Model Predictive Control (Prj.)

1. Modeling.

From the differential equations,

$$\begin{bmatrix} \dot{T}_1^c(t) \\ \dot{T}_2^c(t) \\ \dot{T}_3^c(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{m_1}(\partial_{1,0}+\partial_{1,2}), & \frac{\partial_{1,2}}{m_1}, & 0 \\ \frac{\partial_{1,2}}{m_2}, & -\frac{1}{m_2}(\partial_{1,2}+\partial_{2,3}+\partial_{2,0}), & \frac{\partial_{2,3}}{m_2} \\ 0, & \frac{\partial_{2,3}}{m_3}, & -\frac{1}{m_3}(\partial_{2,3}+\partial_{3,0}) \end{bmatrix}}_{A^c} \cdots$$

$$\cdots \begin{bmatrix} T_1^c(t) \\ T_2^c(t) \\ T_3^c(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{m_1}, & 0 \\ 0, & \frac{1}{m_2} \\ 0 & 0 \end{bmatrix}}_{B^c} \begin{bmatrix} P_1^c(t) \\ P_2^c(t) \end{bmatrix} + \cdots$$

$$\cdots \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix} \underbrace{\begin{bmatrix} \partial_{1,0}T_0 + w_1 \\ \partial_{2,0}T_0 + w_2 \\ \partial_{3,0}T_0 + w_3 \end{bmatrix}}_{d^c}$$

2. Discretization.

$$\dot{T}^c(t) = g(x^c(t), p^c(t))$$

After discretizing,

$$T(k+1) = T(k) + T_s\, g(x^c(t), p^c(t))$$

$$T(k+1) = T(k) + T_s(A^cT(k) + B^c p(k) + B_d^c d)$$

$$= \underbrace{(I + T_sA^c)}_{A} T(k) + \underbrace{T_sB^c}_{B} p(k) + \underbrace{T_sB_d^c}_{B_d} d$$

3. Steady State.

In the steady state:

$$T(k+1) = T(k)$$

$$= (I + T_sA^c)T(k) + T_sB^c p + T_sB_d^c d$$

$$\therefore A^c T_{sp} + B^c p_{sp} + B_d^c d = 0$$

From the equation of zone 3 (3rd row),

$$0 = \partial_{2,3}(T_{2,sp} - T_{3,sp}) + \partial_{3,0}(T_0 - T_{3,sp}) + W_3$$

$$\therefore T_{3,sp} = \frac{1}{\partial_{2,3}+\partial_{3,0}}(\partial_{2,3}T_{2,sp} + \partial_{3,0}T_0 + W_3)$$

The desired temperature for zone 1,2;

$$T_{1,sp} = -21^\circ c, \quad T_{2,sp} = 0.3^\circ c$$

$$T_{sp} = [-21, \ 0.3, \ T_{3,sp}] \ (^\circ c)$$

From the differential equations of zone 1,2,

$$\begin{cases} 0 = -(\partial_{1,0}+\partial_{1,2})T_{1,sp} + \partial_{1,2}T_{2,sp} + p_{1,sp} + (\partial_{1,0}T_0 + w_1) \\ 0 = \partial_{1,2}T_{1,sp} - (\partial_{1,2}+\partial_{2,3}+\partial_{2,0})T_{2,sp} + \partial_{2,3}T_{3,sp} \cdots \\ \quad \cdots + p_{2,sp} + (\partial_{2,0}T_0 + w_2) \end{cases}$$

$$\therefore p_{sp} = \begin{bmatrix} p_{1,sp} \\ p_{2,sp} \end{bmatrix} = \begin{bmatrix} (\partial_{1,0}+\partial_{1,2})T_{1,sp} - \partial_{1,2}T_{2,sp} - (\partial_{1,0}T_0 + w_1) \\ -\partial_{1,2}T_{1,sp} + (\partial_{1,2}+\partial_{2,3}+\partial_{2,0})T_{2,sp} - \partial_{2,3}T_{3,sp} \\ \cdots - (\partial_{2,0}T_0 + W_2) \end{bmatrix}$$

The definition of delta-formulation:

$$\begin{cases} \Delta x(k+1) = T(k+1) - T_{sp} \\ \Delta x(k) = T(k) - T_{sp} \\ \Delta U(k) = p(k) - p_{sp} \end{cases}$$

$$\therefore T(k+1) - T_{sp} = A(T(k) - T_{sp}) + B(p(k) - p_{sp})$$

$$T(k+1) = AT(k) + Bp(k) + (I - A)T_{sp} - Bp_{sp}.$$

4. Constraints.

Zone temperatures and cooling units constraints:

$$T_{min} = \begin{bmatrix} -\infty \\ 0 \\ -\infty \end{bmatrix} \le \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \le \begin{bmatrix} -15 \\ 4 \\ +\infty \end{bmatrix} = T_{max}$$

$$p_{min} = \begin{bmatrix} -2500 \\ -2000 \end{bmatrix} \le \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \end{bmatrix} = p_{max}.$$

Delta-Formulation constraints:

$$\Delta x_{min} = T_{min} - T_{sp} = \begin{bmatrix} -\infty \\ 0 \\ -\infty \end{bmatrix} - \begin{bmatrix} -21 \\ 0.3 \\ T_{3,sp} \end{bmatrix} = \begin{bmatrix} -\infty \\ -0.3 \\ -\infty \end{bmatrix}$$

$$\Delta x_{max} = T_{max} - T_{sp} = \begin{bmatrix} -15 \\ 4 \\ +\infty \end{bmatrix} - \begin{bmatrix} -21 \\ 0.3 \\ T_{3,sp} \end{bmatrix} = \begin{bmatrix} 6 \\ 3.7 \\ +\infty \end{bmatrix}$$

$$\therefore [-\infty, -0.3, -\infty]^T \le \Delta x(k) \le [6, 3.7, +\infty]^T.$$

$\Delta U_{min} = p_{min} - p_{sp} = \begin{bmatrix} -2500 \\ -2000 \end{bmatrix} - \begin{bmatrix} p_{1,sp} \\ p_{2,sp} \end{bmatrix} = \begin{bmatrix} -2500 - p_{1,sp} \\ -2000 - p_{2,sp} \end{bmatrix}$.

$\Delta U_{max} = p_{max} - p_{sp} = -p_{sp}$.

$\therefore \begin{bmatrix} -2500 - p_{1,sp} \\ -2000 - p_{2,sp} \end{bmatrix} \le \Delta U(k) \le -p_{sp}$.

5: LQR

The LQR controller has to satisfy 2 constraints:

(1) Not avoiding state/input constraints. This is achieved since there is no warning running the simulation.

(2) Fast convergence:

$$\| T_{sp} - T(30) \|_2 \le \frac{1}{5} \| x_0^{(1)} \|_2$$

It's realized by gain contraction in varying degrees (∂k).

From the simulation results, $T_1(30) = -20.92\,°C$, $T_2(30) = 0.2525\,°C$, $T_3(30) = 7.411\,°C$;

The set points: $T_{sp} = [-21, 0.3, 7.32]'(°C)$.

$$\| T_{sp} - T(30) \|_2 = \sqrt{[-21 - (-20.92)]^2 + (0.3 - 0.2525)^2 + (7.32 - 7.411)^2} = 0.152 \le \frac{1}{5}\sqrt{3^2 + 1^2 + 0^2} = 0.632$$

Therefore, all the requirements are met by the designed LQR controller

6. LQR cost.

We have the following equations:

$$\begin{cases} U(k) = F_\infty x_{LQR}(k). \\ x_{LQR}(0) = x(0) \\ x_{LQR}(k) = A x_{LQR}(k-1) + B U(k-1). \\ J_{LQR}^{\infty}(x(0)) = \sum_{k=0}^{\infty} x_{LQR}(k)^T Q x_{LQR}(k) + U_{LQR}(k)^T R U_{LQR}(k) \end{cases}$$

$$\begin{aligned} x_{LQR}(k) &= (A + BF_\infty)' x_{LQR}(k-1) \\ &= (A + BF_\infty)^2 x_{LQR}(k-2) \\ &= \cdots \\ &= (A + BF_\infty)^k x_{LQR}(0) \\ &= (A + BF_\infty)^k x(0). \end{aligned}$$

$$\begin{aligned} J_{LQR}^{\infty}(x(0)) &= \sum_{k=0}^{\infty} x_{LQR}(k)^T (Q + F_\infty^T R F_\infty) x_{LQR}(k) \\ &= \sum_{k=0}^{\infty} x(0)^T \underbrace{((A + BF_\infty)^k)^T (Q + F_\infty^T R F_\infty) \cdots (A + BF_\infty)^k}_{M(k)} x(0) \\ &= x(0)^T \left[\sum_{k=0}^{\infty} M(k) \right] x(0) \end{aligned}$$