Point count of the variety of modules over the quantum plane over a finite field

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Background

Given a field $\mathbb{F}$ and $n \geq 0$, define the n-th **commuting variety** over F as

$$K_{1,n}(\mathbb{F}) := \{(A, B) \in \mathrm{Mat}_n(\mathbb{F}) \times \mathrm{Mat}_n(\mathbb{F}) : AB = BA\}.$$

(The meaning of the notation will be clear later.)

What's known:

- When $\mathbb{F} = \mathbb{C}$, the commuting variety $K_{1,n}(\mathbb{C})$ is a complex algebraic variety. Motzkin and Taussky (1955) and Gerstenhaber (1961) showed that $K_{1,n}(\mathbb{C})$ is irreducible.
- When $\mathbb{F} = \mathbb{F}_q$, the finite field of q elements, the set $K_{1,n}(\mathbb{F}_q)$ is a finite set. Feit and Fine (1960) gave its cardinality by the formula:

$$\sum_{n=0}^{\infty} \frac{|K_{1,n}(\mathbb{F}_q)|}{(q^n - 1)(q^n - q) \dots (q^n - q^{n-1})} x^n = \prod_{i,j \geq 1} \frac{1}{1 - x^i q^{2-j}}. \qquad (1)$$

Background

A quantum deformation of the commuting variety has also been considered. Let ζ be a nonzero element of $\mathbb{F}$, define the n-th ζ-**commuting variety** as

$$K_{\zeta,n}(\mathbb{F}) := \{(A,B) \in \mathrm{Mat}_n(\mathbb{F}) \times \mathrm{Mat}_n(\mathbb{F}) : AB = \zeta BA\}.$$

If $\zeta = 1$, then it simply becomes the commuting variety, hence the notation $K_{1,n}$ for the commuting variety.

Efforts have been to spent to extend the work of Motzkin, Taussky and Gerstenhaber to the ζ-commuting variety:

- Chen and Wang (2018) described the irreducible components of the anti-commuting variety $K_{-1,n}(\mathbb{C})$. There are more than one, unlike the $\zeta = 1$ case.
- Chen and Lu (2019) further extended the above result to general ζ.

Main result

We give a direct generalization of Feit–Fine's formula.

Theorem 1 (H., 2021).

Let ζ be a nonzero element of $\mathbb{F}_q$, and let m be the smallest positive integer such that $\zeta^m = 1$. Then

$$\sum_{n=0}^{\infty} \frac{|K_{\zeta,n}(\mathbb{F}_q)|}{(q^n-1)(q^n-q)\dots(q^n-q^{n-1})} x^n = \prod_{i=1}^{\infty} F_m(x^i;q),$$

where

$$F_m(x;q) := \frac{1-x^m}{(1-x)(1-x^m q)} \cdot \frac{1}{(1-x)(1-xq^{-1})(1-xq^{-2})\dots}.$$

When $\zeta = 1$, we have $m = 1$, so $F_1(x^i;q) = \prod_{j\geq 1} \dfrac{1}{1-x^i q^{2-j}}$ and we recover Feit–Fine.

Variety of modules

The commuting variety $K_{1,n}(\mathbb{F})$ parametrizes and classifies finite-$\mathbb{F}$-dimensional modules over the polynomial ring $\mathbb{F}[X,Y]$. So $K_{1,n}(\mathbb{F})$ is also called the **variety of modules** over $\mathbb{F}[X,Y]$. To specify an $\mathbb{F}[X,Y]$-module with underlying space $\mathbb{F}^n$, it suffices to specify the x-action $A : \mathbb{F}^n \to \mathbb{F}^n$ and the y-action $B : \mathbb{F}^n \to \mathbb{F}^n$ under the constraint $AB = BA$. This constraint is because x and y commmute in $\mathbb{F}[X,Y]$.

Similarly, the ζ-commuting variety parametrizes finite-$\mathbb{F}$-dimensional modules over the associative algebra $\mathbb{F}\{X,Y\}/(XY - \zeta YX)$. This algebra is called the **quantum plane**, and is considered as a quantum deformation of $\mathbb{F}[X,Y]$.

Remarks on Theorem 1

$$\sum_{n=0}^{\infty} \frac{|K_{\zeta,n}(\mathbb{F}_q)|}{(q^n-1)(q^n-q)\dots(q^n-q^{n-1})} x^n = \prod_{i=1}^{\infty} F_m(x^i; q),$$

$$F_m(x;q) := \frac{1-x^m}{(1-x)(1-x^m q)} \cdot \frac{1}{(1-x)(1-xq^{-1})(1-xq^{-2})\dots}.$$

- The cardinality of $K_{\zeta,n}(\mathbb{F}_q)$ depends only on the order of ζ as a root of unity of $\mathbb{F}_q$. This is expected.
- The denominator $(q^n-1)(q^n-q)\dots(q^n-q^{n-1})$ is precisely the size of $\mathrm{GL}_n(\mathbb{F}_q)$. This is the natural denominator in this type of generating function. In fact, the coefficient $|K_{\zeta,n}(\mathbb{F}_q)|/|\mathrm{GL}_n(\mathbb{F}_q)|$ is the number of n-dimensional modules over the quantum plane up to isomorphism, each measured with a weight of 1/(size of automorphism group).
- Bavula (1997) classified simple modules over the quantum plane; Theorem 1 should encode statistical information about this classification.

Main result: further breakdown

We now state a refinement of Theorem 1. Let

$$U_{\zeta,n}(\mathbb{F}_q) := \{(A,B) \in \mathrm{GL}_n(\mathbb{F}_q) \times \mathrm{Mat}_n(\mathbb{F}_q) : AB = \zeta BA\},$$

and

$$N_{\zeta,n}(\mathbb{F}_q) := \{(A,B) \in \mathrm{Nilp}_n(\mathbb{F}_q) \times \mathrm{Mat}_n(\mathbb{F}_q) : AB = \zeta BA\}.$$

It turns out that the varieties $U_{\zeta,n}(\mathbb{F}_q)$ and $N_{\zeta,n}(\mathbb{F}_q)$ are building blocks of $K_{\zeta,n}(\mathbb{F}_q)$, in the sense that

$$\sum_{n=0}^{\infty} \frac{|K_{\zeta,n}(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n = \left(\sum_{n=0}^{\infty} \frac{|U_{\zeta,n}(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n\right) \left(\sum_{n=0}^{\infty} \frac{|N_{\zeta,n}(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n\right)$$

Recall that the left-hand side is the content of Theorem 1.

Main result: further breakdown

Theorem 2 (H., 2021)

Let m be the order of ζ. Then

- $$\sum_{n=0}^{\infty} \frac{|U_{\zeta,n}(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n = \prod_{i=1}^{\infty} G_m(x^i; q),$$

  where

  $$G_m(x; q) := \frac{1 - x^m}{(1 - x)(1 - x^m q)}.$$

- $$\sum_{n=0}^{\infty} \frac{|N_{\zeta,n}(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n = \prod_{i=1}^{\infty} H(x^i; q),$$

  where

  $$H(x; q) := \frac{1}{(1 - x)(1 - xq^{-1})(1 - xq^{-2}) \dots}.$$

Remarks on Theorem 2

Theorem 2 can be interpreted as that in the formula

$$F_m(x;q) := \frac{1-x^m}{(1-x)(1-x^m q)} \cdot \frac{1}{(1-x)(1-xq^{-1})(1-xq^{-2})\dots}$$

related to the count of $\{(A,B) : AB = \zeta BA\}$, the factor $\dfrac{1-x^m}{(1-x)(1-x^m q)}$ is the contribution of invertible A, while the factor $\dfrac{1}{(1-x)(1-xq^{-1})(1-xq^{-2})\dots}$ is the contribution of nilpotent A.

Note that the latter does not depend on m, so $|N_{\zeta,n}(\mathbb{F}_q)|$ does not depend on ζ.

Ideas of proof: decomposition

- Given $A, B \in \mathrm{Mat}_n(\mathbb{F}_q)$ such that $AB = \zeta BA$, by Fitting's lemma, there is a unique direct sum decomposition ${\mathbb{F}_q}^n = V \oplus W$ such that $A(V) \subseteq V, A(W) \subseteq W$, $A|_V$ is invertible, and $A|_W$ is nilpotent.
- It turns out that B must satisfy $B(V) \subseteq V, B(W) \subseteq W$. All we need in the proof is that $\zeta \neq 0$.
- This allows $K_{\zeta,n}(\mathbb{F}_q)$ to be "decomposed" into $U_{\zeta,n}(\mathbb{F}_q)$ (requiring invertible A) and $N_{\zeta,n}(\mathbb{F}_q)$ (requiring nilpotent A), in the sense of

$$\sum_{n=0}^{\infty} \frac{|K_{\zeta,n}(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n = \left(\sum_{n=0}^{\infty} \frac{|U_{\zeta,n}(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n \right) \left(\sum_{n=0}^{\infty} \frac{|N_{\zeta,n}(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n \right)$$

Ideas of proof: nilpotent part

- To compute $|N_{\zeta,n}(\mathbb{F}_q)| = |\{(A,B) : AB = \zeta BA, A \text{ nilp}\}|$, we first fix A and count the number of B.
- The number of B only depends on the similarity class of A, so we may assume A is in the Jordan canonical form.
- The general form of B can then be determined entry-wise.
- In particular, the number of B does not depend on ζ (even if $\zeta = 0$).

Ideas of proof: invertible part

- To compute $|U_{\zeta,n}(\mathbb{F}_q)| = |\{(A,B) : AB = \zeta BA, A \text{ invertible}\}|$, we first fix B and count the number of A. (Opposite to the nilpotent case!!)
- Not every B contributes. In order for the number of A to be nonzero, we must have that B is similar to ζB (by the definition of similarity).
- Using the standard orbit-stabilizer argument, it suffices to count the number of similarity classes of B such that B is similar to ζB.
- **This is where m, the order of ζ, matters.** The similarity class corresponds to a finite sequence $(g_1, g_2, \dots)$ of monic polynomials over $\mathbb{F}_q$ such that g_i divides g_{i+1}. Requiring B to be similar to ζB is equivalent to requiring every g_i in the sequence of polynomials associated to B to be of the following form:
  $t^d + c_1 t^{d-m} + c_2 t^{d-2m} + \dots$

An interdependence phenomenon

- The numbers $|U_{\zeta,n}(\mathbb{F}_q)|$ and $|N_{\zeta,n}(\mathbb{F}_q)|$ look like two independent building blocks of $|K_{\zeta,n}(\mathbb{F}_q)|$.
- However, in the commutative case $\zeta = 1$, the data of $|U_{1,n}(\mathbb{F}_q)|$ for all n and $|N_{1,n}(\mathbb{F}_q)|$ for all n recover each other.
- The reason is from (commutative) algebraic geometry. This idea was used in an alternative proof of Feit–Fine formula given by Bryan and Morrison (2015). I will explain it in the next few slides.
- For general ζ, the number $|N_{\zeta,n}(\mathbb{F}_q)|$ does not seem to determine $|U_{\zeta,n}(\mathbb{F}_q)|$ because the former does not depend on ζ while the latter does.

Work of Bryan and Morrison

We sketch their alternative proof of Feit–Fine's formula.

- Compute $|\{(A,B) \in \mathrm{Nilp}_n(\mathbb{F}_q) \times \mathrm{GL}_n(\mathbb{F}_q) : AB = BA\}|$ using the orbit-stabilizer argument.
- Use the notion of "power structure" due to Gusein-Zade, Luengo and Melle-Hernandez to recover $|K_{1,n}(\mathbb{F}_q)|$ for all n from $|\{(A,B) \in \mathrm{Nilp}_n(\mathbb{F}_q) \times \mathrm{GL}_n(\mathbb{F}_q) : AB = BA\}|$ for all n. QED.

Remarks:

- The power structure is in the language of Grothendieck ring of complex varieties, but I will explain its consequence on point counting over finite fields in an elementary way.
- The point is that any one of $|U_{1,n}(\mathbb{F}_q)|$, $|N_{1,n}(\mathbb{F}_q)|$ or their variants determines the rest. No variant is special; the one chosen by Bryan and Morrison is just the easiest to compute.

Work of Bryan and Morrison: elementary interpretation

The reason why these quantities determine each other is that any of these counts and $\nu_{n,q} := |\{(A, B) \in \mathrm{Nilp}_n(\mathbb{F}_q) \times \mathrm{Nilp}_n(\mathbb{F}_q) : AB = BA\}|$ determine each other; so $\nu_{n,q}$ serves as a bridge to connect any two such quantities. I illustrate how $|K_{1,n}(\mathbb{F}_q)|$ and $\nu_{n,q}$ determine each other; other variants work the same way.

Recall that $|K_{1,n}(\mathbb{F}_q)|$ "counts" (finite-$\mathbb{F}_q$-dimensional) modules over $\mathbb{F}_q[X, Y]$. Such a module is determined by its localization at closed points of the affine plane $\mathrm{Spec}\, \mathbb{F}_q[X, Y]$. In other words, to classify $\mathbb{F}_q[X, Y]$-modules, it suffices to classify $\mathbb{F}_q[X, Y]$-modules supported at each given closed point.

On the other hand, $\nu_{n,q}$ "counts" $\mathbb{F}_q[X, Y]$-modules supported at the origin (i.e., the maximal ideal (x, y)).

Work of Bryan and Morrison: elementary interpretation

The key is that every closed point of $\operatorname{Spec}\mathbb{F}_q[X,Y]$ "looks like" the origin, in the sense of Cohen's structure theorem: for any maximal ideal m of $\mathbb{F}_q[X,Y]$, the complete localization of $\mathbb{F}_q[X,Y]$ at m is isomorphic to $\mathbb{F}[[X,Y]]$, where $\mathbb{F}$ is a finite extension of $\mathbb{F}_q$.

Hence $\nu_{n,q}$ determines $|K_{1,n}|$. In terms of formula, given

$$\sum_{n=0}^{\infty} \frac{\nu_{n,q}}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n = \prod_{i,j\geq 1} \frac{1}{1-x^i q^{-j}},$$

the geometric argument above shows that $\sum_{n=0}^{\infty} \frac{|K_{1,n}(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n$ is obtained by replacing $\frac{1}{1-x^i q^{-j}}$ by $Z_{\mathbb{F}_q[X,Y]}(x^i q^{-j})$, where $Z_{\mathbb{F}_q[X,Y]}$ is the Hasse–Weil zeta function of $\operatorname{Spec}\mathbb{F}_q[X,Y]$. The reason why the Hasse–Weil zeta function appears is because its Euler product over all closed points.

Work of Bryan and Morrison: elementary interpretation

Using $Z_{\mathbb{F}_q[X,Y]}(u) = 1/(1-uq^2)$, we get

$$\sum_{n=0}^{\infty} \frac{|K_{1,n}(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n = \prod_{i,j\geq 1} Z_{\mathbb{F}_q[X,Y]}(x^i q^{-j}) = \prod_{i,j\geq 1} \frac{1}{1-x^i q^{2-j}},$$

precisely the formula of Feit and Fine. This is how $\nu_{n,q}$ for all n determines $|K_{1,n}(\mathbb{F}_q)|$ for all n.

In fact, the above process can be reversed, which is not surprising because $|K_{1,n}(\mathbb{F}_q)|$ is determined by $\nu_{n,q}$ alone, again thanks to the homogeneity of $\operatorname{Spec} \mathbb{F}_q[X,Y]$. We can recover $\nu_{n,q}$ from $|K_{1,n}(\mathbb{F}_q)|$ as well.

This finishes the explanation why $K_{1,n}(\mathbb{F}_q)$ and $\nu_{n,q}$ determines each other.

Work of Bryan and Morrison: elementary interpretation

Similarly, $N_{1,n}(\mathbb{F}_q)$ classifies $\mathbb{F}_q[X,Y]$-modules supported on the axis $X=0$, while $U_{1,n}(\mathbb{F}_q)$ classifies $\mathbb{F}_q[X,Y]$-modules supported on the open set $X \neq 0$. Since each of these subsets consist of closed points that "look the same", the same argument applies and we have

$$|N_{1,n}(\mathbb{F}_q)| \text{ for all } n \longleftrightarrow \nu_{n,q} \text{ for all } n \longleftrightarrow |U_{1,n}(\mathbb{F}_q)| \text{ for all } n,$$

where $\longleftrightarrow$ means "determines each other".

We note that the notion of **localization** plays a key role; if we could not break down to closed points, the numbers $|N_{1,n}(\mathbb{F}_q)|$ and $|U_{1,n}(\mathbb{F}_q)|$ would not have been related because $X=0$ and $X \neq 0$ are disjoint! Of course, another necessary ingredient is Cohen's structure theorem: **closed points look the same everywhere**.

Noncommutative case?

Question

Can we find a geometric connection between $|N_{\zeta,n}(\mathbb{F}_q)|$ and $|U_{\zeta,n}(\mathbb{F}_q)|$, similar to the $\zeta = 1$ case explained before?

Recall that m is the order of ζ, and

$$\sum_{n=0}^{\infty} \frac{|N_{\zeta,n}(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n \rightsquigarrow \frac{1}{(1-x)(1-xq^{-1})(1-xq^{-2})\dots}.$$

$$\sum_{n=0}^{\infty} \frac{|U_{\zeta,n}(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n \rightsquigarrow \frac{1-x^m}{(1-x)(1-x^m q)};$$

It seems impossible that $|N_{\zeta,n}(\mathbb{F}_q)|$ determines $|U_{\zeta,n}(\mathbb{F}_q)|$, since the former doesn't depend on m while the latter does. However, it is still possible that $|U_{\zeta,n}(\mathbb{F}_q)|$ can be recoverd from $|N_{\zeta,n}(\mathbb{F}_q)|$ *together with* the geometry of the quantum plane $XY = \zeta YX$ (which depends on m).

Final takeaway

- We extend a formula that counts matrix pairs $AB = BA$ to the case $AB = \zeta BA$ where ζ is nonzero. The answer depends on the order of ζ as a root of unity.
- The count of $AB = \zeta BA$ encodes statistical information about modules over the quantum plane.
- The count in question has two seemingly independent building blocks that turn out to be interdependent in the $\zeta = 1$ case, using ingredients from (commutative) algebraic geometry. I hope that the study of a possible interdependence in the case of general ζ will inspire interesting noncommutative geometry.