \sigma\left[\left(\gamma_i^u u_t + \gamma_i^r r_t^n + \gamma_i^\nu \nu_t\right) - \left(\gamma_\pi^u \rho_u u_t + \gamma_\pi^r \rho_r r_t^n + \gamma_\pi^\nu \rho_\nu \nu_t\right) - r_t^n\right],$$

collecting items, we have

$$\begin{aligned}
\gamma_x^u &= \gamma_x^u \rho_u - \sigma\left(\gamma_i^u - \gamma_\pi^u \rho_u\right), && (12)\\
\gamma_x^r &= \gamma_x^r \rho_r - \sigma\left(\gamma_i^r - \gamma_\pi^r \rho_r - 1\right), && (13)\\
\gamma_x^\nu &= \gamma_x^\nu \rho_\nu - \sigma\left(\gamma_i^\nu - \gamma_\pi^\nu \rho_\nu\right). && (14)
\end{aligned}$$

For (3),

$$\gamma_i^u u_t + \gamma_i^r r_t^n + \gamma_i^\nu \nu_t = \phi_\pi\left(\gamma_\pi^u u_t + \gamma_\pi^r r_t^n + \gamma_\pi^\nu \nu_t\right) + \phi_x\left(\gamma_x^u u_t + \gamma_x^r r_t^n + \gamma_x^\nu \nu_t\right) + \nu_t,$$

so we have

$$\begin{aligned}
\gamma_i^u &= \phi_\pi \gamma_\pi^u + \phi_x \gamma_x^u, && (15)\\
\gamma_i^r &= \phi_\pi \gamma_\pi^r + \phi_x \gamma_x^r, && (16)\\
\gamma_i^\nu &= \phi_\pi \gamma_\pi^\nu + \phi_x \gamma_x^\nu + 1. && (17)
\end{aligned}$$

Note that in the system of equations (9) through (17), there are 9 equations with 9 unknowns, which are entries in the matrix $\mathbf{P}$. We denote the vectorization of the matrix $\mathbf{P}$ as vector $\mathbf{x}$,

$$\mathbf{P} = \begin{bmatrix} \gamma_\pi^u & \gamma_\pi^r & \gamma_\pi^\nu \\ \gamma_x^u & \gamma_x^r & \gamma_x^\nu \\ \gamma_i^u & \gamma_i^r & \gamma_i^\nu \end{bmatrix} \implies \mathbf{x} := \operatorname{vec}(\mathbf{P}) = \left[\gamma_\pi^u,\ \gamma_x^u,\ \gamma_i^u,\ \gamma_\pi^r,\ \gamma_x^r,\ \gamma_i^r,\ \gamma_\pi^\nu,\ \gamma_x^\nu,\ \gamma_i^\nu\right]^\top.$$

The system of linear equations (9) through (17) can be packed into matrix form

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{18}$$

where

$$\mathbf{A}=\begin{bmatrix} 1-\beta\rho_u & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\beta\rho_r & -\kappa & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-\beta\rho_\nu & -\kappa & 0 \\ \sigma\rho_u & \rho_u-1 & -\sigma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma\rho_r & \rho_r-1 & -\sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma\rho_\nu & \rho_\nu-1 & -\sigma \\ \phi_\pi & \phi_x & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_\pi & \phi_x & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \phi_\pi & \phi_x & -1 \end{bmatrix},\ \mathbf{b}=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -\sigma \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

(18) can be easily solved using matrix inversion, so that we get $\text{vec}(\mathbf{P})=\mathbf{A}^{-1}\mathbf{b}$. In this way, $\mathbf{P}$ is solved and known.

1.2.2 State-Space Form

With $\mathbf{P}$, the model is characterized by a linear state-space system with the measurement equation

$$\begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix}=\mathbf{P}\begin{bmatrix} u_t \\ r_t^n \\ \nu_t \end{bmatrix}, \tag{19}$$

and the transition equation

$$\begin{bmatrix} u_t \\ r_t^n \\ \nu_t \end{bmatrix}=\begin{bmatrix} \rho_u & 0 & 0 \\ 0 & \rho_r & 0 \\ 0 & 0 & \rho_\nu \end{bmatrix}\begin{bmatrix} u_{t-1} \\ r_{t-1}^n \\ \nu_{t-1} \end{bmatrix}+\begin{bmatrix} \sigma_u & 0 & 0 \\ 0 & \sigma_r & 0 \\ 0 & 0 & \sigma_\nu \end{bmatrix}\begin{bmatrix} \varepsilon_{ut} \\ \varepsilon_{rt} \\ \varepsilon_{\nu t} \end{bmatrix}. \tag{20}$$

1.2.3 Main Problems to Address

We use Python to process the following procedures and discuss the results.

- Solve $\mathbf{P}$ from (18) and formulate the model solutions as the linear state space system (19)(20).
- Set initial shock $\begin{bmatrix} u_0 \\ r_0^n \\ \nu_0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Run the linear state space system iteratively, record $\mathbf{s_t}=\begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix}$. Stop until

$$\text{abs}\left(\|\mathbf{s_{t+1}}\|_2-\|\mathbf{s_t}\|_2\right)<\epsilon,$$

  where we take $\epsilon=10^{-5}$. In this case, we reach the **rational expectations equilibrium**, denoted as $\mathbf{s}^*=\begin{bmatrix} \pi^* \\ x^* \\ i^* \end{bmatrix}$.
- Start from the rational expectations equilibrium $\mathbf{s_t}=\mathbf{s}^*$. Pose a one standard deviation change in economic shocks $\varepsilon_{rt},\varepsilon_{ut}$ and the monetary policy shock $\varepsilon_{\nu t}$, respectively. Plot and discuss the dynamic responses of $\pi_{t+i},x_{t+i},i_{t+i}$ to the impulse.
- Investigate how the results would change if we change $\kappa,\rho_u,\rho_r,\phi_\pi,\phi_x$. Discuss the results.

1.3 Results

1.3.1 Benchmark Analysis

We take the calibrated parameters from Table 2 to carry out the benchmark analysis.

$$\mathbf{P} = \begin{bmatrix} 1.5156735 & 0.00826731 & -0.00826728 \\ -0.3961419 & 0.22511195 & -0.22511198 \\ 2.0754392 & 0.12495694 & 0.87504311 \end{bmatrix}$$

1.3.2 Parameter Discussion

2 An Optimal Monetary Policy Problem

In this section, we apply Linear-Quadratic Dynamic Programming (LQDP) to study optimal monetary policy in a simple New Keynesian framework. This problem can be formulated as a Linear-Quadratic Regulator (LQR), where the policy-maker minimizes deviations of inflation and output from their targets. It involves formulating the central bank's objective function, setting up the state-space representation, solving the optimal policy problem using numerical methods, analyzing the optimal policy response to economic shocks, and doing comparative statics when the model parameters change.

2.1 Problem Setup

We consider a simple New Keynesian model, where the economy is described by a Phillips curve (1) and IS curve (2), with shocks following AR(1) processes (4)(5).

2.1.1 Central Bank's Loss Function

The central bank seeks to minimize a quadratic loss function:

$$L = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\pi_t^2 + \lambda_x (x_t - x^*)^2 + \lambda_i (i_t - i^*)^2 \right), \tag{21}$$

where weights $\lambda_x, \lambda_i > 0$ reflect relative weight on output stabilization and penalty on interest rate volatility; x^*, i^* represent some optimal levels of output gap and nominal interest rate. We focus on a class of monetary policy rules, the Taylor rule, which takes the following form

$$i_t = \phi_\pi \pi_t + \phi_x x_t + \phi_0. \tag{22}$$

2.1.2 Model Summary

Variables	Meaning	Setting
π_t	inflation	New Keynesian Phillips curve (1)
x_t	output gap	IS curve (2)
i_t	nominal interest rate (policy instrument)	Taylor rule (22)
u_t	cost-push shock	AR(1) process (4)
r_t^n	demand shock	AR(1) process (5)
ε_{ut}	unit white noise for cost-push shock	$N(0,1)$
ε_{rt}	unit white noise for demand shock	$N(0,1)$

Table 3: Variables in the model: (π_t, x_t, i_t) of our interest, two shocks (u_t, r_t^n), with unit white noise $(\varepsilon_{ut}, \varepsilon_{rt})$.

Parameter	Meaning	Benchmark Value
β	discount factor	0.99
σ	sensitivity to the real interest rate	1/6
κ	slope of the Phillips curve	0.024
ρ_r	AR(1) coefficient for demand shock	0.35
ρ_u	AR(1) coefficient for cost-push shock	0.35
σ_r	std for demand shock white noise	13.8
σ_u	std for cost-push shock white noise	0.17
λ_x	relative weight on output stabilization	0.048
λ_i	penalty on interest rate volatility	0.236
x^*	some optimal levels of output gap	0
i^*	some optimal levels of nominal interest rate	$1/\beta - 1$

Table 4: Parameters in the model. In the benchmark analysis, use the calibrated parameters taken from the literature.

Remark. In standard macroeconomic models, particularly within the New Keynesian framework, the optimal output gap x^* is typically set to zero. This specification reflects the central bank's objective of maintaining output at its potential level, thereby minimizing inflationary or deflationary pressures that arise from output deviations. Consequently, we adopt $x^* = 0$ in our optimal monetary policy model.

Denote the inflation target as π^*. Starting from the representative household's Euler equation, $1 = \beta \dfrac{1+i^*}{1+\pi^*}$, the gross nominal risk-free return that is consistent with intertemporal optimality must satisfy $i^* = \left(\frac{1}{\beta} - 1\right) + \pi^*$, where the first term, $r^* \equiv 1/\beta - 1$, is the natural real interest rate. When the central bank targets zero steady-state inflation ($\pi^* = 0$, a standard normalization that keeps the algebra transparent), and isolates the propagation of nominal rigidities—the optimal nominal rate collapses to the real rate, $i^* = 1/\beta - 1$. With the conventional quarterly calibration $\beta = 0.99$, this implies an annualized neutral nominal rate of roughly four percent, though lower values of i^* can be obtained by choosing a higher β to match the persistently low real rates observed in recent decades.

2.2 Problem Solution

2.2.1 State-Space Representation

Let $\mathbf{s_t} = [1,\, \pi_t,\, x_t - x^*,\, u_t,\, r_t^n]'$ be the state vector, and $\mathbf{i_t} = [i_t - i^*]$ be the control vector. From (4)(5), we have

$$u_{t+1} = \rho_u u_t + \sigma_u \varepsilon_{ut+1},$$
$$r_{t+1}^n = \rho_r r_t^n + \sigma_r \varepsilon_{rt+1}.$$

We assume certainty equivalence to take the following approximations in (1)(2)

$$\mathbb{E}_t \pi_{t+1} \approx \pi_{t+1},\ \mathbb{E}_t x_{t+1} \approx x_{t+1}. \tag{23}$$

Certainty equivalence simplifies the decision-making process under uncertainty by replacing uncertain variables with their expected values. So, the expected future inflation and output gap should be, on average, equal to the future inflation and output gap. Then we get:

$$\beta \pi_{t+1} = \pi_t - \kappa x_t - u_t, \tag{24}$$

$$x_{t+1} + \sigma \pi_{t+1} = x_t + \sigma i_t - \sigma r_t^n, \tag{25}$$

Combine (24)(25) with the optimal levels of output gap x^* and nominal interest rate i^*, we get:

$$\beta \pi_{t+1} = \pi_t - \kappa (x_t - x^*) - u_t - \kappa x^*, \tag{26}$$

$$x_{t+1} - x^* = -\sigma\pi_{t+1} + (x_t - x^*) + \sigma(i_t - i^*) - \sigma r_t^n + \sigma i^*, \tag{27}$$

Transfer (26)(27)(4)(5) into matrix form, we get the state dynamics:

$$\mathbf{s_{t+1}} = \mathbf{A}\mathbf{s_t} + \mathbf{B}\mathbf{i_t} + \mathbf{C}\varepsilon_{\mathbf{t+1}}, \tag{28}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{\kappa}{\beta}x^* & \frac{1}{\beta} & -\frac{\kappa}{\beta} & -\frac{1}{\beta} & 0 \\ \sigma(\frac{\kappa}{\beta}x^* + i^*) & -\frac{\sigma}{\beta} & \frac{\beta+\kappa\sigma}{\beta} & \frac{\sigma}{\beta} & -\sigma \\ 0 & 0 & 0 & \rho_u & 0 \\ 0 & 0 & 0 & 0 & \rho_r \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \sigma \\ 0 \\ 0 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sigma_u & 0 \\ 0 & \sigma_r \end{bmatrix}.$$

2.2.2 Value Function and Bellman Equation

Transfer the loss function (21) into quadratic form, we get:

$$L = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\mathbf{s}_\mathbf{t}' \mathbf{Q} \mathbf{s_t} + \mathbf{i}_\mathbf{t}' \mathbf{R} \mathbf{i_t} \right), \tag{29}$$

where:

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{R} = \begin{bmatrix} \lambda_i \end{bmatrix}, \tag{30}$$

The value function is:

$$V(\mathbf{s_t}) = \min_{\mathbf{i_t}} \sum_{t=0}^{\infty} \beta^t \left(\mathbf{s}_\mathbf{t}' \mathbf{Q} \mathbf{s_t} + \mathbf{i}_\mathbf{t}' \mathbf{R} \mathbf{i_t} \right), \tag{31}$$

We define the Bellman Equation:

$$V(\mathbf{s_t}) = \min_{\mathbf{i_t}} (\mathbf{s}_t' \mathbf{Q} \mathbf{s_t} + \mathbf{i}_\mathbf{t}' \mathbf{R} \mathbf{i_t} + \beta \mathbb{E}_t \left[V(\mathbf{s_{t+1}}) \right]) \tag{32}$$

Using the quadratic guess for the value function:

$$V(\mathbf{s_t}) = \mathbf{s}_\mathbf{t}' \mathbf{P} \mathbf{s_t} \tag{33}$$

where $\mathbf{P}$ is a symmetric positive semi-definite matrix that we solve for.

The Bellman equation becomes:

$$\mathbf{s}_\mathbf{t}' \mathbf{P} \mathbf{s_t} = \min_{\mathbf{i_t}} (\mathbf{s}_t' \mathbf{Q} \mathbf{s_t} + \mathbf{i}_\mathbf{t}' \mathbf{R} \mathbf{i_t} + \beta \mathbb{E}_t \left[(\mathbf{A}\mathbf{s_t} + \mathbf{B}\mathbf{i_t} + \mathbf{C}\varepsilon_{\mathbf{t+1}})' \mathbf{P} (\mathbf{A}\mathbf{s_t} + \mathbf{B}\mathbf{i_t} + \mathbf{C}\varepsilon_{\mathbf{t+1}}) \right]) \tag{34}$$

2.2.3 Optimal Policy Solution

Taking the first-order condition of (34) with respect to i_t:

$$\mathbf{2}\mathbf{R}\mathbf{i_t} + \mathbf{2}\beta\mathbf{B}'\mathbf{P}(\mathbf{A}\mathbf{s_t} + \mathbf{B}\mathbf{i_t}) = \mathbf{0} \tag{35}$$

Solving for $\mathbf{i_t}$:

$$\mathbf{i_t} = -(\mathbf{R} + \beta\mathbf{B}'\mathbf{P}\mathbf{B})^{-1}\beta\mathbf{B}'\mathbf{P}\mathbf{A}\mathbf{s_t} \tag{36}$$

which is of the form:

$$\mathbf{i_t} = -\mathbf{F}\mathbf{s_t} \tag{37}$$

where

$$\mathbf{F} = (\mathbf{R} + \beta\mathbf{B}'\mathbf{P}\mathbf{B})^{-1}\beta\mathbf{B}'\mathbf{P}\mathbf{A} = \begin{bmatrix} F_1 & F_2 & F_3 & F_4 & F_5 \end{bmatrix} \tag{38}$$

Substitute (22) into The optimal policy coefficients are:

$$\phi_\pi = -F_2, \quad \phi_x = -F_3, \quad \phi_0 = -F_1 + F_3 x^* + i^* \tag{39}$$

The matrix $\mathbf{P}$ is found by solving the discrete-time Riccati equation:

$$\mathbf{P} = \mathbf{Q} + \beta\mathbf{A}'\mathbf{P}\mathbf{A} - \beta\mathbf{A}'\mathbf{P}\mathbf{B}(\mathbf{R} + \beta\mathbf{B}'\mathbf{P}\mathbf{B})^{-1}\mathbf{B}'\mathbf{P}\mathbf{A} \tag{40}$$

2.3 Results

2.3.1 The Optimal Taylor Rule

We solve for the optimal policy coefficients in (22) with Python:

$$i_t = 1.3349\pi_t - 1.1447x_t - 1.1707, \tag{41}$$

which is the optimal Taylor rule. 41 reflects the central bank's best response under a standard New Keynesian framework with forward-looking expectations and a quadratic loss function. The coefficient on inflation (ϕ_π) exceeds one, satisfying the Taylor principle and ensuring that real interest rates rise in response to higher inflation, thereby anchoring inflation expectations. The negative output gap coefficient ϕ_x implies a countercyclical stance: when the economy operates below potential, the central bank reduces the nominal interest rate to stimulate demand. The intercept term reflects the implied steady-state nominal interest rate, which in this case is negative, suggesting a low or even negative natural real rate given the assumed inflation and output targets.

2.3.2 Central Bank's Optimal Response To Shocks

In the standard New Keynesian framework, a positive demand shock ($\varepsilon_{r_t} > 0$), which raises the natural rate of interest, typically leads to an increase in the output gap and a rise in inflation, as higher demand pressures push prices upward. In response, the central bank raises the nominal interest rate to stabilize inflation, in line with the Taylor principle. However, our results as figure 1 show, while the output gap increases and the interest rate responds procyclically as expected, inflation surprisingly decreases in the short run. This deviation can be attributed to two key features of the model: the extremely flat Phillips curve ($\kappa = 0.024$), which weakens the transmission from real activity to inflation, and a strong forward-looking component in price-setting behavior, whereby agents significantly revise down their expected future inflation following the anticipated monetary tightening. As a result, the expectation term ($\beta E_t \pi_{t+1}$) dominates the direct effect of output on inflation, leading to a net decline in observed inflation. Over time, as the shock decays at rate $\rho_r < 1$ and policy tightening closes the output gap, all three variables—π_t, x_t, and i_t—converge smoothly back to their steady-state targets.

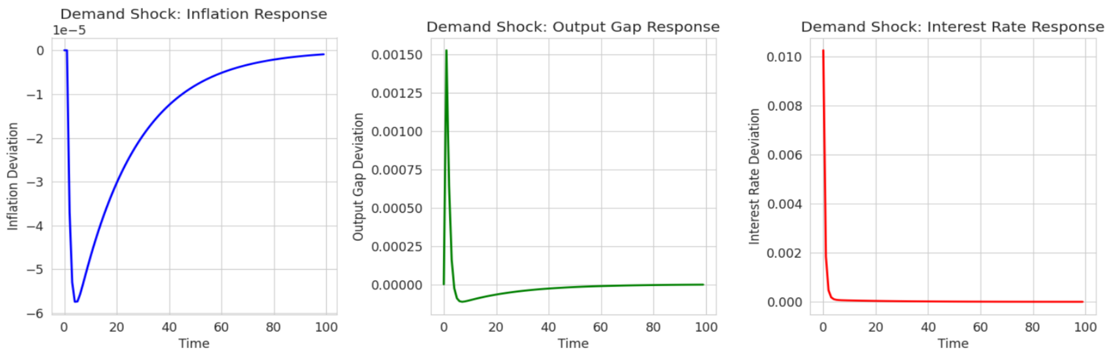

Figure 1: Response to ε_{rt}

Similarly, a positive cost-push shock ($\varepsilon_{u_t} > 0$) would ordinarily be expected to raise inflation due to direct upward pressure on marginal costs, reduce output through higher real interest rates, and elicit a tightening monetary policy response. In contrast, our results shown in figure 2 displays an initial decline in inflation, a fall in output, and a non-monotonic interest rate response, where the policy rate first drops sharply and then rises. This behavior is again explained by the forward-looking nature of inflation expectations, which initially overcompensate for the shock due to strong belief in the central bank's credibility and commitment to price stability, thus pulling down π_t through the expectations channel. Additionally, the relatively high penalty on interest rate volatility in the central bank's loss function ($\lambda_i = 0.236$) encourages a more gradual policy adjustment. The central bank may initially accommodate the shock by lowering the interest rate, especially if the observed inflation does not immediately rise, and only later tightens policy as inflationary expectations adjust. As the cost-push disturbance weakens in an autoregressive manner (with persistence $\rho_u < 1$), monetary policy gradually eases, allowing inflation, output, and the interest rate to return to their respective long-run benchmarks.

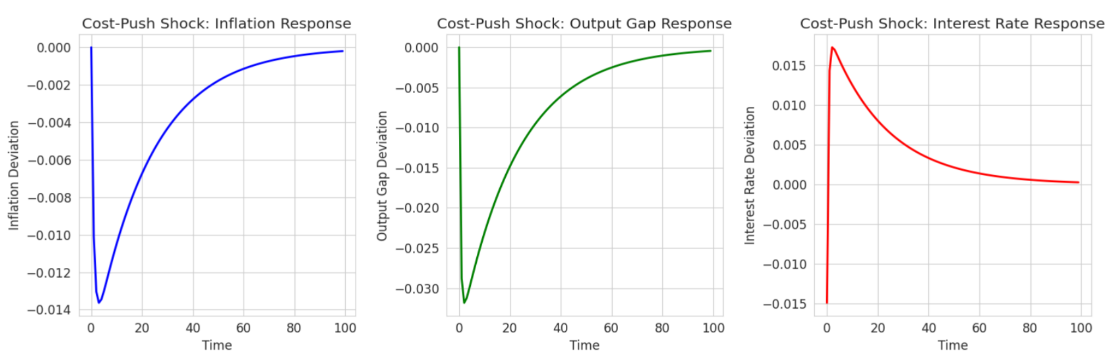

Figure 2: Response to ε_{ut}

2.3.3 Sensitivity To Parameters

The parameter σ captures the sensitivity of the output gap to changes in the real interest rate, reflecting how responsive consumption and investment decisions are to intertemporal price changes. A higher σ indicates that agents adjust their spending behavior more aggressively in response to interest rate shifts, enhancing the effectiveness of monetary policy. Consequently, when σ is large, smaller adjustments in the nominal interest rate are sufficient to stabilize both inflation and the output gap. This reduces the need for strong policy responses, leading to lower absolute values of the optimal coefficients on both inflation ($|\phi_\pi|$) and the output gap ($|\phi_x|$) in the Taylor rule. Since we have positive ϕ_π and negative ϕ_x in the optimal Taylor rule, ϕ_π will decrease while ϕ_x will increase when σ become larger, as shown in figure 3.

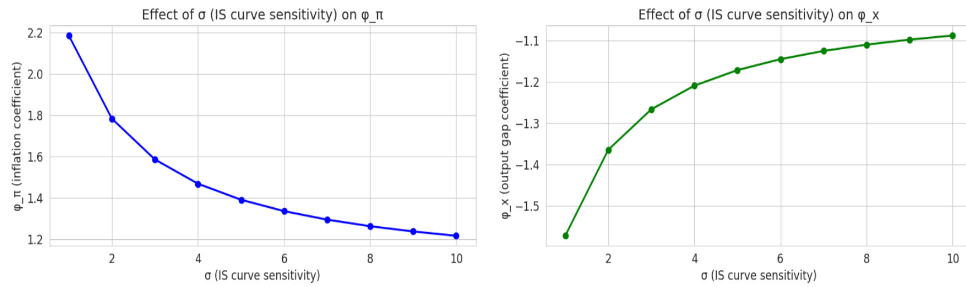

Figure 3: Sensitivity to σ

The parameters ρ_u and ρ_r capture the persistence of cost-push and demand shocks, respectively, representing the extent to which economic disturbances are transitory or long-lasting. Although persistent shocks prolong deviations of inflation and output from their targets, they do not alter the optimal steady-state reaction coefficients in the Taylor rule under discretionary policy. Instead, these parameters affect the dynamic paths of inflation, output, and interest rates following a shock, shaping the impulse responses rather than the structural form of the policy rule itself. As a result, ρ_u and ρ_r have no effect on ϕ_π and ϕ_x, as shown in figure 4 and figure 5.

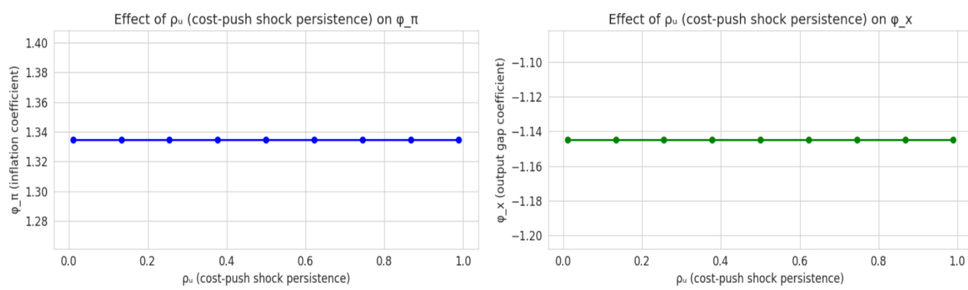

Figure 4: Sensitivity to ρ_u

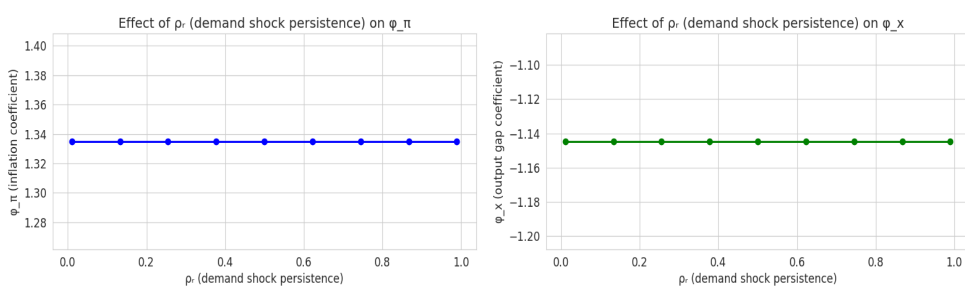

Figure 5: Sensitivity to ρ_r

The parameter λ_x measures the relative importance the central bank assigns to minimizing the output gap, reflecting its commitment to real economic stability, such as employment and growth. A higher λ_x implies greater tolerance for inflation deviations if necessary to support economic activity. Accordingly, the central bank optimally places greater weight on the output gap in the Taylor rule, resulting in a more negative ϕ_x (i.e., stronger countercyclical policy), and less weight on the inflation in the Taylor rule, resulting in a smaller ϕ_π, as shown in figure 6.

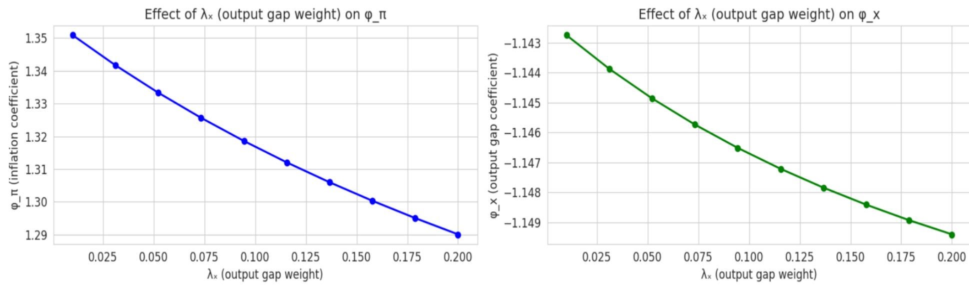

Figure 6: Sensitivity to λ_x

The parameter λ_i represents the weight the central bank places on minimizing deviations of the nominal interest rate from its target. A higher λ_i means that the central bank exhibits a preference for interest rate smoothing, reflecting an aversion to large and abrupt changes in the nominal interest rate. As a result, it optimally attenuates its policy responses to fluctuations in inflation and the output gap, leading to lower Taylor rule coefficients in absolute value, as shown in figure 7. This implies a willingness to tolerate greater volatility in inflation or output in exchange for a more stable and predictable interest rate trajectory.

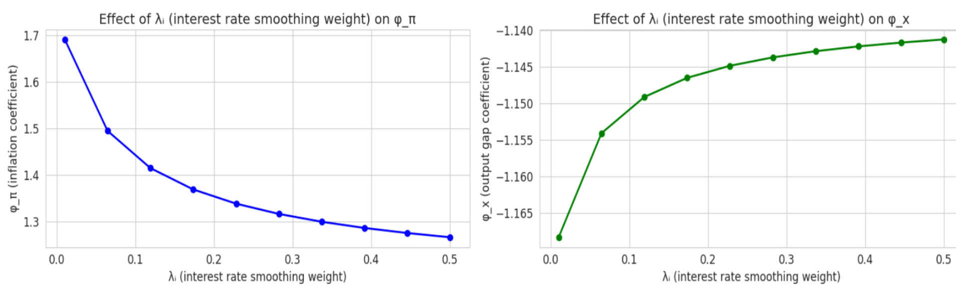

Figure 7: Sensitivity to λ_i

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