Question 1

Part 1: Solution

First compute the marginal distribution of X:

$$P(X=0)=\frac{1}{8},\quad P(X=1)=\frac{7}{8}.$$

Then

$$H(X)=-\sum_{x=0}^{1}P(X=x)\log_2 P(X=x)=-\Big(\tfrac{1}{8}\log_2\tfrac{1}{8}+\tfrac{7}{8}\log_2\tfrac{7}{8}\Big)\approx 0.544\text{ bits.}$$

Next compute the conditional entropy:

$$H(X\mid Y)=\sum_{y=0}^{1}P(Y=y)\,H(X\mid Y=y).$$

For $Y=0$, $P(Y=0)=\frac{1}{8}$ and X is degenerate ($H=0$).
For $Y=1$, $P(Y=1)=\frac{7}{8}$ and

$$P(X=0\mid 1)=\frac{1/8}{7/8}=\frac{1}{7},\quad P(X=1\mid 1)=\frac{6}{7},$$

so

$$H(X\mid Y=1)=-\Big(\tfrac{1}{7}\log_2\tfrac{1}{7}+\tfrac{6}{7}\log_2\tfrac{6}{7}\Big)\approx 0.592\text{ bits.}$$

Hence

$$H(X\mid Y)=\frac{1}{8}\cdot 0+\frac{7}{8}\cdot 0.592\approx 0.519\text{ bits.}$$

Finally, compare:

$$H(X)\approx 0.544\ \geq\ 0.519\approx H(X\mid Y).$$

Answer

$$\boxed{H(X)\approx 0.544\text{ bits},\quad H(X\mid Y)\approx 0.519\text{ bits},\quad H(X)\geq H(X\mid Y).}$$

Part 2: Solution

We compare $H(X\mid Y=1)\approx 0.592$ bits (from above) with the a-priori $H(X)\approx 0.544$ bits.

Answer

$$\boxed{H(X \mid Y=1) \approx 0.592 \text{ bits } > H(X) \approx 0.544 \text{ bits.}}$$

Part 3: Solution

The mutual information is

$$I(X;Y) = H(X) - H(X \mid Y) \approx 0.544 - 0.519 = 0.025 \text{ bits.}$$

Answer

$$\boxed{I(X;Y) \approx 0.025 \text{ bits.}}$$