FE5213: Quantitative Macroeconomics and Finance with Python

Project on Dynamics in Monetary Policy Models

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Overview: Central banks aim to stabilize inflation and output fluctuations using monetary policy instruments such as the interest rate, typically via a Taylor rule. The analysis of monetary policy is an important issue in New Keynesian monetary economics. In this project, we will carry out two analyses using Python: (1) the linear dynamics in a monetary model; (2) the solution to a central bank's optimal policy problem. Our project can be found in the GitHub repository https://github.com/yifandill/Monetary_Policy_Model, contributed by Hui Yangyifan, Huang Anna, He Sixian.

Contents

1 Linear Dynamics in a Monetary Model

In this section, we will analyze the linear dynamics in a monetary model. It involves formulating and setting up the state-space representation, solving impulse responses to economic and monetary shocks, and conducting comparative statics as the model parameters change. Referred to Sargent & Stachurski (2024): Linear State Space Models.

1.1 Problem Setup

1.1.1 Economic Dynamics

We consider a simple New Keynesian model, where the economy is described by a Phillips curve and IS curve. In particular,

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + u_t, \tag{1}$$

$$x_t = \mathbb{E}_t x_{t+1} - \sigma \left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n \right), \tag{2}$$

where (1) is the New Keynesian Phillips curve and (2) is the IS curve. In these equations, π_t denotes inflation, x_t output gap, i_t nominal interest rate (policy instrument), u_t cost-push shock, r_t^n demand shock, β discount factor, κ slope of the Phillips curve, σ sensitivity to the real interest rate.

1.1.2 Central Bank's Policy Rule

The central bank sets the interest rate via the Taylor rule, which takes the following form:

$$i_t = \phi_\pi \pi_t + \phi_x x_t + \nu_t, \tag{3}$$

where ν_t is the monetary policy surprise/shock.

1.1.3 Shocks Assumption

Assume that u_t, r_t^n, and ν_t follow the following AR(1) process:

$$u_t = \rho_u u_{t-1} + \sigma_u \varepsilon_{ut}, \tag{4}$$

$$r_t^n = \rho_r r_{t-1}^n + \sigma_r \varepsilon_{rt}, \tag{5}$$

$$\nu_t = \rho_\nu \nu_{t-1} + \sigma_\nu \varepsilon_{\nu t}, \tag{6}$$

where $\varepsilon_{ut} \sim N(0,1)$, $\varepsilon_{rt} \sim N(0,1)$, and $\varepsilon_{\nu t} \sim N(0,1)$.

1.1.4 Model Summary

Variables	**Meaning**	**Setting**
π_t	inflation	New Keynesian Phillips curve (1)
x_t	output gap	IS curve (2)
i_t	nominal interest rate (policy instrument)	Taylor rule (3)
u_t	cost-push shock	AR(1) process (4)
r_t^n	demand shock	AR(1) process (5)
ν_t	monetary policy shock	AR(1) process (6)
ε_{ut}	unit white noise for cost-push shock	$N(0,1)$
ε_{rt}	unit white noise for demand shock	$N(0,1)$
$\varepsilon_{\nu t}$	unit white noise for monetary policy shock	$N(0,1)$

Table 1: Variables in the model: the output (π_t, x_t, i_t), three shocks (u_t, r_t^n, ν_t), with unit white noise $(\varepsilon_{ut}, \varepsilon_{rt}, \varepsilon_{\nu t})$.

Parameter	Meaning	Benchmark Value
β	discount factor	0.99
σ	sensitivity to the real interest rate	1/6
κ	slope of the Phillips curve	0.024
ϕ_π	sensitivity to the inflation	1.5
ϕ_x	sensitivity to the output gap	0.5
ρ_r	AR(1) coefficient for demand shock	0.35
ρ_u	AR(1) coefficient for cost-push shock	0.35
ρ_ν	AR(1) coefficient for monetary policy shock	0.35
σ_r	std for demand shock white noise	3.7
σ_u	std for cost-push shock white noise	0.4
σ_ν	std for monetary policy shock white noise	1

Table 2: Parameters in the model. In the benchmark analysis, use the calibrated parameters taken from the literature.

1.2 Problem Analysis

The rational expectations equilibrium for this model can be solved as follows.

1.2.1 Model Solution

Conjecture a model solution where the output is linear in the state comprised of three shocks (u_t, r_t^n, ν_t),

$$\begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = \mathbf{P} \begin{bmatrix} u_t \\ r_t^n \\ \nu_t \end{bmatrix} := \begin{bmatrix} \gamma_\pi^u & \gamma_\pi^r & \gamma_\pi^\nu \\ \gamma_x^u & \gamma_x^r & \gamma_x^\nu \\ \gamma_i^u & \gamma_i^r & \gamma_i^\nu \end{bmatrix} \begin{bmatrix} u_t \\ r_t^n \\ \nu_t \end{bmatrix}. \tag{7}$$

With (7), we can write the one-period-ahead expectations accordingly,

$$\begin{aligned} \mathbb{E}_t \pi_{t+1} &= \gamma_\pi^u \rho_u u_t + \gamma_\pi^r \rho_r r_t^n + \gamma_\pi^\nu \rho_\nu \nu_t, \\ \mathbb{E}_t x_{t+1} &= \gamma_x^u \rho_u u_t + \gamma_x^r \rho_r r_t^n + \gamma_x^\nu \rho_\nu \nu_t, \\ \mathbb{E}_t i_{t+1} &= \gamma_i^u \rho_u u_t + \gamma_i^r \rho_r r_t^n + \gamma_i^\nu \rho_\nu \nu_t. \end{aligned} \tag{8}$$

To solve for $\mathbf{P}$, we can substitute items in (1)(2)(3) with (7) and (8) and solve $\mathbf{P}$ using the method of undetermined coefficients. In particular, for (1),

$$\gamma_\pi^u u_t + \gamma_\pi^r r_t^n + \gamma_\pi^\nu \nu_t = \beta\left(\gamma_\pi^u \rho_u u_t + \gamma_\pi^r \rho_r r_t^n + \gamma_\pi^\nu \rho_\nu \nu_t\right) + \kappa\left(\gamma_x^u u_t + \gamma_x^r r_t^n + \gamma_x^\nu \nu_t\right) + u_t,$$

collecting items and re-arranging,

$$\left(\gamma_\pi^u - \beta\gamma_\pi^u\rho_u - \kappa\gamma_x^u - 1\right) u_t + \left(\gamma_\pi^r - \beta\gamma_\pi^r\rho_r - \kappa\gamma_x^r\right) r_t^n + \left(\gamma_\pi^\nu - \beta\gamma_\pi^\nu\rho_\nu - \kappa\gamma_x^\nu\right)\nu_t = 0,$$

This implies that

$$\gamma_\pi^u = \beta\gamma_\pi^u\rho_u + \kappa\gamma_x^u + 1, \tag{9}$$

$$\gamma_\pi^r = \beta\gamma_\pi^r\rho_r + \kappa\gamma_x^r, \tag{10}$$

$$\gamma_\pi^\nu = \beta\gamma_\pi^\nu\rho_\nu + \kappa\gamma_x^\nu. \tag{11}$$

Similarly, for (2)

$$\gamma_x^u u_t + \gamma_x^r r_t^n + \gamma_x^\nu \nu_t = (\gamma_x^u \rho_u u_t + \gamma_x^r \rho_r r_t^n + \gamma_x^\nu \rho_\nu \nu_t) - \sigma\left[(\gamma_i^u u_t + \gamma_i^r r_t^n + \gamma_i^\nu \nu_t) - (\gamma_\pi^u \rho_u u_t + \gamma_\pi^r \rho_r r_t^n + \gamma_\pi^\nu \rho_\nu \nu_t) - r_t^n\right],$$

collecting items, we have

$$\gamma_x^u = \gamma_x^u \rho_u - \sigma\left(\gamma_i^u - \gamma_\pi^u \rho_u\right), \tag{12}$$

$$\gamma_x^r = \gamma_x^r \rho_r - \sigma\left(\gamma_i^r - \gamma_\pi^r \rho_r - 1\right), \tag{13}$$

$$\gamma_x^\nu = \gamma_x^\nu \rho_\nu - \sigma\left(\gamma_i^\nu - \gamma_\pi^\nu \rho_\nu\right). \tag{14}$$

For (3),

$$\gamma_i^u u_t + \gamma_i^r r_t^n + \gamma_i^\nu \nu_t = \phi_\pi\left(\gamma_\pi^u u_t + \gamma_\pi^r r_t^n + \gamma_\pi^\nu \nu_t\right) + \phi_x\left(\gamma_x^u u_t + \gamma_x^r r_t^n + \gamma_x^\nu \nu_t\right) + \nu_t,$$

so we have

$$\gamma_i^u = \phi_\pi \gamma_\pi^u + \phi_x \gamma_x^u, \tag{15}$$

$$\gamma_i^r = \phi_\pi \gamma_\pi^r + \phi_x \gamma_x^r, \tag{16}$$

$$\gamma_i^\nu = \phi_\pi \gamma_\pi^\nu + \phi_x \gamma_x^\nu + 1. \tag{17}$$

Note that in the system of equations (9) through (17), there are 9 equations with 9 unknowns, which are entries in the matrix $\mathbf{P}$. We denote the vectorization of the matrix $\mathbf{P}$ as vector $\mathbf{x}$,

$$\mathbf{P} = \begin{bmatrix} \gamma_\pi^u & \gamma_\pi^r & \gamma_\pi^\nu \\ \gamma_x^u & \gamma_x^r & \gamma_x^\nu \\ \gamma_i^u & \gamma_i^r & \gamma_i^\nu \end{bmatrix} \implies \mathbf{x} := \mathrm{vec}(\mathbf{P}) = \left[\gamma_\pi^u,\ \gamma_x^u,\ \gamma_i^u,\ \gamma_\pi^r,\ \gamma_x^r,\ \gamma_i^r,\ \gamma_\pi^\nu,\ \gamma_x^\nu,\ \gamma_i^\nu\right]'.$$

The system of linear equations (9) through (17) can be packed into matrix form

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{18}$$

where

$$\mathbf{A} = \begin{bmatrix} 1-\beta\rho_u & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\beta\rho_r & -\kappa & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-\beta\rho_\nu & -\kappa & 0 \\ \sigma\rho_u & \rho_u - 1 & -\sigma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma\rho_r & \rho_r - 1 & -\sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma\rho_\nu & \rho_\nu - 1 & -\sigma \\ \phi_\pi & \phi_x & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_\pi & \phi_x & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \phi_\pi & \phi_x & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -\sigma \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

(18) can be easily solved using matrix inversion, so that we get $\mathrm{vec}(\mathbf{P}) = \mathbf{A}^{-1}\mathbf{b}$. In this way, $\mathbf{P}$ is solved and known.

1.2.2 State-Space Form

With $\mathbf{P}$, the **rational expectations equilibrium** for this model is characterized by a linear state-space system with the measurement equation

$$\begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = \mathbf{P} \begin{bmatrix} u_t \\ r_t^n \\ \nu_t \end{bmatrix}, \tag{19}$$

and the transition equation

$$\begin{bmatrix} u_t \\ r_t^n \\ \nu_t \end{bmatrix} = \begin{bmatrix} \rho_u & 0 & 0 \\ 0 & \rho_r & 0 \\ 0 & 0 & \rho_\nu \end{bmatrix} \begin{bmatrix} u_{t-1} \\ r_{t-1}^n \\ \nu_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma_u & 0 & 0 \\ 0 & \sigma_r & 0 \\ 0 & 0 & \sigma_\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{ut} \\ \varepsilon_{rt} \\ \varepsilon_{\nu t} \end{bmatrix}. \tag{20}$$

1.3 Results

We use Python to process the following procedures and discuss the results.

- Solve $\mathbf{P}$ from (18) and formulate the model solutions as the linear state space system (19)(20).
- Pose a one standard deviation change in economic shocks $\varepsilon_{rt}, \varepsilon_{ut}$ and the monetary policy shock $\varepsilon_{\nu t}$, respectively. Plot and discuss the dynamic responses of $\pi_{t+i}, x_{t+i}, i_{t+i}$ to the impulse.
- Investigate how the results would change if we change $\kappa, \rho_u, \rho_r, \phi_\pi, \phi_x$. Discuss the results.

1.3.1 Benchmark Analysis

We take the calibrated parameters from Table 2 to carry out the benchmark analysis. The matrix $\mathbf{P}$ is solved as

$$\mathbf{P} = \begin{bmatrix} 1.5156735 & 0.00826731 & -0.00826728 \\ -0.3961419 & 0.22511195 & -0.22511198 \\ 2.0754392 & 0.12495694 & 0.87504311 \end{bmatrix}.$$

When analyzing impulse responses to economic shocks, we typically isolate the effect of a single shock at $t = 0$ to study how the system responds over time. Here we apply one standard deviation change to these three shocks (cost-push, demand, monetary policy) at $t = 0$ and observe how the curves change and how long it takes for the system to return to equilibrium. We need to set all other shocks to zero in subsequent periods after $t = 0$, because if we introduce continuous shocks in later periods, it becomes difficult to distinguish the effects of the initial shock from the new ones. By setting all subsequent shocks to zero, we ensure that the observed dynamics are due to the initial shock only. Referred to Aiyagari (1994). Once the initial shock is applied, the system evolves based on its own structure, including persistence parameters ρ_r, ρ_u, ρ_ν, feedback rules, and equilibrium relationships. The persistence of the shock ensures that its effects last beyond $t = 0$ without needing additional shocks.

Remark. In other words, to normally research this model, the shocks should be transitory. Otherwise, if the shocks ε_{ut}, ε_{rt}, $\varepsilon_{\nu t}$ are randomly drawn from $N(0, 1)$ at every time t, i.e., the noises (diffusion terms) are permanent, the dynamics of π, x, i evolving with time would be totally fluctuating and irregular.

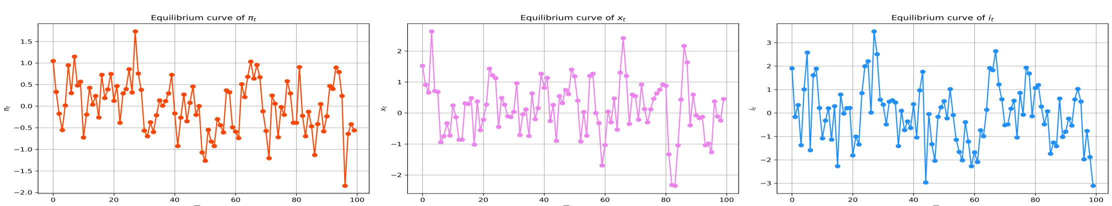


Figure 1: Demonstration of dynamics under permanent shocks

Dynamic Responses to Demand Shock

A demand shock r_t^n directly affects the output gap x_t and indirectly influences inflation π_t and nominal interest rates i_t through monetary policy responses.

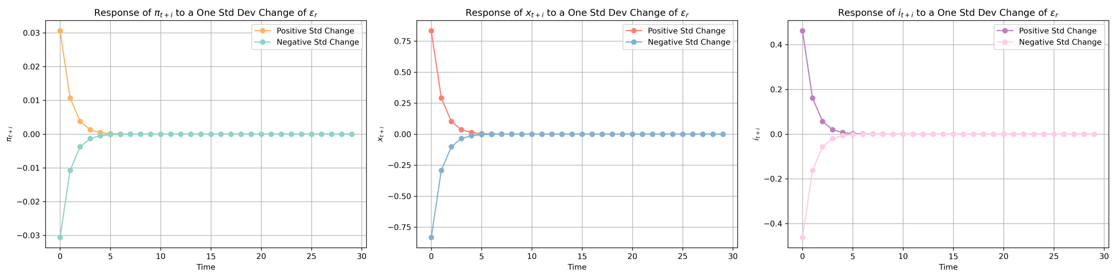


Figure 2: Dynamic responses to one standard deviation change of **demand shock**.

We can see when the demand shock has a positive change, it leads to increasing aggregate demand and thus a higher output gap. In the figure, this corresponds to the immediately rising x_t. And more demand pushes the prices of goods and services up, because the demand exceeds supply, leading to higher inflation π_t. Therefore, the central bank reacts by raising interest rates i_t to control inflation. When the demand shock has a negative change, vice versa.

After the initial shock, they gradually return to equilibrium as the shock fades. The system still evolves only through its internal mechanisms, such as the persistence parameters ρ_r, ρ_u, ρ_ν and feedback relationship.

Dynamic Responses to Cost-Push Shock

A cost-push shock u_t directly raises inflation π_t by increasing firms' marginal costs, and indirectly influences the output gap x_t and the nominal interest rate i_t through the central bank's policy response to counteract the higher price pressure.

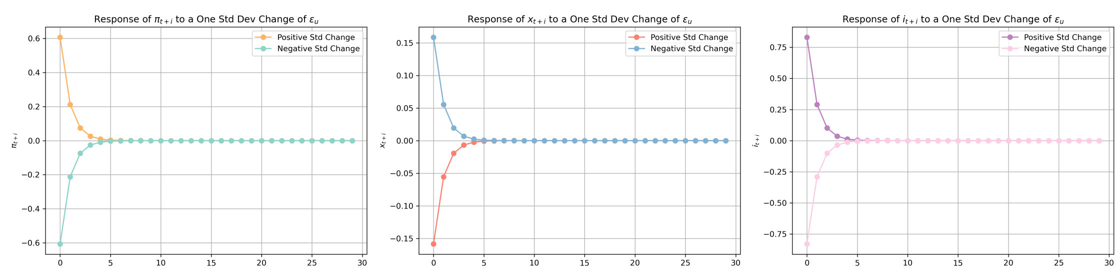


Figure 3: Dynamic responses to one standard deviation change of **cost-push shock**.

We can see when the cost-push shock has a positive change, it leads to higher inflation π_t, because firms face higher costs and tend to pass them on to consumers as higher prices. For the nominal interest rate i_t, since it reflects monetary policy reaction, it responds by increasing value to counteract inflation, because the central bank reacts by raising interest rates. And the output gap x_t will be lower, because higher inflation and the tendency to higher prices of goods and services reduce demand. When the cost-push shock has a negative change, vice versa.

After the initial shock, the system evolves only through its internal mechanisms, such as the persistence parameters ρ_r, ρ_u, ρ_ν. Since there are no new shocks interfering, the effect of the original shock gradually decays over time. Eventually, all variables return to steady state as the shock dissipates.

Dynamic Responses to Monetary Policy Shock

A monetary policy shock ν_t represents unexpected changes in the central bank's interest rate decision. It directly affects the nominal interest rate i_t and influences inflation π_t and the output gap x_t through supply-demand side.

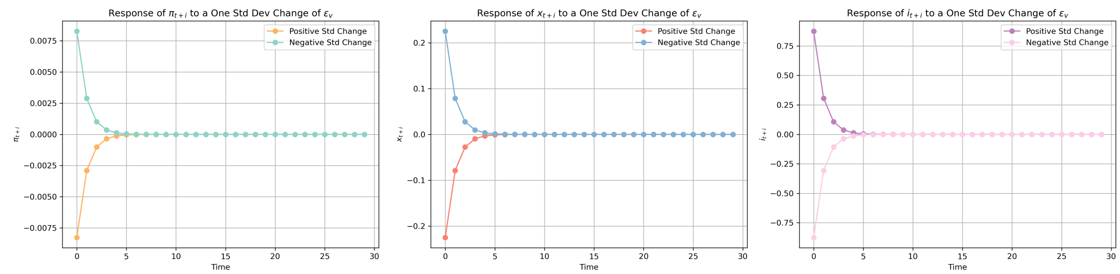


Figure 4: Dynamic responses to one standard deviation change of **monetary policy shock**.

When the monetary policy shock has a positive change, it leads to unexpected increase in interest rate i_t. Firms will cut back on investments in new projects, hiring, and expansion because of higher borrowing cost, leading to a lower output gap x_t. For people, higher interest rates make loans more expensive, weakening the consumption power. Therefore, the lower demand reduces inflation π_t over time, as firms adjust prices downward. When the demand shock has a negative change, vice versa.

After the initial shock, they gradually return to equilibrium as the shock fades. The system still evolves only through its internal mechanisms, such as the persistence parameters ρ_r, ρ_u, ρ_ν and feedback relationship.

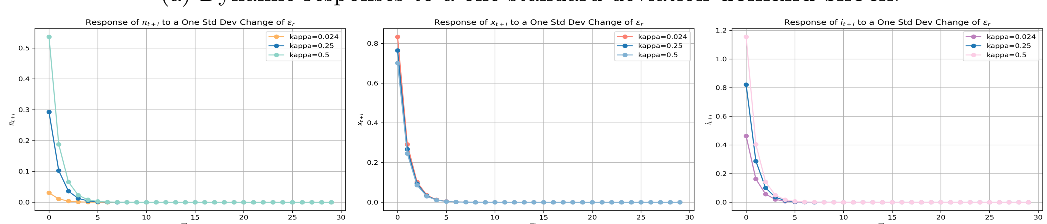
1.3.2 Parameter Discussion

The parameters κ, ρ_u, ρ_r, ϕ_π, ϕ_x influence how economic shocks impact inflation π_t, the output gap x_t, and the interest rate i_t. We isolate the change of every parameter to see how they influence the effect of a single shock at $t = 0$ on system.

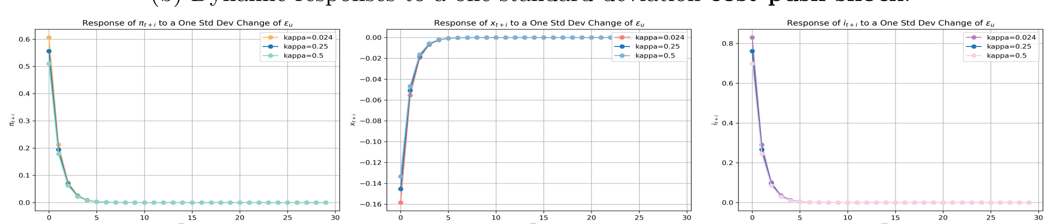
Effect of Changing Phillips Curve Slope κ

The slope of Phillips Curve κ is related to the price stickiness, and it captures the sensitivity of inflation to the output gap in the Phillips curve equation. Now we increase the κ from 0.024 to 0.25 and 0.5 to see how a weaker price stickiness affects the result in previous task. Here we set the one standard deviation change of three shocks to positive.

(a) Dynamic responses to a one-standard-deviation **demand shock**.



(b) Dynamic responses to a one-standard-deviation **cost-push shock**.



(c) Dynamic responses to a one-standard-deviation **monetary-policy shock**.

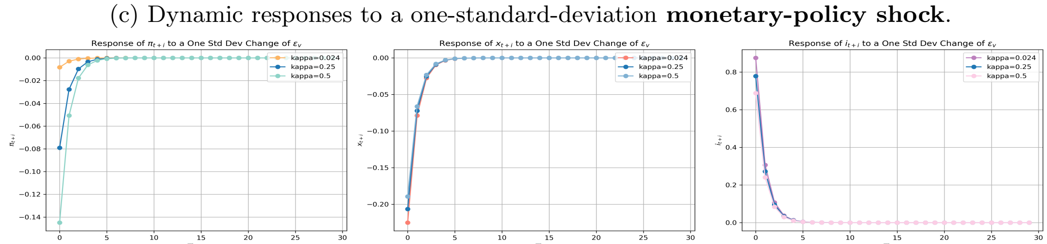


Figure 5: Impulse-response functions to one-standard-deviation structural shocks, different κ values.

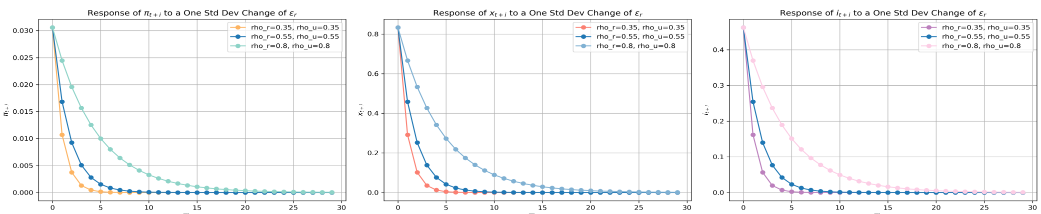
For the change of cost-push shock, increasing slope of Phillips Curve κ has a minor impact on its effect on system. But for the change of demand shock, a small change in demand shock will cause a bigger positive inflation response, as inflation reacts more strongly to the output gap if kappa increases. For the change of monetary policy shock, it leads to a bigger negative inflation response, because the monetary policy must act aggressively to stabilize inflation.

If κ Decreases (Stronger Price Stickiness): Inflation adjusts more slowly, leading to more persistent inflationary pressures after a shock. Central bank actions (monetary policy shocks) have a weaker immediate effect on inflation, requiring more aggressive policy moves.

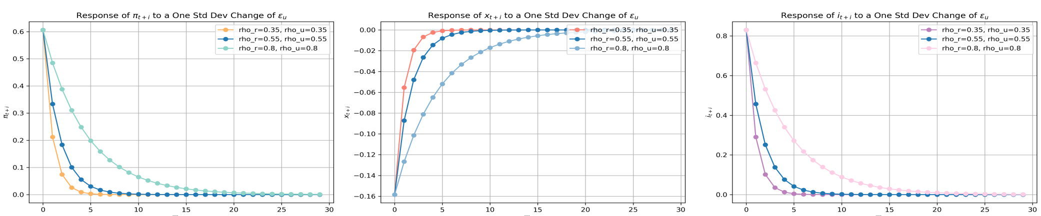
Effect of Changing Shock Persistence ρ_u and ρ_r

The parameters ρ_u and ρ_r serve as the auto-regressive (or persistence) coefficients for the cost-push shock and demand shock respectively. Now we increase the ρ_u and ρ_r from 0.35 to 0.8 to see how a strong shock persistence affects the result in previous task. Here we set the one standard deviation change of three shocks to positive.

(a) Dynamic responses to a one-standard-deviation **demand shock**.



(b) Dynamic responses to a one-standard-deviation **cost-push shock**.



(c) Dynamic responses to a one-standard-deviation **monetary-policy shock**.

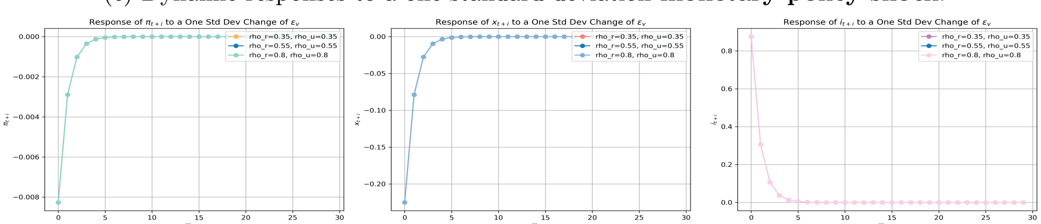


Figure 6: Impulse-response functions to one-standard-deviation structural shocks, different ρ_u and ρ_r values.

For cost-push shock, a larger ρ_u makes the cost-push shock more persistent. As a cost-push shock directly raises production costs, which firms pass on as higher prices, the cost-push disturbance lingers longer, meaning inflation remains elevated over more periods and sustained inflationary pressures. The higher interest rate means the central bank tightens policy aggressively to counteract inflation, so the output gap can turn negative (output falls below potential), leading to a stagflation-like scenario.

For demand shock, a higher ρ_r means that a one-time demand shock lasts longer. As a positive demand shock pushes up consumption and aggregate demand. It can lead to a more persistent increment in inflation via the Phillips curve. Besides, a positive demand shock usually drives the economy above its potential output, creating a positive output gap. Under the Taylor-rule framework, the central bank responds to higher inflation and a positive output gap by raising interest rates.

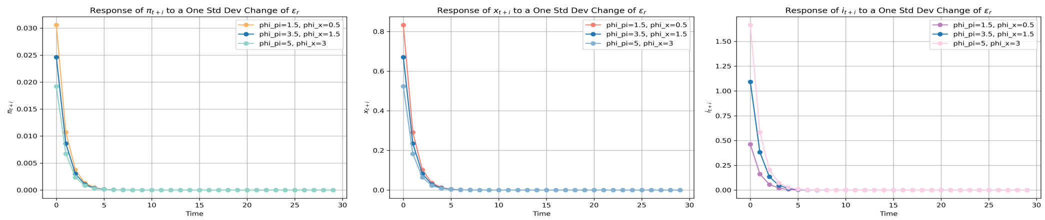
The degree of the persistence effect depends on how strongly these shocks translates into price pressures in the model. In the figure we can see the number of periods to steady state changes from five to twenty.

For monetary policy shock, since the persistence parameter ρ_ν remains unchanged, the curve remains the same as before.

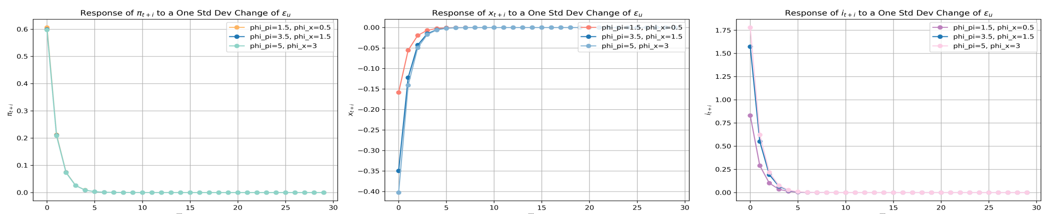
Effect of Changing Sensitivity ϕ_π and ϕ_x

ϕ_π and ϕ_x are the key coefficients in the Taylor rule that determine how strongly monetary policy reacts to inflation and the output gap, respectively. Now we increase the ϕ_π from 1.5 to 3.5 and 5, ϕ_x from 0.15 to 1.5 and 3 to see how a strong shock monetary policy affects the result in previous task. Here we set the one standard deviation change of three shocks to positive.

(a) Dynamic responses to a one-standard-deviation **demand shock**.



(b) Dynamic responses to a one-standard-deviation **cost-push shock**.



(c) Dynamic responses to a one-standard-deviation **monetary-policy shock**.

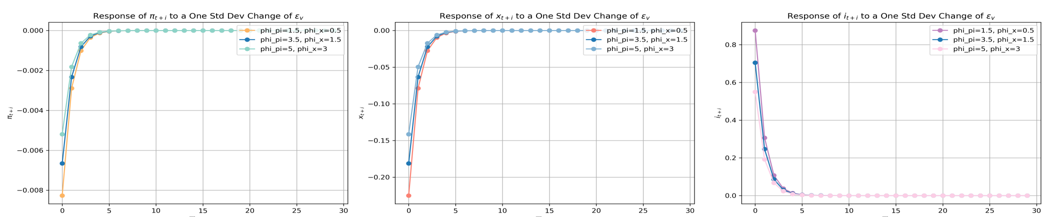


Figure 7: Impulse-response functions to one-standard-deviation structural shocks, different ϕ_π and ϕ_x values.

A cost-push shock raises production costs, putting upward pressure on inflation. With a higher ϕ_π, the central bank reacts more aggressively to rising inflation by increasing the nominal interest rate. And the tighter monetary policy can cool economic activity. This may lead to a contraction in the output gap, as higher interest rates reduce spending and investment. The inflation, however, remains unchanged under the Taylor Rule structure.

A positive demand shock directly raises aggregate demand, influencing the inflation and output gap. Since a demand shock directly affects the output gap, a higher ϕ_x leads to a quicker and more forceful adjustment. The central bank raises rates more sharply to counteract an excess demand situation, which tends to reduce the output gap faster. Then by stabilizing the output gap, the secondary effect is a moderation of inflationary pressures. With a higher ϕ_π, the central bank responds more forcefully to any resulting inflation, thereby curbing the inflation rise.

When faced with a monetary shock, a higher ϕ_π and ϕ_x indicate that the monetary authority is quick to correct any deviations in both inflation and the output gap. This typically means a minor impact on the result due to the change of ϕ_π and ϕ_x.

2 An Optimal Monetary Policy Problem

In this section, we apply Linear-Quadratic Dynamic Programming (LQDP) to study optimal monetary policy in a simple New Keynesian framework. This problem can be formulated as a Linear-Quadratic Regulator (LQR), where the policy-maker minimizes deviations of inflation and output from their targets. It involves formulating the central bank's objective function, setting up the state-space representation, solving the optimal policy problem using numerical methods, analyzing the optimal policy response to economic shocks, and doing comparative statics when the model parameters change. Referred to Sargent & Stachurski (2024): LQ Control: Foundations.

2.1 Problem Setup

We consider a simple New Keynesian model, where the economy is described by a Phillips curve (1) and IS curve (2),

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + u_t, \tag{1}$$

$$x_t = \mathbb{E}_t x_{t+1} - \sigma \left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n \right), \tag{2}$$

with shocks following AR(1) processes (4)(5),

$$u_t = \rho_u u_{t-1} + \sigma_u \varepsilon_{ut}, \tag{4}$$

$$r_t^n = \rho_r r_{t-1}^n + \sigma_r \varepsilon_{rt}, \tag{5}$$

where $\varepsilon_{ut} \sim N(0,1)$, $\varepsilon_{rt} \sim N(0,1)$.

2.1.1 Central Bank's Loss Function

The central bank seeks to minimize a quadratic loss function:

$$L = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\pi_t^2 + \lambda_x (x_t - x^*)^2 + \lambda_i (i_t - i^*)^2 \right), \tag{21}$$

where weights $\lambda_x, \lambda_i > 0$ reflect relative weight on output stabilization and penalty on interest rate volatility; x^*, i^* represent some optimal levels of output gap and nominal interest rate. We focus on a class of monetary policy rules, the Taylor rule, which takes the following form

$$i_t = \phi_\pi \pi_t + \phi_x x_t + \phi_0. \tag{22}$$

2.1.2 Model Summary

Variables	Meaning	Setting
π_t	inflation	New Keynesian Phillips curve (1)
x_t	output gap	IS curve (2)
i_t	nominal interest rate (policy instrument)	Taylor rule (22)
u_t	cost-push shock	AR(1) process (4)
r_t^n	demand shock	AR(1) process (5)
ε_{ut}	unit white noise for cost-push shock	$N(0,1)$
ε_{rt}	unit white noise for demand shock	$N(0,1)$

Table 3: Variables in the model: (π_t, x_t, i_t) of our interest, two shocks (u_t, r_t^n), with unit white noise $(\varepsilon_{ut}, \varepsilon_{rt})$.

Parameter	Meaning	Benchmark Value
β	discount factor	0.99
σ	sensitivity to the real interest rate	1/6
κ	slope of the Phillips curve	0.024
ρ_r	AR(1) coefficient for demand shock	0.35
ρ_u	AR(1) coefficient for cost-push shock	0.35
σ_r	std for demand shock white noise	13.8
σ_u	std for cost-push shock white noise	0.17
λ_x	relative weight on output stabilization	0.048
λ_i	penalty on interest rate volatility	0.236
x^*	some optimal levels of output gap	0
i^*	some optimal levels of nominal interest rate	$1/\beta - 1$

Table 4: Parameters in the model. In the benchmark analysis, use the calibrated parameters taken from the literature.

Remark. In standard macroeconomic models, particularly within the New Keynesian framework, the optimal output gap x^* is typically set to zero. This specification reflects the central bank's objective of maintaining output at its potential level, thereby minimizing inflationary or deflationary pressures that arise from output deviations. Consequently, we adopt $x^* = 0$ in our optimal monetary policy model.

Denote the inflation target as π^*. Starting from the representative household's Euler equation, $1 = \beta \dfrac{1+i^*}{1+\pi^*}$, the gross nominal risk-free return that is consistent with intertemporal optimality must satisfy $i^* = \left(\frac{1}{\beta} - 1\right) + \pi^*$, where the first term, $r^* \equiv 1/\beta - 1$, is the natural real interest rate. When the central bank targets zero steady-state inflation ($\pi^* = 0$, a standard normalization that keeps the algebra transparent), and isolates the propagation of nominal rigidities—the optimal nominal rate collapses to the real rate, $i^* = 1/\beta - 1$. With the conventional quarterly calibration $\beta = 0.99$, this implies an annualized neutral nominal rate of roughly four percent, though lower values of i^* can be obtained by choosing a higher β to match the persistently low real rates observed in recent decades. Referred to Debortoli *et al.* (2017).

2.2 Problem Solution

2.2.1 State-Space Representation

Let $\mathbf{s_t} = [1,\, \pi_t,\, x_t - x^*,\, u_t,\, r_t^n]'$ be the state vector, and $\mathbf{i_t} = [i_t - i^*]$ be the control vector. From (4)(5), we have

$$u_{t+1} = \rho_u u_t + \sigma_u \varepsilon_{ut+1},$$
$$r_{t+1}^n = \rho_r r_t^n + \sigma_r \varepsilon_{rt+1}.$$

We assume **certainty equivalence** to take the following approximations in (1)(2)

$$\mathbb{E}_t \pi_{t+1} \approx \pi_{t+1},\, \mathbb{E}_t x_{t+1} \approx x_{t+1}. \tag{23}$$

Certainty equivalence simplifies the decision-making process under uncertainty by replacing uncertain variables with their expected values. So, the expected future inflation and output gap should be, on average, equal to the future inflation and output gap. Then (1)(2) becomes

$$\beta \pi_{t+1} = \pi_t - \kappa x_t - u_t, \tag{1'}$$

$$x_{t+1} + \sigma \pi_{t+1} = x_t + \sigma i_t - \sigma r_t^n, \tag{2'}$$

Transforming using state and control variables, we get

$$\pi_{t+1} = \frac{1}{\beta}\pi_t - \frac{\kappa}{\beta}(x_t - x^*) - \frac{1}{\beta}u_t - \frac{\kappa}{\beta}x^*, \tag{24}$$

$$x_{t+1} - x^* = -\frac{\sigma}{\beta}\pi_{t+1} + \left(\frac{\sigma\kappa}{\beta} + 1\right)(x_t - x^*) + \frac{\sigma}{\beta}u_t - \sigma r_t^n + \sigma(\frac{\kappa}{\beta}x^* + i^*) + \sigma(i_t - i^*). \tag{25}$$

Let $\boldsymbol{\varepsilon}_t = [\varepsilon_{ut}, \varepsilon_{rt}]'$. Transfer (24)(25)(4)(5) into matrix form, we get the state dynamics

$$\mathbf{s_{t+1}} = \mathbf{As_t} + \mathbf{Bi_t} + \mathbf{C}\boldsymbol{\varepsilon}_{t+1}, \tag{26}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{\kappa}{\beta}x^* & \frac{1}{\beta} & -\frac{\kappa}{\beta} & -\frac{1}{\beta} & 0 \\ \sigma(\frac{\kappa}{\beta}x^* + i^*) & -\frac{\sigma}{\beta} & \frac{\sigma\kappa}{\beta}+1 & \frac{\sigma}{\beta} & -\sigma \\ 0 & 0 & 0 & \rho_u & 0 \\ 0 & 0 & 0 & 0 & \rho_r \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \sigma \\ 0 \\ 0 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sigma_u & 0 \\ 0 & \sigma_r \end{bmatrix}.$$

2.2.2 Value Function and Bellman Equation

Transforming the loss function (21) into quadratic form of state and control vectors, we get

$$L = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\mathbf{s_t'Rs_t} + \mathbf{i_t'Qi_t}\right), \tag{27}$$

where

$$\mathbf{R} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} \lambda_i \end{bmatrix}.$$

We define the value function

$$V(\mathbf{s_t}) = \min_{\mathbf{i_t}} \sum_{t=0}^{\infty} \beta^t \left(\mathbf{s_t'Rs_t} + \mathbf{i_t'Qi_t}\right), \tag{28}$$

and the Bellman Equation is

$$V(\mathbf{s_t}) = \min_{\mathbf{i_t}}(\mathbf{s}_t'\mathbf{Rs_t} + \mathbf{i_t'Qi_t} + \beta\mathbb{E}_t\left[V(\mathbf{s_{t+1}})\right]. \tag{29}$$

Using the quadratic guess for the value function:

$$V(\mathbf{s_t}) = \mathbf{s_t'Ps_t}, \tag{30}$$

where $\mathbf{P}$ is a symmetric positive semi-definite matrix that we solve for.

Plugging (30) in (29), the Bellman equation becomes

$$\mathbf{s_t'Ps_t} = \min_{\mathbf{i_t}}(\mathbf{s}_t'\mathbf{Rs_t} + \mathbf{i_t'Qi_t} + \beta\mathbb{E}_t\left[(\mathbf{As_t} + \mathbf{Bi_t} + \mathbf{C}\boldsymbol{\varepsilon}_{t+1})'\mathbf{P}(\mathbf{As_t} + \mathbf{Bi_t} + \mathbf{C}\boldsymbol{\varepsilon}_{t+1})\right]). \tag{31}$$

2.2.3 Stationary Optimal Policy

The first-order condition of (31) with respect to $\mathbf{i_t}$ gives

$$\mathbf{2Qi_t} + \mathbf{2}\beta\mathbf{B'P}(\mathbf{As_t} + \mathbf{Bi_t}) = \mathbf{0}. \tag{32}$$

Solving for $\mathbf{i_t}$, we get

$$\mathbf{i_t} = -\mathbf{Fs_t}, \tag{33}$$

where

$$\mathbf{F} = (\mathbf{Q} + \beta\mathbf{B'PB})^{-1}\beta\mathbf{B'PA}. \tag{34}$$

(33) and (34) give the **stationary optimal policy** for this model.

Remark. The objective function given by (27) is the infinite horizon case of the LQ control problem. In the infinite horizon case, optimal policies can depend on time only if time itself is a component of the state vector $\mathbf{s_t}$. In other words, there exists a fixed matrix $\mathbf{F}$ such that $\mathbf{i_t} = -\mathbf{Fs_t}$ for all t. That decision rules are constant over time is intuitive — after all, the decision-maker faces the same infinite horizon at every stage, with only the current state changing. The stationary matrix $\mathbf{P}$ is also constant, found by solving the discrete-time Riccati equation

$$\mathbf{P} = \mathbf{Q} + \beta\mathbf{A'PA} - \beta\mathbf{A'PB}(\mathbf{R} + \beta\mathbf{B'PB})^{-1}\mathbf{B'PA}. \tag{35}$$

Now, Denote $\mathbf{F} \equiv \begin{bmatrix} F_1 & F_2 & F_3 & F_4 & F_5 \end{bmatrix}$. Substituting (22) and (34) into (33), we have

$$\begin{aligned} \phi_\pi \pi_t + \phi_x x_t + \phi_0 &= -F_2\pi_t - F_3 x_t - F_4 u_t - F_5 r_t^n - F_1 + F_3 x^* + i^* \\ &\approx -F_2\pi_t - F_3 x_t - F_1 + F_3 x^* + i^*. \end{aligned} \tag{36}$$

Same as the Remark in 1.3.1, the shocks ε_{ut} and ε_{rt} for u_t and r_t^n should be transitory, posing at $t = 0$, and set as 0 afterwards. At stationary states, the u_t and r_t^n terms can be omitted. From (36), the optimal policy coefficients are

$$\phi_\pi = -F_2, \quad \phi_x = -F_3, \quad \phi_0 = -F_1 + F_3 x^* + i^*. \tag{37}$$

2.3 Results

We use Python to process the following procedures and discuss the results. Referred to Woodford (2003).

- Solve the optimal Taylor rule (22) using (37).
- Plot and discuss the dynamic responses to a one standard deviation change in shocks ε_{rt} and ε_{ut}.
- Investigate how the results would change if we change $\sigma, \rho_u, \rho_r, \lambda_x, \lambda_i$. Discuss the results.

2.3.1 Benchmark Analysis

We take the calibrated parameters from Table 4 to carry out the benchmark analysis. Feeding the prepared matrices **A**, **B**, **R**, **Q** into the **LQ** class from **QuantEcon**, we can get the stationary optimal policy (33) for this model. The **stationary_values** method from the LQ class can return the matrix **F** in (34) for us, which is

$$\mathbf{F} = \begin{bmatrix} 2.28317256 & -3.11539053 & 2.2376021 & 4.52954715 - 0.4352364 \end{bmatrix}.$$

With **F**, the optimal policy coefficients in (37) are known.

The Optimal Taylor Rule

The optimal Taylor rule in (22) is given by

$$i_t = 3.1154\pi_t - 2.2376x_t - 2.2731. \tag{38}$$

(38) reflects the central bank's best response under a standard New Keynesian framework with forward-looking expectations and a quadratic loss function. The coefficient on inflation (ϕ_π) exceeds one, satisfying the Taylor principle and ensuring that real interest rates rise in response to higher inflation, thereby anchoring inflation expectations. The negative output gap coefficient ϕ_x implies a counter-cyclical stance: when the economy operates below potential, the central bank reduces the nominal interest rate to stimulate demand. The intercept term reflects the implied steady-state nominal interest rate, which in this case is negative, suggesting a low or even negative natural real rate given the assumed inflation and output targets.

Central Bank's Optimal Response to Shocks

In the standard New Keynesian framework, a positive demand shock ($\varepsilon_{r_t} > 0$), which raises the natural rate of interest, typically leads to an increase in the output gap and a rise in inflation, as higher demand pressures push prices upward. In response, the central bank raises the nominal interest rate to stabilize inflation, in line with the Taylor principle. However, our results as Figure 8 show, while the inflation increases and the interest rate responds pro-cyclically as expected, the output gap surprisingly decreases in the short run. This deviation can be attributed to two key features of the model: the unusually low intertemporal substitution parameter ($\sigma = 0.167$) and very flat Phillips slope ($\kappa = 0.024$) render both the output gap and inflation remarkably insensitive to changes in the real interest rate, yet the optimal rule coefficients ($\phi_\pi \approx 3.12$ and $|\phi_x| \approx 2.24$) remain extremely large. The steep ϕ_π and ϕ_x "over-amplify" even a small demand shock into a forceful interest rate increase, squeezing the output gap negative, resulting in a decrease in the output gap. Over time, as the shock decays at rate $\rho_r < 1$ and policy tightening closes the output gap, all three variables—π_t, x_t, and i_t—converge smoothly back to their steady-state targets.

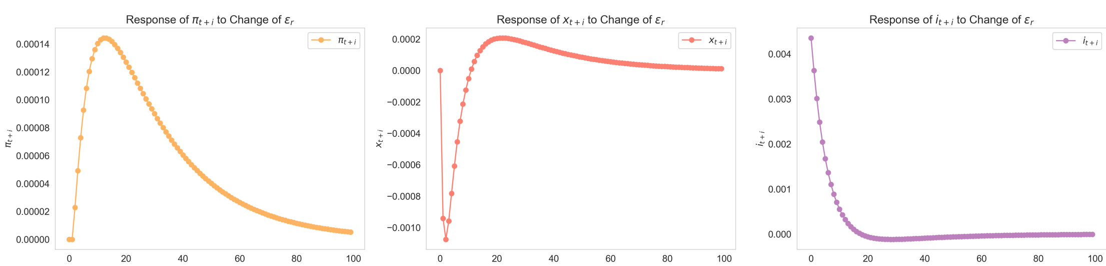


Figure 8: Dynamic responses to a one-standard-deviation demand shock ε_{rt}.

Similarly, a positive cost-push shock ($\varepsilon_{u_t} > 0$) would ordinarily be expected to raise inflation due to direct upward pressure on marginal costs, reduce output through higher real interest rates, and elicit a tightening monetary policy response. In contrast, our results shown in Figure 9 display an initial decline in inflation, a fall in output, and a non-monotonic interest rate response, where the policy rate first drops sharply and then rises. This behavior is again explained by the forward-looking nature of inflation expectations, which initially overcompensate for the shock due to a strong belief in the central bank's credibility and commitment to price stability, thus pulling down π_t through the expectations channel. Additionally, the relatively high penalty on interest rate volatility in the central bank's loss function ($\lambda_i = 0.236$) encourages a more gradual policy adjustment. The central bank may initially accommodate the shock by lowering the interest rate, especially if the observed inflation does not immediately rise, and only later tightens policy as inflationary expectations adjust. As the cost-push disturbance weakens in an autoregressive manner (with $\rho_u < 1$), monetary policy gradually eases, allowing inflation, output, and the interest rate to return to their respective long-run benchmarks.

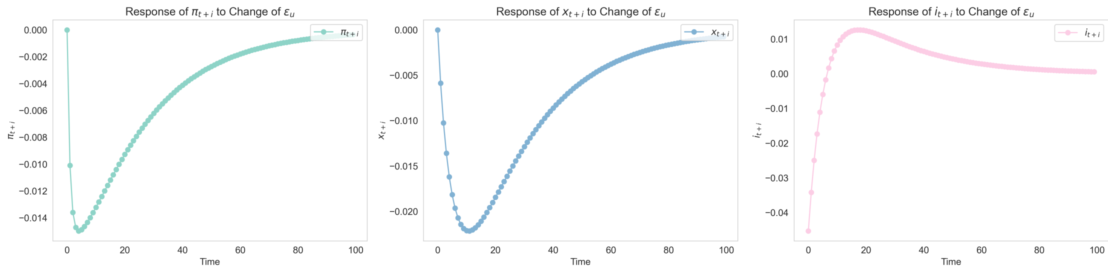


Figure 9: Dynamic responses to a one-standard-deviation cost-push shock ε_{ut}.

2.3.2 Parameter Sensitivity

The parameter σ captures the sensitivity of the output gap to changes in the real interest rate, reflecting how responsive consumption and investment decisions are to intertemporal price changes. A higher σ indicates that agents adjust their spending behavior more aggressively in response to interest rate shifts, enhancing the effectiveness of monetary policy. Consequently, when σ is large, smaller adjustments in the nominal interest rate are sufficient to stabilize both inflation and the output gap. This reduces the need for strong policy responses, leading to lower absolute values of the optimal coefficients on both inflation ($|\phi_\pi|$) and the output gap ($|\phi_x|$) in the Taylor rule, and consequently, a higher intercept ϕ_0. Since we have positive ϕ_π and negative ϕ_x in the optimal Taylor rule, ϕ_π will decrease while ϕ_x will increase when σ becomes larger, as shown in Figure 10.

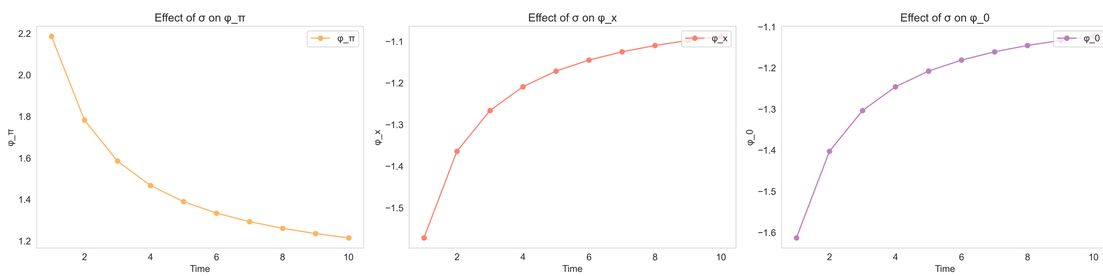


Figure 10: Sensitivity of the optimal policy coefficients to σ.

The parameters ρ_u and ρ_r capture the persistence of cost-push and demand shocks, respectively, representing the extent to which economic disturbances are transitory or long-lasting. Although persistent shocks prolong deviations of inflation and output from their targets, they do not alter the optimal steady-state reaction coefficients in the Taylor rule under discretionary policy. Instead, these parameters affect the dynamic paths of inflation, output, and interest rates following a shock, shaping the impulse responses rather than the structural form of the policy rule itself. As a result, ρ_u and ρ_r have no effect on ϕ_π, ϕ_x and ϕ_0, as shown in Figure 11 and Figure 12.

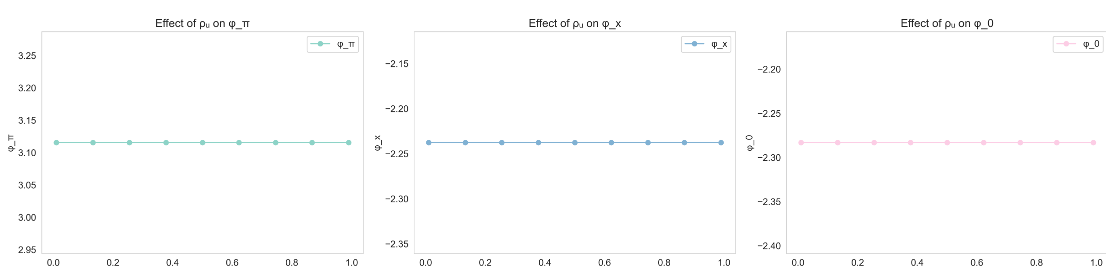


Figure 11: Sensitivity of the optimal policy coefficients to ρ_u.

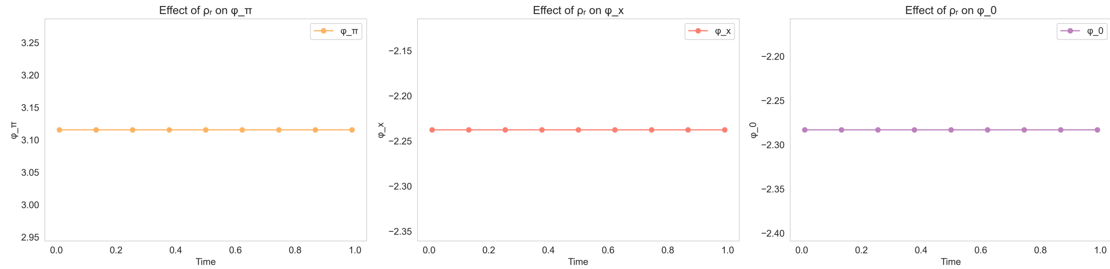


Figure 12: Sensitivity of the optimal policy coefficients to ρ_r.

The parameter λ_x measures the relative importance the central bank assigns to minimizing the output gap, reflecting its commitment to real economic stability, such as employment and growth. When the central bank raises the weight on output gap stabilization (ϕ_x), it must steepen its policy response to deviations of output from potential—hence the optimal coefficient on the output gap ($|\phi_x|$) increases—to more decisively counteract cyclical fluctuations. However, because aggressive output gap interventions propagate through the Phillips and IS curves to generate second-round effects on inflation and real rates, the optimal inflation coefficient (ϕ_π) must also rise, effectively acting as a "brake" that prevents overly vigorous demand-management from destabilizing price dynamics. In this way, an elevated λ_x yields a coordinated amplification of both $|\phi_x|$ and ϕ_π, and results in a more negative ϕ_0, as shown in Figure 13.

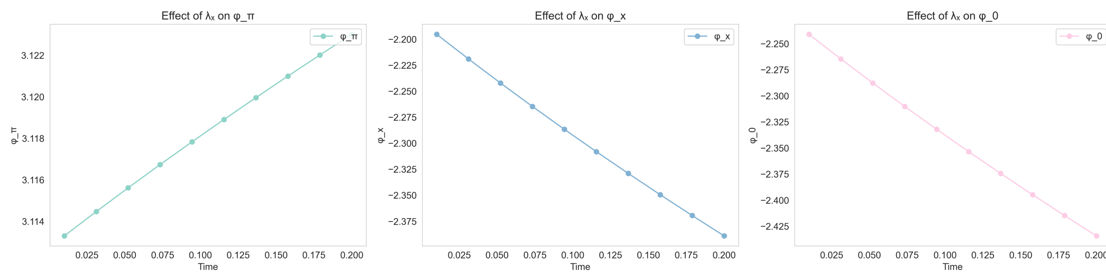


Figure 13: Sensitivity of the optimal policy coefficients to λ_x.

The parameter λ_i represents the weight the central bank places on minimizing deviations of the nominal interest rate from its target. A higher λ_i means that the central bank exhibits a preference for interest rate smoothing, reflecting an aversion to large and abrupt changes in the nominal interest rate. As a result, it optimally attenuates its policy responses to fluctuations in inflation and the output gap, leading to lower Taylor rule coefficients in absolute value, and results in a higher ϕ_0, as shown in Figure 14. This implies a willingness to tolerate greater volatility in inflation or output in exchange for a more stable and predictable interest rate trajectory.

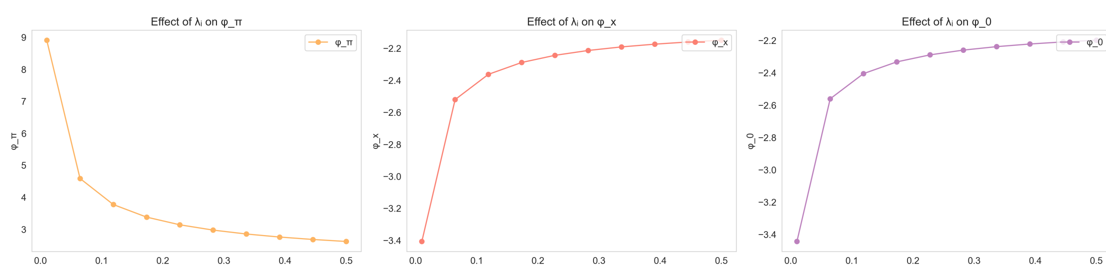


Figure 14: Sensitivity of the optimal policy coefficients to λ_i.

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