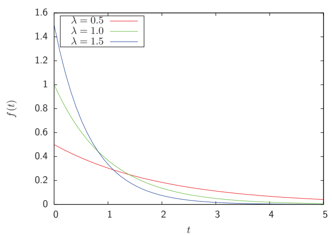
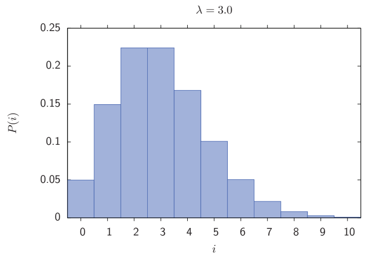
Poisson Processes

- "Random" arrival processes are ubiquitous in the real world and are often extremely well approximated by so-called Poisson processes
- First, here are two close friends:



Exp. rv, X: $f(t) = \lambda e^{-\lambda t}, \quad t \geq 0$



Poisson rv X: $P(X = n) = \frac{(\lambda t)^n e^{-(\lambda t)}}{n!}$

Properties of the exponential distribution

- The exponential distribution is memoryless, in that the future is independent of the past: if $X \sim exp(\lambda)$ then for $s, t > 0$ then

$$\begin{aligned} P(X \leq t+s \mid X > t) &= 1 - P(X > t+s \;\&\; X > t)/P(X > t) \\ &= 1 - P(X > t+s)/P(X > t) \\ &= 1 - e^{-\lambda(t+s)}/e^{-\lambda t} \;=\; 1 - e^{-\lambda s} \\ &= P(X \leq s) \end{aligned}$$

- If $X_1 \sim exp(\lambda_1)$ and $X_2 \sim exp(\lambda_2)$ then $\min(X_1, X_2) \sim exp(\lambda_1 + \lambda_2)$:

$$\begin{aligned} P(\min(X_1, X_2) \leq t) &= 1 - P(\min(X_1, X_2) > t) \\ &= 1 - P(X_1 > t \;\&\; X_2 > t) \\ &= 1 - e^{-\lambda_1 t} e^{-\lambda_2 t} = 1 - e^{-(\lambda_1 + \lambda_2)t} \end{aligned}$$

Poisson processes

- A Poisson arrival process is an arrival "stream" where the inter-arrival times, T, are exponentially-distributed, i.e. for "rate" parameter λ,

$$P(T \leq t) = 1 - e^{-\lambda t}$$

- Alternatively, it is a process where the number of arrivals, A, in time interval t has a *Poisson* distribution:

$$P(A = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

- A Poisson process implies exponentially-distributed inter-arrival times and vice-versa, e.g.

$$\begin{aligned} P(T \leq t) &= 1 - P(\text{no arrivals in } (0, t)) \\ &= 1 - P(A = 0) \;=\; 1 - \frac{(\lambda t)^0 e^{-\lambda t}}{0!} \\ &= 1 - e^{-\lambda t} \end{aligned}$$

Because the exponential distribution is memoryless we can "reset" time to 0 at each arrival instant; thus *all* inter-arrival times are exponential

- Arrival processes in the real world are often extremely well approximated by Poisson processes because arrivals are typically i. independent ii. ignorant of previous arrivals and the state of the system they are arriving at

Merging Poisson processes

- If we merge two Poisson processes with rates λ_1 and λ_2, the merged process is Poisson with rate $\lambda_1 + \lambda_2$:
- Let T_1, T_2 and T be the time to the next arrival in the two processes and merged process respectively; then

$$
\begin{aligned}
P(T \le t) &= P(\min(T_1, T_2) \le t) \\
&= 1 - e^{-(\lambda_1+\lambda_2)t}
\end{aligned}
$$

  from the above.
- The inter-arrival times are thus exponentially distributed with rate $\lambda_1 + \lambda_2$, so the merged process is Poisson
- Of course, this generalises to $n > 2$ processes, e.g. by forming a binary "merge tree"

- Thus:

$$
\begin{aligned}
P(N_1(t) = m) &= \sum_{n=0}^{\infty} P(N_1(t) = m, N_2(t) = n) \\
&= \frac{(\lambda t p)^m e^{-\lambda t p}}{m!} \sum_{n=0}^{\infty} \frac{(\lambda t(1-p))^n e^{-\lambda t(1-p)}}{n!} \\
&= \frac{(\lambda t p)^m e^{-\lambda t p}}{m!} \times 1 \\
&= \frac{(\lambda t p)^m e^{-\lambda t p}}{m!}
\end{aligned}
$$

  i.e. stream 1 is a Poisson process with "rate" λp
- Similarly,

$$
P(N_2(t) = n) = \frac{(\lambda t(1-p))^n e^{-\lambda t(1-p)}}{n!}
$$

Forking Poisson processes

- If we fork a Poisson process of rate λ into two streams by assigning arrivals to the first with probability p and the second with probability $1-p$ then the resulting processes are Poisson with rates λp and $\lambda(1-p)$ respectively:
- Let $N_1(t), N_2(t)$, $N(t)$ be the number of arrivals seen on the two output streams and input stream in time interval t respectively

$$
\begin{aligned}
&P(N_1(t) = m, N_2(t) = n) \\
&\quad = \sum_{k=0}^{\infty} P(N_1(t) = m, N_2(t) = n \mid N(t) = k) \times P(N(t) = k) \\
&\quad = P(N_1(t) = m, N_2(t) = n \mid N(t) = m+n) \times P(N(t) = m+n) \\
&\quad = \binom{m+n}{m} p^m (1-p)^n \times \frac{(\lambda t)^{m+n} e^{-\lambda t}}{(m+n)!} \\
&\quad = \frac{(\lambda t p)^m e^{-\lambda t p}}{m!} \times \frac{(\lambda t(1-p))^n e^{-\lambda t(1-p)}}{n!}
\end{aligned}
$$