Autocorrelation in Time Series

KNNL – Chapter 12

Issues in Autocorrelated Data

One of the standard assumptions in the regression model is that the error terms ε_i and ε_j are uncorrelated for $i \neq j$. When error terms are correlated, problems occur when using OLS estimates:

- Regression Coefficients are still unbiased
- MSE *underestimates* σ^2
- Standard errors of regression coefficients based on OLS *underestimate* the true standard error
- Inflated t and F statistics and *artificially narrow* confidence intervals

Autocorrelated errors of 1st order:

$$\varepsilon_t = \rho\, \varepsilon_{t-1} + u_t$$

where u_t are uncorrelated disturbances assumed normally distributed.

First-Order Model - I

First-Order Autoregressive Model ($AR(1)$):

Simple Regression: $Y_t = \beta_0 + \beta_1 X_t + \varepsilon_t \quad t = 1,\ldots,n \qquad \varepsilon_t = \rho\varepsilon_{t-1} + u_t$

$\rho \equiv$ autoregression parameter with $|\rho| < 1$

$u_t \sim N\left(0, \sigma^2\right)$ and independent

Generalizes to Multiple Regression:

$$Y_t = \beta_0 + \beta_1 X_{t1} + \ldots + \beta_{p-1} X_{t,p-1} + \varepsilon_t \quad t = 1,\ldots,n \qquad \varepsilon_t = \rho\varepsilon_{t-1} + u_t$$

Properties of Errors (assumption regarding ε_1 for model consistency):

$$\varepsilon_1 \sim N\left(0, \frac{\sigma^2}{1-\rho^2}\right)$$

$$\varepsilon_2 = \rho\varepsilon_1 + u_2 \quad \Rightarrow \quad E\left\{\varepsilon_2\right\} = \rho E\left\{\varepsilon_1\right\} + E\left\{u_2\right\} = 0 \qquad \sigma^2\left\{\varepsilon_2\right\} = \rho^2\sigma^2\left\{\varepsilon_1\right\} + \sigma^2\left\{u_2\right\} = \rho^2\left(\frac{\sigma^2}{1-\rho^2}\right) + \sigma^2 = \frac{\sigma^2}{1-\rho^2}$$

Covariance: $\sigma\left\{\varepsilon_2, \varepsilon_1\right\} = \sigma\left\{\rho\varepsilon_1 + u_2, \varepsilon_1\right\} = \rho\sigma^2\left\{\varepsilon_2\right\} + \sigma\left\{u_2, \varepsilon_1\right\} = \rho\sigma^2\left\{\varepsilon_2\right\} + 0 = \dfrac{\rho\sigma^2}{1-\rho^2}$

Corrrelation: $\rho\left\{\varepsilon_2, \varepsilon_1\right\} = \dfrac{\sigma\left\{\varepsilon_2, \varepsilon_1\right\}}{\sigma\left\{\varepsilon_2\right\}\sigma\left\{\varepsilon_1\right\}} = \dfrac{\dfrac{\rho\sigma^2}{1-\rho^2}}{\sqrt{\dfrac{\sigma^2}{1-\rho^2}}\sqrt{\dfrac{\sigma^2}{1-\rho^2}}} = \rho$

First-Order Model - II

In General:

$$\varepsilon_t = \rho\varepsilon_{t-1} + u_t = \rho\left(\rho\varepsilon_{t-2} + u_{t-1}\right) + u_t = \rho^2\varepsilon_{t-2} + \rho u_{t-1} + u_t = \ldots = \sum_{s=0}^{\infty} \rho^s u_{t-s}$$

$$E\left\{\varepsilon_t\right\} = 0 \qquad \sigma^2\left\{\varepsilon_t\right\} = \sigma^2\left\{\sum_{s=0}^{\infty} \rho^s u_{t-s}\right\} = \sum_{s=0}^{\infty} \rho^{2s}\sigma^2\left\{u_{t-s}\right\} = \sigma^2 \sum_{s=0}^{\infty} \rho^{2s} = \frac{\sigma^2}{1-\rho^2}$$

Covariance: $\sigma\left\{\varepsilon_t, \varepsilon_{t-s}\right\} = \dfrac{\rho^s \sigma^2}{1-\rho^2} \quad s \geq 0$

Correlation: $\rho\left\{\varepsilon_t, \varepsilon_{t-s}\right\} = \rho^s \quad s \geq 0$

$$\sigma^2\left\{\varepsilon\right\} = \frac{\sigma^2}{1-\rho^2} \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{bmatrix}$$

$AR(2): \quad \varepsilon_t = \rho_1\varepsilon_{t-1} + \rho_2\varepsilon_{t-2} + u_t$ Even Higher or models can be fit as well.

Test For Independence : Durbin-Watson Test

$Y_t = \beta_0 + \beta_1 X_{t1} + \ldots + \beta_{p-1} X_{t,p-1} + \varepsilon_t \quad \varepsilon_t = \rho \varepsilon_{t-1} + u_t \quad u_t \sim NID\left(0, \sigma^2\right) \quad |\rho| < 1$

$H_0 : \rho = 0 \quad \Rightarrow$ Errors are uncorrelated over time

$H_A : \rho > 0 \quad \Rightarrow$ Positively correlated

1) Obtain Residuals from Regression

2) Compute Durbin-Watson Statistic (given below)

3) Obtain Critical Values from Table B.7, pp. 1330-1331 (R will provide a p-value)

If $DW < d_L\left(p-1, n\right) \Rightarrow$ Reject H_0 If $DW > d_U\left(p-1, n\right) \Rightarrow$ Conclude H_0 Otherwise Inconclusive

Test Statistic: $DW = \dfrac{\sum_{t=2}^{n}\left(e_t - e_{t-1}\right)^2}{\sum_{t=1}^{n} e_t^2}$

$$E\left\{\varepsilon_t\right\} = 0 \quad E\left\{\varepsilon_t \varepsilon_{t-1}\right\} = \frac{\rho\sigma^2}{1-\rho^2}$$

$$\Rightarrow \quad \sum_{t=2}^{n}\left(e_t - e_{t-1}\right)^2 = \sum_{t=2}^{n} e_t^2 + \sum_{t=2}^{n} e_{t-1}^2 - 2\sum_{t=2}^{n} e_t e_{t-1} \approx 2\sum_{t=1}^{n} e_t^2 - 2n\frac{\rho\sigma^2}{1-\rho^2}$$

$\Rightarrow$ Under H_0, expect DW ≈ 2

Autocorrelation - Remedial Measures

- Determine whether a missing predictor variable can explain the autocorrelation in the errors
- Include a linear (trend) term if the residuals show a consistent increasing or decreasing pattern
- Include seasonal dummy variables if data are quarterly or monthly and residuals show cyclic behavior
- Use transformed Variables that remove the (estimated) autocorrelation parameter (Cochrane-Orcutt and Hildreth-Lu Procedures)
- Use First Differences
- Estimated Generalized Least Squares

Transformed Variables

Suppose ρ is known: $Y_t = \beta_0 + \beta_1 X_t + \varepsilon_t \qquad \varepsilon_t = \rho\varepsilon_{t-1} + u_t$

Let $Y_t^{'} = Y_t - \rho Y_{t-1} = \left(\beta_0 + \beta_1 X_t + \varepsilon_t\right) - \rho\left(\beta_0 + \beta_1 X_{t-1} + \varepsilon_{t-1}\right) =$

$\beta_0\left(1-\rho\right) + \beta_1\left(X_t - \rho X_{t-1}\right) + \left(\varepsilon_t - \rho\varepsilon_{t-1}\right) = \beta_0\left(1-\rho\right) + \beta_1\left(X_t - \rho X_{t-1}\right) + u_t$

$\Rightarrow \quad Y_t^{'} = \beta_0^{'} + \beta_1^{'} X_t^{'} + u_t$ (Standard Simple linear regression with independent errors)

where:

$Y_t^{'} = Y_t - \rho Y_{t-1} \qquad X_t^{'} = X_t - \rho X_{t-1} \qquad \beta_0^{'} = \beta_0\left(1-\rho\right) \qquad \beta_1^{'} = \beta_1$

In Practice, we need to estimate ρ with a sample based value r

$Y_t^{'} = Y_t - rY_{t-1} \qquad X_t^{'} = X_t - rX_{t-1}$

Fit: $\hat{Y}' = b_0^{'} + b_1^{'} X'$ and if errors are uncorrelated, back transform to:

$\hat{Y} = b_0 + b_1 X$ where: $b_0 = \dfrac{b_0^{'}}{1-r} \qquad s\left\{b_0\right\} = \dfrac{s\left\{b_0^{'}\right\}}{1-r} \qquad b_1 = b_1^{'} \qquad s\left\{b_1\right\} = s\left\{b_1^{'}\right\}$

Cochrane-Orcutt Method

- Start by estimating ρ in Model: $\varepsilon_t = \rho\varepsilon_{t-1} + u_t$ by regression through the origin for residuals (see below)
- Fit transformed regression model (previous slide)
- Check to see if new residuals are uncorrelated (Durbin-Watson test), based on the transformed model
- If uncorrelated, stop and keep current model
- If correlated, repeat process with new estimate r based on current regression residuals from the original (back transformed) model

$$r = \frac{\sum_{t=2}^{n} e_{t-1} e_t}{\sum_{t=2}^{n} e_{t-1}^2}$$

Hildreth-Lu and First Difference Methods

- Hildreth-Lu Method
  - Find value of *r* (between 0 and 1) that minimizes the SSE for the transformed model by grid search
  - Apply the transformed analysis based on the estimated *r*

- First Differences Method
  - Uses $\rho = 1$ in transformed model ($Y_t' = Y_t - Y_{t-1} \quad X_t' = X_t - X_{t-1}$)
  - Set $b_0' = 0$ and fits regression through origin of Y' on X'
  - When back-transforming:

$$b_0 = \overline{Y} - b_1' \overline{X} \qquad b_1 = b_1'$$

Forecasting with Autocorrelated Errors

Makes use of any of the 3 estimation techniques (C-O, H-L, First Differences):

$$Y_t = \beta_0 + \beta_1 X_t + \varepsilon_t \qquad \varepsilon_t = \rho\varepsilon_{t-1} + u_t$$

$$\Rightarrow \quad Y_t = \beta_0 + \beta_1 X_t + \rho\varepsilon_{t-1} + u_t \quad \Rightarrow \quad \Rightarrow \quad Y_{n+1} = \beta_0 + \beta_1 X_{n+1} + \rho\varepsilon_n + u_{n+1}$$

3 Elements:

1. Expected Value: $\beta_0 + \beta_1 X_{n+1}$ Estimated as $\hat{Y}_{n+1} = b_0 + b_1 X_{n+1}$
2. Multiple of period n Error Term: $\rho\varepsilon_n$ Estimated as re_n
3. Current disturbance $u_{n+1} \sim N\left(0, \sigma^2\right)$

Forecast for period $n+1$ (note the notation is "Forecast", not F-distribution :

$$F_{n+1} = \hat{Y}_{n+1} + re_n$$

Standard Error of the Prediction (based on transformed model):

$$s^2\{\text{pred}\} = MSE'\left[1 + \frac{1}{n-1} + \frac{\left(X'_{n+1} - \overline{X}'\right)}{\sum_{i=2}^{n}\left(X'_i - \overline{X}'\right)^2}\right]$$

Approximate 95% PI: $F_{n+1} \pm t\left(1-(\alpha/2); n-3\right)s\{\text{pred}\}$ (First Differences has n - 2 df)