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THE NONHYPERBOLICITY OF MULTIPHASE FLOW EQUATIONS: A NONLINEAR NONPROBLEM?

A. THYAGARAJA and D.F. FLETCHER

UKAEA, Culham Laboratory, Abingdon, Oxon, OX14 3DB, UK

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The nonhyperbolicity of the simplest, single pressure, formulation of the multiphase flow equations has apparently raised concerns in the literature about their usefulness. It is shown, by a number of examples, that when properly interpreted these equations may be solved without explicit regularizing terms by the use of faithful numerical schemes, which preserve the positivity and conservation properties of the physical system. Specifically, nonlinear systems with suitable structure are considered with respect to a class of finite-difference schemes. When the mesh size Δx (say) satisfies the inequalities $1/k_{\max} \approx \Delta x \ll L$, where $k_{\max}$ is the largest limiting wave-number for the validity of the equations and L is a typical linear dimension, such schemes are shown to produce accurate, stable results irrespective of the hyperbolicity or otherwise of the governing partial differential equations.

1. Introduction

Multiphase fluid dynamics is a relatively new branch of fluid mechanics of great practical importance. As a discipline of continuum mechanics it also poses fascinating theoretical and numerical challenges. In recent years, much analytical and computational effort has been devoted to the solution of practical problems using the multiphase equations reviewed by Harlow and Amsden [11], Spalding [23], and many other authors. As a result of this work, it has become clear that in engineering applications (see, for example, Bankoff and Han [1], and Thyagaraja and Fletcher [27]) that the key limitations of these equations are contained in uncertainties associated with the constitutive relations (e.g. for interphase drag) and the correct determination of suitable boundary and initial conditions relevant to "real life". It is also clear that progress in these areas can only come about by a very close mutual fertilization of theory (inevitably computational, although analysis does play a role) and experiments. It is therefore somewhat surprising to find that in parallel with the "engineering type" calculations mentioned above, a large body of authors (e.g. Holm and Kupershmidt [13], Drew [4], Gidaspow [8], Ransom and Hicks [22], and to a lesser extent Stewart and Wendroff [24]) seem to hold the view that the simplest single pressure formulations of the multiphase equations are ill-posed in a specific way (described below) and are consequently useless for practical computation. In our opinion such a view is extreme, and unwarranted by the actual facts.

If such arguments were to be logically followed, one would not use steady, inviscid gas dynamics in transonic flows simply because the system is mixed and hyperbolic equations are, in general, ill-posed for boundary value problems. As it will turn out, this view fails to take account of the fact that multiphase flow is a singular perturbation problem for which the standard equations are asymptotic outer limits, valid and relevant over most of the flow domain except in boundary and shock layers. They are no less useful than the incompressible Euler equations of classical hydrodynamics which also have various "pathologies" associated with them (i.e. nonexistence, nonuniqueness, violent instability, etc. in many situations of practical interest). See Birkhoff [2] for some examples.

The purpose of the present paper is to demonstrate by a number of concrete examples, that hyperbolicity of a system of nonlinear partial differential equations with conservation properties is not a useful concept for many practical purposes. We shall also demonstrate that equations which are not hyperbolic can nevertheless be extremely useful in engineering applications in the sense of an asymptotic expansion. We would like to stress that many of the basic ideas in this paper can be traced to the excellent papers of Ramshaw and Trapp [21] and Lax [16,18], although the emphasis and some of the conclusions are decidedly different. Indeed we should mention that Harlow [10] has expressed very similar views. In particular, he distinguishes between the possibility of linear instabilities implied under certain circumstances by nonhyperbolic systems. To paraphrase Harlow, the occurrence of complex characteristics in the equations is a manifestation of genuine features that actually occur in nature and consequently suitably interpreted, nonhyperbolic equations can be useful and no more alarming than laminar instability in single-phase flows.

2. Hyperbolicity and well-posedness

For simplicity we consider problems in a single spatial dimension x and the time t. Suppose u is a real column vector function of x and t, with n components. Suppose further (we are specializing somewhat to avoid fussy and irrelevant details) that the following system of quasilinear equations govern the evolution of u,

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = f, \tag{1}$$

where A is a real $n \times n$ matrix function of u, x and t (but *not* the derivatives of u!) and f is an arbitrary real vector function of u, x and t. Suppose A is regarded as a function of the parameters u at some x and t and possesses n linearly independent real eigenvectors and n real, not necessarily distinct, eigenvalues. The system (1) is said to be hyperbolic at (x, t) for the values of u considered, if A has n linearly independent real eigenvectors and n real eigenvalues. If it is hyperbolic for *all* x, t, u (this will happen, for example, if A is real symmetric) then the system (1) is simply referred to as hyperbolic. If A is diagonal, clearly (1) is hyperbolic. To see the significance of this concept, suppose $f = 0$ and that $u = u_0$ is a steady, homogeneous (i.e. independent of x) solution of (1). For simplicity, we shall assume A is a function only of u. The linearized equations about this "equilibrium" state may be written as

$$\frac{\partial \tilde{u}}{\partial t} + A(u_0)\frac{\partial \tilde{u}}{\partial x} = 0, \tag{2}$$

where $\tilde{u}$ represents the perturbation vector whilst $A(u_0)$ is the matrix evaluated for $u = u_0$. Since $A(u_0)$ has n real eigenvectors and real eigenvalues, for any given perturbation $\exp[\mathrm{i}(\omega t + kx)]$, ω/k is always real. Hence we conclude *linear* stability of the basic equilibrium for arbitrary wavenumber k. Hyperbolicity, together with certain assumptions about A and f, is seen to assure linear stability.

The next notion to be dealt with is well-posedness in the sense of Hadamard [9] (see also Garabedian [7], for example, p. 109). Let us first discuss this in connection with the *linear* system (2). If $\tilde{u}(x, 0)$ is specified, eq. (2) describes its time evolution. For definiteness, consider the bounded domain $x \epsilon [0, L]$ and impose periodic boundary conditions on $\tilde{u}(x, t)$. Fourier analysis and the knowledge of the eigenvectors and eigenvalues of $A(u_0)$ gives the solution $\tilde{u}(x, t)$. It is easy to verify that the hyperbolicity guarantees that the solution vector $\tilde{u}(x, t)$ depends continuously on $\tilde{u}(x, 0)$. In particular, if a very short wavelength disturbance of small amplitude (relative to $\tilde{u}(x, 0)$ itself) is added, the disturbance is no larger for any finite value of T. All this can be explicitly verified, if desired, by the one-component form of eq. (2) which is *always* hyperbolic. In a well-known paper Lax [17] demonstrates this in greater detail. With this preparation, we are ready to formulate Hadamard's concept of well-posedness (cf. Hadamard [9], Garabedian [7]).

Definition. A linear homogeneous system, such as system (2), is well-posed in Hadamard's sense, if the solution vector $\tilde{u}(x, t)$ depends continuously on the initial vector $\tilde{u}(x, 0)$.

The key point of this definition is exemplified by Hadamard's classic example of an ill-posed system. Let $u(x, t)$, $v(x, t)$ be two scalar functions satisfying

$$\frac{\partial u}{\partial t} = c\frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial t} = -c\frac{\partial u}{\partial x} \quad (c = \text{constant}). \tag{3}$$

Consider the initial function $u(x, 0) = \sin(2\pi x/L)$; $v(x, 0) = \cos(2\pi x/L)$. The solution is $u(x, t) = \exp(-2\pi ct/L)\sin(2\pi x/L)$; $v(x, t) = \exp(-2\pi ct/L)\cos(2\pi x/L)$. Unfortunately, the system (3) (none other than the Cauchy–Riemann equations) admits solutions $u_n^*(x, t) = \exp(2n\pi ct/L)\sin(2n\pi x/L)$; $v_n^*(x, t) = -\exp(2n\pi ct/L)\cos(2n\pi x/L)$ for arbitrarily large integers n. This means that for given t and ϵ (however small) we can always find n such that $\epsilon\exp(2n\pi ct/L) > 1$. Thus an arbitrarily small (but sufficiently high n) disturbance "overwhelms" the solution at any given time t. It is plain that hyperbolicity of linear systems is equivalent to Hadamard's well-posedness and in this case the system is not hyperbolic.

Note that so far we have only considered well-posedness for *linear*, homogeneous systems. What does it mean for linear inhomogeneous or more importantly, nonlinear systems? It is our thesis, that in so far as this concept can be defined at all in a meaningful way for nonlinear systems, it is inequivalent to hyperbolicity and so restrictive that it is of little use in physics and engineering. We propose in this paper a new definition of *asymptotic well-posedness* of certain classes of conservative nonlinear systems. In order to motivate our definition and its origin in multiphase flow dynamics, we consider some examples.

3. Some examples

We return to the inhomogeneous form (1) and show that even a linear, hyperbolic system can behave "badly". Thus, consider the infinite domain $(-\infty, \infty)$ and the equations

$$\frac{1}{c}\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = v \tag{4}$$

and

$$\frac{1}{c}\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = -\lambda u. \tag{5}$$

Eliminating v, we get the familiar, strictly hyperbolic, Klein–Gordon equation,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -\lambda u. \tag{6}$$

Solution by Fourier analysis leads to the dispersion relation

$$\frac{\omega^2}{c^2} = k^2 + \lambda. \tag{7}$$

If $\lambda < 0$, there is a *low-k* instability. Unless the initial function is restricted so that $k^2 - |\lambda| \geqslant 0$, the solution is unbounded (in time). If the time interval is *fixed*, the problem may be well-posed in Hadamard's sense. However, if the problem is one of obtaining the solution for all t, an initial "error" can, and will, dominate the true solution after a sufficiently long time. This example illustrates the fact that if a system of equations describes a *physically* unstable system, only existence and uniqueness of solutions to the initial value problem can be expected in general. Thus, well-posedness in the sense of Hadamard requires the existence, uniqueness and continuity of solutions with respect to initial data for all $t > 0$. In a physically unstable system, even if existence and uniqueness can be guaranteed, instability in the sense of Lyapunov (Nemystkii and Stepanov [20], p. 152) implies that the solutions are not continuous with respect to initial data for all t.

Suppose now we consider a more general case where instead of $-\lambda u$ in eq. (5) we have $-V'(u)$, where V is some differentiable, even function of u. For example, consider

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -V'(u). \tag{8}$$

Assuming appropriate boundary conditions, we may obtain the conservation law,

$$\int_{-\infty}^{\infty}\left(\frac{1}{c^2}\frac{u_t^2}{2} + \tfrac{1}{2}u_x^2 + V(u)\right)\mathrm{d}x = \text{constant}. \tag{9}$$

Let us consider $V(u) = -\frac{1}{2}u^2 + \frac{1}{4}\mu^2 u^4$, ($\mu$ real). Equation (8) has a low-k linear instability about

$u=0$, but is linearly stable about $u=\pm 1/\mu$. Furthermore (9) can be written in the form,

$$\int_{-\infty}^{\infty}\left(\frac{1}{c^2}\frac{u_t^2}{2}+\tfrac{1}{2}u_x^2+\frac{\mu^2}{4}\left(u^2-\frac{1}{\mu^2}\right)^2\right)\mathrm{d}x=\text{const.},\tag{10}$$

showing that it has a positive definite integral invariant implying nonlinear stability. This demonstrates that linear low-k instability may be a rather artificial phenomenon under certain circumstances. It is also clear that systems with positive invariants can be linearly unstable, but may be nonlinearly stable (another well-known example is provided by the nonlinear Schrödinger equation and the Benjamin–Feir instability; Thyagaraja [25]).

Our next example shows that even (quasilinear) hyperbolic equations require a wider concept of well-posedness than that provided by the Hadamard definition. Consider the well-known (cf. Whitham [29]) "kinematic wave" equation,

$$u_t+uu_x=0.\tag{11}$$

This has the implicit "exact" solution

$$u=F(x-ut),\tag{12}$$

where

$$u(x,0)=F(x).\tag{13}$$

Clearly, $u=c_0$ (a constant) are equilibrium solutions of eq. (11). Such states are linearly stable. In addition, eq. (11) has an infinity of integral invariants $I(f)$ of the form

$$I(f)=\int_{-\infty}^{\infty}f(u)\,\mathrm{d}x,\tag{14}$$

where f is an "arbitrary" (subject to fairly weak restrictions) function. Yet all of these properties are insufficient to prevent the well-known "shock" catastrophe. Arbitrarily small (in amplitude) perturbations of certain types "blow-up" in arbitrarily short times in the sense that u_x becomes unbounded. This is an "ultra-violet" or high-k catastrophe. The system does not even *have* a solution (in the usual sense that is used by Hadamard) beyond times which depend not on the amplitude of the initial perturbation (which can be arbitrarily small), but on the *slope* of the initial function. Thus hyperbolicity has no real bearing on the classical concept of well-posedness (i.e. existence, uniqueness and continuity with respect to initial data) in this case. The resolution of this problem is of course extremely well-known. Since the problem involves an ultra-violet catastrophe, Burgers simply added νu_{xx} to the right side of eq. (11) originating Lax's [16] "viscosity method" of regularization. In this view, eq. (11) is regarded as the asymptotic outer limit (in the sense of singular perturbation theory, see Van Dyke [28]) of the Burger's equation as $\nu\to 0$. Its *weak* solutions (for a definition see Whitham [29]) are generated from the latter. They are asymptotic to the proper solutions of the parabolic Burger's equation. All of this has been explicitly demonstrated in the literature using the exact solutions (due to Cole [3] and Hopf [14]) of Burger's equation.

This single example proves that neither hyperbolicity nor the existence of nonnegative invariant functionals can by themselves guarantee the absence of high-k nonlinear instabilities in a given system of nonlinear equations. A less trivial example is provided by the inviscid gasdynamic equations. Finally, as Whitham [29] remarks, the general water-wave equations are a curious mixture of an elliptic spatial equation (Laplace's equation for the velocity potential) and a time dependent boundary condition which cannot be put in the form of a hyperbolic system. This system has positive integral invariants, but needs regularization by imbedding (in the full Navier–Stokes system with surface tension) to describe "breaking" waves and bores. These examples indicate that dismissing certain idealized sets of evolutionary equations (hyperbolic or otherwise) as "ill-posed", and therefore practically useless simply because their high-k behaviour could lead to linear or nonlinear instability or even non-existence of solutions beyond a certain time, could be gravely misleading. This is exactly the point made by Harlow [10].

We now apply these ideas to the multiphase flow equations and show how physically relevant results can be obtained from them in spite of their well-known nonhyperbolicity.

4. Multiphase flow equations

We consider the simple, source-free, dissipationless, two component, one-dimensional incompressible flow. The single pressure formulation (see Harlow and Amsden [11]) leads to

$$\begin{aligned}
&\frac{\partial \alpha_1}{\partial t} + \frac{\partial}{\partial x}(\alpha_1 v_1) = 0,\\
&\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x} + \frac{1}{\rho_1}\frac{\partial p}{\partial x} = 0,\\
&\frac{\partial \alpha_2}{\partial t} + \frac{\partial}{\partial x}(\alpha_2 v_2) = 0,\\
&\frac{\partial v_2}{\partial t} + v_2 \frac{\partial v_2}{\partial x} + \frac{1}{\rho_2}\frac{\partial p}{\partial x} = 0, \quad \alpha_1 + \alpha_2 = 1.
\end{aligned} \tag{15}$$

These equations are supplemented by the constitutive relations $\rho_1 =$ constant, $\rho_2 =$ constant, characteristic of incompressible flow.

The above system is unusual in several respects. The hydrodynamic pressure is determined by the constraint

$$\frac{\partial}{\partial x}\left(\alpha_1 v_1^2 + \alpha_2 v_2\right) = 0, \tag{16}$$

rather than a time evolution equation. However, two nonnegative (see Lax [18]) conservation laws can be derived. Assuming periodic boundary conditions on a bounded domain $[0, L]$ (other possibilities can also be considered), we find that

$$\int_0^L \alpha_1 \, \mathrm{d}x = I_1, \quad \int_0^L \left(\tfrac{1}{2}\rho_1 \alpha_1 v_1^2 + \tfrac{1}{2}\rho_2 \alpha_2 v_2^2\right) \mathrm{d}x = I_2,$$

both constant in t. (17)

It is well known that if $\alpha_1 \geqslant 0$ at $t = 0$, the equation of continuity implies that α_1 is always nonnegative. To prove this, suppose $v_1(x, t)$ is given in $x \in [0, L]$, $t \in [0, T]$. The linear hyperbolic equation

$$\frac{\partial \Psi}{\partial t} + v_1 \frac{\partial \Psi}{\partial x} + \frac{\Psi}{2}\frac{\partial v_1}{\partial x} = 0, \tag{18}$$

with the initial condition $\Psi(x, 0) = +\sqrt{2\alpha_1(x, 0)}$ defines a real function $\Psi(x, t)$ in the same domain. We find that $\frac{1}{2}\Psi^2$ satisfies the continuity equation and matches the initial function $\alpha_1(x, 0)$. Thus the result is established.

We further note that the variables α_2, v_2 and p may all be eliminated. The functions α ($\equiv \alpha_1$) and v_1 are seen to satisfy the equations

$$\begin{aligned}
&\frac{\partial \alpha}{\partial t} + \frac{\partial}{\partial x}(\alpha v_1) = 0, \quad \text{and}\\
&\frac{\partial v_1}{\partial t} + \frac{\partial}{\partial x}\left(\frac{v_1^2}{2}\right)\\
&\qquad - \left(\frac{\rho_2}{\rho_2 \alpha + \rho_1(1-\alpha)}\right)\frac{\partial}{\partial x}\left(\frac{\alpha v_1^2}{1-\alpha}\right) = 0.
\end{aligned} \tag{19}$$

The second equation can also be put in a conservation form,

$$\frac{\partial}{\partial t}\left(\frac{\alpha v_1^2}{(1-\alpha)}\left(\rho_2 \alpha + \rho_1(1-\alpha)\right)\right) + \frac{\partial}{\partial x}\left(\frac{\alpha v_1^3}{(1-\alpha)^2}\left(\rho_2 \alpha^2 + \rho_1 (1-\alpha)^2\right)\right) = 0.$$

Alternatively, the system (19) is in the normal form (1) with $f = 0$ and the two-by-two matrix A,

$$A = \begin{bmatrix} v_1 & \alpha \\ \dfrac{-(\rho_2 v_1^2)}{(\rho_2 \alpha + \rho_1(1-\alpha))(1-\alpha)^2} & v_1 - \dfrac{2\rho_2 v_1}{(\rho_2 \alpha + (1-\alpha)\rho_1)}\left(\dfrac{\alpha}{1-\alpha}\right) \end{bmatrix}. \tag{20}$$

It turns out that the system is always (provided $v_1 \neq 0$) nonhyperbolic (Gidaspow [8]). However, this is very far from saying that the eqs. (19) are not useful as they stand. The integral invariants are nonnegative and hence ensure nonlinear stability of the *low-k* (just as in the nonlinear Klein–Gordon equation). It is not obvious, though probable, that the "shock" catastrophe is inherent in the above system. Thus if $\rho_2 \ll \rho_1$, (this can occur in a water–steam system), the v_1 equation is well approximated (at least initially) by the "kinematic wave" equation. An arbitrarily small perturbation can lead to a "shock" catastrophe. Thus, the system (19) is perfectly adequate as it stands for low-k evolution, whereas it requires viscous "regulation" for determining shock-like structures. The addition of *drag* between the species leads to a decay of I_2 in time (hence stabilization of low-k instabilities), but does not ensure the existence of the solutions for all t. This is analogous to adding $-\nu u$ to eq. (11). The exact solution shows that the shocks still occur. Thus, drag is not a gradient effect like viscosity and, although dissipative, does not regularize shock-like behaviour. Physically, the correct regularization is achieved by writing eqs. (15) in the conservation forms

$$\frac{\partial}{\partial t}(\rho_1\alpha_1 v_1) + \frac{\partial}{\partial x}(\rho_1\alpha_1 v_1^2) + \alpha_1\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(\alpha_1\mu_1\frac{\partial v_1}{\partial x}\right), \tag{21}$$

$$\frac{\partial}{\partial t}(\rho_2\alpha_2 v_2) + \frac{\partial}{\partial x}(\rho_2\alpha_2 v_2^2) + \alpha_2\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(\alpha_2\mu_2\frac{\partial v_2}{\partial x}\right). \tag{22}$$

It then becomes possible to discuss the initial boundary value problem of this parabolic system along the lines of Ladyzhenskaya [15]. It is believed, but not yet proved rigorously, that there is a unique solution to this system valid for all $t > 0$. We note however that nonlinear parabolic equations can exhibit *linear* instability. But this is *not* ill-posedness! It is also probable that in the asymptotic limit of μ_1, $\mu_2 \to 0$, the system (19) represents correctly the "outer" solutions (apart from singularities like shock layers). Thus, provided we mean by "solution" a *weak* solution of (19), which is an asymptotic outer limit of a solution of the full nonlinear parabolic system, the system (19) is just as "well-posed" as the full system. This suggests the following definition of an asymptotically well-posed system.

Definition. A system of evolutionary partial differential equations with nonnegative integral invariants is said to be *asymptotically well-posed*, if it is the outer limit of a system of parabolic equations which imply the decay of the invariants (in the absence of external sources) and which uniquely identify a class of weak solutions of the original system in the sense of Lax [16]. The solutions of the full system must exist for all time and must be unique for given initial data, but are not required to be linearly stable.

What are the implications for practical multiphase calculations? We note that not all numerical methods are suitable for handling systems like (19). It is clearly necessary that over and above the usual definition of consistency, we must require faithfulness or qualitative consistency if erroneous results are to be avoided. Any finite difference method (for example) for solving the system (15) is said to be *faithful* if for arbitrary Δx (Δt appropriately chosen) the approximants to α are maintained nonnegative and the invariant I_2 is only allowed to decrease. The scheme must be devised to keep the decrease as small as possible. It is obvious that requiring the scheme to be *convergent* to a solution of (15) as Δx, $\Delta t \to 0$ is far too strong, since the equations themselves are not valid for arbitrarily large frequencies and wave numbers. At best we can compute a weak solution. There is no point in refining Δx beyond the underlying high-k limit on the asymptotic validity of (15). If such refinement is wanted then the parabolic, viscosity terms *must* be included in the numerical analysis. For larger Δx (still satisfying $\Delta x \ll L$) the numerical method (provided it is faithful to the conservation and positivity constraints) itself provides a dissipative regularization. This is an implicit regularization valid for $L \gg \Delta x \geqslant 1/k_{\max}$, where $k_{\max}$ is the limiting wave number above which the equations (and their exact solutions, if they exist) cease to have physical validity.

There are unfortunately no general rules to determine the smallest length-scale beyond which the outer equations, such as (15), cease to be valid. In any given case, both theoretical and experimental considerations suggest at what point new physics, such as viscosity, must be included in the equations.

Harlow and Besnard [12] have proposed that the representation of the instability of the differential equations can be accomplished by means of auxiliary turbulence transport equations, which result in Reynolds stress components, whose incorporation into the mean equations render them exactly neutrally stable. Thus in this view, a suitably chosen "turbulent viscosity" regularizes the high-k instabilities. The difficulty with this approach consists in identifying the instability (linear or nonlinear) and in representing the flow as the sum of a mean and fluctuating part. The latter decomposition is not valid in many transient multiphase flows of interest (e.g. detonations, vapour explosions, etc). Furthermore, as yet no direct experimental evidence has been advanced in support of such a regularization scheme.

We remark that high-k instabilities may also be suppressed by dispersive physical effects, such as surface tension (Ramshaw and Trapp [21]). Such conservative regularizations are analogous to the Korteweg–De Vries or the nonlinear Schrödinger equations which are dispersive regularizations of eq. (11). As Whitham [29] remarks, in general, both dispersive and dissipative effects are encountered in water wave theory. In the present work we do not consider dispersive regularization.

5. Numerical simulations of multiphase equations

It remains to consider whether the nonhyperbolicity of multiphase flow equations in the common pressure formulation constitutes a serious obstacle to numerical calculations. Before we consider this topic, it is useful to clarify certain issues. In a concise but deep paper, Lax [18] considered real systems of the form

$$u_t + A u_x = 0, \tag{23}$$

where u is a column vector of n components and A is an $n \times n$ matrix function of u, x, and t. He makes a set of plausible assumptions and shows that nonnegative conservation laws of the system can be derived for the system if and only if the system is hyperbolic. How can one reconcile this result with the manifest nonhyperbolicity of the system (19) satisfied by α and v_1? The answer lies in the fact that physically α and v_1 are very different kinds of variable. We have seen that while the A matrix for the system is not hyperbolic, it has the property that α must lie in the interval [0,1]. Furthermore, in this instance, the conservation laws (I_1, I_2) can be shown to exist independently of Lax's general method! This simply means that one or more of his assumptions are not valid for the multiphase equations. His results, valuable though they are, do not apply in our case. In particular, Lax considers general systems, where nothing is assumed about the nonnegativity or otherwise of some of the variables. Our example shows that if the system of equations is such that it preserves the nonnegativity of a subset of the variables, hyperbolicity is not a necessary condition for the existence of nonnegative integral invariants. It is interesting to note that Lax [18] himself observed that provided certain variables (like the density) could be shown to be nonnegative, hyperbolicity may not be necessary. It is precisely this a priori inequality that was demonstrated using eq. (18). As stated earlier, whether or not the system is hyperbolic, the mere existence of nonnegative integral invariants is not sufficient to guarantee high-k stability. We might mention that armed with the example (1) it is relatively easy to "manufacture" systems of nonhyperbolic nonlinear equations, which nevertheless possess nonnegative conservation laws. The mathematical theory of such systems seems nonexistent at present. In particular, it would be very useful to have a theory of explicit regularization of such systems by the "viscosity" method.

Returning to multiphase numerical simulations, we present two examples to illustrate our view that transient initial-boundary value problems using these equations yield perfectly acceptable solutions in practice. Care must of course by exercised in ensuring the faithfulness or qualitative consistency of the numerical schemes. However, this is no different from the situation in transonic turbomachine hydrodynamics (see Thyagaraja [26]) where mixed systems (i.e. elliptic–hyperbolic) and shocks are encountered.

The first example considered is the well-known, (Liepmann and Roshko [19]) shock-tube problem. This problem can be solved by writing two-fluid gas dynamics equations in conservation form to ensure shock capturing. Using the suffices 1 and 2 to denote the two gases, the interacting gas-dynamic equations are the following:

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x}(\rho_1 v_1) = 0, \tag{24}$$

$$\frac{\partial \rho_2}{\partial t} + \frac{\partial}{\partial x}(\rho_2 v_2) = 0, \tag{25}$$

$$\frac{\partial}{\partial t}(\rho_1 v_1) + \frac{\partial}{\partial x}(\rho_1 v_1^2) + \frac{\partial p_1}{\partial x} = D_v(v_2 - v_1), \tag{26}$$

$$\frac{\partial}{\partial t}(\rho_2 v_2) + \frac{\partial}{\partial x}(\rho_2 v_2^2) + \frac{\partial p_2}{\partial x} = D_v(v_1 - v_2), \tag{27}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\rho_1\left(e_1 + \tfrac{1}{2}v_1^2\right)\right) + \frac{\partial}{\partial x}\left(\rho_1 v_1\left(e_1 + \frac{p_1}{\rho_1} + \tfrac{1}{2}v_1^2\right)\right) \\ &\quad = D_e(T_2 - T_1) + \tfrac{1}{2}D_v(v_1 - v_2)^2 \\ &\qquad + v_1 D_v(v_2 - v_1), \end{aligned} \tag{28}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\rho_2\left(e_2 + \tfrac{1}{2}v_2^2\right)\right) + \frac{\partial}{\partial x}\left(\rho_2 v_2\left(e_2 + \frac{p_2}{\rho_2} + \tfrac{1}{2}v_2^2\right)\right) \\ &\quad = D_e(T_1 - T_2) + \tfrac{1}{2}D_v(v_1 - v_2)^2 \\ &\qquad + v_2 D_v(v_1 - v_2), \end{aligned} \tag{29}$$

$$p_1 = \rho_1 R_1 T_1, \quad p_2 = \rho_2 R_2 T_2, \tag{30}$$

$$e_1 = C_{v_1} T_1, \quad e_2 = C_{v_2} T_2. \tag{31}$$

The coefficients D_v, D_e are the momentum and energy equilibration constants which are chosen as follows:

$$D_v = \frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)\tau_D} \quad \text{and} \quad D_e = \frac{\rho_1 C_{v_1} \rho_2 C_{v_2}}{(\rho_1 C_{v_1} + \rho_2 C_{v_2})\tau_T} \tag{32}$$

where the parameters R_1, R_2, C_{v_1}, C_{v_2}, τ_D and τ_T are suitably chosen, as in Fletcher and Thyagaraja [6].

Note that the equations do not involve any explicit viscosity. Although they are irreversible (due to D_v and D_e), they are still hyperbolic since the A matrix is unaffected by D_v and D_e. In accordance with kinetic considerations, each gas "feels" only its partial pressure. All the interactions are due to momentum and energy equilibration mechanisms. The shock-tube problem can be considered as a valid initial value problem for a two-component "multiphase" fluid mixture governed by the compressible generalizations of (15). In the following, α_1, α_2 are the volume fractions and ρ_1, ρ_2 refer to the thermodynamic densities of the two components (i.e. quantities which occur in the equations of state). The common pressure is denoted by p. The governing equations are:

$$\alpha_1 + \alpha_2 = 1, \tag{33}$$

$$\frac{\partial}{\partial t}(\rho_1 \alpha_1) + \frac{\partial}{\partial x}(\rho_1 \alpha_1 v_1) = 0, \tag{34}$$

$$\frac{\partial}{\partial t}(\rho_2 \alpha_2) + \frac{\partial}{\partial x}(\rho_2 \alpha_2 v_2) = 0, \tag{35}$$

$$\frac{\partial}{\partial t}(\rho_1 \alpha_1 v_1) + \frac{\partial}{\partial x}(\rho_1 \alpha_1 v_1^2) + \alpha_1 \frac{\partial p}{\partial x} = D_v(v_2 - v_1), \tag{36}$$

$$\begin{aligned} &\frac{\partial}{\partial t}(\rho_2 \alpha_2 v_2) + \frac{\partial}{\partial x}(\rho_2 \alpha_2 v_2^2) + \alpha_2 \frac{\partial p}{\partial x} \\ &\quad = D_v(v_1 - v_2), \end{aligned} \tag{37}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\alpha_1 \rho_1\left(e_1 + \tfrac{1}{2}v_1^2\right)\right) \\ &\quad + \frac{\partial}{\partial x}\left(\alpha_1 \rho_1 v_1\left(e_1 + \frac{p}{\rho_1} + \tfrac{1}{2}v_1^2\right)\right) \\ &\quad = D_e(T_2 - T_1) + \tfrac{1}{2}D_v(v_1 - v_2)^2 \\ &\qquad + v_1 D_v(v_2 - v_1) - p\frac{\partial \alpha_1}{\partial t}, \end{aligned} \tag{38}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\alpha_2 \rho_2\left(e_2 + \tfrac{1}{2}v_2^2\right)\right) \\ &\quad + \frac{\partial}{\partial x}\left(\alpha_2 \rho_2 v_2\left(e_2 + \frac{p}{\rho_2} + \tfrac{1}{2}v_2^2\right)\right) \\ &\quad = D_e(T_1 - T_2) + \tfrac{1}{2}D_v(v_1 - v_2)^2 \\ &\qquad + v_2 D_v(v_1 - v_2) - p\frac{\partial \alpha_2}{\partial t}, \end{aligned} \tag{39}$$

$$p = \rho_1 R_1 T_1 = \rho_2 R_2 T_2, \tag{40}$$

$$e_1 = C_{v_1} T_1, \quad e_2 = C_{v_2} T_2. \tag{41}$$

The coefficients D_v and D_e in the gas dynamic formulation were functions of ρ_1 and ρ_2. In the multiphase formulation the *same* functional forms are used except that the independent variables are $\rho_1\alpha_1$ and $\rho_2\alpha_2$. Just as in the gas dynamics case the 6 evolution equations are sufficient to determine the 6 unknown ρ_1, ρ_2, v_1, v_2, T_1 and T_2, the 6 multiphase evolution equations determine $\rho_1\alpha_1$, $\rho_2\alpha_2$, v_1, v_2, T_1 and T_2. The pressure is obtained from the symmetrical relation

$$p = \rho_1 \alpha_1 R_1 T_1 + \rho_2 \alpha_2 R_2 T_2, \tag{42}$$

since $\rho_1\alpha_1$, $\rho_2\alpha_2$, T_1 and T_2 can be simply evolved. From p, we get $\rho_1 = p/R_1T_1$, $\rho_2 = p/R_2T_2$. The α's are now simply calculated. No explicit evolutionary equation for p is needed. The details of the numerical methods needed for solving these systems ensuring faithfulness have been published elsewhere (see Fletcher and Thyagaraja [5,6]). We merely remark that whilst the gasdynamic equations are always hyperbolic (in this instance leading to a shock structure determined by Rankine–Hugoniot conditions and regularized by "numerical viscosity"), the multiphase equations are well-known to be nonhyperbolic (Steward and Wendroff [24]), if the following inequalities hold:

$$0 < (v_1 - v_2)^2 < \left(\frac{c_1^2 c_2^2}{\rho_2\alpha_1 c_2^2 + \rho_1\alpha_2 c_1^2}\right)\left((\rho_1\alpha_2)^{1/3} + (\rho_2\alpha_1)^{1/3}\right)^3, \tag{43}$$

where c_1 and c_2 are the usual sound speeds defined by

$$c_1^2 = \left.\frac{\partial p}{\partial \rho_1}\right|_{S_1}, \quad c_2^2 = \left.\frac{\partial p}{\partial \rho_2}\right|_{S_2}.$$

In the present examples, this condition is satisfied almost everywhere.

In fig. 1 we show the initial data for the shock-tube problem. For illustrative purposes we present two cases: a "high-drag" case in which D_v is set so large that $(v_1 - v_2)^2 \ll v_1^2$, v_2^2; the second case involves a smaller value of D_v so that $(v_1 - v_2)^2 \approx v_1^2$, v_2^2. When the drag is high, both gases move with a common velocity which is determined by the total momentum conservation equation. The shock-tube problem can be solved analytically in this case. In fig. 2 we present the results obtained. Thus, fig. 2a shows the combined pressure $p_1(x, t) + p_2(x, t)$ obtained from the gas dynamic equations 8 ms after $t = 0$. The agreement with the analytic solution is excellent, showing rather good shock resolution and monotonic shock pressure rise. Figure 2b shows the results from the multiphase equations; this time the common pressure $p(x, t)$ is the function plotted. The characteristic relaxation time $\tau_D = 5$ μs in this case. When v_1

High pressure section	Low pressure section
$p_1 = 20$ MPa, $p_2 = 0$ $T_1 = 414$ K $v_1 = 0$	$p_1 = 0$, $p_2 = 0.1$ MPa $T_2 = 300$ K $v_2 = 0$

(a) Multigas case

High pressure section	Low pressure section
$p = 20$ MPa $\alpha_1 = 1$ $T_1 = 414$ K $v_1 = 0$	$p = 0.1$ MPa $\alpha_2 = 1$ $T_2 = 300$ K $v_2 = 0$

(b) Multiphase case

$C_{v_1} = 13550$ J/kg K, $R_1 = 4517$ J/kg K
$C_{v_2} = 1355$ J/kg K, $R_2 = 451.7$ J/kg K
length of shock tube = 44 m
position of diaphragm = 14.6 m
comparison time = 0.008 s
number of grid points = 960
time-step = 1 μs

Fig. 1. Details of the shock-tube simulation.

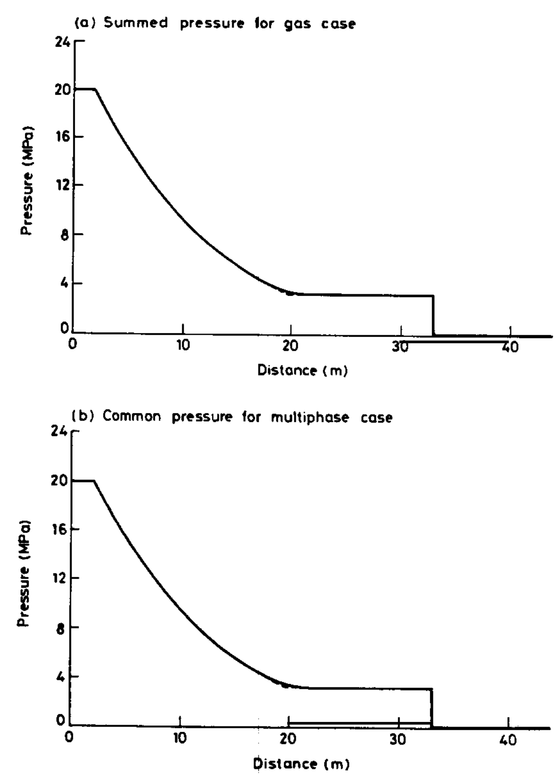

Fig. 2. Shock-tube simulations ($\tau_D = \tau_T = 5$ μs) (dashed curve shows the analytic solution).

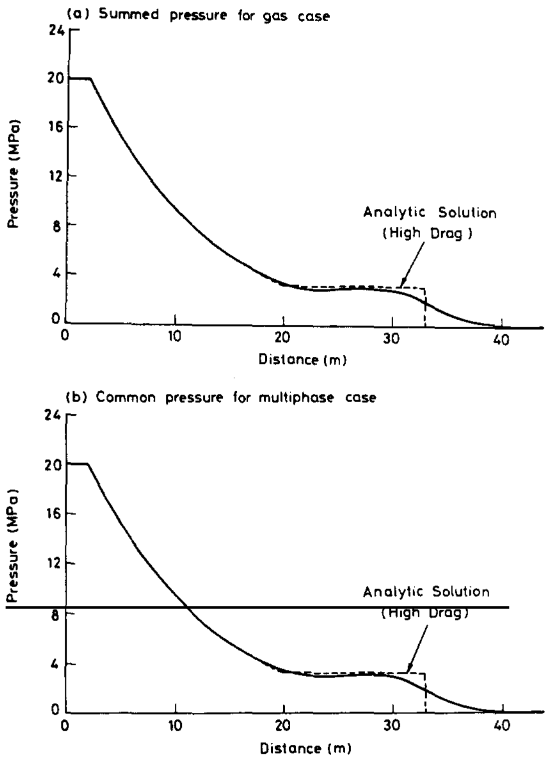

Fig. 3. Shock-tube simulations ($\tau_D = \tau_T = 1$ ms).

and v_2 are nearly equal equation (43) shows that nonhyperbolicity is marginal.

When the drag is high, both solutions coincide with each other and the exact analytic solution of the gas dynamic equations. When the drag is low, the two gases can have a substantial relative velocity. Figure 3a shows that even in the gas dynamic case, there can be no simple shock structure and the combined pressure is smeared relative to the analytic solution assuming a common velocity. Figure 3b shows that the common pressure obtained from the multiphase equations agrees quite well with the gas dynamic combined pressure. $\tau_D = 1$ ms in this low drag case.

We must emphasize that the gas dynamic and multiphase equations are physically and mathematically inequivalent models, valid generally under different conditions. We make no claim that they lead to the same solutions under all conditions (indeed our next example is designed to illustrate the differences). We do claim however that the well-known nonhyperbolicity per se of the multiphase equations do not lead to numerical difficulties which might vitiate numerical simulations. It must be remembered that the comparisons are made at the same accuracy level and that the nonhyperbolic results are no worse than the hyperbolic ones.

The above example also illustrates that although D_v has no effect on the hyperbolicity or otherwise of the equations, under suitable conditions, the system mimics the behaviour of a hyperbolic equation closely. Since both systems are ultimately regularized by imbedding in a parabolic equation, it would appear that the nature of the outer equations as far as the reality of their characteristics are concerned is irrelevant. We do not present plots of other variables and the time evolution in this problem for lack of space. These and results for other values of D_v are given in Fletcher and Thyagaraja [5,6]. Suffice it to say that the conserved quantities (mass and *total* energy) are preserved to an accuracy of better than 0.1% of their initial values.

The second example involves finite amplitude, compessible subsonic motion in a tube with both ends closed. In this case, we have deliberately set D_v and D_e to zero. The initial conditions and parameters used in the simulation are given in fig. 4. Figure 5 shows the total pressure in the gas case and the common pressure in the multiphase case

$v_1 = v_2 = 0, \quad T_1 = T_2 = 413$ K
$p_1 = 12 \times (1 + \sin(\pi x/L)) \times (\frac{1}{2} + 0.1 \sin(\pi x/L))$ MPa
$p_2 = 12 \times (1 + \sin(\pi x/L)) \times (\frac{1}{2} - 0.1 \sin(\pi x/L))$ MPa

(a) Multigas case

$v_1 = v_2 = 0, \quad T_1 = T_2 = 413$ K
$\alpha_1 = 0.5 + 0.1 \sin(\pi x/L)$
$p = 12(1 + \sin(\pi x/L))$ MPa

(b) Multiphase case

$C_{v_1} = 1355$ J/kg K, $R_1 = 452$ J/kg K
$C_{v_2} = 2710$ J/kg K, $R_2 = 903$ J/kg K
length of tube (L) $= 50$ m
number of grid points $= 100$
time-step $= 5\ \mu$s
$D_v = D_e$ $= 0$

Fig. 4. Initial conditions and parameters used in the subsonic simulations.

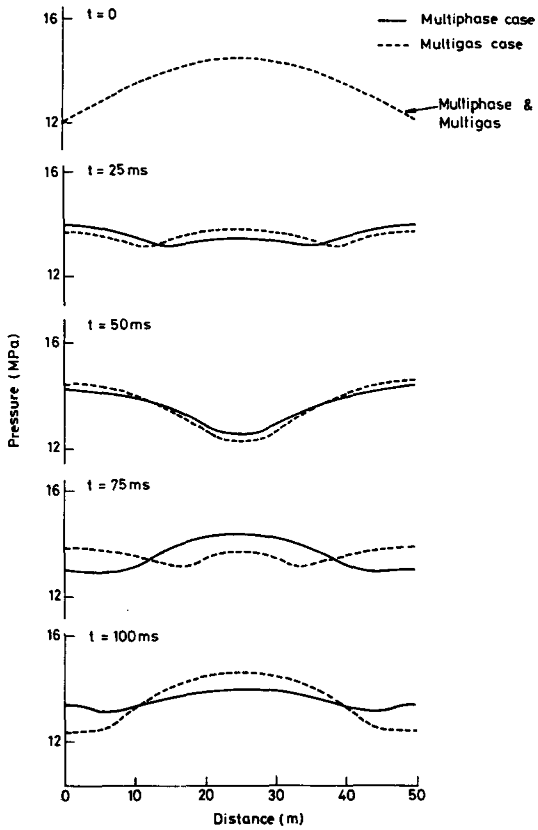

Fig. 5. Pressure distribution at various times.

at a number of different times. The velocity distributions at these times are shown in fig. 6. The figure show that the motion is regular (i.e. no evidence of high-k instability) and that symmetry about the mid point is preserved.

The two gases have no dynamic interaction at all. Both gases are governed by the standard Euler equations. In the gas case, since v_1 and v_2 "feel" different partial pressures, they can sometimes be in opposite directions, as shown in fig. 6a. In the multiphase case, even when D_v and D_e are zero, the two components are always coupled by the common pressure. This is reflected in the fact that v_1 and v_2 are in the same direction in this case. The accelerations are always in the same direction. Yet, remarkably enough, the qualitative features of the solutions as far as pressure is concerned are quite similar. There is no evidence of a high-k instability characteristic of "ill-posed equations" spoiling the symmetry in space or periodicity in time of the multiphase solution. It has been checked that the multiphase equations are nonhyperbolic everywhere.

The constants of the motion are conserved in both codes to a high accuracy. In the gas case the individual energies are conserved as reguired. Figure 7 shows the change in the different energy sums (from their initial value) as functions of time. We see that each gas does indeed behave as though the other gas was absent, but in the multiphase flow case clearly energy is transferred between the species due to the common pressure. In both cases the periodicity of the motion is evident.

6. Conclusions

The notions of hyperbolicity, well-posedness and regularization have been examined with particular reference to the single pressure formulation of multiphase flow. The equations which govern such flows are generally not hyperbolic, but are known to possess nonnegative conserved constants in the absence of dissipation and sources. In this latter respect, they are in exactly the same position as the compressible inviscid gas dynamics equations which are, of course, hyperbolic. In principle, both the multiphase flow equations and the gas dynamics equations must, of course, be regarded as outer limit of equations of a parabolic character involving viscosity and thermal conduction. These effects, which introduce higher derivatives of field variables, are sufficient to resolve high-k instabilities, such as those caused by shocks and boundary layers. In general, nonlinear physical systems can only be expected to have existence and uniqueness of solutions, but not continuity with respect to initial data. Thus, the concept of well-posedness developed for linear systems is too restrictive, in that nonlinear physical systems are often unstable in the sense of Lyapunov (i.e. solutions are not continuous with respect to initial data for all t).

We have shown by specific numerical examples that both hyperbolic nonlinear evolutionary systems and nonhyperbolic ones, such as those arising in multiphase flow, can be solved by the same

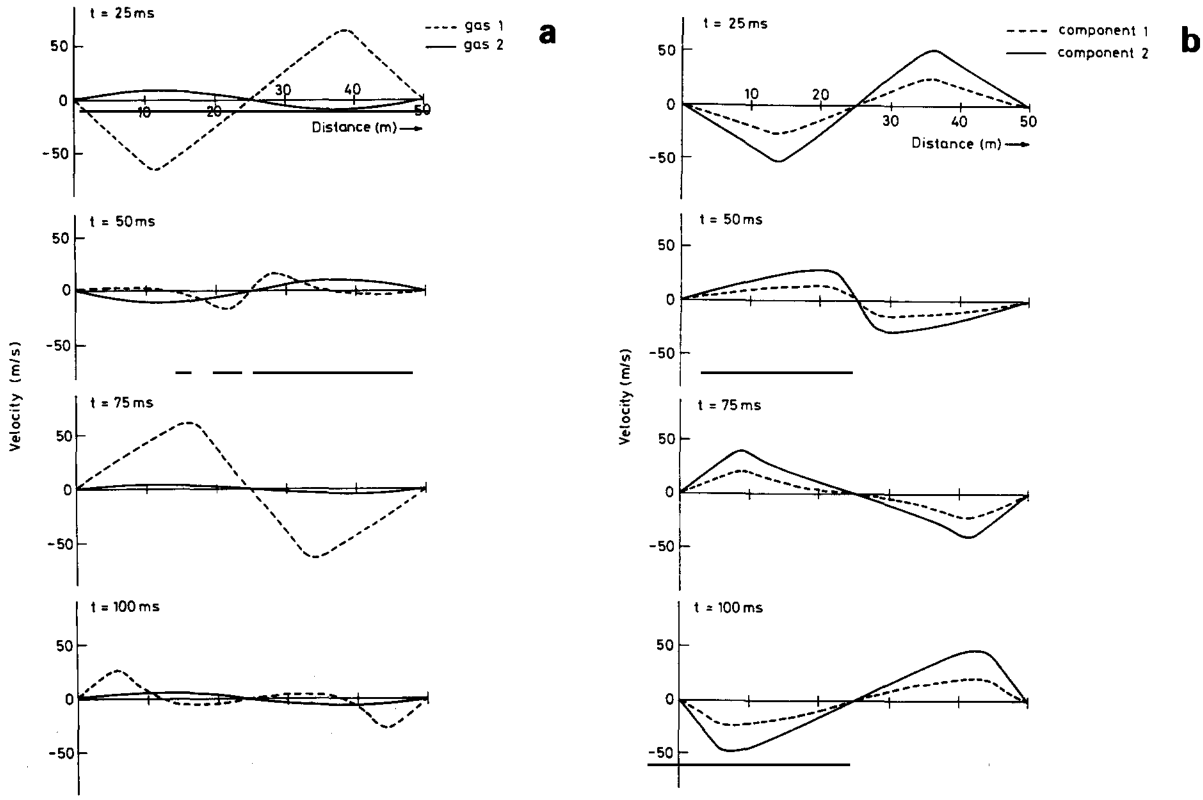

Fig. 6. Transient velocity plots for the multigas case (a) and the multiphase case (b).

finite difference methods, to the same accuracy without explicit viscous regularizations. Our finite difference scheme contains within it an implicit regularization, which is thermodynamically consistent in the sense that the entropy rise across a shock is always monotonic, although the shock itself can only be resolved to within three grid points. The supposed difficulties due to nonhyperbolicity simply do not arise in practical calculations, provided that care is taken in making the numerical method qualitatively "faithful" (in addition to being consistent in the usual sense) to the differential equations. This means that positive quantities, such as volume fractions and densities, must always be kept positive and integral invariants must be conserved by the numerical scheme to a sufficient accuracy. Given these conditions the nonhyperbolic multiphase equations are no more difficult to solve than the hyperbolic gas dynamics equations. It is also our experience that unless care is taken with proper definitions of D_v and D_e numerical simulations sometimes lead to spurious high-k instabilities in regions where one of the species is nearly depleted. In such regions the thermodynamic density, velocity and temperature of the depleted species need to be properly defined even though the volume fraction approaches zero.

Our views are illustrated by comparing the numerical simulations of two test problems (the first is analytically solvable) using multi-gas equations (hyperbolic, multi-pressure), and multiphase equations (non-hyperbolic, single pressure). The latter present no special difficulties. To avoid confusion, it must be remembered that these two sets of equations represent *different* physical models, which are generally applicable under different physical conditions. The fact that they produce similar (in the first case the same) results does *not* imply that the models are isomorphic. However, it

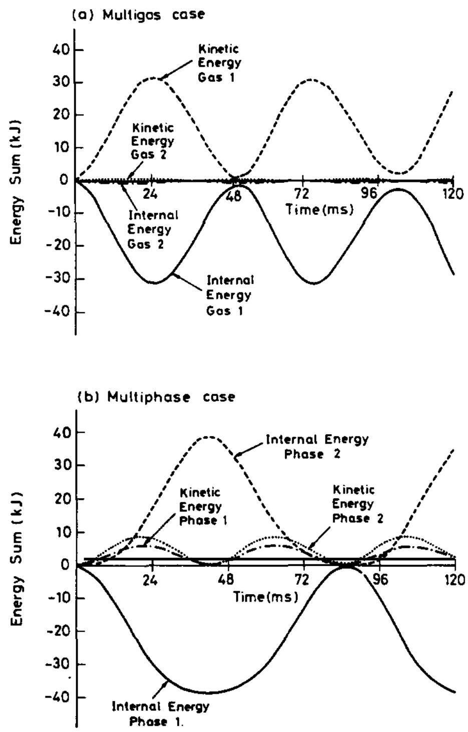

Fig. 7. Time variation of the energy integrals.

does demonstrate that the numerical treatment (provided it is faithful) does not distinguish between hyperbolic and nonhyperbolic conservative nonlinear systems, and that the claim that the later are useless for practical computations cannot be sustained. It is appropriate to end this paper with a quotation from Lax (see Harlow [10]) "all guidance to correct formulation must come from a careful consideration of the *physical principles* involved" (our emphasis).

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