MAST218 FINAL

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1.

(a) $\frac{d^2y}{dx^2} = \frac{d(dy/dx)}{dx} = \frac{d(dy/dx)}{dt} \Big/ \frac{dx}{dt}$

and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin(2t)}{\cos t}$

$\frac{d(dy/dx)}{dt} = \frac{-(2\cos(2t)\cos t + \sin(2t)\sin t)}{\cos^2(t)}$

$\frac{dx}{dt} = \cos t.$

$\frac{d^2y}{dx^2} = \frac{-2\cos(2t)\cos t - \sin(2t)\sin(t)}{\cos^3(t)}$

(b) concave downward means $\frac{d^2y}{dx^2} < 0$

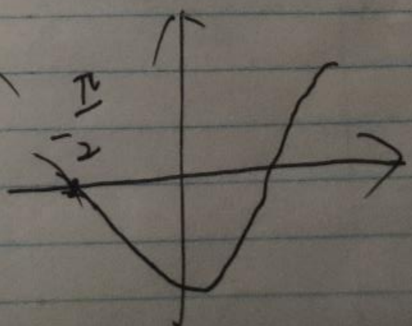
$$\frac{-2\cos(2t)\cos t - \sin(2t)\sin(t)}{\cos^3(t)} < 0.$$

From $(-\frac{\pi}{2}, \frac{\pi}{2})$, $\cos(t) > 0$

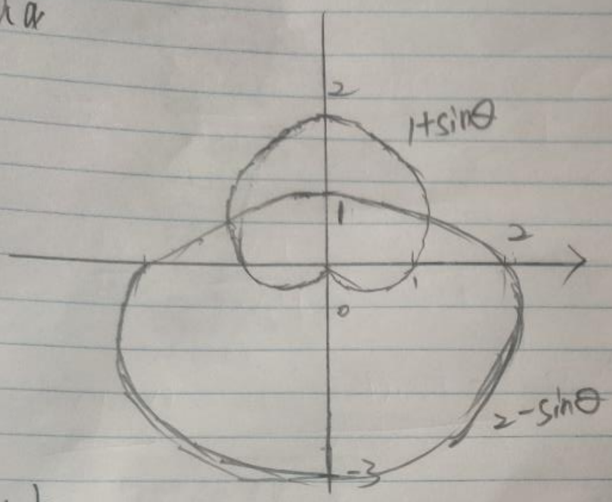
$-2\cos(2t)\cos t - \sin(2t)\sin(t) < 0$

Solution: $(0, \frac{\pi}{2})$

$(-\frac{\pi}{2}, \frac{\pi}{2})$



2. a



b

step1: $1+\sin\theta = 2-\sin\theta$

$\sin\theta = \frac{1}{2}$

so from $-\frac{\pi}{2}$, to $\frac{\pi}{6}$, $2-\sin\theta$ has bigger r

$\frac{\pi}{6}$ to $\frac{\pi}{2}$ $1+\sin\theta$ has bigger r

$$\text{Area} = 2\times\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \frac{(1+\sin(\theta))^2}{2}\,d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{(2-\sin(\theta))^2}{2}\,d\theta\right)$$

$$= 2\times\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \frac{1+2\sin\theta+\sin^2\theta}{2}\,d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{4-4\sin\theta+\sin^2\theta}{2}\,d\theta\right)$$

$$= 2\times\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \frac{1}{2}\,d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \sin\theta\,d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \frac{\sin^2\theta}{2}\,d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\,d\theta\right.$$

$$\left. + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} -2\sin\theta\,d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\sin^2\theta}{2}\,d\theta\right)$$

$$= \frac{5\pi - 6\sqrt{3}}{2} = 2.65782$$

3 (a). direction $l_1 = \langle -2,3,2 \rangle$
$l_2 = \langle 2,-4,3 \rangle$

normal vector should perpendicular both direction ⇒

$$\begin{vmatrix} i & j & k \\ -2 & 3 & 2 \\ 2 & -4 & 3 \end{vmatrix} = 17i + 10j + 2k.$$

random point $\langle 1,1,2 \rangle$

$17(x-1) + 10(y-1) + 2z = 0$

(b) $3x-2y+5z-6=0$
$2x-2y-2z+17=0$.

normal vector
$\langle 3,-2,5 \rangle$
$\langle 2,-2,-2 \rangle$

$\langle 3,-2,5 \rangle \cdot \langle 2,-2,-2 \rangle = 6+4-10=0.$

so their normal vector are ~~perpendicular~~ orthogonal.

so they are orthogonal

Bonus.

gradient

$yz = \frac{1}{1+xyz} = \lambda 2x$

$\lambda = y/z$

(4.) $v(t) = \int a(t)\,dt$

$= \int \langle 3t, -4e^{-t}, 12t^2 \rangle \,dt$

$= \langle 1.5t^2 + c_1, 4e^{-t} + c_2, 4t^3 + c_3 \rangle$

$= \langle 1.5t^2, 4e^{-t} - 3, 4t^3 - 3 \rangle$

$r(t) = \int v(t)\,dt$

$= \int \langle 1.5t^2, 4e^{-t} - 3, 4t^3 - 3 \rangle \,dt$

$= \langle \frac{1}{2}t^3 + c_1, -4e^{-t} - 3t + c_2, t^4 - 3t + c_3 \rangle \,dt$

$= \langle \frac{1}{2}t^3 - 5, -4e^{-t} - 3t + 6, t^4 - 3t - 3 \rangle$

(1)

a. $b^2x^2+a^2y^2=a^2b^2$

$r(t)=a\cos t\,i+b\sin t\,j$

b: $r'(t)=(-a\sin t, b\cos t)$

$r''(t)=(-a\cos t, -b\sin t)$

$\|r'\|=\sqrt{a^2\sin^2 t+b^2\cos^2 t}$

$$r'\times r''=\begin{vmatrix} i & j & k \\ -a\sin t & b\cos t & 0 \\ -a\cos t & -b\sin t & 0 \end{vmatrix}=(0,0,\ ab\sin^2(t)+ab\cos^2 t)$$

$\|r'\times r''\|=ab$

$$k=\frac{\|r'\times r''\|}{\|r'\|^3}=\frac{ab}{\left(\sqrt{a^2\sin^2 t+b^2\cos^2 t}\right)^3}$$

⑥ (a) Squeeze theorem

$$0 \le \frac{x^4+y^4}{x^2+y^2} = \frac{x^4}{x^2+y^2} + \frac{y^4}{x^2+y^2} \le \frac{x^4}{x^2} + \frac{y^4}{y^2} = x^2+y^2 = 0.$$

$$0 \le \frac{x^4+y^4}{x^2+y^2} \le 0. \quad \text{So} \lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2} = 0$$

(b).
$$\lim_{(x,y)\to(2,4)} \frac{\sqrt{x+2}-\sqrt{y}}{x-y+2}$$

$$= \lim_{(x,y)\to(2,4)} \frac{(\sqrt{x+2}-\sqrt{y})(\sqrt{x+2}+\sqrt{y})}{(x-y+2)(\sqrt{x+2}+\sqrt{y})}$$

$$= \lim_{(x,y)\to(2,4)} \frac{\cancel{x-y+2}\ 1}{\cancel{(x-y+2)}(\sqrt{x+2}+\sqrt{y})}$$

$$= \frac{1}{4}.$$

⑦

1) gradient $= \langle -400x\, e^{-2x^2-3y^2-4z^2}, -600y\, e^{-2x^2-3y^2-4z^2}, -800z\, e^{-2x^2-3y^2-4z^2} \rangle$

$\nabla(2,-1,2) = \langle -800 e^{-27}, 600 e^{-27}, -1600 e^{-27} \rangle$

unit vector $= \frac{v}{\|v\|} = \frac{\langle -1, 2, -2 \rangle}{3} = \langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \rangle$

$D_u f(2,-1,2) = \frac{800 e^{-27}}{3} + 400 e^{-27} + \frac{3200 e^{-27}}{3}$

$= \frac{5200 e^{-27}}{3}$

2) $D_u f(2,-1,2) = \nabla f(2,-1,2) \cdot u = \nabla f(2,-1,2) \cdot |u| \cos\theta$

when $\cos\theta = 1$, $\theta = 0$; max

$u = \langle -8, 6, -16 \rangle$

$Max = |\nabla(2,-1,2)| = 1788.86\, e^{-27}$

(8) $f(x,y) = y^3 - x^2 + 7x - 8y + 2xy - 25$

$f_x(x,y) = -2x + 7 + 2y$

$f_y(x,y) = 3y^2 - 8 + 2x$

Critical point: $\begin{cases} -2x + 2y + 7 = 0 \\ 3y^2 + 2x - 8 = 0. \end{cases}$

$3y^2 + 2y - 1 = 0.$

$(3y-1)(y+1) = 0.$

$\begin{cases} x = \frac{23}{6} \\ y = \frac{1}{3} \end{cases}$ $\begin{cases} x = 2.5 \\ y = -1 \end{cases}$

Step 2: $f_{xx}(x,y) = -2$

$\begin{cases} f_{yy}(x,y) = 6y \\ f_{xy}(x,y) = 2 \end{cases}$

for $\begin{cases} x = \frac{23}{6} \\ y = \frac{1}{3} \end{cases}$ $f_{xx} < 0$. $f_{yy} > 0$, don't need it.

for $\begin{cases} x = 2.5 \\ y = -1 \end{cases}$ $f_{xx} < 0$. $f_{yy} < 0$, $D(x,y) = 12 - 2 = 10 > 0$.

and $f_{xx} < 0$.

so $(2.5, -1)$ is maximum point.

(f)

$f_x(x,y) = 2x$

$f_y(x,y) = 2y - 4$

$x = 0,\ y = 2.$

The value at critical point $= 4 - 8 + 100 = 96.$

Boundary consists of three line.

$L_1 =$ from $(2,0)$ to $(2,2)$, $x = 2.$

$L_2 =$ from $(2,0)$ to $(4,2)$, $y = x - 2.$

$L_3 =$ from $(2,2)$ to $(4,2)$, $y = 2.$

for L_1: $y^2 - 4y < 0$, decreasing. max 104. min. 100.

for $L_2 = (y+2)^2 + y^2 - 4y + 100 = 2y^2 + 104$. max 112, min 104

for $L_3 = x^2 + 96$, increasing. min 100, max 112.

So maximum is 112, minimum is 96.

(10.) $f_x = \lambda g_x$

$yz \cdot \frac{1}{1+xyz} = \lambda 2x$

$\lambda = \frac{yz}{2x+2x^2yz}$

$f_y = \lambda g_y$

$\lambda = \frac{xz}{2y+2y^2xz}$

$f_z = \lambda g_z$

$\lambda = \frac{xy}{2z+2z^2xy}$

$\frac{yz}{2x+2x^2yz} = \frac{xz}{2y+2y^2xz} = \frac{xy}{2z+2z^2xy}$.

$\frac{yz}{2x+2x^2yz} = \frac{xz}{2y+2y^2xz}$

$y^3xz + y^2 = x^2 + x^3yz$

$y^2(xyz+1) = x^2(1+xyz)$

$y^2 = x^2$, easily, we can get $y^2 = x^2 = z^2$.

$3x^2 = 1$

$x^2 = \frac{1}{3}$ $x = \pm\frac{\sqrt{3}}{3}$

when $x = y = z = \frac{\sqrt{3}}{3}$, max = $\ln\left(1 + \frac{\sqrt{3}}{9}\right)$

$x = y = z = -\frac{\sqrt{3}}{3}$, min $\ln\left(1 - \frac{\sqrt{3}}{9}\right)$

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Bonus.

~~gra~~ ~~gradient 1 = (2x, 2y, 2z)~~

$\nabla_1 = (2x, 4y, 6z) = (2, 4, 6)$

$\nabla_2 = (2xyz^3, x^2z^3, 3x^2yz^2) = (2, 1, 3)$

direction vector $= \nabla_1 \times \nabla_2 = \begin{vmatrix} i & j & k \\ 2 & 4 & 6 \\ 2 & 1 & 3 \end{vmatrix} = 6i + 6j - 2k.$

$= 3i + 3j - k$

$x = 1 + 3t,\ y = 1 + 3t,\ z = 1 - t.$