HW1

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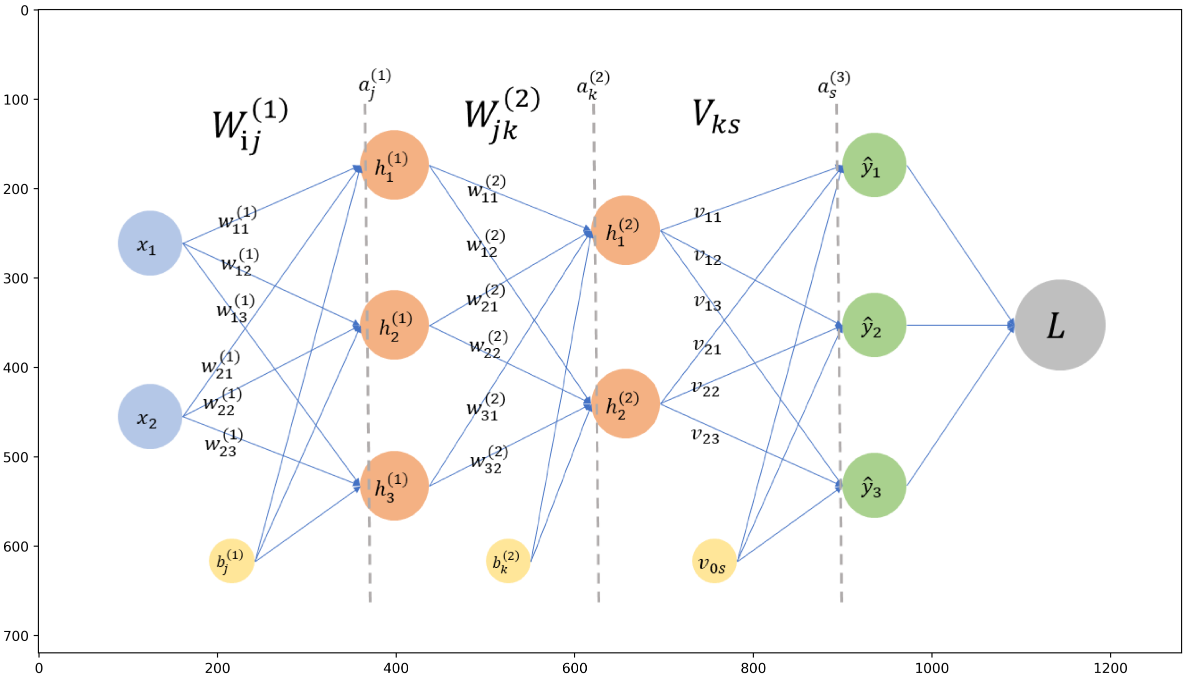
1. Feedforward: Building a ReLu 2 Layer neural network

1. Plot (draw) a network with:

> **• 2 inputs,**
>
> **• 2 hidden layers (where the first layer contains 3 hidden units and the second contains 2 hidden units) and a**
>
> **• 3-class output (use a softmax function)**

In [2]:
```
import matplotlib.pyplot as plt
import matplotlib.image as mpimg
%matplotlib inline
import matplotlib as mpl
mpl.rcParams['figure.dpi']= 500
img=mpimg.imread("my_network.png")
plt.figure(figsize = (15,15))
plt.imshow(img)
```

Out[2]: <matplotlib.image.AxesImage at 0x1e5a60d8128>



2. Write out the mathematical equation for this network

$$i = 1, 2$$

$$j = 1, 2, 3$$

$$k = 1, 2$$

$$s = 1, 2, 3$$

$$h_j^{(1)} = max(0, W_{1j}^{(1)} x_1 + W_{2j}^{(1)} x_1 + b_j^{(1)})$$

$$h_k^{(2)} = max(0, W_{1k}^{(2)} h_1^{(1)} + W_{2k}^{(2)} h_2^{(1)} + W_{3k}^{(2)} h_3^{(1)} + b_k^{(2)})$$

$$\hat{y}_s = \frac{exp(V_{1s} h_1^{(2)} + V_{2s} h_2^{(2)} + V_{0s})}{\sum_s exp(V_{1s} h_1^{(2)} + V_{2s} h_2^{(2)} + V_{0s}))}$$

3. Write out the function in python, call it ff_nn_2_ReLu(...)

In [3]:
```
import matplotlib.pyplot as plt
%matplotlib inline
import numpy as np

def softmax(A):
    e = np.exp(A)
    return e / e.sum(axis=1).reshape((-1,1))

def ff_nn_2_ReLu(x, w_1, w_2, v, b_1, b_2, c):

    a_1 = np.dot(x, w_1) + b_1
    h_1 = np.maximum(0, a_1)

    a_2 = np.dot(h_1, w_2) + b_2
    h_2 = np.maximum(0, a_2)

    y = softmax(np.dot(h_2,v) + c)

    return np.array(y)
```

4. what are the class probabilities associated with the forward pass of each sample?

In [4]:
```
x = np.array([[1., 0., 0.],
              [-1., -1., 1.]]).T

w_2 = np.array([[1, 0],
               [-1, 0],
               [0, 0.5]])

w_1 = np.array([[1., 0., 0.],
               [-1., -1., 0.]])

v = np.array([[1., 1.],
             [0., 0.],
             [-1., -1.]]).T

b_1 = np.array([0., 0., 1.]).T
b_2 = np.array([1., -1.]).T
c = np.array([1., 0., 0.]).T

result = ff_nn_2_ReLu(x, w_1, w_2, v, b_1, b_2, c)
result
```

Out[4]:
```
array([[0.94649912, 0.04712342, 0.00637746],
       [0.84379473, 0.1141952 , 0.04201007],
       [0.84379473, 0.1141952 , 0.04201007]])
```

In [5]:
```
np.sum(result, axis = 1)
```

Out[5]:
```
array([1., 1., 1.])
```

2 Gradient Descent

1. What are the partial derivatives of f with respect to x and to y?

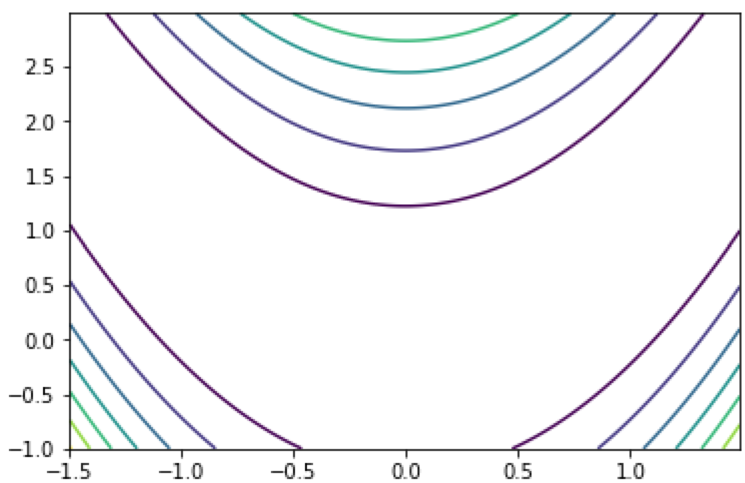
$f = (1 - x)^2 + 100((y - x^2)^2)$

$\frac{df}{dx} = 2(-1 + x + 200x^3 - 200xy)$

$\frac{df}{dy} = 200(-x^2 + y)$

2. Create a visualization of the contours of the Rosenbrock function.

In [6]:
```
# Gradient Descent 2.2 Create a visualization of the contours of the Rosenbroc
k function.

delta = 0.01
x = np.arange(-1.5, 1.5, delta)
y = np.arange(-1, 3, delta)
X, Y = np.meshgrid(x, y)
Z = (1 - X)**2  + 100 * (Y - X**2)**2
fig, ax = plt.subplots()
CS = ax.contour(X, Y, Z)
```



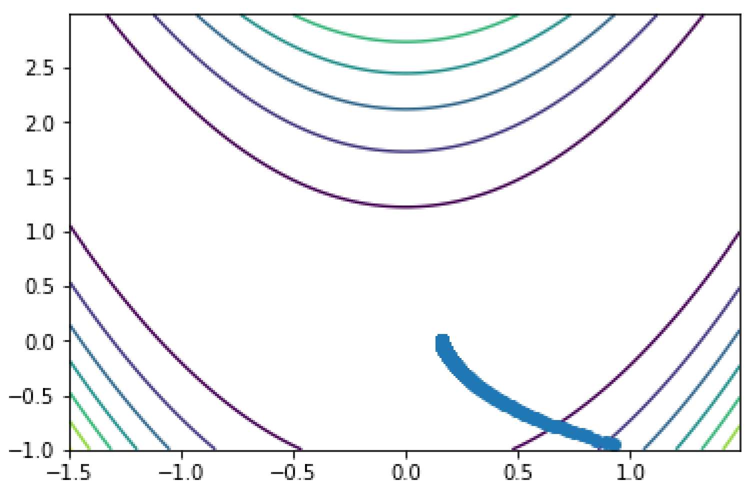
3. Write a Gradient Descent algorithm for finding the minimum of the function. Visualize your results with a few different learning rates.

In [7]:
```
def grad_f(vector):
    x, y = vector
    df_dx = 2 * (-1 + x + 200 * x**3 - 200 * x * y)
    df_dy = 200 * (-x**2 + y)
    return np.array([df_dx, df_dy])
```

In [8]:
```
%matplotlib inline

def grad_descent(starting_point = None, iterations = 5, learning_rate = 1):
    if starting_point:
        point = starting_point
    else:
        point = np.random.uniform(-1, 1.5, size = 2)
    trajectory = [point]
    print(trajectory)
    for i in range(iterations):
        grad = grad_f(point)
        point = point - learning_rate * grad
        trajectory.append(point)
    return np.array(trajectory)

np.random.seed(10)
traj = grad_descent(iterations = 300, learning_rate = 5**(-6))

fig, ax = plt.subplots()
CS = ax.contour(X, Y, Z)
x = traj[:, 0]
y = traj[:, 1]
plt.plot(x, y, '-o')
```

```
[array([ 0.92830161, -0.94812013])]
```

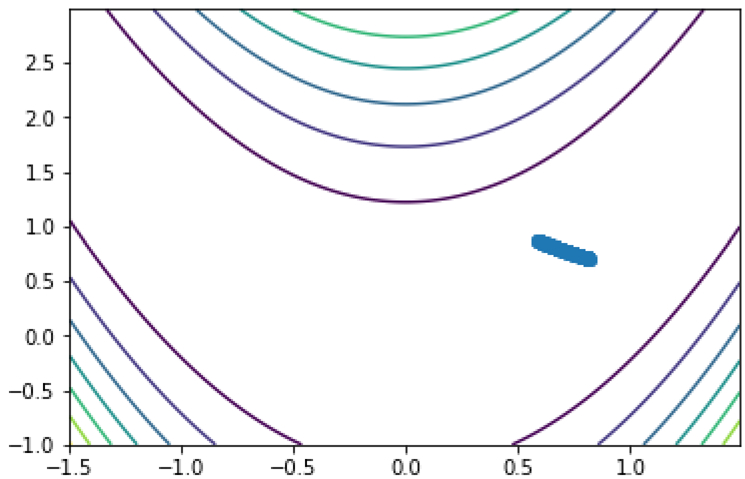
Out[8]:
```
[<matplotlib.lines.Line2D at 0x1e5a6be89b0>]
```



In [9]:
```
traj = grad_descent(iterations = 300, learning_rate = 5**(-7))

fig, ax = plt.subplots()
CS = ax.contour(X, Y, Z)
x = traj[:, 0]
y = traj[:, 1]
plt.plot(x, y, '-o')
```

```
[array([0.58412059, 0.87200971])]
```

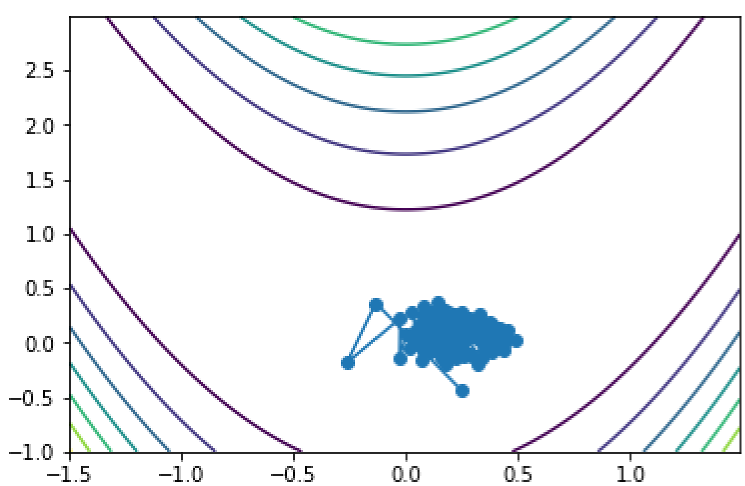
Out[9]:
```
[<matplotlib.lines.Line2D at 0x1e5a7d39da0>]
```



In [10]:
```
traj = grad_descent(iterations = 300, learning_rate = 5**(-3))

fig, ax = plt.subplots()
CS = ax.contour(X, Y, Z)
x = traj[:, 0]
y = traj[:, 1]
plt.plot(x, y, '-o')
```

```
[array([ 0.24626753, -0.43800839])]
```

Out[10]:
```
[<matplotlib.lines.Line2D at 0x1e5a7cf4d68>]
```

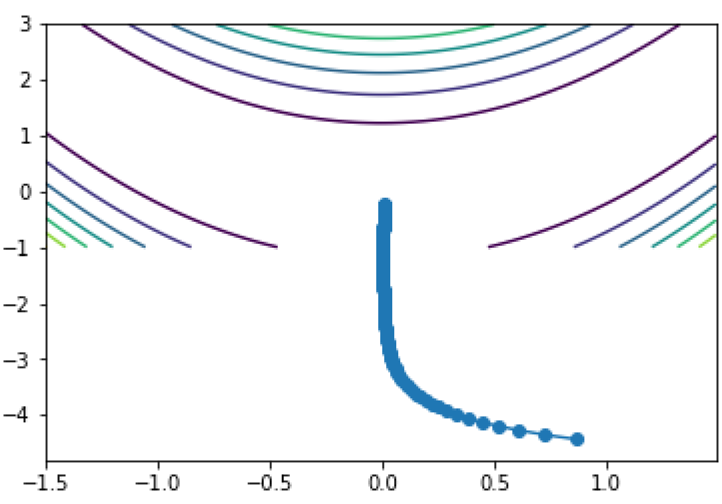


4. Write a Gradient Descent With Momentum algorithm for finding the minimum. Visualize your results with a few different settings of the algorithm's hyperparameters.

In [11]:
```
# --- Gradient with momentum ---
def grad_descent_with_momentum(starting_point=None, iterations=10, alpha=.0009
, epsilon=10):
    if starting_point:
        point = starting_point
    else:
        point = np.random.uniform(-10,10,size=2)
    trajectory = [point]
    v = np.zeros(point.size)

    for i in range(iterations):
        grad = grad_f(point)
        v = alpha*v + epsilon*grad
        point = point - v
        trajectory.append(point)
    return np.array(trajectory)
```

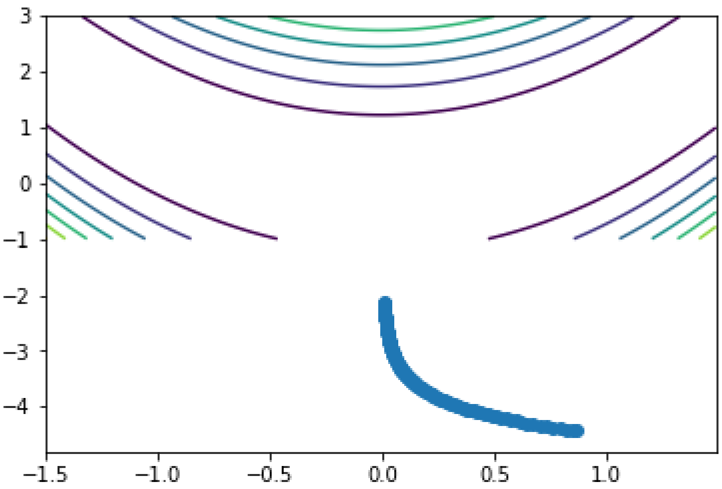
In [12]:
```
# --- Visualizing trajectory --
np.random.seed(100)
traj = grad_descent_with_momentum(iterations=180, epsilon=8*10**(-5), alpha=5*
*(-6))

fig, ax = plt.subplots()
CS = ax.contour(X, Y, Z)
x= traj[:,0]
y= traj[:,1]
plt.plot(x,y,'-o')
```

Out[12]: [<matplotlib.lines.Line2D at 0x1e5a7db62b0>]



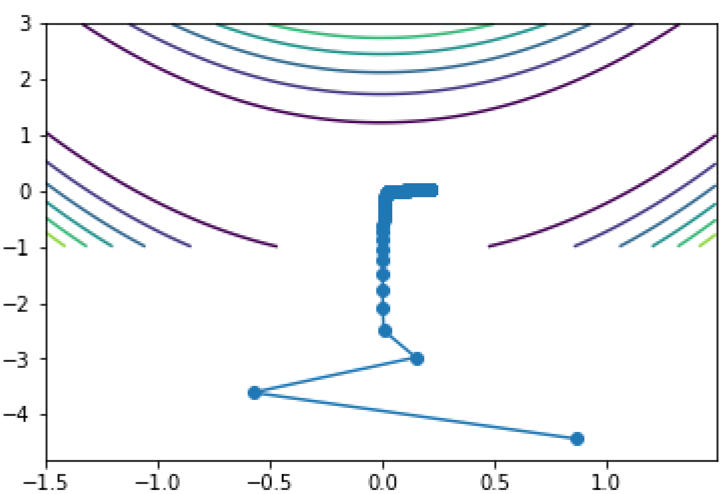
In [13]:
```
np.random.seed(100)
traj = grad_descent_with_momentum(iterations=180, epsilon=10**(-5), alpha=0.5)

fig, ax = plt.subplots()
CS = ax.contour(X, Y, Z)
x= traj[:,0]
y= traj[:,1]
plt.plot(x,y,'-o')
```

Out[13]: [<matplotlib.lines.Line2D at 0x1e5a7e208d0>]



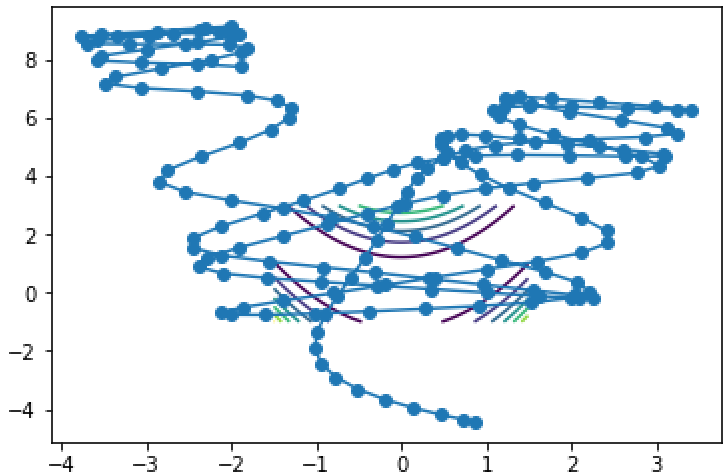
In [14]:
```
np.random.seed(100)
traj = grad_descent_with_momentum(iterations=180, epsilon=8*10**(-4), alpha=5*
*(-6))

fig, ax = plt.subplots()
CS = ax.contour(X, Y, Z)
x= traj[:,0]
y= traj[:,1]
plt.plot(x,y,'-o')
```

Out[14]: [<matplotlib.lines.Line2D at 0x1e5a7e8d940>]



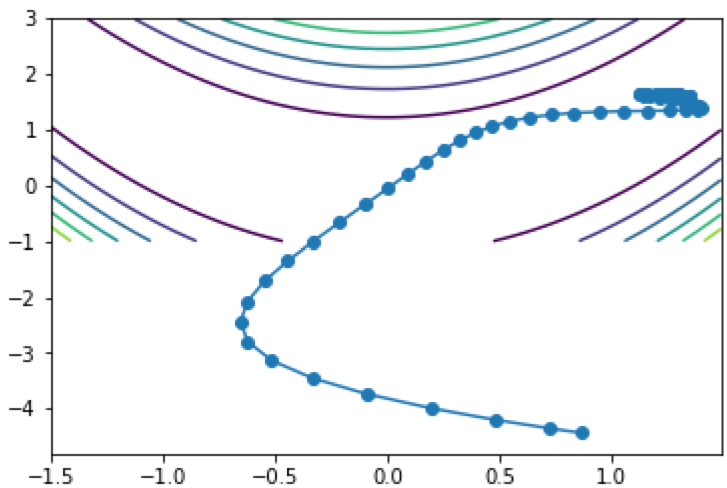
In [15]:
```
np.random.seed(100)
traj = grad_descent_with_momentum(iterations=180, epsilon=8*10**(-5), alpha=1)

fig, ax = plt.subplots()
CS = ax.contour(X, Y, Z)
x= traj[:,0]
y= traj[:,1]
plt.plot(x,y,'-o')
```

Out[15]: [<matplotlib.lines.Line2D at 0x1e5a7f03160>]



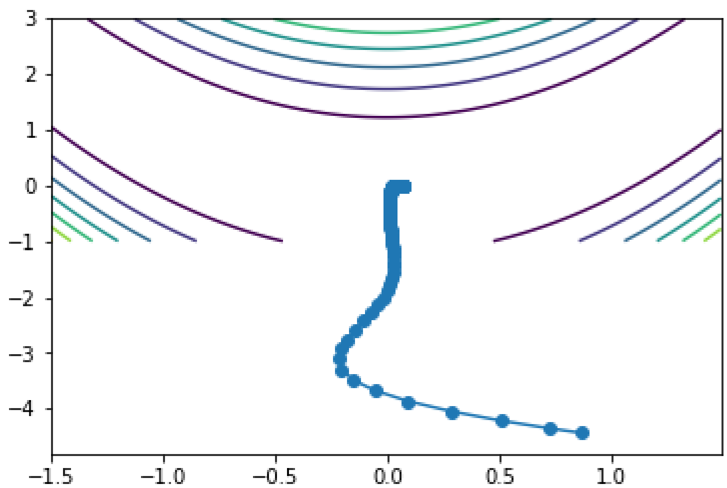
In [16]:
```
np.random.seed(100)
traj = grad_descent_with_momentum(iterations=180, epsilon=8*10**(-5), alpha=0.
9)

fig, ax = plt.subplots()
CS = ax.contour(X, Y, Z)
x= traj[:,0]
y= traj[:,1]
plt.plot(x,y,'-o')
```

Out[16]: [<matplotlib.lines.Line2D at 0x1e5a7f6e5f8>]



In [17]:
```
np.random.seed(100)
traj = grad_descent_with_momentum(iterations=180, epsilon=8*10**(-5), alpha=0.
7)

fig, ax = plt.subplots()
CS = ax.contour(X, Y, Z)
x= traj[:,0]
y= traj[:,1]
plt.plot(x,y,'-o')
```

Out[17]: [<matplotlib.lines.Line2D at 0x1e5a7fe4550>]



3 Backprop

1. For the same network as in Number 1, derive expressions of the gradient of the Loss function with respect to each of the model parameters.

The loss function:

$$L(y,\hat{y}) = y_1 \log \hat{y}_1 + y_2 \log \hat{y}_2 + y_3 \log \hat{y}_3$$

And:

$$h_j^{(1)} = max(0, W_{1j}^{(1)} x_1 + W_{2j}^{(1)} x_1 + b_j^{(1)})$$

$$h_k^{(2)} = max(0, W_{1k}^{(2)} h_1^{(1)} + W_{2k}^{(2)} h_2^{(1)} + W_{3k}^{(2)} h_3^{(1)} + b_k^{(2)})$$

$$\hat{y}_s = \frac{exp(V_{1s}h_1^{(2)} + V_{2s}h_2^{(2)} + V_{0s})}{\sum_s exp(V_{1s}h_1^{(2)} + V_{2s}h_2^{(2)} + V_{0s}))}$$

The gradients:

$$\begin{aligned}\frac{\partial L}{\partial V_{ks}} &= \sum_s \frac{\partial L}{\partial \hat{y}_s} \frac{\partial \hat{y}_s}{\partial a_s^{(3)}} \frac{\partial a_s^{(3)}}{\partial V_{ks}} \\ &= (1_{(s=true\ class)} - \hat{y}_s) h_k^{(2)}\end{aligned} \quad \text{, s=1,2,3}$$

$$\begin{aligned}\frac{\partial L}{\partial v_{0s}} &= \sum_s \frac{\partial L}{\partial \hat{y}_s} \frac{\partial \hat{y}_s}{\partial a_s^{(3)}} \frac{\partial a_s^{(3)}}{\partial v_{0s}} \\ &= \sum_s (1_{(s=true\ class)} - \hat{y}_s)\end{aligned} \quad \text{, s=1,2,3}$$

$$\begin{aligned}\frac{\partial L}{\partial W_{jk}^{(2)}} &= \sum_s \frac{\partial L}{\partial \hat{y}_s} \frac{\partial \hat{y}_s}{\partial a_s^{(3)}} \frac{\partial a_s^{(3)}}{\partial h_k^{(2)}} \frac{\partial h_k^{(2)}}{\partial a_k^{(2)}} \frac{\partial a_k^{(2)}}{\partial W_{jk}^{(2)}} \\ &= \sum_s (1_{(s=true\ class)} - \hat{y}_s) V_{ks} 1_{(a_k^{(2)}>0)} h_j^{(1)}\end{aligned} \quad \text{, s=1,2,3; j = 1,2,3; k = 1, 2;}$$

$$\begin{aligned}\frac{\partial L}{\partial b_k^2} &= \sum_s \frac{\partial L}{\partial \hat{y}_s} \frac{\partial \hat{y}_s}{\partial a_s^{(3)}} (\sum_k \frac{\partial a_s^{(3)}}{\partial h_k^{(2)}} \frac{\partial h_k^{(2)}}{\partial a_k^{(2)}} \frac{\partial a_k^{(2)}}{\partial b_k^2}) \\ &= \sum_s (1_{(s=true\ class)} - \hat{y}_s)(\sum_k V_{ks} 1_{(a_k^{(2)}>0)})\end{aligned} \quad \text{, s=1,2,3; j = 1,2,3; k = 1, 2;}$$

$$\begin{aligned}\frac{\partial L}{\partial W_{ij}^{(1)}} &= \sum_s \frac{\partial L}{\partial \hat{y}_s} \frac{\partial \hat{y}_s}{\partial a_s^{(3)}} (\sum_k \frac{\partial a_s^{(3)}}{\partial h_k^{(2)}} \frac{\partial h_k^{(2)}}{\partial a_k^{(2)}} \frac{\partial a_k^{(2)}}{\partial h_j^{(1)}} \frac{\partial h_j^{(1)}}{\partial a_j^{(1)}} \frac{\partial a_j^{(1)}}{\partial W_{ij}^{(1)}}) \\ &= \sum_s (1_{(s=true\ class)} - \hat{y}_s)(\sum_k V_{ks} 1_{(a_k^{(2)}>0)} W_{jk}^{(2)} 1_{(a_j^{(1)}>0)} x_i)\end{aligned} \quad \text{, s=1,2,3; j = 1,2,3; k = 1, 2; i = 1,2}$$

$$\begin{aligned}\frac{\partial L}{\partial b_j^1} &= \sum_s \frac{\partial L}{\partial \hat{y}_s} \frac{\partial \hat{y}_s}{\partial a_s^{(3)}} (\sum_k \frac{\partial a_s^{(3)}}{\partial h_k^{(2)}} \frac{\partial h_k^{(2)}}{\partial a_k^{(2)}} (\sum_j \frac{\partial a_k^{(2)}}{\partial h_j^{(1)}} \frac{\partial h_j^{(1)}}{\partial a_j^{(1)}} \frac{\partial a_j^{(1)}}{\partial b_j^1})) \\ &= \sum_s (1_{(s=true\ class) - \hat{y}_s})(\sum_k V_{ks} 1_{(a_k^{(2)}>0)} (\sum_j W_{jk}^{(2)} 1_{(a_j^{(1)}>0)}))\end{aligned} \quad \text{, s=1,2,3; j = 1,2,3; k = 1, 2; i = 1,2}$$

2. Write a function grad_f(...) that takes in a weights vector and returns the gradient of the Loss at that location.

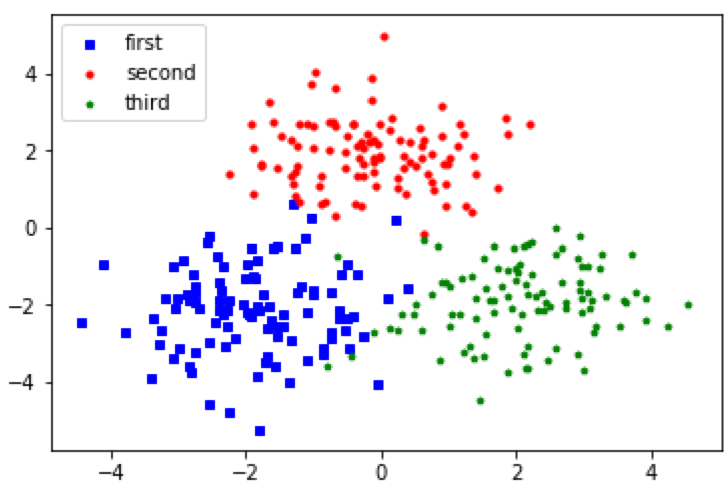
In [18]:
```
# Write a function grad_f(...) that takes in a weights vector and returns the
 gradient of the Loss at that location.

def grad_f(data, T, Y, H1, H2, V, W2, W1, b1, b2):

    d_v = H2.T.dot(T - Y)
    d_v0 = (T - Y).sum(axis = 0)

    d_w2 = H1.T.dot((T - Y).dot(V.T) * (H2 > 0))
    d_b2 = ((T - Y).dot(V.T) * (H2 > 0)).sum(axis=0)

    d_w1 = data.T.dot(((T - Y).dot(V.T)*(H2 > 0)).dot(W2.T)*(H1 > 0))
    d_b1 = (((T - Y).dot(V.T)*(H2 > 0)).dot(W2.T)*(H1 > 0)).sum(axis=0)

    return d_v, d_v0, d_w2, d_b2, d_w1, d_b1
```

3. Generate a synthetic dataset of 3 equally sampled bivariate Gaussian distributions with parameters $\mu_1 = (0, 2), \mu_2 = (2, -2), \mu_3 = (-2, -2)$; **$Σ_i = \left[**

$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

\right]$; i = 1,2,3 that you'll use for fitting your network. Plot your sample dataset, coloring data points by their respective class.

In [19]:
```
def generate_data(N = 100):
    np.random.seed(123)
    # Setting up 3 clusters with centers at [-2,-2],[0,2],[2,-2]
    c1 = np.random.multivariate_normal([-2,-2], np.eye(2), size=N)
    c2 = np.random.multivariate_normal([0,2], np.eye(2), size=N)
    c3  = np.random.multivariate_normal([2,-2], np.eye(2), size=N)
    data = np.vstack((c1,c2,c3))
    # Classes of the three clusters
    classes = np.array([0]*N + [1]*N + [2]*N)
    fig = plt.figure()
    ax1 = fig.add_subplot(111)
    ax1.scatter(c1[:,0],c1[:,1], s=10, c='b', marker="s", label='first')
    ax1.scatter(c2[:,0],c2[:,1], s=10, c='r', marker="o", label='second')
    ax1.scatter(c3[:,0],c3[:,1], s=10, c='g', marker="p", label='third')
    plt.legend(loc='upper left');
    plt.show()
    # One hot encoding for T:
    T = np.zeros((N * 3, 3))
    for n in range(N * 3):
        T[n, classes[n]] = 1

    return T, data

T, data = generate_data()
```

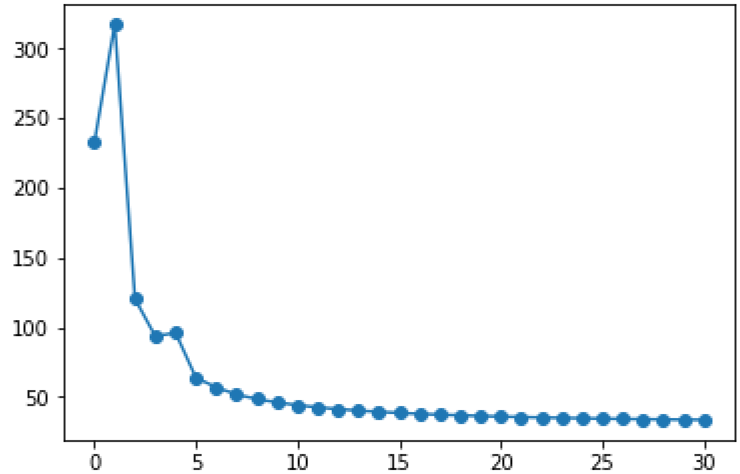


4. Fit your network using Gradient Descent. Keep track of the total Loss at each iteration and plot the result.

In [20]:
```
def softmax(A):
    e = np.exp(A)
    return e / e.sum(axis=1).reshape((-1,1))

def ff_nn_2_ReLu(data, W1, W2, V, b1, b2, v0):

    a_1 = np.dot(data, W1) + b1
    h_1 = np.maximum(0, a_1)

    a_2 = np.dot(h_1, W2) + b2
    h_2 = np.maximum(0, a_2)

    y = softmax(np.dot(h_2, V) + v0)

    return h_1, h_2, np.array(y)

def cost(T, Y):
    tot = T * np.log(Y)
    return tot.sum()
```

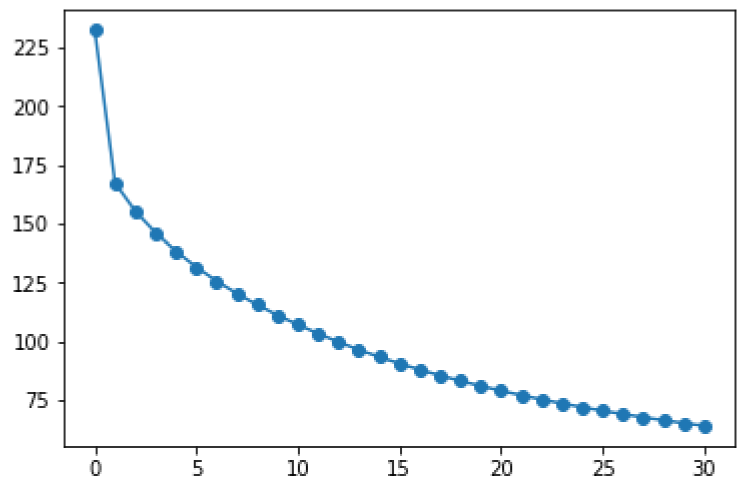
In [69]:
```
def gradient_descent(iterations = 500, learning_rate = 2*10e-4):

    # Intilaizing weights
    np.random.seed(8)
    V = np.random.randn(2 * 3).reshape(2,3)
    v0= np.random.randn(3).reshape(1,3)
    W2 = np.random.randn(3 * 2).reshape(3,2)
    b2 = np.random.randn(2).reshape(1,2)
    W1 = np.random.randn(2 * 3).reshape(2,3)
    b1 = np.random.randn(3).reshape(1,3)
    H1, H2, Y = ff_nn_2_ReLu(data, W1, W2, V, b1, b2, v0) #feed forward to get
 initial H1, H2, and Y

    costs = [-cost(T, Y)]


    for i in range(iterations):

        d_v, d_v0, d_w2, d_b2, d_w1, d_b1 = grad_f(data, T, Y, H1, H2, V, W2,
W1, b1, b2)

        V = V + learning_rate * d_v
        v0 = v0 + learning_rate * d_v0
        W2 = W2 + learning_rate * d_w2
        b2 = b2 + learning_rate * d_b2
        W1 = W1 + learning_rate * d_w1
        b1 = b1 + learning_rate * d_b1

        H1, H2, Y = ff_nn_2_ReLu(data, W1, W2, V, b1, b2, v0)

        c = -cost(T, Y)
        costs.append(c)

    plt.plot(range(iterations+1), costs, '-o')
    print("The loss dropped from initial " + str(round(costs[0],2)) + " to fin
ally: " + str(round(costs[-1],2))
          + " after " + str(iterations) + " iterations.")


gradient_descent(iterations = 30, learning_rate = 6*10e-4)
```

```
The loss dropped from initial 232.49 to finally: 33.51 after 30 iterations.
```



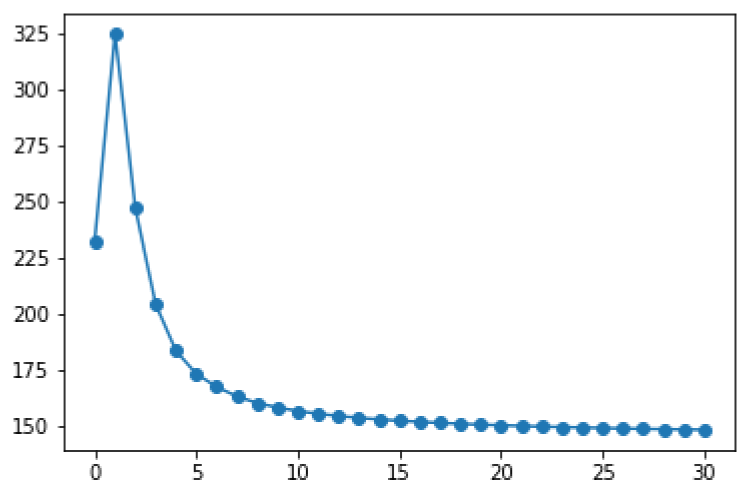
In [70]:
```
gradient_descent(iterations = 30, learning_rate = 1*10e-4)
```

```
The loss dropped from initial 232.49 to finally: 63.92 after 30 iterations.
```



In [73]:
```
gradient_descent(iterations = 30, learning_rate = 6.5*10e-4)
```

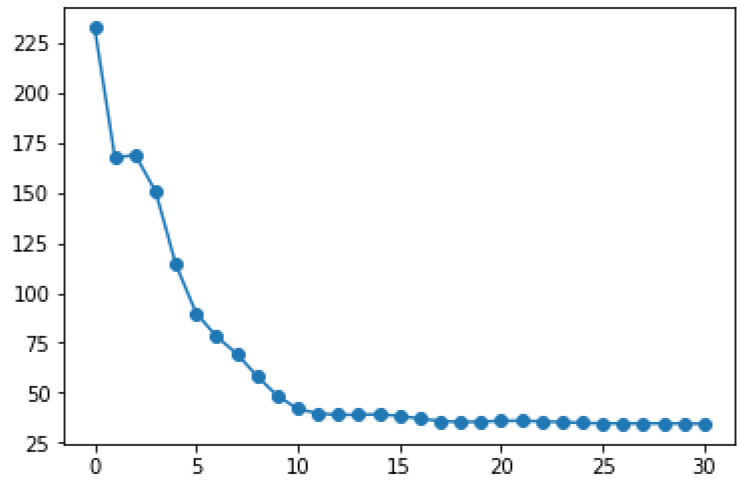
```
The loss dropped from initial 232.49 to finally: 148.33 after 30 iterations.
```



5. Repeat the exercise above using Momentum. Comment on whether your algorithm seems to converge more efficiently.

In [48]:
```
def grad_descent_with_momentum(iterations=500, alpha=.9, epsilon=0.001):

    # Intilaizing weights
    np.random.seed(8)
    V = np.random.randn(2 * 3).reshape(2,3)
    v0 = np.random.randn(3).reshape(1,3)
    W2 = np.random.randn(3 * 2).reshape(3,2)
    b2 = np.random.randn(2).reshape(1,2)
    W1 = np.random.randn(2 * 3).reshape(2,3)
    b1 = np.random.randn(3).reshape(1,3)
    H1, H2, Y = ff_nn_2_ReLu(data, W1, W2, V, b1, b2, v0) #feed forward to get
 initial H1, H2, and Y
    v_V, v_v0, v_W2, v_b2, v_W1, v_b1 = (np.zeros_like(x) for x in [V, v0, W2,
 b2, W1, b1]) #initializing velocity vectors

    costs = [-cost(T, Y)]
    for i in range(iterations):

        d_v, d_v0, d_w2, d_b2, d_w1, d_b1 = grad_f(data, T, Y, H1, H2, V, W2,
W1, b1, b2)

        v_V = alpha*v_V + epsilon*d_v
        V = V + v_V

        v_v0 = alpha*v_v0 + epsilon*d_v0
        v0 = v0 + v_v0

        v_W2 = alpha*v_W2 + epsilon*d_w2
        W2 = W2 + v_W2

        v_b2 = alpha*v_b2 + epsilon*d_b2
        b2 = b2 + v_b2

        v_W1 = alpha*v_W1 + epsilon*d_w1
        W1 = W1 + v_W1

        v_b1 = alpha*v_b1 + epsilon*d_b1
        b1 = b1 + v_b1

        H1, H2, Y = ff_nn_2_ReLu(data, W1, W2, V, b1, b2, v0) #feed forward

        c = -cost(T, Y)
        costs.append(c)

    plt.plot(range(iterations+1), costs, '-o')
    print("The loss dropped from initial " + str(round(costs[0],2)) + " to fin
ally: " + str(round(costs[-1],2))
          + " after " + str(iterations) + " iterations.")


grad_descent_with_momentum(iterations=30, alpha=0.9, epsilon = 0.001)
```

```
The loss dropped from initial 232.49 to finally: 34.47 after 30 iterations.
```



As we can see from the trajectory of the loss function between gradient descent/without gradient descent, there is a significant change in the convergence rate. With the help of momentum update, at 30 iteration, the loss function was able to drop to 34. The best choice of leanring rate will only achive the same result as with momentum update.