Model Predictive Control (Prj.)

1. Modeling.

From the differential equations,

$$\begin{bmatrix} \dot{T}_1^c(t) \\ \dot{T}_2^c(t) \\ \dot{T}_3^c(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{m_1}(\partial_{1,0}+\partial_{1,2}), & \frac{\partial_{1,2}}{m_1}, & 0 \\ \frac{\partial_{1,2}}{m_2}, & -\frac{1}{m_2}(\partial_{1,2}+\partial_{2,3}+\partial_{2,0}), & \frac{\partial_{2,3}}{m_2} \\ 0, & \frac{\partial_{2,3}}{m_3}, & -\frac{1}{m_3}(\partial_{2,3}+\partial_{3,0}) \end{bmatrix}}_{A^c} \cdots$$

$$\cdots \begin{bmatrix} T_1^c(t) \\ T_2^c(t) \\ T_3^c(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{m_1}, & 0 \\ 0, & \frac{1}{m_2} \\ 0 & 0 \end{bmatrix}}_{B^c} \begin{bmatrix} p_1^c(t) \\ p_2^c(t) \end{bmatrix} + \cdots$$

$$\cdots \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix} \underbrace{\begin{bmatrix} \partial_{1,0}T_0 + w_1 \\ \partial_{2,0}T_0 + w_2 \\ \partial_{3,0}T_0 + w_3 \end{bmatrix}}_{d^c}$$

2. Discretization.

$$\dot{T}^c(t) = g(x^c(t), p^c(t))$$

After discretizing,

$$T(k+1) = T(k) + T_s\, g(x^c(t), p^c(t))$$

$$T(k+1) = T(k) + T_s(A^c T(k) + B^c p(k) + B_d^c d)$$

$$= \underbrace{(I + T_s A^c)}_{A} T(k) + \underbrace{T_s B^c}_{B} p(k) + \underbrace{T_s B_d^c}_{B_d} d$$

3. Steady State.

In the steady state:

$$T(k+1) = T(k)$$

$$= (I + T_s A^c) T(k) + T_s B^c p + T_s B_d^c d$$

$$\therefore A^c T_{sp} + B^c p_{sp} + B_d^c d = 0$$

From the equation of zone 3 (3rd row),

$$0 = \partial_{2,3}(T_{2,sp} - T_{3,sp}) + \partial_{3,0}(T_0 - T_{3,sp}) + W_3$$

$$\therefore T_{3,sp} = \frac{1}{\partial_{2,3}+\partial_{3,0}}(\partial_{2,3}T_{2,sp} + \partial_{3,0}T_0 + W_3)$$

The desired temperature for zone 1,2;

$$T_{1,sp} = -21°c, \quad T_{2,sp} = 0.3°c$$

$$T_{sp} = [-21,\ 0.3,\ T_{3,sp}]\ (°c)$$

From the differential equations of zone 1,2,

$$\begin{cases} 0 = -(\partial_{1,0}+\partial_{1,2})T_{1,sp} + \partial_{1,2}T_{2,sp} + p_{1,sp} + (\partial_{1,0}T_0 + w_1) \\ 0 = \partial_{1,2}T_{1,sp} - (\partial_{1,2}+\partial_{2,3}+\partial_{2,0})T_{2,sp} + \partial_{2,3}T_{3,sp} \cdots \\ \quad \cdots + p_{2,sp} + (\partial_{2,0}T_0 + w_2) \end{cases}$$

$$\therefore p_{sp} = \begin{bmatrix} p_{1,sp} \\ p_{2,sp} \end{bmatrix} = \begin{bmatrix} (\partial_{1,0}+\partial_{1,2})T_{1,sp} - \partial_{1,2}T_{2,sp} - (\partial_{1,0}T_0 + w_1) \\ -\partial_{1,2}T_{1,sp} + (\partial_{1,2}+\partial_{2,3}+\partial_{2,0})T_{2,sp} - \partial_{2,3}T_{3,sp} \\ \cdots - (\partial_{2,0}T_0 + w_2) \end{bmatrix}$$

The definition of delta-formulation:

$$\begin{cases} \Delta x(k+1) = T(k+1) - T_{sp} \\ \Delta x(k) = T(k) - T_{sp} \\ \Delta u(k) = p(k) - p_{sp} \end{cases}$$

$$\therefore T(k+1) - T_{sp} = A(T(k) - T_{sp}) + B(p(k) - p_{sp})$$

$$T(k+1) = A\,T(k) + B\,p(k) + (I - A)T_{sp} - B\,p_{sp}.$$

4. Constraints.

Zone temperatures and cooling units constraints:

$$T_{min} = \begin{bmatrix} -\infty \\ 0 \\ -\infty \end{bmatrix} \le \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \le \begin{bmatrix} -15 \\ 4 \\ +\infty \end{bmatrix} = T_{max}$$

$$p_{min} = \begin{bmatrix} -2500 \\ -2000 \end{bmatrix} \le \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \end{bmatrix} = p_{max}.$$

Delta-Formulation constraints:

$$\Delta x_{min} = T_{min} - T_{sp} = \begin{bmatrix} -\infty \\ 0 \\ -\infty \end{bmatrix} - \begin{bmatrix} -21 \\ 0.3 \\ T_{3,sp} \end{bmatrix} = \begin{bmatrix} -\infty \\ -0.3 \\ -\infty \end{bmatrix}$$

$$\Delta x_{max} = T_{max} - T_{sp} = \begin{bmatrix} -15 \\ 4 \\ +\infty \end{bmatrix} - \begin{bmatrix} -21 \\ 0.3 \\ T_{1,sp} \end{bmatrix} = \begin{bmatrix} 6 \\ 3.7 \\ +\infty \end{bmatrix}$$

$$\therefore [-\infty,\ -0.3,\ -\infty]^T \le \Delta x(k) \le [6,\ 3.7,\ +\infty]^T.$$

$$\Delta U_{min} = p_{min} - p_{sp} = \begin{bmatrix} -2500 \\ -2000 \end{bmatrix} - \begin{bmatrix} p_{1,sp} \\ p_{2,sp} \end{bmatrix} = \begin{bmatrix} -2500 - p_{1,sp} \\ -2000 - p_{2,sp} \end{bmatrix}.$$

$$\Delta U_{max} = p_{max} - p_{sp} = -p_{sp}.$$

$$\therefore \begin{bmatrix} -2500 - p_{1,sp} \\ -2000 - p_{2,sp} \end{bmatrix} \le \Delta U(k) \le -p_{sp}.$$

5: LQR

The LQR controller has to satisfy 2 constraints:

(1) Not avoiding state/input constraints. This is achieved since there is no warning running the simulation.

(2) Fast convergence:

$$\| T_{sp} - T(30) \|_2 \le \frac{1}{5} \| x_0^{(1)} \|_2$$

It's realized by gain contraction in varying degrees (∂k).

From the simulation results, $T_1(30) = -20.92\,°C$, $T_2(30) = 0.2525\,°C$, $T_3(30) = 7.411\,°C$;

The set points: $T_{sp} = [-21, 0.3, 7.32]'\,(°C)$.

$$\| T_{sp} - T(30) \|_2 = \sqrt{[-21 - (-20.59)]^2 + (0.3 - 0.5859)^2 + (7.32 - 7.477)^2} = 0.524 \le \frac{1}{5}\sqrt{3^2 + 1^2 + 0^2} = 0.632$$

Therefore, all the requirements are met by the designed LQR controller.

6. LQR cost.

We have the following equations:

$$\begin{cases} u(k) = F_\infty x_{LQR}(k). \\ x_{LQR}(0) = x(0) \\ x_{LQR}(k) = A x_{LQR}(k-1) + B u(k-1). \\ J^\infty_{LQR}(x(0)) = \sum_{k=0}^{\infty} x_{LQR}(k)^T Q x_{LQR}(k) + u_{LQR}(k)^T R u_{LQR}(k) \end{cases}$$

$$\begin{aligned} x_{LQR}(k) &= (A + BF_\infty)' x_{LQR}(k-1) \\ &= (A + BF_\infty)^2 x_{LQR}(k-2) \\ &= \cdots \\ &= (A + BF_\infty)^k x_{LQR}(0) \\ &= (A + BF_\infty)^k x(0). \end{aligned}$$

$$\begin{aligned} J^\infty_{LQR}(x(0)) &= \sum_{k=0}^{\infty} x_{LQR}(k)^T (Q + F_\infty^T R F_\infty) x_{LQR}(k) \\ &= \sum_{k=0}^{\infty} x(0)^T \underbrace{((A+BF_\infty)^k)^T (Q + F_\infty^T R F_\infty) \cdots (A+BF_\infty)^k}_{M(k)} x(0) \\ &= x(0)^T \left[\sum_{k=0}^{\infty} M(k) \right] x(0) \end{aligned}$$

20. Offset-free Control.

$$\begin{bmatrix} x(k+1) \\ d(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A & B_d \\ 0_{3\times3} & I_{3\times3} \end{bmatrix}}_{A_{aug}} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0_{3\times2} \end{bmatrix}}_{B_{aug}} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} I_{3\times3} & 0_{3\times3} \end{bmatrix}}_{C_{aug}} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} + \underbrace{0_{3\times2}}_{D_{aug}} u(k).$$

21.

From the results of matlab parameter tuning, $L = 0.05 \times 1_{6\times3}$.

In the problem, we track constant reference

$$\begin{cases} r = T_{sp} \\ z(k) = H y(k) \to r, \text{ for } k \to \infty. \\ H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{cases}$$

The expression for computing steady-state (x_s, u_s)

$$\begin{bmatrix} A - I & B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} -B_d \hat{d} \\ r \end{bmatrix}$$

Estimator error dynamics

$$\begin{aligned} \begin{bmatrix} x(k+1) - \hat{x}(k+1) \\ d(k+1) - \hat{d}(k+1) \end{bmatrix} &= A_{aug} \begin{bmatrix} x(k) - \hat{x}(k) \\ d(k) - \hat{d}(k) \end{bmatrix} - L C_{aug} \begin{bmatrix} x(k) - \hat{x}(k) \\ d(k) - \hat{d}(k) \end{bmatrix} \\ &= \left[A_{aug} - L C_{aug} \right] \begin{bmatrix} x(k) - \hat{x}(k) \\ d(k) - \hat{d}(k) \end{bmatrix} \end{aligned}$$

eigenvalues of $(A_{aug} - L C_{aug})$ are

$[1, 0.8312, 0.9145, 0.9998, 0.9628, 1]^T$.