Equilibrium vs Transient Analysis

- Let $p_s(t)$ be the probability that a system is in state s at time t, $s \in S, t \geq 0$ for state space S
- We can estimate $p_s(t)$ from a simulation by accumulating T_s – the time spent in state s – and dividing by t
- Transient analysis concerns the analysis of $p_s(t)$, and/or associated measures, for some interval (t, t'), where $t, t' > 0$ (typically $t = 0$)
- A system is "stationary", or "in equilibrium" or "in steady state" if the state probabilities $p_s(t)$ are independent of time:

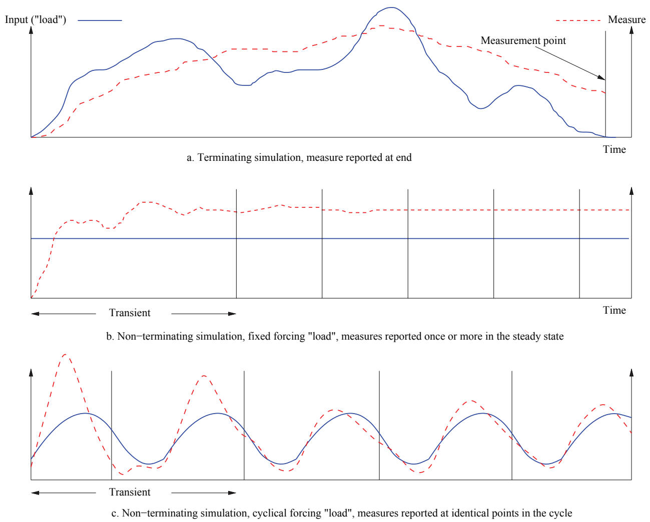
$$p_s(t) \rightarrow p_s \text{ as } t \rightarrow \infty$$

- Equilibrium Analysis or Steady State Analysis seeks to estimate $p_s, s \in S$ and/or other equilibrium measures, e.g. U, R, N... derived from them

Terminating vs Non-terminating Simulation

- A terminating simulation models a system over a specified period during which there is no notion of equilibrium, e.g.
  - A call centre which opens at 9-00am and closes at 10-00pm with a time-varying arrival rate of calls
  - An overnight batch processing system that has to clear a specified number of pre-queued jobs
- A non-terminating simulation runs indefinitely and seeks to model a system at equilibrium, e.g.
  - A network in which there is a statistically stable pattern of traffic (this may be static or seasonally varying, i.e. cyclical)
  - A factory that operates continuously or for very long periods of time
  - An A&E ward in a hospital

Simulation Types/Scenarios

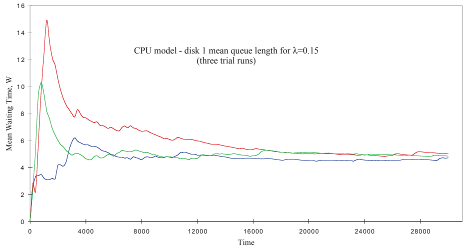


- To avoid measurement bias we must either
  1. Discard the "warm-up" transient by resetting measures
  2. Measure for long enough to render any bias insignificant
- Important: If we reset measures at time w, say, the aim is to ensure that $p_s(w)$, $s \in S$, is approximately the equilibrium probability p_s
- Note: The state at time w can be any $s \in S$: there is no such thing as an "equilibrium state"!
- How do we pick w? We can *only* do this by inspecting pilot runs, e.g.
  - Plot running mean or moving average over one run
  - Plot measures having discarded $p\%$ of the initial observations
  - Plot means/distributions of n^{th} observation of many runs

See, for example Welch's Method

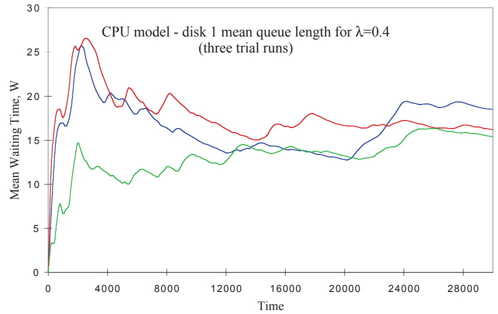
Example: A Simulation Model of the Central Server

- Here are three example time series plots for the mean response time (moving average) from a *simulation* of the central server system (Tutorial sheet 2) when $\lambda = 0.15$:



- It looks as if the system is approaching equilibrium after around 15000 time units ("ish"!)

- Now let's up the load to $\lambda = 0.4$...



- As λ increases the system takes longer to reach equilibrium (why?)
- An an example, we'd expect measurements taken between 15000 and 30000 time units, say, to have wider confidence intervals for $\lambda = 0.4$ than for $\lambda = 0.15$. Let's see...

- Recall (UNASSESSED): If $X_1, X_2, ..., X_n$ are iid normally-distributed observations from a simulation (e.g. from an ensemble of runs) then a $p\%$ confidence interval for the sample mean $\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ is:

$$\overline{X} \pm \frac{t_{1-p,n-1}S}{\sqrt{n}}$$

where S is the sample standard deviation and $t_{1-p,n-1}$ comes from tables of the Student's 't' distribution

λ	W (exact)	Point Estimate	90% Confidence Interval (10 independent replications)
0.1	4.86	4.839	(4.776, 4.902)
0.15	5.42	5.267	(5.044, 5.490)
0.20	6.16	6.198	(5.790, 6.607)
0.25	7.17	6.679	(6.158, 7.199)
0.30	8.68	8.662	(7.784, 9.541)
0.35	11.25	11.726	(10.767, 12.685)
0.40	16.86	16.295	(12.297, 20.293)
0.45	42.45	30.765	(18.621, 42.909)

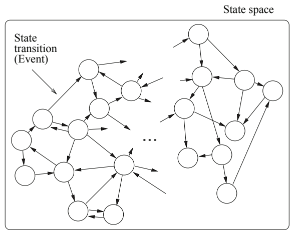
Q: 1. What's going on?

Q: 2. How do we reduce the width of the confidence intervals when λ increases?

Q: 3. Where did the exact results come from?

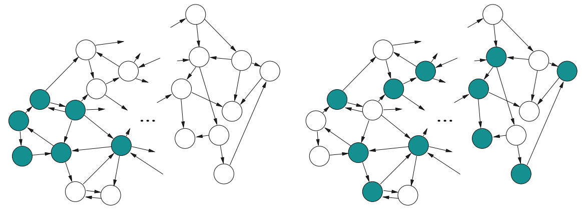
Answer to Q1: State Space Coverage

- Consider the underlying state transition system



- Suppose the simulation is run for time T at (approximate) equilibrium and spends time T_s in state s
- T_s/T is then an estimate, $\hat{p}_s$, of (the unknown) p_s
- Remark: Measures such as W could be computed directly, or in terms of $\hat{p}_s$ (e.g. using Little's Law) – the answer will be the same

- How good is the estimate $\hat{p}_s = T_s/T$ (similarly other measures)? It clearly depends on T, but also on the distribution $p_s, s \in S$ itself:



Parameterisation 1: Simulation time spent in a small number of states

Parameterisation 2: Simulation time spent covering a large number of states

Filled states: largest subset $S' \subseteq S$ s.t.
$inf\{p_i \mid i \in S'\} \geq sup\{p_j \mid j \in S - S'\}$ and $\sum_{i \in S'} p_i \leq p_{max}$ for some p_{max}

- For the same simulation time the C.I. for the left model (light load) should be narrower than the right (heavy load)