Question 4

Consider a binary communication channel with input $X \sim \mathrm{Bern}(p)$, output $Y \sim \mathrm{Bern}(q)$, and crossover (error) probability ε. It is known that

$$q \;=\; p + \varepsilon \;-\; 2\varepsilon p.$$

The channel capacity is

$$C \;=\; \max_{0 \le p \le 1} I(X;Y).$$

Find the value of p that maximises C (i.e. maximises $I(X;Y)$).

Solution

The mutual information can be written

$$I(X;Y) = H(Y) - H(Y \mid X) = H\big(q\big) - H\big(\varepsilon\big),$$

where $H(\varepsilon) = -\varepsilon \log_2 \varepsilon - (1-\varepsilon)\log_2(1-\varepsilon)$ is constant in p, and

$$q(p) = p + \varepsilon - 2\varepsilon p = (1-2\varepsilon)p + \varepsilon.$$

Differentiate with respect to p:

$$\frac{dI}{dp} = \frac{d}{dp} H(q) = (1-2\varepsilon)\,\big[-\log_2 q + \log_2(1-q)\big] \stackrel{!}{=} 0.$$

Hence

$$\log_2 \frac{1-q}{q} = 0 \implies q = \tfrac{1}{2}.$$

Solving $q = (1-2\varepsilon)p + \varepsilon = \frac{1}{2}$ gives

$$(1-2\varepsilon)p = \tfrac{1}{2} - \varepsilon \implies p = \frac{\frac{1}{2} - \varepsilon}{1-2\varepsilon} = \tfrac{1}{2},$$

for all $\varepsilon \neq \frac{1}{2}$.

Answer

$$\boxed{p = \frac{1}{2}.}$$