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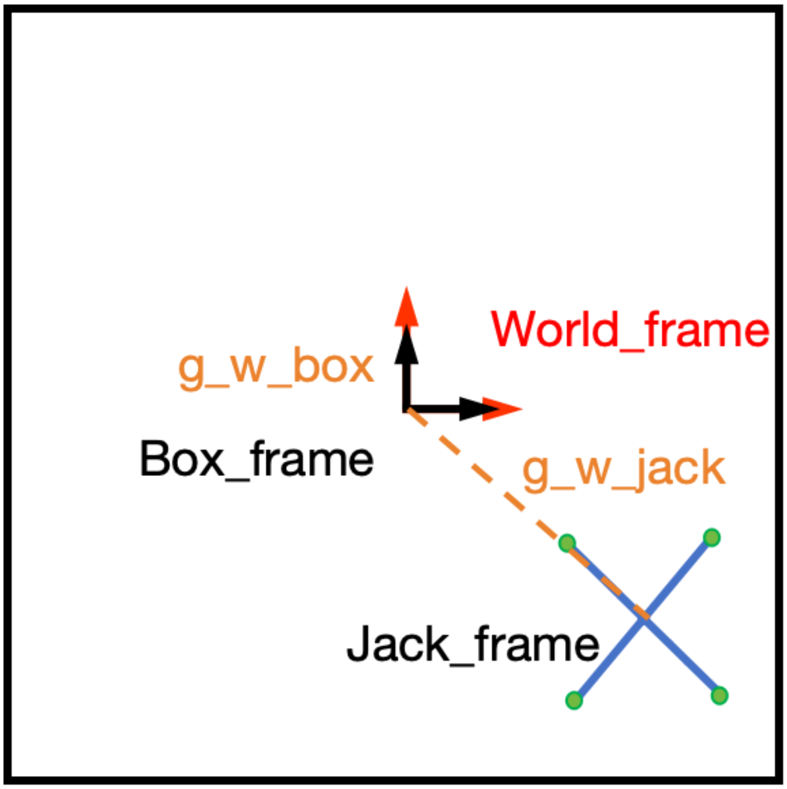
Overview

For the final project, I choose the default project: Jack-in-Box, which simulate a dice bouncing in a box. In this simulation, external force is applied on the box as the input to the whole system. The applied external force is a vertical force which prevent the box from falling and a sinusoidal force to rotate the box. The jack in the box is free-fall.

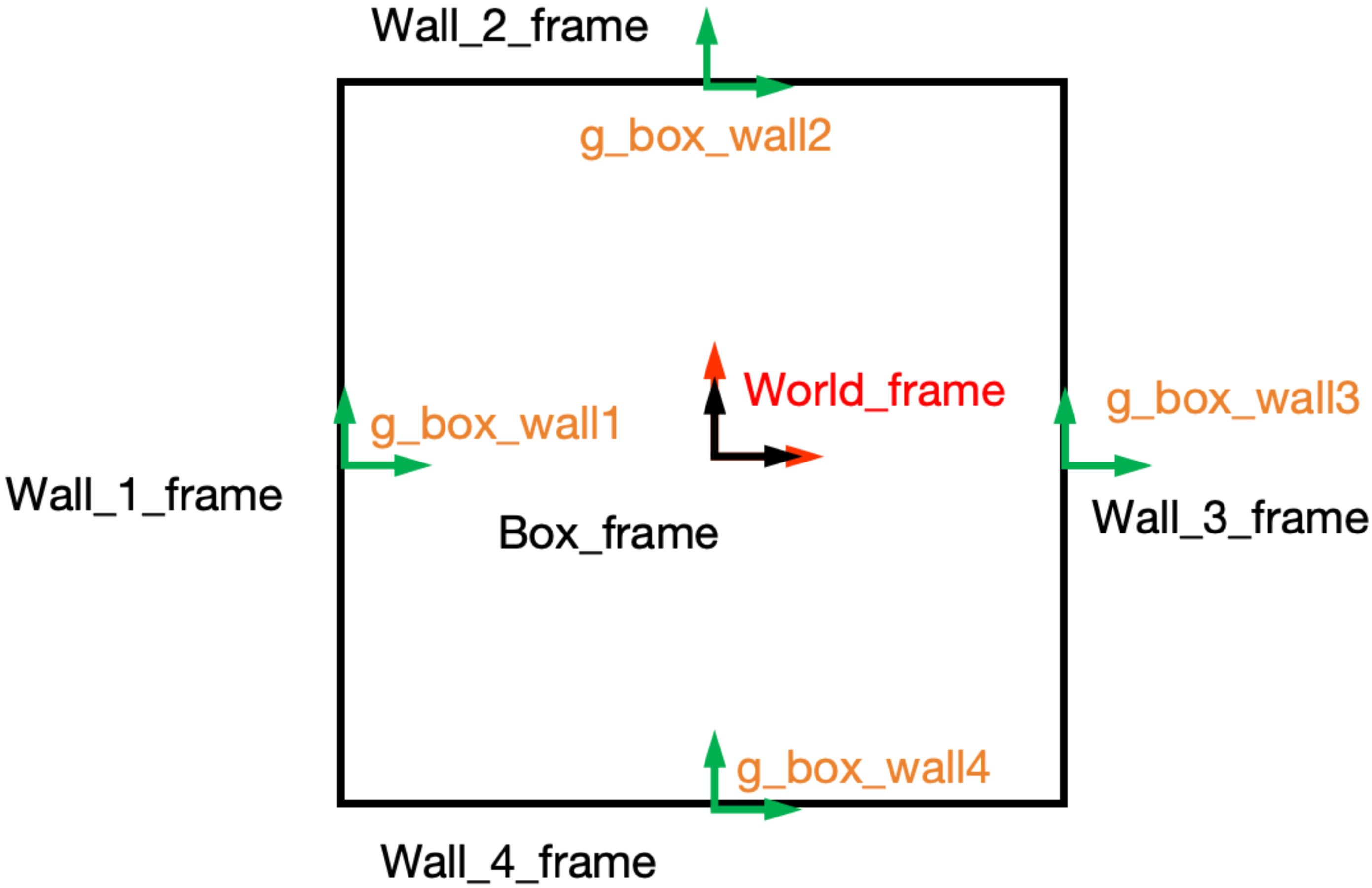
System Diagram

The system diagram for the system and the corresponding frame. Three diagrams show the box and jack related frame and each of them.
For the whole system the frame are shown in the figure below.



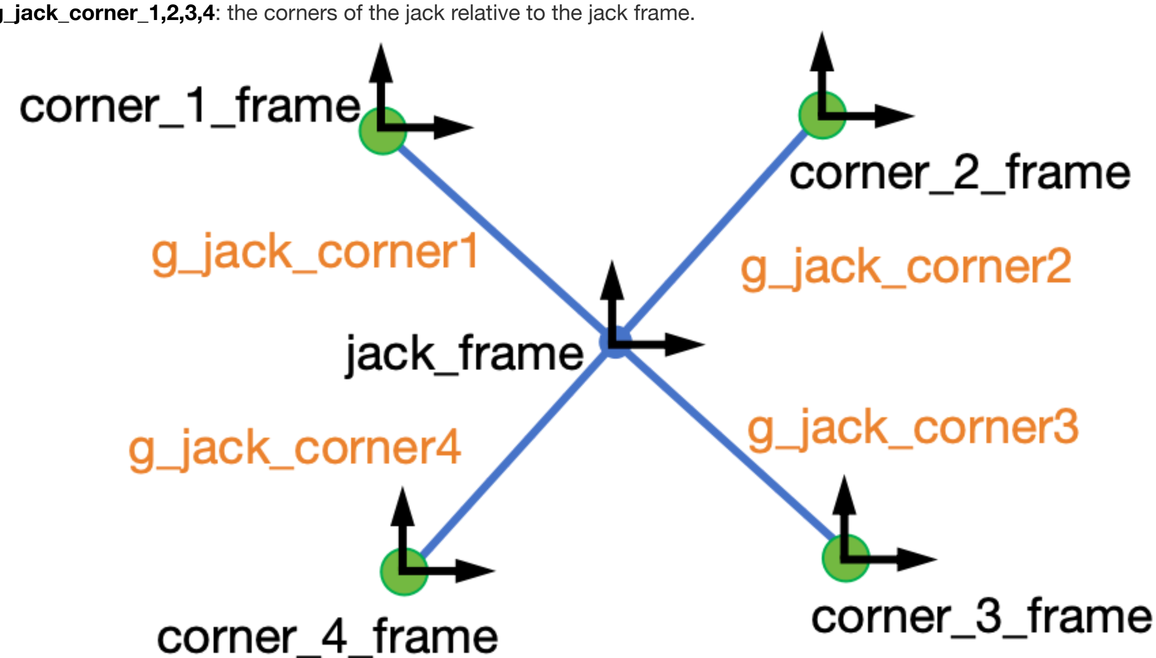
For simulate the box, the frames are defined:

- **World_frame**
- **Box_frame**: the center of the box
    - **Wall_1_frame, Wall_2_frame, Wall_3_frame, Wall_4_frame**: Walls of the box, the middle point of the line.
      The transformation for the box are:
- **g_wb**: the center of the box relative to the world_frame
- **g_box_wall1,2,3,4**: the walls of the box relative to the Box_frame.



For simulate the jack, the frame are defined:

- **Jack_frame**: The center of the jack.
- **Corner_1_frame, Corner_2_frame, Corner_3_frame, Corner_4_frame**: Corner point of the jack.
  The transformation for the jack are:
- **g_w_jack**: the center of the jack relative to the world_frame.
- **g_jack_corner_1,2,3,4**: the corners of the jack relative to the jack frame.



Algorithm Description

State Variables

The state variables contains the position and orientation of the box and the jack relative to the world frame. All these variables are stored in `q` . Velocities are stored in the `qdot` and accelerations in the vector `qddot` .

- `x_box` : the x coordinate of the center of the box.
- `y_box` : the y coordinate of the center of the box.
- `theta_box` : the current angle of the box about the z-axis(out of the plane)
- `x_jack` the x coordinate of the center of the jack.
- `y_jack` : the y coordinate of the center of the jack.
- `theta_jack` : the current angle of the jack about the z-axis(out of the plane)

Euler Lagrange

To model the dynamics of the objects in the system, the Euler-Lagrange equations and external forces were derived using the following steps

1. **Calculate Body Velocities**:
    - Determine the velocities of each object in the system using the reference frames placed at the center of the box and the jack.
2. **Determine the 6×6 Mass-Inertia Matrices**:
    - Compute the mass-inertia matrices for both the jack and the box. Each object is approximated using a simplified model of four point masses, whose configuration defines the overall mass distribution.
3. **Compute the Total Kinetic Energy (KE)**:
    - Since both objects exhibit translational and rotational motion, calculate:
        - **Linear Kinetic Energy** for both objects.
        - **Rotational Kinetic Energy** for both objects.
    - Sum these contributions to obtain the total kinetic energy of the system.
4. **Calculate the Total Potential Energy (V)**:
    - The only source of potential energy in this system is gravity.
    - Compute the gravitational potential energy for both objects relative to the origin of the world frame and sum these values to obtain the total potential energy.
5. **Formulate the Lagrangian**:
    - Using the total kinetic energy and potential energy, compute the Lagrangian using the equation:

$$L = KE - V$$

6. **Compute the Force Matrix**:
    - Define a force matrix containing the external forces acting on the system:
        - Forces in the x and y directions.
        - Torques about the z-axis applied to the box and the jack.
7. **Derive the Euler-Lagrange Equations**:
    - Use the Lagrangian and force matrix to derive the equations of motion using the formula `F=ddLdqdot-dLdq` :

$$F = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q}$$

        - $\frac{\partial L}{\partial q}$ Derivative of the Lagrangian with respect to the generalized coordinates q.
        - $\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}})$ Time derivative of the partial derivative of the Lagrangian with respect to the generalized velocities $\dot{q}$.

Constraint Equations

The system constraints prevent the corners of the jack from passing through the walls of the box. These constraints are defined as follows:

1. **Defining the Constraints**:
    - A total of 16 constraint equations are defined, one for each corner of the jack relative to each wall of the box.
    - The constraint variable, ϕ, represents the distance between a specific corner of the jack and the corresponding wall of the box.
2. **Collision Detection**:
    - During the simulation, as the jack's corner and the box's wall approach each other, the associated ϕ value decreases.

- When ϕ falls below a threshold, the objects are considered to have collided.

Impact Updates

When a collision occurs, the system's states are updated to reflect the post-impact dynamics. The following steps are used to compute the impact updates:

1. **Introduce Dummy Variables**:
   - Define two sets of dummy variables:
     - One set for the state variables **before impact**. Variables denoted with a "minus" subscript $(-)$.
     - One set for the state variables **after impact**. Variables denoted with a "plus" subscript $(+)$.
2. **Calculate Key Components**:
   - Compute the following terms for the system:
     - Hamiltonian (H): the total energy of the system.
     - $\frac{\partial L}{\partial \dot{q}}$: Partial derivative of the Lagrangian with respect to generalized velocities.
     - $\frac{\partial \phi}{\partial q}$: Partial derivative of the constraint variable with respect to generalized coordinates.
3. **Impact Equations**:
   - The equations are based on conservation laws and constraint relationships, expressed as follows (The left hand side):

$$\begin{bmatrix} \frac{\partial L}{\partial \dot{q}_1^+} - \frac{\partial \phi}{\partial \dot{q}_1^-} \\ \frac{\partial L}{\partial \dot{q}_2^+} - \frac{\partial \phi}{\partial \dot{q}_2^-} \\ \vdots \\ \frac{\partial L}{\partial \dot{q}_6^+} - \frac{\partial \phi}{\partial \dot{q}_6^-} \\ H^+ - H^- \end{bmatrix}$$

   - Constraint forces applied at impact (The right hand side):

$$\begin{bmatrix} \lambda \frac{\partial \phi}{\partial q_1^-} \\ \lambda \frac{\partial \phi}{\partial q_2^-} \\ \vdots \\ \lambda \frac{\partial \phi}{\partial q_6^-} \\ 0 \end{bmatrix}$$

Solving the Impact Equations

- For the whole impact equation:

$$LHS = RHS$$

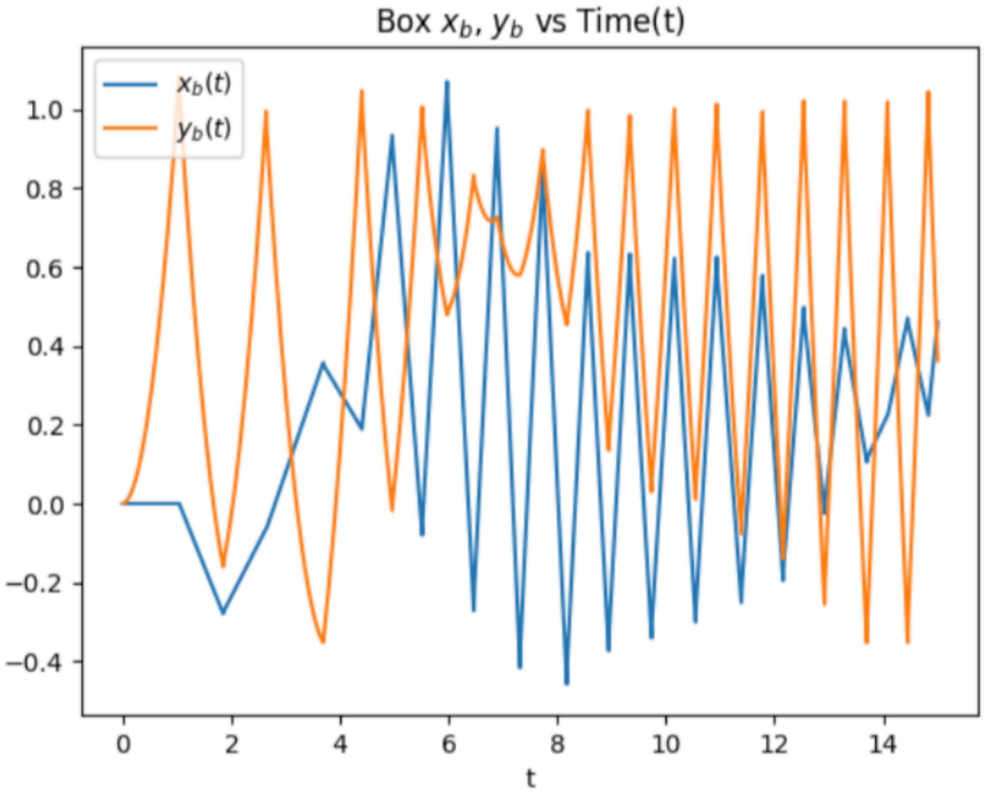
Simulation Results

The initial condition are `[0, 0, np.pi/4, 2, 2, np.pi/4, 0, 0, 0, 0, 0, 0]`. The box is centered at $(0, 0)$ with a center angle of $\pi/4$, the jack starts at $(2, 2)$ with initial angle $\pi/4$. The velocities and accelerations are set to zero.

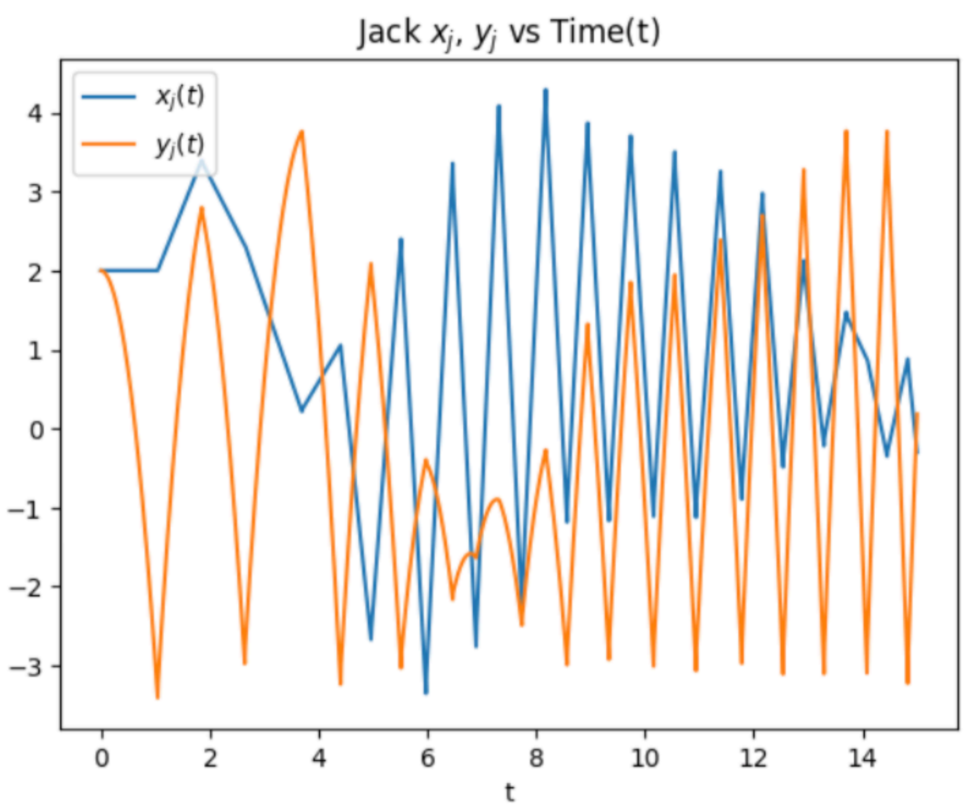
The system starts with both the jack and the box at rest. Under the influence of gravity, the jack begins to fall until one of its corners comes into contact with the inner wall of the box. Upon impact, both the jack and the box respond dynamically, with the jack rotating and translating while the box shifts slightly in response to the collision forces. As the simulation progresses, multiple collisions occur between the corners of the jack and the box walls. The constraints ensure that the jack's corners remain within the box and do not pass through the walls. The simulation demonstrates realistic physical interactions, with both objects exhibiting translational and rotational motion consistent with their mass-inertia properties and the applied forces.

For the simulation, I plot the the positions and angles for the box and jack.

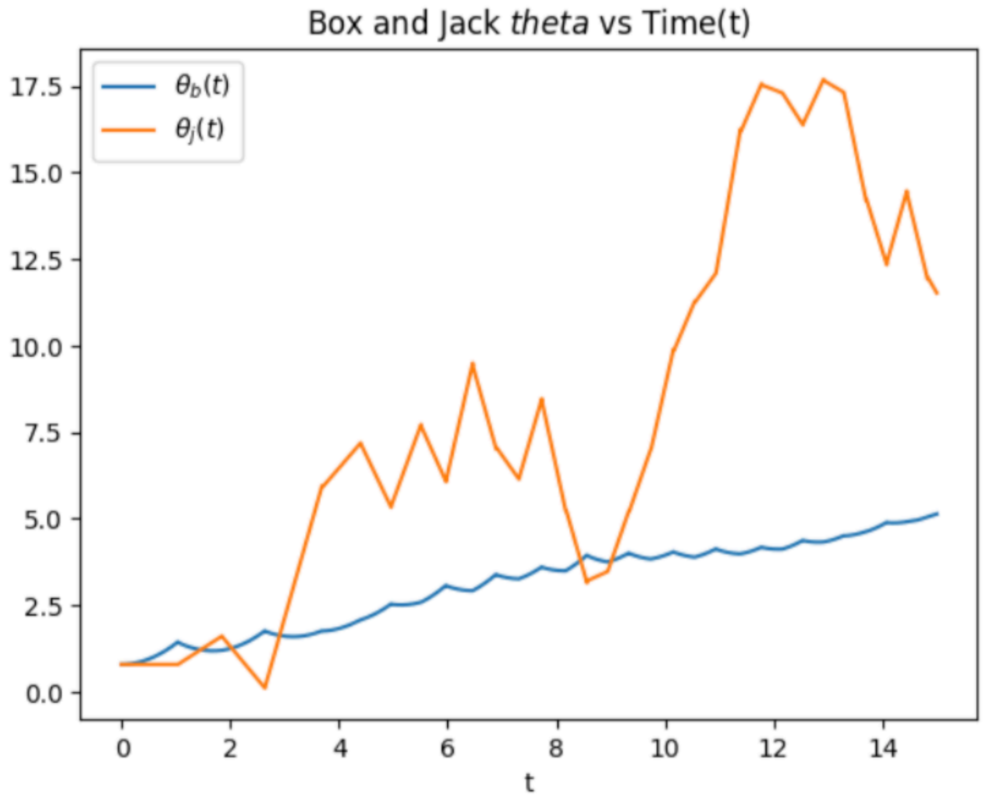
- Box x_b, y_b vs Time(t)



- Jack x_j, y_j vs Time(t)



- Box and Jack $theta$ vs Time(t)



- Box and Jack x_b, y_b, x_j, y_j vs Time(t)