Tutorial Exercises (Sheet 5)

Integer Linear Programming

Exercise 1. Consider the following (non-standard) Pure Integer Program:

$$\begin{array}{ll} \text{maximise} & \frac{1}{2}x_1 + \frac{2}{3}x_2 \\ \text{subject to} & x_1 - \frac{1}{3}x_2 = -2 \\ & x_1 \geq -\frac{2}{5} \\ & \frac{1}{3}x_1 \leq \frac{1}{4}x_2 \\ & x_1 \geq 0,\, x_2 \geq 0 \\ & x_1,\ x_2 \in \mathbb{N}_0 \end{array}$$

(a) Reformulate the problem as a Mixed-Integer Linear Program (MILP) in standard form.

(b) Reformulate the problem as a Pure Integer Linear Program (Pure ILP) in standard form.

Solution.

(a) To reformulate the problem as an MILP in standard form, we

(i) multiply the objective function with -1 to obtain a minimisation objective,

(ii) multiply the first constraint with -1 to obtain a non-negative right-hand side,

(iii) multiply the second constraint with -1 to obtain a non-negative right-hand side,

(iv) add a (continuous!) slack variable x_3 to reformulate the second constraint as an equality,

(v) bring all variables in the third constraint to the left-hand side, and

(vi) add a (continuous!) slack variable x_4 to reformulate the third constraint as an equality.

The resulting MILP in standard form is:

$$
\begin{array}{rl}
-\text{ minimise} & -\frac{1}{2}x_1 - \frac{2}{3}x_2 \\
\text{subject to} & -x_1 + \frac{1}{3}x_2 = 2 \\
& -x_1 + x_3 = \frac{2}{5} \\
& \frac{1}{3}x_1 - \frac{1}{4}x_2 + x_4 = 0 \\
& x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0 \\
& x_1,\ x_2 \in \mathbb{N}_0
\end{array}
$$

(b) To reformulate the problem as a Pure ILP in standard form, we multiply the first constraint with 3, the second constraint with 5 and the third constraint with $3 \cdot 4 = 12$ in order to obtain integer coefficients for all constraints. We also scale the objective function by 6. The resulting problem is:

$$
\begin{array}{rl}
\text{maximise} & z = \frac{1}{2}x_1 + \frac{2}{3}x_2 \\
\text{subject to} & 3x_1 - x_2 = -6 \\
& 5x_1 \geq -2 \\
& 4x_1 \leq 3x_2 \\
& x_1 \geq 0,\ x_2 \geq 0 \\
& x_1,\ x_2 \in \mathbb{N}_0
\end{array}
$$

Now we take the same steps as in part (a) of the exercise to obtain the following Pure ILP in standard form:

$$
\begin{array}{rl}
-\text{ minimise} & z' = -3x_1 - 4x_2 \\
\text{subject to} & -3x_1 + x_2 = 6 \\
& -5x_1 + x_3 = 2 \\
& 4x_1 - 3x_2 + x_4 = 0 \\
& x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0 \\
& x_1,\ x_2,\ x_3,\ x_4 \in \mathbb{N}_0
\end{array}
$$

where we will remember that $z = z'/6$. Note that – contrary to part (a) of this exercise – all variables, including the auxiliary slack variables x_3 and x_4, are integer. This is a requirement for Pure ILPs, and we are allowed to impose that requirement on x_3 and x_4 since all coefficients in the constraints are integer.

Exercise 2. Reformulate the requirement

$$x \in \{1, 3, 7\}$$

as a set of linear constraints using auxiliary binary variables.

Solution. This is a special instance of the 'k-allowed-values' type of logical constraints (see lecture notes):

$$\begin{aligned} & x = 1 \\ \text{or} \quad & x = 3 \\ \text{or} \quad & x = 7. \end{aligned}$$

Following the approach presented in the lecture, we can reformulate this requirement as follows.

$$\begin{aligned} & x = 1y_1 + 3y_2 + 7y_3 \\ & y_1 + y_2 + y_3 = 1 \\ & y_1, y_2, y_3 \in \{0, 1\}\,. \end{aligned}$$

By the way: we can actually eliminate one of the three binary variables in this constraint set. Can you see how this is possible?

Exercise 3. Consider the following set of constraints:

$$\begin{aligned} 2x_1 - x_2 &\leq 8 \\ 3x_1 + x_2 &\geq 4 \\ x_1 - 2x_2 &\leq 1 \end{aligned}$$

(a) Formulate a set of linear constraints (using auxiliary binary variables) that ensures that *at least* two of the above-mentioned constraints are satisfied.

(b) Formulate a set of linear constraints (using auxiliary binary variables) that ensures that *at most* two of the above-mentioned constraints are satisfied as strict inequalities (that is, the first and last constraint as '$<$', and the second constraint as '$>$'). *Hint:* Rewrite this first as a requirement to satisfy *at least* some constraints!

Solution. Let M be a sufficiently large number.

(a) We follow the approach presented in the lecture notes and obtain:

$$\begin{aligned} 2x_1 - x_2 &\leq 8 + My_1 \\ 3x_1 + x_2 &\geq 4 - My_2 \\ x_1 - 2x_2 &\leq 1 + My_3 \\ y_1 + y_2 + y_3 &\leq 1 \end{aligned}$$

The logic is the following: if $y_i = 1$, then the ith constraint is essentially 'switched off.' Since we should satisfy at least two of the three constraints, we are allowed to 'switch off' at most one constraint – hence the requirement that $y_1 + y_2 + y_3 \leq 1$. Note that the second constraint is a '$\geq$'-inequality, which is different to the setting in the lecture notes.

In that case, we must *subtract* My_2 to switch off the second constraint. Equally, we could have first multiplied the second constraint with -1 to bring the constraint into the form considered in the lecture notes.

(a) We want to satisfy *at most* two of these strict inequalities:

$$\begin{aligned} 2x_1 - x_2 &< 8 \\ 3x_1 + x_2 &> 4 \\ x_1 - 2x_2 &< 1 \end{aligned}$$

Note that this is equivalent to the requirement that we satisfy *at least* one of the complementary inequalities:

$$\begin{aligned} 2x_1 - x_2 &\geq 8 \\ 3x_1 + x_2 &\leq 4 \\ x_1 - 2x_2 &\geq 1 \end{aligned}$$

Now we can use the technique from part (a) of the exercise to reformulate this constraint set:

$$\begin{aligned} &2x_1 - x_2 \geq 8 - My_1 \\ &3x_1 + x_2 \leq 4 + My_2 \\ &x_1 - 2x_2 \geq 1 - My_3 \\ &y_1 + y_2 + y_3 \leq 2 \end{aligned}$$

Exercise 4. You are given the task of solving a mixed-integer linear program (MILP) involving the decision variable $x \in \{0, 1, \ldots, \delta\}$, for some $\delta \in \mathbb{N}$. Unfortunately, you only possess a solver for mixed-binary linear programs (MBLPs).

(a) How can you convert your MILP into an MBLP involving δ auxiliary binary variables?

(b) **Challenge:** Assume that $\delta = 2^\theta - 1$ for some $\theta \in \mathbb{N}$ (for example, $\delta = 7 = 2^3 - 1$ for $\theta = 3$ or $\delta = 15 = 2^4 - 1$ for $\theta = 4$). How can you reduce the number of binary variables in the resulting MBLP?

Even more challenging: Is it possible to generalise this result even when $\delta \neq 2^\theta - 1$ for all $\theta \in \mathbb{N}$?

Solution.

(a) In order to convert the MILP into an MBLP involving δ binary variables, we can replace the integer variable x by a sum of δ binary variables $y_i \in \{0, 1\}$, $i = 1, \ldots, \delta$ as follows:

$$x = \sum_{i=1}^{\delta} y_i$$

We remark that using this representation, there are $\binom{\delta}{n}$ ways to model $x = n$ for $n \in \{0, \ldots, \delta\}$. This 'redundancy' typically slows down solution techniques (such as the branch-and-bound and cutting planes techniques discussed later in the lecture), and it indicates that we may be able to save some binary variables with a more clever formulation.

(b) **Challenge:** If $\delta = 2^\theta - 1$ for some $\theta \in \mathbb{N}$, then we can use the binary numeral system in order to represent the integer variable x using only θ binary variables $y_i \in \{0, 1\}$, $i = 0, \ldots, \theta - 1$:

$$x = \sum_{i=0}^{\theta-1} 2^i y_i.$$

For example, $x \in \{0, 1, \ldots, 15\}$ can be reformulated using $\theta = 4$ binary variables $y_i \in \{0, 1\}$, $i = 0, \ldots, 3 = \theta - 1$ and

$$x = 1y_0 + 2y_1 + 4y_2 + 8y_3.$$

The choice $x = 7$, for example, is then represented by $y_0 = y_1 = y_2 = 1$ and $y_3 = 0$.

Even more challenging: When $\delta \neq 2^\theta - 1$ for all $\theta \in \mathbb{N}$, then the binary numeral system can still be used by representing x as the weighted sum of $\lfloor \log_2 \delta \rfloor + 1$ binary variables $y_i \in \{0, 1\}$, $i = 0, \ldots, \lfloor \log_2 \delta \rfloor$:

$$x = \sum_{i=0}^{\lfloor \log_2 \delta \rfloor} 2^i y_i.$$

Let's have a look at some examples to see how it works:

$$\begin{array}{lclcll}
x \in \{0,1\} & \rightsquigarrow & \lfloor \log_2 \delta \rfloor + 1 = \lfloor 0 \rfloor + 1 = 1 & \rightsquigarrow & x = 1y_0, & y_0 \in \{0,1\} \\
x \in \{0,1,2\} & \rightsquigarrow & \lfloor \log_2 \delta \rfloor + 1 = \lfloor 1 \rfloor + 1 = 2 & \rightsquigarrow & x = 1y_0 + 2y_1, & y_0, y_1 \in \{0,1\} \\
x \in \{0,1,\ldots,3\} & \rightsquigarrow & \lfloor \log_2 \delta \rfloor + 1 \approx \lfloor 1.58 \rfloor + 1 = 2 & \rightsquigarrow & x = 1y_0 + 2y_1, & y_0, y_1 \in \{0,1\} \\
x \in \{0,1,\ldots,4\} & \rightsquigarrow & \lfloor \log_2 \delta \rfloor + 1 = \lfloor 2 \rfloor + 1 = 3 & \rightsquigarrow & x = 1y_0 + 2y_1 + 4y_2, & y_0, y_1, y_2 \in \{0,1\} \\
x \in \{0,1,\ldots,5\} & \rightsquigarrow & \lfloor \log_2 \delta \rfloor + 1 \approx \lfloor 2.32 \rfloor + 1 = 3 & \rightsquigarrow & x = 1y_0 + 2y_1 + 4y_2, & y_0, y_1, y_2 \in \{0,1\}
\end{array}$$

Note that replacing x with $\sum_{i=0}^{\lfloor \log_2 \delta \rfloor} 2^i y_i$ everywhere is *not* enough in this case. Indeed, when $x \in \{0, 1, \ldots, 4\}$, replacing x with $1y_0 + 2y_1 + 4y_2$, $y_0, y_1, y_2 \in \{0, 1\}$, actually allows us to choose $x \in \{0, 1, \ldots, 7\}$! So we need the additional constraint that $x \leq \delta$, which – after replacing x – is equivalent to

$$\sum_{i=0}^{\lfloor \log_2 \delta \rfloor} 2^i y_i \leq \delta.$$

We remark that using this alternative representation, there is a unique way to choose $x = n$ for any integer $n \in \{0, \ldots, \delta\}$.

Exercise 5. In this exercise we illustrate an important property of *totally unimodular* matrices. A *unimodular* matrix is a matrix whose determinant A is $+1$ or -1. A totally unimodular matrix is a matrix for which every square non-singular submatrix is unimodular.

Assume the coefficient matrix A of an ILP to be a $m \times n$ matrix whose rows can be partitioned into two disjoint sets B and C. If the following sufficient conditions hold, we say that A is totally unimodular:

- Every column of A contains at most two non-zero entries;
- Every entry in A is 0, +1, or -1;
- If two non-zero entries in a column of A have the same sign, then the row of one is in B, and the other in C;
- If two non-zero entries in a column of A have opposite signs, then the rows of both are in B, or both in C.

A surprising property of ILPs with constraits $Ax = b$, where A is totally unimodular and x integer valued, is that the solution of the ILP and the solution of its LP relaxation are identical! This is therefore a special class of ILPs that can be solved using (continuous) LPs.

Tell if the following matrices respect the sufficient conditions for total unimodularity:

(a)
$$A = \begin{bmatrix} +3 & 0 \\ 0 & -1 \end{bmatrix}$$

(b)
$$A = \begin{bmatrix} 0 & +1 & 0 \\ +1 & -1 & 0 \\ +1 & 0 & 0 \end{bmatrix}$$

(c)
$$A = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & +1 \\ +1 & 0 & -1 & -1 & 0 & 0 \\ 0 & +1 & +1 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 & +1 & -1 \end{bmatrix}.$$

(d)
$$A = \begin{bmatrix} +1 & +1 & 0 & 0 & 0 & +1 \\ +1 & 0 & -1 & -1 & 0 & 0 \\ 0 & +1 & +1 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 & +1 & +1 \end{bmatrix}.$$

(e)

$$A = \begin{bmatrix} +1 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & +1 \end{bmatrix}$$

(f)

$$A = \begin{bmatrix} +1 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & +1 & -1 \\ +1 & 0 & 0 & +1 \end{bmatrix}$$

Solution.

(a) No, one entry is neither +1, 0, -1.

(b) Yes. The indexes of the rows for the two sets are $B = \{1, 2\}$ and $C = \{3\}$.

(c) Yes. The indexes of the rows for the two sets are $B = \{1, 2, 3, 4\}$ and $C = \emptyset$.

(d) Yes. The indexes of the rows for the two sets are $B = \{1\}$ and $C = \{2, 3, 4\}$.

(e) Yes. The indexes of the rows for the two sets are $B = \{1, 2, 3, 4\}$ and $C = \emptyset$.

(f) No. This is similar to the previous case, but rows 1 and 4 would need to be in separate partitions due to column 1. This would violate the requirement that the first 4 rows need to be in the same partition, in order to match the conditions for non-zero entries with opposite signs.