1/

a) Let G be an undirected graph with n vertices. If G is isomorphic to its own complement $\overline{G}$, how many edges must G have? (Such a graph is called *self-complementary*.)

b) Find an example of a self-complementary graph on four vertices and one on five vertices.

c) If G is a self-complementary graph on n vertices, where $n > 1$, prove that $n = 4k$ or $n = 4k + 1$, for some $k \in \mathbf{Z}^+$.

a.

---

let edge of G=E1, edge of $\bar{G}$ =E2

Firstly, if G and $\bar{G}$ are isomorphic, then they have same numbers of edges, E1=E2

Secondly, edge of G + edge of $\bar{G}$ = edge of compete graph ,E1+E2= nC2

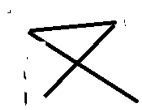
So, E1=nC2/2= n!/(2*(n-2!)*2!)=n(n-1)/4

b.

---

b.1: based on part a, 4*3/4=3 , so it have 3 edges

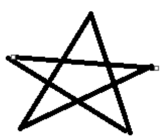
G:  2 vertices have 1 edge, 2 vertices have 2 edge

$\bar{G}$:  2 vertices have 1 edge, 2 vertices have 2 edge

True

b.2: 5*4/4=5, it have 5 edges

G:  5 vertices have 2 edge

$\bar{G}$:  5 vertices have 2 edges

True

c.

---

based on part a

n(n-1)/4, n and n-1 must be one odd one even, so the even one must be the multiple of 4, then n(n-1)/4 can be integer.

So n=4k or n=4k+1 //n-1=4k

Problem2

a) Find the number of edges in Q_8.

b) Find the maximum distance between pairs of vertices in Q_8. Give an example of one such pair that achieves this distance.

c) Find the length of a longest path in Q_8.

a/

There are 2^8 vertices, the degree of every vertice is 8, and the edge = the sum of all degree/2

So the answer is 8*2^8/2=8*2^7

b/when every digit is completely different,like 00000000 to 11111111 , one diffrent digit means one distance .So the maximum distance is 8

c/

It will cover all vertices, which is 2^8 -1

PROBLEM 3

For $n \in \mathbf{Z}^+$, how many distinct (though isomorphic) paths of length 2 are there in the n-dimensional hypercube Q_n?

Firstly, there are 2^n vertices in Qn hybercube. assume it is vertex X

Then, for any vertex, there are n vertices adjacent it

So we can pick 2 vertices Y Z from n adjacent vertices, with the vertex X, we can build a path {Y,X}, {X,Z}

So there are nC2 *2^n paths.

PROBLEM 4

Prove that for each $n \in \mathbf{Z}^+$ there exists a loop-free connected undirected graph $G = (V, E)$, where $|V| = 2n$ and which has two vertices of degree i for every $1 \leq i \leq n$.

Step1: When n=1, we can build a path with 2 vertices, degree i=1,true

Step2: Assume for all k<=n, there are 2 vertices X, Y of degree i 1<=i<=k

Step3: Then for we can add 2 new vertices A, B to this graph, now we have 2(k+1) vertices which is , we connect vertice A to X, B to Y, now the graph is still connected, and the degree of X and Y will become 1<=i<=k+1 because the new vertices

PROBLEM 5

Let k be a fixed positive integer and let $G = (V, E)$ be a loop-free undirected graph, where $\deg(v) \geq k$ for all $v \in V$. Prove that G contains a path of length k.

Step 1: When k=1, cause deg>=1, there must be at least one path, true

Step 2: Assume for all m<=k, there is a graph with deg(v)>=m have a path of length m, and because the degree can >=m, there are at least m+1 vertices in this graph

Step 3: then we add a new vertice to the graph, and connect the new vertex to all old vertices, then deg(v) will >=m+1 for all old v, and the degree of new vertex will be the number of old vertices which is also >=m+1

And there will be an edge between the end vertex of length-m-path to the new vertex (cause it is connected to all vertice) , we add this edge to the length-m-path, and we will get a new path of length m+1.

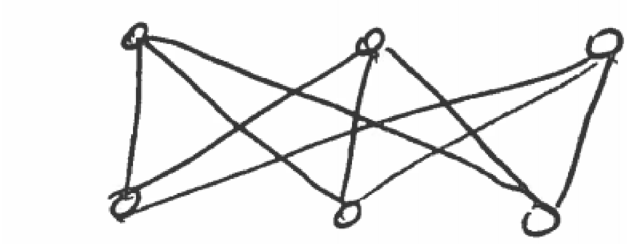
Problem 6

What is the length of a longest path in each of the following graphs?

a) $K_{1,4}$ b) $K_{3,7}$ c) $K_{7,12}$

d) $K_{m,n}$, where $m, n \in \mathbf{Z}^+$ with $m < n$.

For this question, I will divide vertices into top part {top1,top2,top3} and bottom part {bot1,bo2,bot3 } like k3,3



a/ 2 , bot1->top1->bot2

b/ the path will be bot1->top1->bot2->top2->bot3…..

and we can divide the path by top point, then the path will be bot1<-top1->bot2 , bot2<-top2->bot3…the length of path will equal 2* the number of top(or bot if the number of bot vertices is smaller) , The answer is 2*3=6

c/ 2*7=14

d/2*m

PROBLEM 7

a) Find all the nonisomorphic complete bipartite graphs $G = (V, E)$, where $|V| = 6$.

b) How many nonisomorphic complete bipartite graphs $G = (V, E)$ satisfy $|V| = n \geq 2$?

A/K15 K24 K33

B/

If n is even, it will be K(1,n-1) K(2,n-2),…………..k(n/2,n/2) So there are n/2 graphs

If n is odd, it will be K(1,n-1) ………….K((n-1)/2) so there are (n-1)/2 graphs

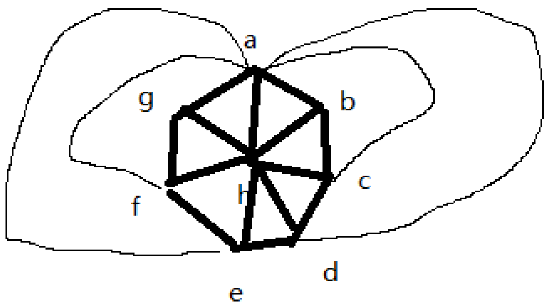
PROBLEM 8

a) Let $G = (V, E)$ be a loop-free connected graph with $|V| \geq 11$. Prove that either G or its complement $\overline{G}$ must be nonplanar.

b) The result in part (a) is actually true for $|V| \geq 9$, but the proof for $|V| = 9, 10$, is much harder. Find a counterexample to part (a) for $|V| = 8$.

a/assume V=11, the edge of G is E1, edge of $\bar{G}$ is E2, then if they are both planar, then E1<=3v-6 ,E2<=3V-6 , E1<= 27 ,E2<=27, . And K11 edge will be nC2= 11C2=55 >=e1+e2, so it is impossible, so one of them must be nonplanar

b/ E1 E2<=18, 8C2=28, possible



G edge =18 max

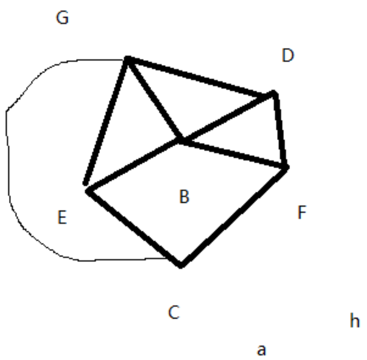
Point A is complete, H is compelete,

B need bd be bf bg

C need CE CF CG

D need df dg

E need eg

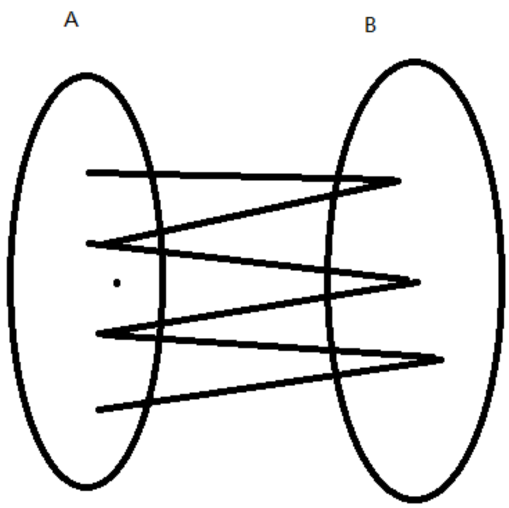


$\bar{G}$ edge =10 , perfect

PROBLEM 9

Can a bipartite graph contain a cycle of odd length? Explain.

No



if we start from vertice set A , it must be one a->b and one b->a loop to come back to setA if we want to build a loop, it must start from A(B), end at A(B), which means that the loop will be made up of these come-back loops which are 2-edge-long(even)

PROBLEM 10

Let $G = (V, E)$ be a loop-free connected planar graph. If G is isomorphic to its dual and $|V| = n$, what is $|E|$?

cause its dual graph is isomorphic to G, then they have same |V|, and |V| of dual graph = r of G, so |V|-|E|+r=2

2|V|-|E|=2

|E|=2n-2

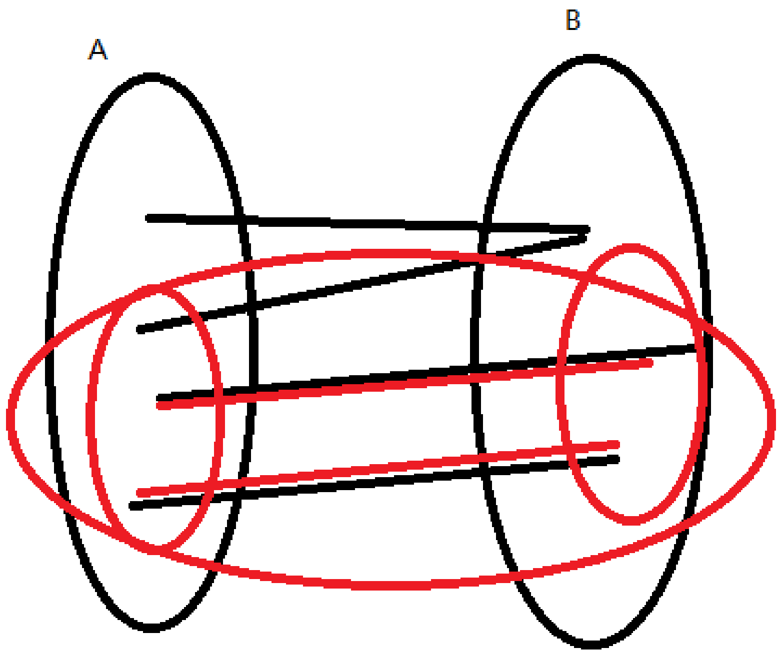
PROBLEM 11

If $G = (V,E)$ is a connected graph with $|E| = 17$ and $deg(v) > 2$ for *all* vertices of graph G, what is the maximum value for $|V|$.

sum of degree=2*E=34, and every vertice degree>=3, so 34/3=11

PROBLEM 12

Prove that any subgraph of a bipartite graph is bipartite.



Let the vertices of orginal graph G be devided into set A and set B , pick random vertices from G to build subgraph S, and we can find every vertex in subgraph is a member of A or B, so we can divided the vertices of S into set C and set D , and C D is subset of A and B, and the old {A,B} edge will also be {C,D} edge in subgraph, so S is still bipartite

PROBLEM 13

QUESTION 18

Let $G = (V,E)$ be an undirected connected loop-free planar graph. Suppose G determines 53 regions. If, for some planar embedding of G, each region has at least five edges in its boundary, prove that $|V| > 81$.

each region has at least five edges, which means the degree of each region >=5,

5*53<=The sum of degree= 2|E|

|E|>=132.5 , |E|>=133

|V|-|E|+r=2

|v|=|E|-51>=82

PROBLEM14

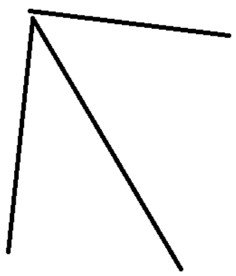
(a) If graph G is self-complementary (see Problem 1) (i) determine $|E|$ if $|V| = n$; (ii) Prove that G is connected. b) Let $n = 4k$ or $n = 4k + 1$ for non- negative number k. Prove that there exist a self-complementary graph $G = (V,E)$, where $|V| = n$.

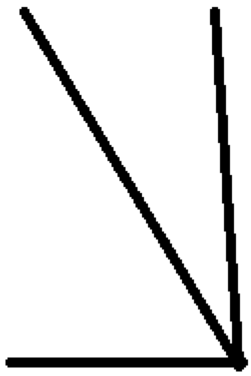
a/

1/so there are 2n vertices in a complete graph, then each vertice has 2n-1 degree, there are 2n*(2n-1)/2 =n(2n-1)edges in complete graph(divide 2 because repetition), and edge of G and $\bar{G}$ are equal, so |E|=n(2n-1)/2

2/if G is not connected ,then $\bar{G}$ will be connected because it is undirected graph, and cause they are isomorphic ,so if $\bar{G}$ is connected ,G will be connected

b/when k=1, n=4,





G $\bar{G}$

There exists one example ,true.