Question 4

Given the joint PMF:

	$X=-2$	$X=0$	$X=2$
$Y=1$	0	$2a$	a
$Y=2$	$2a$	0	$2a$
$Y=4$	a	$2a$	0

Let $S = X + Y$, $Z = X - Y$.

Part 1: Find the value of a and the marginal probability density function of X.

Solution

Summing all entries: $(3a) + (4a) + (3a) = 10a = 1$, so $a = \frac{1}{10}$. Marginally,

$$\begin{aligned} P(X=-2) = 3a &= 0.3 \\ P(X=0) = 4a &= 0.4 \\ P(X=2) = 3a &= 0.3 \end{aligned}$$

Answer

$$\boxed{a = 0.1, \quad P_X(-2) = 0.3,\ P_X(0) = 0.4,\ P_X(2) = 0.3}$$

Part 2: Are X and Y independent?

Solution

For example, $P(X=-2, Y=1) = 0 \neq P(X=-2)P(Y=1) = 0.3 \cdot 0.3 = 0.09$, so not independent.

Answer

$$\boxed{No.}$$

Part 3: Compute the covariance of S and Z.

Solution

Note $S = X + Y$, $Z = X - Y$, so

$$\begin{aligned}\mathrm{Cov}(S, Z) &= \mathrm{Cov}(X + Y, X - Y) \\ &= \mathrm{Var}(X) - \mathrm{Var}(Y)\end{aligned}$$

We have

$$\begin{aligned}E[X] &= 0 \\ \mathrm{Var}(X) &= 4(0.3 + 0.3) = 2.4 \\ E[Y] &= 2.3 \\ \mathrm{Var}(Y) &= 6.7 - (2.3)^2 = 1.41\end{aligned}$$

Thus

$$\mathrm{Cov}(S, Z) = 2.4 - 1.41 = 0.99$$

Answer

$$\boxed{\mathrm{Cov}(S, Z) = 0.99}$$

Part 4: Are S and Z independent?

Solution

Since $\mathrm{Cov}(S, Z) = 0.99 \neq 0$, they cannot be independent.

Answer

$$\boxed{No.}$$