Exercise 8.1
#6
a/
19 item 3 divider
C(19-1+4,3)=1540

b/
S=1549 ways that all x>=0
S1: one solution>=8
Assume x1 is the one >=8, let y1=x1-8 >=0
Then y1+x2+x3+x4 =11
4*C(11-1+4,3)=1456 // 4 is N(x1)+N(x2)+N(x3)+N(x4)
S2: two solution >=8
Y1+Y2+X3+X4= 3
6* C(3-1+4,3)=120 //6 is x1x2 x1x3 x1x4 x2x3 x2x4 x3x4

N($\overline{x_1 x_2 x_3 x_4}$) =1540-1456+120=204

c/ let y3= x3-3 y4= x4-3 ,then y3<=4, y4<=5
then the question becomes
X1+X2+Y3+Y4=13

S=C(13+4-1,3)=560

ForS1,
N(x1)= C(7-1+4,7)=120 // x1>=6 y1=x1-6>=0 y1+x2+y3+y4=7
N(x2)=C(6-1+4,6)=84
N(x3)=C(8-1+4,8)=165
N(x4)= C(7-1+4,7)=120
S1=120+84+165+120=489

ForS2

N(x1x2) =1 //X1>=6,X2>=7
N(X1X3)=C(2-1+4,2)=10 //X1>=6, Y3>=5
N(X1X4)=C(1-1+4,1)=4 ///X1>=6, Y4>=6
N(X2X3)=C(1-1+4,1)=4
N(X2X4)=1
N(X3X4)=C(2-1+4,2)=10
S2=1+10+4+4+1+10=30

N($\overline{x_1 x_2 x_3 x_4}$)=560-489+30=101

#10
Every question is 5k ,where k>=1
Assume yi=xi/5
Then the question become
Y1+y2+….y12=40   where 2<=yi<=5
Then Assume Zi=Yi-2
Z1+Z2+…Z12=16   where 0<=zi<=3
S=   C(16-1+12,16)=13037895
One zi >=4,   C(12,1)*C(12-1+12,12)=16224936
Two zi>=4   C(12,2)*C(8-1+12,8)=4988412
Three zi>=4   C(12,3)*C(4-1+12,4)=300300
4zi>=4   C(12,4)*C(0-1+12,0)=495

N($\overline{x_1x_2x_3x_4}$) =13037895-16224936+4988412-300300+495=1501566

#20
S=84
S1=C(7,1)*35=245   //C(n1)+C(n2)….+C(n7)
S2=C(7,2)*16=336
S3=C(7,3)*8=280
S4=C(7,4)*4=140
S5=C(7,5)*2=42
S6=C(7,6)*1=7
S7=0
N($\overline{x_1x_2x_3x_4x_5x_6x_7}$)=84-245+336-280+140-42+7=0

So she never had lunch alone

Exercise 8.2

#2

a/

S2

Let a pair of letter such as AA be a new type

Then S2 =C(4,2)* 9!/2!2!=544320 // For C(4,2),pick 2 kinds of consecutive letters from 4.

For 9!/2!2! ,this is permutation of different kinds of same item (sequence important), there are still 2 groups of repetitive letters (size is 2).

S3= C(4,3)*8!/2!=80640

S4=C(4,4)*7!=5040

E2=S2-C(3,1)S3+C(4,2)S4=332640

L2=S2-C(2,1)S3+C(3,1)S4 =398160

b/

E3= S3-C(4,1)S4= 80640-4*5040=60480

L3=S3-C(3,2)S4=65520

#6

S4:4 at right place ,then permutation the rest 6 P(6, 6)

S5: P(5, 5)

S6: P(4,4)

……

E4=S4-C(5, 1) *S5 + C(6,2) *S6 -C(7,3)*S7+C(8,4)*S8-C(9,5)*S9+C(10,6)S10

=6!-5*5!+15*4!-35*3!+70*2!-126*1! +210

=494

L4=S4-C(4, 3)*S5+C(5, 3)*S6-C(6,3)*S7+C(7,3)*S8-C(8,3)*S9+C(9,3)S10

=6!-4*5!+10*4!-20*3!+35*2!-56*1!+84

=458

Exercise 9.1

#2

a/$(1+x+x^2+x^3+\cdots)^5$ //1 means this child has no penny x^0

b/$(x+x^2+x^3.....)^5$ //

c/$(x^2+x^3+....)^5$

d/$(x^{10}+x^{11}+\cdots)*(1+x+x^2\cdots)^4$

e/$(x^{10}+x^{11}+\cdots)^2*(1+x+x^2\cdots)^3$

Exercise9.2

#20

a/Start with 1

Then the palindromes of 11 will be

1 9 1

So the question is in fact the number or palindromes of 9

9-2-2-2-2=1>0

2^4=16

272

7-2-2-2=1>0

2^3=8

353

5-2-2=1>0

2^2=4

434

3-2=1>0

2^1=2

So there are 16 palindromes of 11 start with 1,8 with 2, 4 with 3,2 with 4

b/

1 10 1

10/2=5

2^5=32

2 8 2

8/2=4

2^4=16

3 6 3

6/2=3

2^3=8

4 4 4

4/2=2

2^2=4

So there are 32 palindromes of 12 start with 1,16 with 2, 8 with 3,4 with 4

#30

Step1:
Assume a nonconsecutive example
3, 5, 7, 9,11,13,15
Then 3-1=2 ,5-3=2……. 50-15=35
2*7+35=49 //coefficient of x^49 in step2

Step2:
So we need to assum
C1 and C8>=0
C2 to C7>=2 // no consecutive
Generating function
(1+x+x^2+…..)^2 * (x^2+x^3+…)^6
=X^12 * (1+x+x^2+…..)^8
=x^12 * (1-x)^-8
We need find the coefficient of x^49

Step 3
(1-x)^-8= $\sum_{i=0}^{\infty} C(8+i-1, i)$*x^i
We need to find the coefficient of x^37
So I=37
C(8+37-1,37)=44C37=38320568

Exercise 9.3

#4

a/

2w=2*k which k >=0, = 1+k^2+k^4+k^6..

The rest are same

So the generating function= (1+k^2+k^4…)(1+k^3+k^6…)(1+k^5+k^10…)(1+k^7+k^14…)

=(1/(1-k^2)) *(1/(1-k^3)) *(1/(1-k^5))* (1/(1-k^7))

b/

the generating function is (1+k^2+k^4….)(k^12+k^15+k^18…)*(k^20+k^25+k^30…)(k^35+k^40….)

=(1+k^2+k^4…)*k^12*(1+k^3+….) *k^20* (1+k^5+k^10…)*k^35*(1+k^5+k^10…)

=k^67 *(1/(1-k^2)) *(1/(1-k^3)) *(1/(1-k^5))* (1/(1-k^7))

#6

a/

take 1 for example

1 cannot exceed 5 times means its generating function is

1+x+x^2+x^3+x^4+x^5 //1 means x^0

So the generating function is

(1+x+x^2+x^3+x^4+x^5)(1+x^2+x^4+x^6+x^8+x^10)……

=[(1-x^6)/(1-x)] * [(1-x^12)/(1-x^2)] * ……[1-x^(6i)/(1-x^i)]

b/[(1-x^6)/(1-x)] * [(1-x^12)/(1-x^2)] * ……[1-x^(6*12)/(1-x^12)]

9.4
#2
a/
3e^3x=3(1+3x+(3x)^2/2!....)=3+9x+27x^2/2!+⋯..
So the sequence is
3,9,27⋯..3^i
b/
6(1,5,25,⋯)-3(1,2,4,8⋯.)
=6*5^i-3*2^i
So the sequence is
3,24,⋯..6*5^i-3*2^i
c/
1+x+x^2/2!+x^2+x^3/3!⋯⋯.
=1+x+3x^2/2!.....
So the sequence is
1 1 3 1 1 1 1 1
d/
1+2x+7x+(2x)^2/2!+5x^2 +8x^3/3! -3x^3
=1+9x +14x^2/2!-10x^3/3!.........
So the sequence is
1 9 14 -10 2^4⋯⋯.. 2^I
e/
1/1-x=1+x+x^2+x^3+x^4⋯.
= 1+1/1! *x +2!* x^2/2!.....
So the sequence is
0! 1! 2! 3!,,,,,,i!
f/
3/(1-2x)=3(1+2x+4x^2+8x^3⋯.)+ 1+x/1!+x^2/2!.....
=4+7x+ (3*2!*2^2 +1) *x^2/2!..............
So the sequence is
4,7,25⋯⋯⋯⋯⋯⋯⋯ 3*i!*2^i+1)

#6
1/
2*A,2*I
(1+X+X^2/2!)^2 *(1+X)^2
2/
4*I,4*S,2*P,1*M
(1+X)(1+X+X^2/2!) (1+X+X^2/2!+X^3/3!+X^4/4!)^2
3/
2*I,2*O,2*S,2*M,1*R,1*P,1*H
(1+X)^3*(1+X+X^2/2!)^4
b/
(1+X)(1+X+X^2/2!) (1+X+X^2/2!+X^3/3!+X^4/4!) *(X^2/2!+X^3/3!+X^4/4!)

10.4

#1

a/

step1:

$a_{n+1}X^{n+1}$-anX^{n+1}=3^n X^{n+1}

Step2:

Sum: a1x-a0x^1=3^1x^1

....

Then:F(x)-a0 -x(f(x))= x/(1-3x)

(1-x)f(x)-1=x/(1-3x)

F(x)=x/(1-3x)(1-x) +1/1-x

Step3:

x/(1-3x)(1-x)= A/(1-3X)+B/(1-X)

A-AX+B-3BX=X

A=-B -2BX=X, B=-0.5, A=0.5

Then F(X)=0.5/(1-3X)+0.5/(1-X)

Step4;

F(X)=0.5(1+3X+9X^2…)+0.5(1+X+X^2…)

Then a_n=coefficient of x^n =0.5*3^n+ 0,5=(3^n+1)/2

b/

step1:

$a_{n+1}X^{n+1}$-anX^{n+1}=n^2 X^{n+1}

step2:

Sum: a1x-a0x^1=0

….

Then:F(x)-a0 -x(f(x))= 0x+1x^2+4x^3+9X^4…..

For 0x+1x^2+4x^3+9X^4=g(x)

g(x)-x(g(x))=0+0x+x^2+3x^3+5x^4+7x^5+9x^6….=h(x)

For h(x)= 0+0x+x^2+3x^3+5x^4+7x^5+9x^6….

H(x)-xh(x)=x^2+2x^3+2x^4+2x^5…..

=x^2(-1+2+2x+2x^2+2x^3…)

=x^2(2/(1-x) -1)

=x^2(1+x/1-x)

=(x^2+x^3)/(1-x)

H(x)=(x^2+x^3)/(1-x)^2

G(x)=(x^2+x^3)/(1-x)^3

F(x)=(x^2+x^3)/(1-x)^4 +1/1-x

=4x^2-3x+1/(1-x)^4

Then we need to get the coefficient of x^n

Step 3:

4x^nC(4+n-2-1,3)-3x^n(4+n-1-1,3)+1x^n(4+n-1,3)

An=4(n+1,3)-3(n+2,3)+(n+3,3)

c/

step1:

multiply x^(n+2)

$a_{n+2}X^{n+2}$ -3$a_{n+1}X^{n+2}$+2anX^{n+2}=0

step2

sum

F(n)-a0-a1x-3x(f(n)-a0)+2x^2 *f(n)=0

F(n)-1-6x-3x(f(n)-1)+2x^2(f(n))=0

F(n)(1-3x+2x^2) -1-6x+3x=0

F(n)=3x+1/(1-3x+2x^2)

=(3x+1)/(2x-1)(x-1)

Step3:

A/2X-1 +B/X-1 =(3x+1)/(2x-1)(x-1)

AX-A+2BX-B=3X+1

A+2B=3, B+A=-1, B=4,A=-5

F(n)=4/x-1 -5/(2x-1)=5/(1-2x)-4/(1-x)

Step4:

Fn=5(1+2x+4x^2+……)-4(1+x+x^2,,,,,,,)

An=5*2^n-4

d/

step1:

multiply x^(n+2)

$a_{n+2}X^{n+2}$ -2$a_{n+1}X^{n+2}$+anX^{n+2}= 2^n *x^(n+2)

step2

sum

F(n)-a0-a1x-2x(f(n)-a0)+x^2 *f(n)=x^2 /(1-2x)

F(n)-1-2x-2x(f(n)-1)+x^2(f(n))= x^2 /(1-2x)

F(n)(1-2x+x^2) -1-2x+2x= x^2 /(1-2x)

F(n)(1-2x+x^2)= (1-2x+x^2)/(1-2x)

F(n)= 1/1-2x

Step3

F(n)=1+2x+4x^2+8x^3

An=2^n

#3

a/

step1:multiply x^n+1

a(n+1) *x^(n+1)=x(-2anx^n-4bnx^n)

F(x)-a0=x(-2f(x)-4g(x))

b(n+1)x^(n+1)=x(4anx^n+6bnx^n)

g(x)-b0=x(4f(x)+6g(x))

f(x)(1+2x)+4xg(x)=1

f(x)(-4x)+(1-6x)g(x)=0

step2

f(x)= | 1 4x | / |1+2x 4x| =1-6x/[(1+2x)(1-6x)+16x^2]=1-6x/[4x^2-4x+1]
|0 1-6x| |-4x 1-6x|

=(1-6x)/(1-2x)^2

F(x)=(1-6x)g(x)/4x

(1-2x)^2=4x/g(x)

G(x)=4x/(1-2x)^2

Step3:

F(x)=(1-6x) [⋯.C(1+n,1)X^N+C(N,1)X^(N-1)⋯⋯⋯⋯⋯]

An=2^n(1-2n)

Bn=n2^(n+1)

B/

STEP1

Multiply Xn+1

F(x)=2xf(x)-xg(x)+2x^(n+1)

G(x) -1=-x(fx)+2xg(x)-x^(n+1)

Step2

f(x)+2g(x)-2=3xg(x)

(2-3x)g(x)=2-f(x)

G(x)=(2-f(x))/(2-3x)

(1-2x)f(x)=x(f(x)-2)/2-3x +2x^n+1

15.

a)

Q:Pick k people from n and one will be leader, how many ways?

LHS: Firstly, pick k group members from n people ,which is $C(n,k)$, and then pick 1 leader from k group members which is $C(K,1)=K$

RHS: pick the leader firstly from n people, which is $C(n,1)=n$, then we can pick k-1 group members from n-1 people, which is $C(N-1,K-1)$

b)

LHS:$k* n!/[k!(n-k!)]$

$=n!/[(k-1)!(n-k)!]$

RHS:$n* (n-1)!/[(k-1)!*(n-k)!]$

$=n!/[(K-1)!*(N-K)!]$

LHS=RHS

16.

N balls be a line, one leader ball is red, the rest can be black or white. How many ways?

○○○○○○○○

RHS:

Take a ball from n to be red, then rest n-1 balls is random white or black, which is 2^n-1

LHS:

Take k balls from n to be white, which is $C(n,k)$ and then pick a white from k white ball to be red, which is $C(k,1)=k$, when k=1 this is the situation that one red ball ,the others are black ball. When k =n ,it means one is red ball ,the others are white ball. Then we use summation symbol to sum all situations between all black->all white