PROBLEM 1

a) Show that the Petersen graph [Fig. 11.52(a)] has no Hamilton cycle but that it has a Hamilton path.

b) Show that if any vertex (and the edges incident to it) is removed from the Petersen graph, then the resulting subgraph has a Hamilton cycle.

a/hamilton path: a->b->c->d->e->j->g->i->f->h

b/if we remove an inner point like f, we can have cycle

a->b->g->i->d->c->h->j->e->a ,successfully

if we remove a outer point like a, we can have cycle

f->h->c->b->g->j->e->d->i->f, successfully

PROBLEM2

Take problem 1 for example , there exists a hamilton path, however ,the sum of any two point'degree is 6, smaller than n-1 which is 9

PROBLEM 3

Let $G = (V, E)$ be a loop-free undirected n-regular graph with $|V| \geq 2n + 2$. Prove that $\overline{G}$ (the complement of G) has a Hamilton cycle.

cause it is n-regular, and there are more than 2n+2 vertices ,

then for complement of G, the degree of every vertice will >= 2n+2-n=n+2 >(2n+2)/2=n+1,

(collary 11.5) so it will the complement of G will have hamilton cycle.

PROBLEM 4

Let $n \in \mathbf{Z}^+$ with $n \geq 4$, and let the vertex set V' for the complete graph K_{n-1} be $\{v_1, v_2, v_3, \ldots, v_{n-1}\}$. Now construct the loop-free undirected graph $G_n = (V, E)$ from K_{n-1} as follows: $V = V' \cup \{v\}$, and E consists of all the edges in K_{n-1} except for the edge $\{v_1, v_2\}$, which is replaced by the pair of edges $\{v_1, v\}$ and $\{v, v_2\}$.

a) Determine $\deg(x) + \deg(y)$ for all nonadjacent vertices x and y in V.

b) Does G_n have a Hamilton cycle?

c) How large is the edge set E?

d) Do the results in parts (b) and (c) contradict Corollary 11.6?

so the change is add a new vertex ,and {v1 v2} is replaced by {v1v} {vv2}

a/ if x!=v, y!=v, then x and y must be v1 v2 if they want to be nonadjacent. Then deg(x)+deg(y)=deg(v1)+deg(v2)=2*(n-2)=2n-4 // though edge {v1,v2} is lost, a new edge {v1v}/{v2v} is added

if x=v, y can be any point except a/b , then deg(x)+deg(y)= 2+n-2=n //2 is {v1v},{v2v}

b/

Yes, take any nonadjacent vertices, the sum of their degree >=n (theorem 11.9)

c/ for K(n-1): n*(n-1)/2 ,then we remove one edge, add 2 edge, so the final answer is n*(n-1)/2+1

d/ no. E= n*(n-1)/2+1 and has hamilton cycle doesn't contradict corolary 11.6 (because 11.6 doesn't have if and only if)

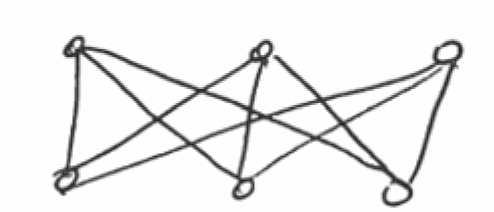
PROBLEM 5

If G is a loop-free undirected graph with at least one edge, prove that G is bipartite if an only if $\chi(G) = 2$.

part 1: if it is bipartite, and we divide the vertices to top side and bottom side, then just give top side one color, bottom side one color, $\chi(G) = 2$. ,true

part 2: if $\chi(G) = 2$. ,then we can only give vertices one color of two, if a vertice have first color, divide it to top side (set V1), if a vertice have second color, divide it to bottom side (set V2), cause there is no edge between vertices with same color, then it is bipartite.

Like this



PROBLEM 6

Consider the complete graph K_n for $n \geq 3$. Color r of the vertices in K_n red and the remaining $n - r$ $(= g)$ vertices green. For any two vertices v, w in K_n color the edge $\{v, w\}$ (1) red if v, w are both red; (2) green if v, w are both green; or (3) blue if v, w have different colors. Assume that $r \geq g$.

a) Show that for $r = 6$ and $g = 3$ (and $n = 9$) the total number of red and green edges in K_9 equals the number of blue edges in K_9.

b) Show that the total number of red and green edges in K_n equals the number of blue edges in K_n if and only if $n = r + g$, where g, r are consecutive triangular numbers. [The triangular numbers are defined recursively by $t_1 = 1$, $t_{n+1} = t_n + (n + 1)$, $n \geq 1$; so $t_n = n(n + 1)/2$. Hence $t_1 = 1, t_2 = 3, t_3 = 6, \ldots$.]

a/ red: 6*5/2=15 //one vertex of 6 * other 5 red vertex/ repeat

green: 3*2/2=3 // one green vertex of 3* other 3 green vertex/repeat

blue: 3*6=18 // one green vertex of 3* one red vertex of 6 (no repeat)

3+15=18

b/ blue= r*g

red = r*(r-1)/2

green= g*(g-1)/2

cause we want blue=red+green

r*g= r*(r-1)/2+ g*(g-1)/2

r^2-r+g^2-g-2gr=0

(r-g)^2 -r-g=0

cause r always >g ,assume r=g+k

k^2- 2g-k=0

g=(k^2-k)/2 =(k-1)k/2 which is triangular number

r=g+k= (k^2+k)/2= k(k+1)/2 which is next triangular number

PROBLEM 7

Let G be a loop-free undirected graph, where $\Delta = \max_{v \in V}\{\deg(v)\}$. (a) Prove that $\chi(G) \leq \Delta + 1$. (b) Find two types of graphs G, where $\chi(G) = \Delta + 1$.

a/pick a vertex from the graph, we can there are most Δ vertices adjacent to it. Even all of them are different color, we can still pick the rest 1 from Δ+1 color. It is true for all vertices in G.

b/ K4/K5

PROBLEM 8

a) If a tree has four vertices of degree 2, one vertex of degree 3, two of degree 4, and one of degree 5, how many pendant vertices does it have?

b) If a tree $T = (V, E)$ has v_2 vertices of degree 2, v_3 vertices of degree 3, . . . , and v_m vertices of degree m, what are $|V|$ and $|E|$?

a/ let x= pendant tree which has degree 1

2E= x+8+3+8+5 =2V-2

X+24=2*(x+4+1+2+1)-2=2x+14

x=10

b/

2E=v1+2v2+3v3+4v4+....mvm

E=V-1=v1+...vm-1

V=V1+V2+....VM

PROBLEM 9

The connected undirected graph $G = (V, E)$ has 30 edges. What is the maximum value that $|V|$ can have?

31 (tree is the critical point between connected and disconnected)

PROBLEM 10

Let $G = (V, E)$ be a loop-free connected undirected graph where $V = \{v_1, v_2, v_3, \ldots, v_n\}$, $n \geq 2$, $\deg(v_1) = 1$, and $\deg(v_i) \geq 2$ for $2 \leq i \leq n$. Prove that G must have a cycle.

if G is connected undirected graph and also have no cycle, then G is a tree, then B must have at least two vertices whose degree is one, while $\deg(v_1) = 1,$,contradiction

Problem 11

Does there exist a graph on 5 vertices with the vertex degrees 4, 4, 3, 2, 2 ?

No, degV=2E, however 4+4+3+2+2 is not even

Problem 12

Prove that in any tree there are two vertices of the same degree.

a tree has at least 2 leaves(pendant vertex) which degree both 1

Problem 13

Prove that any undirected, loop-free graph on *n* vertices with at least *(n-1)(n-2)/2 + 1* edges is connected. Also, give an example of a disconnected *n*-vertex graph with one fewer edge.

A/if G is not connected , then G has at least 2 subgraphs, one have k vertices, then the rest one have n-k vertices.

Then the edge will be at most k(k-1)/2 + (n-k)(n-k-1)/2 //COMPLETE GRAPH

=k^2-k+n^2+k^2-2nk-n+k /2

=2k^2+n^2-2nk-n/2

n^2-n is constant, 2k^2-2nk<0

so to maximum edge, we need to maximum 2k^2-2nk

b^2-4ac=0, k should be -b/2a=0.5n

(1)then at most 1.5n^2- n^2-n/2 =0,5n^2-n/2=0.25n^2-0.5n

(2) *(n-1)(n-2)/2 + 1 e*=0.5n^2-1.5n+2

use (2) minus (1) = 0.25n^2 -n+2 , b^2-4ac=1-2<1, which means (2)minus (1) always >0

which means G is must connected because even the most of disconnected edges still cannot satisfy at least *(n-1)(n-2)/2 +* , contradiction

B/based on question 1,

use (2) minus (1)=1

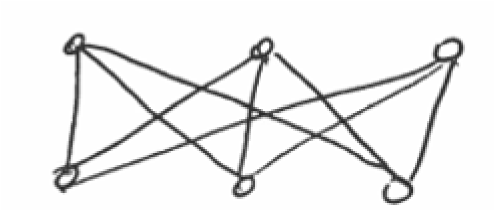
n^2-4n+4=0

n=2

so the answer is , the original graph is 2 vertices with one edge, the disconnected graph is 2 disconnected vertices

Problem 14.

Which complete bipartite graphs $K_{m,n}$ have Hamilton cycles? Which have Hamilton paths?

a/ when m=n,   in this way

b/ m=n+-1 or m=n