Question 3

Suppose that X and Y are independent and uniformly distributed on $\{0, 1, 2\}$. Let

$$S = X + Y, \quad W = X \cdot Y.$$

Part 1: Solution

First list the probability mass function of S:

s	0	1	2	3	4
$P(S = s)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

and of W:

w	0	1	2	4
$P(W = w)$	$\frac{5}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

Therefore

$$H(S) = -\sum_{s=0}^{4} P(S = s) \log_2 P(S = s) \approx 2.1972 \text{ bits},$$

$$H(W) = -\sum_{w\in\{0,1,2,4\}} P(W = w) \log_2 P(W = w) \approx 1.6577 \text{ bits}.$$

Answer

$$\boxed{H(S) \approx 2.1972 \text{ bits}, \quad H(W) \approx 1.6577 \text{ bits}.}$$

Part 2: Solution

The mutual information is

$$I(S; W) = H(S) + H(W) - H(S, W),$$

where the joint entropy

$$H(S, W) = -\sum_{s,w} P(S = s, W = w) \log_2 P(S = s, W = w) \approx 2.5033 \text{ bits}.$$

Hence

$$I(S; W) \approx 2.1972 + 1.6577 - 2.5033 \approx 1.3516 \text{ bits}.$$

Answer

$$\boxed{I(S; W) \approx 1.3516 \text{ bits}.}$$