

# Model Predictive Control (Prj.).

## 1. Modeling

From the differential equations.

$$\begin{bmatrix} \dot{T}_1^c(t) \\ \dot{T}_2^c(t) \\ \dot{T}_3^c(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{m_1}(\partial_{1,0} + \partial_{1,2}), & \frac{\partial_{1,2}}{m_1}, & 0 \\ \frac{\partial_{1,2}}{m_2}, & -\frac{1}{m_2}(\partial_{1,2} + \partial_{2,1} + \partial_{2,0}), & \frac{\partial_{2,3}}{m_2} \\ 0, & \frac{\partial_{2,3}}{m_3}, & -\frac{1}{m_3}(\partial_{2,3} + \partial_{3,0}) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T_{1,sp} = -21^\circ C, \quad T_{2,sp} = 0.3^\circ C$$

$$T_{sp} = [-21, 0.3, T_{3,sp}]^T$$

From the differential equations of zone 1,2,

$$\begin{bmatrix} \dot{T}_1^c(t) \\ \dot{T}_2^c(t) \\ \dot{T}_3^c(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{m_1}, 0 \\ 0, \frac{1}{m_2} \\ 0, 0 \end{bmatrix} \begin{bmatrix} P_1^c(t) \\ P_2^c(t) \end{bmatrix} + \dots$$

$$\begin{aligned} 0 &= -(\partial_{1,0} + \partial_{1,2}) T_{1,sp} + \partial_{1,2} T_{2,sp} + P_{1,p} (\partial_{1,0} T_0 + w_1) \\ 0 &= \partial_{1,2} T_{1,sp} - (\partial_{1,2} + \partial_{2,1} + \partial_{2,0}) T_{2,sp} + \partial_{2,1} T_{3,sp} \dots \\ &\dots + P_{2,p} + (\partial_{2,1} T_0 + w_2) \\ P_{sp} &= \begin{bmatrix} P_{1,sp} \\ P_{2,sp} \end{bmatrix} = \begin{bmatrix} (\partial_{1,0} + \partial_{1,2}) T_{1,sp} - \partial_{1,2} T_{2,sp} - (\partial_{1,0} T_0 + w_1) \\ -\partial_{1,2} T_{1,sp} + (\partial_{1,2} + \partial_{2,1} + \partial_{2,0}) T_{2,sp} - \partial_{2,1} T_{3,sp} \dots \\ \dots - (\partial_{2,1} T_0 + w_2) \end{bmatrix} \end{aligned}$$

The definition of delta-formulation:

$$\Delta x(k+1) = T(k+1) - T_{sp}$$

$$\Delta x(k) = T(k) - T_{sp}$$

$$\Delta u(k) = p(k) - p_{sp}$$

$$\therefore T(k+1) - T_{sp} = A(T(k) - T_{sp}) + B(p(k) - p_{sp})$$

$$T(k+1) = A T(k) + B p(k) + (I - A) T_{sp} - B p_{sp}$$

## 4. Constraints

Zone temperatures and cooling units

constraints:

$$T_{min} = \begin{bmatrix} -\infty \\ 0 \\ -\infty \end{bmatrix} \leq \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \leq \begin{bmatrix} -15 \\ 4 \\ +\infty \end{bmatrix} = T_{max}$$

$$P_{min} = \begin{bmatrix} -2500 \\ -2000 \end{bmatrix} \leq \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix} = P_{max}$$

Delta-Formulation constraints:

$$\Delta x_{min} = T_{min} - T_{sp} = \begin{bmatrix} -10 \\ 0 \\ -\infty \end{bmatrix} - \begin{bmatrix} -21 \\ 0.3 \\ -\infty \end{bmatrix} = \begin{bmatrix} -10 \\ 0.3 \\ -\infty \end{bmatrix}$$

$$\Delta x_{max} = T_{max} - T_{sp} = \begin{bmatrix} -15 \\ 4 \\ +\infty \end{bmatrix} - \begin{bmatrix} -21 \\ 0.3 \\ -\infty \end{bmatrix} = \begin{bmatrix} 6 \\ 3.7 \\ +\infty \end{bmatrix}$$

$$\therefore [-\infty, -0.3, -\infty]^T \leq \Delta x(k) \leq [6, 3.7, +\infty]^T$$

## 3. Steady State.

In the steady state:

$$T(k+1) = T(k)$$

$$= (I + T_s A^c) T(k) + T_s B^c p + T_s B_d^c d$$

$$\therefore A^c T_{sp} + B^c p_{sp} + B_d^c d = 0$$

From the equation of zone 3 (3rd row),

$$0 = \partial_{2,3} (T_{2,sp} - T_{3,sp}) + \partial_{3,0} (T_0 - T_{3,sp}) + w_3$$

$$\therefore T_{3,sp} = \frac{1}{\partial_{2,3} + \partial_{3,0}} (\partial_{2,3} T_{2,sp} + \partial_{3,0} T_0 + w_3)$$

The desired temperature for zone 1,2;

$$\Delta U_{min} = P_{min} - P_{sp} = \begin{bmatrix} -2500 \\ -2000 \end{bmatrix} - \begin{bmatrix} P_{1,sp} \\ P_{2,sp} \end{bmatrix} = \begin{bmatrix} -2500 - P_{1,sp} \\ -2000 - P_{2,sp} \end{bmatrix}.$$

$$\Delta U_{max} = P_{max} - P_{sp} = -P_{sp}.$$

$$\therefore \begin{bmatrix} -2500 - P_{1,sp} \\ -2000 - P_{2,sp} \end{bmatrix} \leq \Delta U(k) \leq -P_{sp}.$$

$$x_{LQR}(k) = (A + BF_{\infty})' x_{LQR}(k-1)$$

$$= (A + BF_{\infty})^2 x_{LQR}(k-2)$$

$$= \dots$$

$$= (A + BF_{\infty})^k x_{LQR}(0)$$

$$= (A + BF_{\infty})^k x(0).$$

$$J_{LQR}(x(0)) = \sum_{k=0}^{\infty} x_{LQR}(k)^T (Q + F_{\infty}^T R F_{\infty}) x_{LQR}(k)$$

$$= \sum_{k=0}^{\infty} x(0)^T ((A + BF_{\infty})^k)^T (Q + F_{\infty}^T R F_{\infty}) \dots$$

$$\dots (A + BF_{\infty})^k x(0) \quad M(k)$$

$$= x(0)^T \left[ \sum_{k=0}^{\infty} M(k) \right] x(0)$$

## 5.1 LQR

The LQR controller has to satisfy 2 constraints:

(1) Not violating state/input constraints. This is achieved since there is no warning running the simulation.

(2) Fast convergence:

$$\| T_p - T(s_0) \|_2 \leq \frac{1}{5} \| x^{(0)} \|_2$$

It's realized by gain contraction in varying degrees ( $\omega k$ ).

From the simulation results,  $T_1(s_0) = -20.92^\circ C$ ,

$T_2(s_0) \approx 0.2523^\circ C$ ,  $T_3(s_0) \approx 7.411^\circ C$ ;

The set points:  $T_{sp} = [-21, 0.3, 7.32]'^\circ C$ .

$$\| T_{sp} - T(s_0) \|_2 = \sqrt{[-21 - (-20.59)]^2 + (0.3 - 0.2523)^2}$$

$$+ (7.32 - 7.411)^2 = 0.524 \leq \frac{1}{5} \sqrt{3^2 + 1^2 + 0^2} = 0.622$$

Therefore, all the requirements are met.

by the designed LQR controller

## 6. LQR cost.

We have the following equations:

$$\begin{cases} u(k) = F_{\infty} x_{LQR}(k), \\ x_{LQR}(0) = x(0), \\ x_{LQR}(k) = Ax_{LQR}(k-1) + Bu(k-1), \\ J_{LQR}(x(0)) = \sum_{k=0}^{\infty} x_{LQR}(k)^T Q x_{LQR}(k) + u_{LQR}(k)^T R u_{LQR}(k) \end{cases}$$