Mass Formula — Bhargava

Ref: • Bhargava – Mass Formulae for Extensions of Local Fields ...

• Serre (1978) – Une "Formule de masse" pour ...

<u>Def</u>. An <u>S_n-number field</u> is K with $[K:\mathbb{Q}]=n$ and $\mathrm{Gal}(L/\mathbb{Q})=S_n$, $L=$ Gal closure of $K/\mathbb{Q}$

* Paper didn't make it clear, but I believe so. I will show a formula for # of fields in a fixed $\bar{\mathbb{Q}}$.

Goal: Let $N_n(X)=\#$ of iso-classes of S_n-number fields of abs disc $\le X$.

Conjecture that $N_n(X)\sim c_n X$ and give an explicit formula for c_n.

$$\boxed{\begin{array}{l}\text{Known up to } n=5.\\ \text{e.g. } c_2=\zeta(2)^{-1},\ c_3=\frac{1}{3}\zeta(3)^{-1}\\ \text{* Paper says } c_2=\frac{1}{2}\zeta(2)^{-1}.\ \text{I think it is a subtle error.}\end{array}}$$

Input: 1. (Local)

<u>Thm (Serre)</u> F – local field with residue field $\mathbb{F}_q$.

$$\sum_K \frac{1}{q^{v_F(\operatorname{disc} K/F)}}\frac{1}{\mathrm{Aut}(K/F)}=\frac{1}{q^{n-1}}$$

Rmk. This is finite sum.

where K runs over isom classes of tot ram ext of F of deg n. [Idea: Parametrize tot ram ext by Eisenstein poly. It turns out the Haar measure of the set of polys defining K is proportional to the weight $\frac{1}{q^{v_F(\operatorname{disc} K/F)}}\frac{1}{\#\mathrm{Aut}(K/F)}$]

2. (Local-$S_n$global) Bhargava's Assumptions

For an S_n-field K, consider its completion
$K_v := K\otimes \mathbb{Q}_v$ (v fin or inf)

We expect one can recover "a lot" about K from local data $(K_v)_v$

Note: K_v is in gen. an étale alg $/\mathbb{Q}_v$, i.e. prod of $\underbrace{\text{field ext}}_{\text{fin}}$ of $\mathbb{Q}_v$.

<u>Discriminant in general</u>

Recall: defined as $\det(\mathrm{Tr}(e_ie_j))$ for a basis

· $K/\mathbb{Q}_p$ field ext

(computed by integral basis)

Then $\mathrm{disc}(K/\mathbb{Q}_p)$ is well def as an ideal of $\mathbb{Z}_p$

Fix a represent. $\mathrm{Disc}(K/\mathbb{Q}_p) := p^{v_p(\mathrm{disc}(K/\mathbb{Q}_p))}$

$v_p(\mathrm{disc}(K/\mathbb{Q}_p)) = e-1$ if $p \nmid e$ (tamely ram.), and $\geq e$ if $p \mid e$ (wild)

- $K = \mathbb{R}$ or $\mathbb{C}$ / $\mathbb{R}$

  Then $\mathrm{disc}(K/\mathbb{R})$ (computed by any basis) is well-defined up to sign.

  Define $\mathrm{Disc}(K/\mathbb{R}) := \begin{cases} 1, & K = \mathbb{R} \\ -1, & K = \mathbb{C} \end{cases}$ [The defining quadratic eq. of any non-real number is < 0 !]

- K étale alg / $\mathbb{Q}_v$

  $K = K_1 \oplus \cdots \oplus K_r$, K_i are field ext of $\mathbb{Q}_v$, then define

  $\mathrm{Disc}(K) := \prod_{i=1}^{r} \mathrm{Disc}(K_i)$

- K number field, then

$$\mathrm{Disc}\, K/\mathbb{Q} = \prod_{v \leq \infty} \mathrm{Disc}(K \otimes \mathbb{Q}_v)$$

i.e. We can recover the prime factorization of $\mathrm{Disc}\, K$ by looking at $K \otimes \mathbb{Q}_p$, and the sign of $\mathrm{Disc}\, K$ from $K \otimes \mathbb{R}$.

- $\mathrm{Disc}\, K/\mathbb{Q} = D$ iff $\mathrm{Disc}(K \otimes \mathbb{Q}_v)$ $\underbrace{\text{is compatible to}}_{\text{written "}\sim\text{" in future}}$ D $\forall v$.

Assumption (Bhargava)

(weight) (a) Weight the global fields and local étale algs by $\frac{1}{\#\mathrm{Aut}}$

(indep) (b) $K \otimes \mathbb{Q}_v$ behaves independently among all places v, when K ranges over S_n fields.

(normalization) (c) The weighted number of S_n fields satisfying some reasonable local conditions is roughly the weighted number of local data $\vec{K} = (K_v)_v$ (K_v étale alg/$\mathbb{Q}_v$ of deg n) satisfying the local conditions.

For my purposes denote $\Omega_v = \{$iso classes of deg n étale alg / $\mathbb{Q}_v\}$, $\Omega = \prod_v \Omega_v$. Say $\Sigma_v \subseteq \Omega_v$ and $\Sigma = \prod_v \Sigma_v \subseteq \Omega$ is a local condition, e.g. $\Sigma_v = \{K_v : \mathrm{Disc}\, K_v \sim D\}$.

Then $\displaystyle\sum_{(K \otimes \mathbb{Q}_v)_v \in \Sigma} \frac{1}{\#\mathrm{Aut}\, K} = \text{"} \sum_{\vec{K} \in \Sigma} \frac{1}{\#\mathrm{Aut}\, \vec{K}} \text{"} := \prod_v \sum_{K_v \in \Sigma_v} \frac{1}{\#\mathrm{Aut}\, K_v}$ (or $\mathrm{vol}(\Sigma)$, where Ω = prod measure space of Ω_v)

My picture: set of S_n-fields is a "lattice" in Ω.

Rmk (1). Assumptions (b)(c) are only true if K ranges over S_n-fields

(2) For S_n-field K, $\# \mathrm{Aut} K = 1$ $(n>2)$ or 2 $(n=2)$. [So the weighted count immediately give the unweighted count]

(3) Assumption (c) is actually quite strong in the sense that it is not just a randomness statement, but also assents that $\frac{1}{\# \mathrm{Aut} K}$ is the correct weight given to global fields rather than $c / \# \mathrm{Aut} K$ for $c \neq 1$. [(b) predicts the number of S_n-field satisfying some local conditions up-to proportionality]

Also note that there is no Gal gp condition on the local side, so it would stop to be true if "S_n-field" requirement is changed.

Any reason why (c) should hold?

Local Mass Formulae for Étale Algs

Prop 2.3
$$\sum_{\substack{[K:\mathbb{Q}_p]=n \\ \text{étale}}} \frac{1}{\mathrm{Disc}_p K} \frac{1}{\# \mathrm{Aut} K} = \sum_{k=0}^{n-1} q(k, n-k)\, p^{-k},$$

where $q(k, n-k)$ is the number of partitions of k into at most $n-k$ parts

Pf. Next section. Use Serre + basic NT + combinatorics.

Prop 2.4
$$\sum_{\substack{[K:\mathbb{R}]=n \\ \text{étale}}} \frac{1}{\# \mathrm{Aut} K} = \frac{r_2(S_n)}{n!}, \text{ where}$$

$r_2(S_n) = \#$ of 2-torsions in S_n

Pf. Noting $K = \mathbb{R}^? \times \mathbb{C}^?$ and pure combinatorics.
Details see next section if time permits.

Conj 1.2 $\quad N_n(X) :=$ # of iso classes of S_n-fields w/ $|\mathrm{Disc}| \le X$.

Then

$$\lim_{X\to\infty} \frac{N_n(X)}{X} = \begin{cases} \dfrac{r_2(S_n)}{2\, n!} \prod_p \overbrace{\left(\sum_{k=0}^{n} \dfrac{q(k,n-k) - q(k-1,n-k+1)}{p^k} \right)}^{\frac{p-1}{p}\cdot \text{RHS of Prop 2.3}}, & n > 2 \\ \boxed{2} \cdot \text{above} = \zeta(2)^{-1}, & n = 2 \end{cases}$$

missing in paper ; this factor comes from $\#\mathrm{Aut}(K)$, K S_n-field.

Equivalently,

$$\lim_{X\to\infty} \frac{\#\text{ of } S_n \text{ fields in a fixed } \bar{\mathbb{Q}} \text{ having } |\mathrm{Disc}| \le X}{X} = \frac{r_2(S_n)}{2\,(n-1)!} \prod_p \sum_{k=0}^{n} \frac{q(k,n-k) - q(k-1,n-k+1)}{p^k}$$

"Proof" given heuristic assumptions

$$\text{weighted \# of fields} = \sum_{\substack{D=-X \\ D\equiv 0,1 \bmod 4}}^{X} \sum_{\substack{K\ S_n\text{-field} \\ \mathrm{Disc}\,K = D}} \frac{1}{\#\mathrm{Aut}\,K}$$

(Δ_X = set of legal disc in $[-X,X]$)

$$\overset{(c)}{=} \sum_{D\in\Delta_X} \text{''} \sum_{\substack{\vec{K}\in\Omega \\ \mathrm{Disc}\,\vec{K}=D}} \frac{1}{\#\mathrm{Aut}\,\vec{K}} \text{''}$$

$$= \text{''} \sum_{\vec{K}\in\Omega} \sum_{\substack{D\in\Delta_X \\ D=\mathrm{Disc}\,\vec{K}}} \frac{1}{\#\mathrm{Aut}\,\vec{K}} \text{''}$$

$$= \text{''} \sum_{\vec{K}} \frac{1}{\#\mathrm{Aut}\,\vec{K}} \cdot \#\{D\in\Delta_X : D = \mathrm{Disc}\,\vec{K}\} \text{''}$$

$$\approx \text{''} \sum_{\vec{K}} \frac{1}{\#\mathrm{Aut}\,\vec{K}} |\Delta_X| \,\mathrm{Prob}(D = \mathrm{Disc}\,\vec{K}) \text{''}$$

$$\approx \text{''} \sum_{\vec{K}} \frac{1}{\#\mathrm{Aut}\,\vec{K}} \cdot X \cdot \prod_v \mathrm{Prob}(\underset{\in\Delta_X}{D} \text{ compatible with } \mathrm{Disc}\,K_v) \text{''}$$

$[-X,X]$ is $\approx 2X$ numbers, but half the numbers are $\equiv 0,1 \bmod 4$

Chinese Remainder Thm

- $v=\infty$, $D \sim \mathrm{Disc}\, K_v$ means D has prescribed sign. Prob $= \frac{1}{2}$
- $v=p$, $D \sim \mathrm{Disc}\, K_v$ means $v_p(D) = v_p(\mathrm{Disc}\, K_v)$

$$\mathrm{Prob} = \underbrace{\frac{1}{\mathrm{Disc}\, K_v}}_{\text{divided by } \mathrm{Disc}\, K_v} \underbrace{\frac{p-1}{p}}_{\text{exactly}}$$

Drop quotation mark! $\longrightarrow$ $= X \cdot \prod_v \sum_{K_v \in \Omega_v} \frac{1}{\# \mathrm{Aut}\, K_v} \mathrm{Prob}(D \sim \mathrm{Disc}\, K_v)$

$$= X \left(\sum_{K_\infty} \frac{1}{2} \frac{1}{\# \mathrm{Aut}\, K_\infty} \right) \prod_p \frac{p-1}{p} \sum_{K_p} \frac{1}{\mathrm{Disc}\, K_p} \frac{1}{\# \mathrm{Aut}\, K_p}$$

Local Mass Formulae $\longrightarrow$ $= X \cdot \frac{1}{2} \frac{r_2(S_n)}{n!} \cdot \prod_p (1-p^{-1}) \sum_{k=0}^{n-1} q(k, n-k)\, p^{-k}$

$$= X \cdot \frac{r_2(S_n)}{2n!} \prod_p \sum_{k=0}^{n} p^{-k} \big(q(k, n-k) - q(k-1, n-k+1) \big)$$

[simplify as poly of p^{-1}; note $q(n,0)=0$]

Finally, since $\forall$ S_n-field K, $\# \mathrm{Aut}\, K = \begin{cases} 1, & n>2 \\ 2, & n=2 \end{cases}$, we get the unweighted count from the weighted count. $\square$

Proof of Local Mass Formulae: finite prime

Goal: $\displaystyle\sum_{\substack{[K:\mathbb{Q}_p]=n \\ \text{étale}}} \frac{1}{\mathrm{Disc}(K)} \cdot \frac{1}{\# \mathrm{Aut}\, K}$

First, classify K based on $\pi(K)$, a partition asso. to K as follows:

Write $K = K_1 \oplus \cdots \oplus K_r$, K_i: field ext of $\mathbb{Q}_p$.

e_i, f_i = ram index, inertia deg of K_i
(so $\sum e_i f_i = n$)

$\pi(K)$ is the partition $f_1 \cdot [e_1 - 1] + \cdots + f_r \cdot [e_r - 1]$
of $k := \sum f_i(e_i - 1)$.

Note: the number of parts of $\pi(K)$ is at most
$f_1 + \cdots + f_r = \sum f_i e_i - \sum f_i(e_i - 1) = n - k$

<u>Prop 2.2</u> Let $0 \le k < n$, and π_0 be any partition of k of at most $n-k$ parts, then

$$\sum_{\substack{[K:\mathbb{Q}_p]=n \\ \pi(K)=\pi_0}} \frac{1}{\mathrm{Disc}(K)} \frac{1}{\# \mathrm{Aut}\, K} = \frac{1}{p^k}$$

<u>Prop 2.2 $\Rightarrow$ Mass Formula</u>

$$\sum_{[K:\mathbb{Q}_p]=n} \frac{1}{\mathrm{Disc}\, K} \frac{1}{\# \mathrm{Aut}\, K}$$

$$= \sum_{k=0}^{n-1} \sum_{\substack{\pi_0 \text{ part of } k \\ \text{of } \le n-k \text{ parts}}} \sum_{\substack{[K:\mathbb{Q}_p]=n \\ \pi(K)=\pi_0}} \frac{1}{\mathrm{Disc}\, K} \frac{1}{\# \mathrm{Aut}\, K}$$

$$= \sum_{k=0}^{n-1} \# \left\{ \pi_0 : \begin{array}{c} \text{partition of } k \\ \text{of} \le n-k \text{ parts} \end{array} \right\} \frac{1}{p^k} \qquad \square$$

Proof of 2.2

For a field ext $K_i/\mathbb{Q}_p$ of splitting type (e_i, f_i),

consider F_i = max unram subext of $K_i/\mathbb{Q}_p$, then

$$\begin{array}{ccl} & K_i & \\ e_i & | & \text{tot ram} \\ & F_i & \\ f_i & | & \text{unram} \\ & \mathbb{Q}_p & \end{array}$$

Note: F_i is only determined by f_i.

Upshot: Picking K of splitting type $(e_1, f_1), \cdots, (e_r, f_r)$ is the same as picking a tot. ram ext K_i for each of F_i, where F_i is $\boxed{\text{the}}$ unram ext of $\mathbb{Q}_p$ of deg f_i. $\to$ Have hope to use Serre formula!

Main concerns: ① Serre formula gives totally number of K_i weighted by $\frac{1}{\mathbb{N}(\mathrm{disc}(K_i/F_i))} \cdot \frac{1}{\# \mathrm{Aut}(K_i/F_i)}$

abs norm, i.e. index in val. ring

We want to count with weight

$$\frac{1}{\mathbb{N}(\mathrm{disc}(K_i/\mathbb{Q}_p))} \cdot \frac{1}{\# \mathrm{Aut}(K_i/\mathbb{Q}_p)}$$

<u>Sol.</u> $\mathbb{N}(\mathrm{disc}\, K_i/F_i) = \mathbb{N}(\mathrm{disc}(K_i/\mathbb{Q}_p))$

because $F_i/\mathbb{Q}_p$ is unram.

Also $\# \mathrm{Aut}(K_i/\mathbb{Q}_p) = f_i \cdot \# \mathrm{Aut}(K_i/F_i)$ (Exercise)

(Use $F_i/\mathbb{Q}_p$ is Galois)

② It could be that $K_i, K_i'/F_i$ are not isom as F_i-ext, but are isom as $\mathbb{Q}_p$-ext (e.g. via $\sigma: K_i \to K_i'$ not fixing F_i)

Sol. Keeping track of overcounting factor is basic Galois theory.

A computation leads to the formula:

Prop 2.1
$$\sum_{\substack{[K:\mathbb{Q}_p]=n \\ \text{splitting type is} \\ \alpha=\{(e_1,f_1),\dots,(e_r,f_r)\}}} \frac{1}{\operatorname{Disc} K}\frac{1}{\# \operatorname{Aut} K} = \frac{1}{p^k}\cdot\frac{1}{\boxed{\left(\prod_{i=1}^r f_i\right) N}}$$

as multiset →

call it $\#\operatorname{Aut}(\alpha)$

where · $k = \sum f_i(e_i - 1)$ is as in Prop 2.2

· N is the number of ways to permute the terms (e_i, f_i) that leave the ordered tuple $(e_1, f_1), \dots, (e_r, f_r)$ unchanged.

e.g. Splitting type $(1,2),(1,3),(2,1),(2,1),(1,3),(1,3)$

$N = 1!\,2!\,3!$

Rmk. $\dfrac{1}{\left(\prod_{i=1}^r f_i\right)N}$ can be understood as $\dfrac{1}{\#\operatorname{Aut}}$ of the following datum:

Think of $F = (F_1)_{e_1} \times \cdots \times (F_r)_{e_r}$, where $(F_i)_{e_i}$ is just the field F_i, with a label e_i specifying that we look for deg e_i tot ram ext of F_i. The étale alg F with labeling specifies which K we are looking for.

Consider Aut F as étale alg. There are two ways to make an automorphism: ① Auto. within each F_i (there are $f_i = \#\operatorname{Gal}(F_i/\mathbb{Q}_p)$ many) ② Permuting factors that are the same.

Now we add the restriction that only factors with same label are allowed to be permuted. Then the # of ways of permuting the factors is N.

Prop 2.1 ⇒ Prop 2.2 (Pure combinatorics)

First, $$\sum_{\substack{[K:\mathbb{Q}_p]=n \\ \pi(K)=\pi_0}} \frac{1}{\operatorname{Disc} K} \frac{1}{\# \operatorname{Aut} K} = \Bigg(\sum_{\substack{\text{splitting} \\ \text{types } \alpha \\ \text{with } \pi(\alpha)=\pi_0}} \frac{1}{\# \operatorname{Aut}(\alpha)} \Bigg) \frac{1}{p^k}$$

Σ depends on π_0 only

[Recall dependence graph: $K \xrightarrow{\text{decides}} \alpha \longrightarrow \pi_0 \longrightarrow k = \text{size of } \pi_0$]

Write $\pi_0 = n_1[1-1] + n_2[2-1] + \cdots$, written as a partition of k of exactly $n-k$ parts with 0 allowed [i.e. n_1 is determined s.t. $\sum n_i = n-k$]

Then n_i = total f-values of the terms in α whose e-value is i.

To determine α s.t. $\pi(\alpha) = \pi_0$ is the same as to give a partition of n_i for each i.

e.g. $\pi_0 = 3[1-1] + 2[2-1] + 2[3-1]$

If we give partitions $n_1 = 2+1$, $n_2 = 2$, $n_3 = 1+1$, then

$\alpha = (1,2), (1,1), (2,2), (3,1), (3,1)$

Each partition determines a cycle type using lengths, and pick a permutation having the cycle type:

$\lambda = \big((12)(3), \ (12), \ (1)(2) \big)$

$\in G_{\pi_0} := S_{n_1} \times S_{n_2} \times S_{n_3} \times \cdots$

$(123) = (231) = ($

Observe: $\# \mathrm{Aut}\, \alpha = \#$ Centralizer of λ in G_{π_0}

$\therefore \# \mathrm{Aut}\, \alpha$ = stabilizer of λ as G_{π_0} acts on G_{π_0} by conjugation

$$= \frac{\# G_{\pi_0}}{\# \text{ conj. of } \lambda}$$

$$\therefore \frac{1}{\# \mathrm{Aut}\, \alpha} = \frac{\# \text{ of elts whose cycle type is given by } \alpha}{\# G_{\pi_0}}$$

$$\therefore \sum_{\substack{\alpha \\ \pi(\alpha) = \pi_0}} \frac{1}{\# \mathrm{Aut}\, \alpha} = \frac{\# G_{\pi_0}}{\# G_{\pi_0}} = 1 \qquad \square$$

Prop 2.4. Goal: want to find $\displaystyle\sum_{\substack{[K:\mathbb{R}]=n \\ \text{étale}}} \frac{1}{\# \mathrm{Aut}\, K}$

Pf. Given K, say $n=6$, $K = \mathbb{R} \times \mathbb{R} \times \mathbb{C} \times \mathbb{C}$

Associate a perm. in S_n with $\mathbb{R}$ corresponding to 1-cycle and $\mathbb{C}$ corresponding to a 2-cycle

e.g. $\lambda = (1)(2)(34)(56) \in S_n$

Note that $\mathrm{Aut}\, K = \#$ Centralizer of λ in S_n

[To make an automorphism of K, either use conjugation within each $\mathbb{C}$, or permute the $\mathbb{R}$s, or permute the $\mathbb{C}$s]

So $\displaystyle\sum_K \frac{1}{\# \mathrm{Aut}\, K} = \sum_K \frac{\# \text{ of elts of cycle type asso to } K}{\# S_n}$

$$= \frac{\# \text{ of elts with only 1- and 2-cycle}}{\# S_n} = \frac{\# \{2\text{-torsion}\}}{\# S_n} \qquad \square$$

Appendix: Automorphisms of S_n fields

K - S_n-field, $n \geq 2$

$K = \mathbb{Q}(x_1)$, $L = \mathbb{Q}(x_1, \cdots, x_n)$ where x_i are Galois conjugates of x_1, $\mathrm{Gal}(L/\mathbb{Q}) = S_n$.

Any automorphism of K is given by $x_1 \mapsto x_i$, where $x_i \in K$. $\therefore \# \mathrm{Aut}\, K = \# \{x_1, \cdots, x_n\} \cap K$

Claim: $\# \mathrm{Aut}\, K = 1$ for $n > 2$.

<u>Pf.</u> If not, say $x_2 \in K$. Then $\forall\, \sigma \in \mathrm{Gal}(L/\mathbb{Q}) = S_n$, $\sigma(x_1)$ determines $\sigma(x_2)$.

But σ is arbitrary, this is absurd

... except for $n=2$!

doesn't matter how you interpret the !

[Given $\sigma(x_1)$, $\sigma(x_2)$ always has $n-1$ places to go)

For $n=2$, $n = n!$, $K = L$, $\# \mathrm{Aut}\, K = 2$.

Finally, each iso. class of S_n fields correspond to $\frac{n}{\# \mathrm{Aut}\, K}$ fields in a fixed $\overline{\mathbb{Q}}$. [They are $\mathbb{Q}(x_1), \cdots, \mathbb{Q}(x_n)$, with over-counting factor $\# \mathrm{Aut}\, K$.]

So field count in fixed $\overline{\mathbb{Q}}$ $= n \cdot$ weighted count

Final Remark

The most mysteric object is $\mathrm{Disc}(K_p)$, esp. when ramification is wild. First reduce it to tot. ram. case, because the unram part of an ext has no disc. Then, we can sample tot ram ext by Eisenstein poly, and $1/\mathrm{Disc}$ naturally occurs in the vol in sample space. (Serre)