A generating function for counting mutually annihilating matrices over a finite field

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Introduction

Let $\mathbb{F}_q$ be a finite field of q elements, and write $\mathrm{Mat}_n(\mathbb{F}_q)$ for the set of $n \times n$ matrices over $\mathbb{F}_q$. A typical random matrix problem or matrix enueration problem takes one of the following forms:

1. What is the probability that a random matrix in $\mathrm{Mat}_n(\mathbb{F}_q)$ satisfies a given property?
2. How many matrices (or tuples of matrices) satisfy a given property?
3. How do invariants of a random matrix (such as rank, eigenvalues) distribute?
4. For each of the above, how does the answer depend on n, either precisely or asymptotically?

We often care about the subset $\mathrm{GL}_n(\mathbb{F}_q)$ of invertible matrices and the subset $\mathrm{Nilp}_n(\mathbb{F}_q)$ of nilpotent matrices.

History

Classical. $|\mathrm{Mat}_n(\mathbb{F}_q)| = q^{n^2}$,
$|\mathrm{GL}_n(\mathbb{F}_q)| = q^{n^2}(1-q^{-1})(1-q^{-2})\dots(1-q^{-n})$.
Fine–Herstein '58 There are q^{n^2-n} nilpotent matrices in $\mathrm{Mat}_n(\mathbb{F}_q)$.
Feit–Fine '60

$$\sum_{n=0}^{\infty} \frac{|\{A, B \in \mathrm{Mat}_n(\mathbb{F}_q) : AB = BA\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n = \prod_{i\geq 1, j\geq 0} (1 - q^{1-j}x^i)^{-1}$$

Fulman–Guralnick '18, also Bryan–Morrison '12

$$\sum_{n=0}^{\infty} \frac{|\{A, B \in \mathrm{Nilp}_n(\mathbb{F}_q) : AB = BA\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n = \prod_{i\geq 1, j\geq 2} (1 - q^{1-j}x^i)^{-1}$$

Note the factor $|\mathrm{GL}_n(\mathbb{F}_q)|$ in the denominator – there are good reasons why we need this to make the generating function nice.

Main Result

Theorem (H.)

$$\sum_{n=0}^{\infty} \frac{|\{A, B \in \mathrm{Mat}_n(\mathbb{F}_q) : AB = BA = 0\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n = \big((1-x)(1-q^{-1}x)(1-q^{-2}x)\dots\big)^{-2} H_q(x) \quad (1)$$

$$\sum_{n=0}^{\infty} \frac{|\{A, B \in \mathrm{Nilp}_n(\mathbb{F}_q) : AB = BA = 0\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n = \big((1-q^{-1}x)(1-q^{-2}x)\dots\big)^{-2} H_q(x) \quad (2)$$

for an explicit power series $H_q(x)$ with infinite radius of convergence.
More interesting analytic properties of $H_q(x)$ later!
Proof. Combinatorial techniques involving Young diagrams and Durfee squares. Details at the end of the talk.

Random matrix consequences

These generating functions (and their meromorphic continuations!) give rich asymptotic information about matrix counting. For example, if $\sum a_n x^n$ has convergence radius x_0, and has a meromorphic continuation $\frac{1}{1-x/x_0} h(x)$ where $h(x)$ converges at x_0, then $a_n \sim h(x_0) x_0^{-n}$.

Feit–Fine $\implies$ $|\{A, B \in \mathrm{Mat}_n(\mathbb{F}_q) : AB = BA\}| \sim C_1 q^{n^2+n}$, where $C_1 = \prod_{j=1}^{\infty} (1 - q^{-j})^{-j}$, using the pole at q^{-1}.

Nilpotent Feit–Fine $\implies$ $|\{A, B \in \mathrm{Nilp}_n(\mathbb{F}_q) : AB = BA\}| \ll q^{n^2+\varepsilon n}$ for all $\varepsilon > 0$, using the holomorphy on the unit disk.

H. (1) $\implies$ $|\{A, B \in \mathrm{Mat}_n(\mathbb{F}_q) : AB = BA = 0\}| \sim C_2 n q^{n^2}$, where $C_2 = \dfrac{H_q(1)}{(1-q^{-1})(1-q^{-2})\dots}$, using the double pole at 1.

H. (2) $\implies$ $|\{A, B \in \mathrm{Nilp}_n(\mathbb{F}_q) : AB = BA = 0\}| \sim C_3 n q^{n^2-n}$, where $C_3 = \dfrac{H_q(q)}{(1-q^{-1})(1-q^{-2})\dots}$, using the double pole at q.

Probablistic interpretation. In average, a random matrix in $\mathrm{Mat}_n(\mathbb{F}_q)$ mutually annihilates around $C_2 n$ matrices (and C_2 nilpotent matrices).

A related problem in commutative algebra

Let R be a commutative ring whose quotients by maximal ideals are finite fields. For example, $\mathbb{Z}$, $\mathbb{F}_q[t]$, $\mathbb{Z}_p$, $\mathbb{F}_q[[x,y]]$, $\mathbb{F}_q[[x,y]]/(xy)$.

An Artinian R-module is (in our case) an R-module of finite cardinality. For example, an Artinian $\mathbb{Z}$-module is a finite abelian group, an Artinian $\mathbb{Z}_p$-module is a finite abelian p-groups, and an Artinian $\mathbb{F}_q[t]$-module is what amounts to a rational canonical form of a matrix.

We care about the classification of Artinian modules.

1. An Artinian $\mathbb{Z}$-module corresponds to a partition $(\lambda_1 \geq \lambda_2 \geq \dots)$ via $M = \frac{\mathbb{Z}}{p^{\lambda_1}} \oplus \frac{\mathbb{Z}}{p^{\lambda_2}} \oplus \dots$. Similarly for $\mathbb{F}_q[t], \mathbb{Z}_p, \mathbb{F}_q[[t]]$ or any Dedekind domain.
2. An Artinian $\mathbb{F}_q[[x,y]]$-module is a pair of commuting matrices up to similarity. Classified by Moschetti–Ricolfi '18 up to 4×4.
3. An Artinian $\mathbb{F}_q[[x,y]]/(xy)$-module is a pair of mutually annihilating matrices up to similarity. Classified by Laubenbacher–Sturmfels '95.

A related problem in commutative algebra

Such a classification or "matrix canonical form" problem is hard in general. However, it is often possible to do a weighted count of the number of Artinian modules according to size, inversely weighted by the size of the automorphism group.

Definition

Let R be a ring as before. Define a Dirichlet series

$$\widehat{\zeta}_R(s) := \sum_M \frac{1}{|\mathrm{Aut}\, M|} |M|^{-s} = \sum_{N \geq 1} N^{-s} \sum_{|M|=N} \frac{1}{|\mathrm{Aut}\, M|}$$

where M ranges over Artinian modules of R.

We will see why we use this weight. We remark that studying $\zeta_R(s)$ is still nontrivial even when a classification of M is known.

A related problem in commutative algebra

Recall

$$\widehat{\zeta}_R(s) := \sum_M \frac{1}{|\operatorname{Aut} M|} |M|^{-s}$$

If R is an $\mathbb{F}_q$-algebra, then M can be viewed as an $\mathbb{F}_q$-vector space. Write $\deg M = \dim_{\mathbb{F}_q} M$ and define a power series

$$\widehat{Z}_R(x) := \sum_M \frac{1}{|\operatorname{Aut} M|} x^{\deg M}.$$

Then $\widehat{Z}_R(x) = \widehat{\zeta}_R(s)$ if $x = q^{-s}$.
Compare the Dedekind zeta function $\zeta_R(s) := \sum_{M=R/\mathfrak{m}} |M|^{-s}$, where $\mathfrak{m}$ ranges over maximal ideals of R.

Cohen–Lenstra

In 1984, Cohen and Lenstra computed $\widehat{\zeta}_R(s)$ for a Dedekind domain R in terms of its Dedekind zeta function. In particular,

$$\widehat{\zeta}_{\mathbb{Z}}(s) = \prod_{i=1}^{\infty} \zeta(s+i)$$

$$\widehat{Z}_{\mathbb{F}_q[t]}(x) = \big((1-x)(1-q^{-1}x)(1-q^{-2}x)\dots\big)^{-1}$$

They also gave a functional equation for such $\widehat{\zeta}_R(s)$.
They used the formula for $\widehat{\zeta}_{\mathbb{Z}}$ to prove that if an abelian group with size bounded by N is taken at random according to the Cohen–Lenstra distribution (i.e., with probability inversely proportional to the number of automorphisms), then as $N \to \infty$, its p-Sylow subgroup distributes like a random p-group following the Cohen–Lenstra distribution.

Connection with matrix enumeration

An $\mathbb{F}_q[u,v]$-module structure on a given underlying vector space ${\mathbb{F}_q}^n$ is determined by two commuting matrices given by how u, v act on ${\mathbb{F}_q}^n$.

$$\widehat{Z}_{\mathbb{F}_q[t]}(x) = \sum_{n=0}^{\infty} \frac{|\mathrm{Mat}_n(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n, \ \widehat{Z}_{\mathbb{F}_q[[t]]}(x) = \sum_{n=0}^{\infty} \frac{|\mathrm{Nilp}_n(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n$$

$$\widehat{Z}_{\mathbb{F}_q[u,v]}(x) = \sum_{n=0}^{\infty} \frac{|\{A, B \in \mathrm{Mat}_n(\mathbb{F}_q) : AB = BA\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n$$

$$\widehat{Z}_{\mathbb{F}_q[[u,v]]}(x) = \sum_{n=0}^{\infty} \frac{|\{A, B \in \mathrm{Nilp}_n(\mathbb{F}_q) : AB = BA\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n$$

$$\widehat{Z}_{\mathbb{F}_q[u,v]/(uv)}(x) = \sum_{n=0}^{\infty} \frac{|\{A, B \in \mathrm{Mat}_n(\mathbb{F}_q) : AB = BA = 0\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n$$

$$\widehat{Z}_{\mathbb{F}_q[[u,v]]/(uv)}(x) = \sum_{n=0}^{\infty} \frac{|\{A, B \in \mathrm{Nilp}_n(\mathbb{F}_q) : AB = BA = 0\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n$$

We need $|\mathrm{Aut}\, M|$ and $|\mathrm{GL}_n(\mathbb{F}_q)|$ in the denominator for it to work.

History revisited

The most important property of $\widehat{\zeta}_R(s)$ is the Euler product formula:

$$\widehat{\zeta}_R(s) = \prod_{\mathfrak{m}} \widehat{\zeta}_{R_\mathfrak{m}}(s) = \prod_{\mathfrak{m}} \widehat{\zeta}_{\widehat{R_\mathfrak{m}}}(s),$$

where $\mathfrak{m}$ ranges over maximal ideals of R, and $R_\mathfrak{m}$ is its localization, whose completion is $\widehat{R_\mathfrak{m}}$.

Cohen–Lenstra's original computation of $\widehat{\zeta}_\mathbb{Z}(s)$ essentially requires the classification of finite abelian groups and combinatorial information about the size of automorphism groups.

Alternatively, we can use the Euler product to reduce the problem to $\widehat{\zeta}_{\mathbb{Z}_p}(s)$. But the classification of Artinian $\mathbb{Z}_p$-modules has the same combinatorics as Artinian $\mathbb{F}_p[[t]]$-modules, so $\widehat{\zeta}_{\mathbb{Z}_p}(s) = \widehat{\zeta}_{\mathbb{F}_p[[t]]}(s)$, while the latter is counting nilpotent matrices, solved by Fine–Herstein.

History revisited

Proofs of Fine–Herstein and Feit–Fine (commuting matrices) have both been simplified later. Original proofs require some hard counting.

Nilpotent matrices can be immediately counted using

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Nilp}_n(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n \cdot \sum_{n=0}^{\infty} \frac{|\mathrm{GL}_n(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n = \sum_{n=0}^{\infty} \frac{|\mathrm{Mat}_n(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n,$$

a consequence of the Euler product formula of $\widehat{Z}_{\mathbb{F}_q[t]}(x)$ or a classically known "fitting lemma".

Feit–Fine can be reduced to the unweighted count of similarity classes of one nilpotent matrix, by an idea of Bryan–Morrison that is essentially "inverting the Euler product", or classically, by Möbius inversion. The combinatorial input is ultimately $\sum_{n=0}^{\infty} p(n)t^n = \big((1-t)(1-t^2)\dots\big)^{-1}$. With Bryan–Morrison's idea, one can even simplify Fine–Herstein further: the counting of nilpotent matrices can be "recovered" by $\sum_{n=0}^{\infty} \frac{|\mathrm{GL}_n(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n$ alone – only a trivial combinatorial input, if any!

What is known?

Let R be an $\mathbb{F}_q$-algebra. We summarize all cases where $\widehat{\zeta}_R(s)$ has been understood.

If $\operatorname{Spec} R$ is a smooth curve, then for a maximal ideal P (also a closed point), $\widehat{R_P}$ is a complete regular local ring of dimension 1, so it is isomorphic to $\kappa[[t]]$ by the Cohen structure theorem, where κ is the residue field at P. Fine–Herstein's result for $\mathbb{F}_q[[t]]$ and the Euler product give

$$\widehat{\zeta}_R(s) = \prod_{i=1}^{\infty} \zeta_R(s+i).$$

Similarly, if $\operatorname{Spec} R$ is a smooth surface, the known result for $\mathbb{F}_q[[u,v]]$ gives

$$\widehat{\zeta}_R(s) = \prod_{i,j \geq 1} \zeta_R(is+j),$$

where $\zeta_R(s)$ is the Hasse–Weil zeta function of $\operatorname{Spec} R$.

What is known?

In particular, we know the $1/|\mathrm{Aut}|$-weighted number of Artinian modules of $\mathbb{F}_q[u,v]$ of degree n without needing to know a classification of such modules and their automorphism groups.

We restate what is known purely in the language of matrix enumeration.

Let $f_1, \ldots, f_r \in \mathbb{F}_q[t_1, \ldots, t_m]$ be polynomials in m variables. Consider the problem of counting the numbers

$$N_n = |\{(A_1, \ldots, A_m) : A_i \in \mathrm{Mat}_n(\mathbb{F}_q), A_i A_j = A_j A_i, f_s(A_1, \ldots, A_m) = 0\}|$$

The numbers are understood if $R := \mathbb{F}_q[t_1, \ldots, t_m]/(f_1, \ldots, f_r)$ defines a smooth curve or a smooth surface. We have

$$\sum_{n=0}^{\infty} \frac{N_n}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n = \widehat{Z}_R(x).$$

Homework 1. Can you find applications of this? Note: you can replace some or all of $\mathrm{Mat}_n(\mathbb{F}_q)$ by Nilp or GL and the numbers are still known.

What is unknown?

By this point, we have realized that $\widehat{\zeta}_R(s)$ is determined by $\operatorname{Spec} R$. We can define $\widehat{\zeta}_X(s)$ for any $\mathbb{F}_q$-variety X (for example, using the Euler product), such that $\widehat{\zeta}_{\operatorname{Spec} R}(s) := \widehat{\zeta}_R(s)$.

We have understood $\widehat{\zeta}_X(s)$ when X is a smooth curve or a smooth surface. The next cases to consider are:

1. X is a smooth 3-fold. It is enough to study the case of affine 3-space, namely $\widehat{\zeta}_{\mathbb{A}^3}(s) = \widehat{\zeta}_{\mathbb{F}_q[u,v,w]}(s)$, or counting three commuting matrices. Using the idea of Bryan–Morrison, this is equivalent to counting similarity classes of pairs of commuting matrices. This problem was claimed solved but the paper turned out wrong; see `https://mathoverflow.net/questions/370269`.
2. X is a singular curve. By the Euler product, it suffices to study $\widehat{\zeta}_R(s)$ for a complete local ring R, i.e., one singularity at a time.

My main result is about the simplest singularity – the node $\{uv = 0\}$.

Analytic properties of $\widehat{Z}_X(x)$

We note that $\widehat{Z}_X(x)$ is meromorphic on $\mathbb{C}$ if X is a smooth curve (or a nodal curve, as per my result), and $\widehat{Z}_X(x)$ is meromorphic on the unit disk if X is a smooth surface, and the domain cannot be extended further. If $\dim X \geq 3$, we don't know much about $\widehat{Z}_X(x)$ except it has zero radius of convergence. Dimension is more important than singularity!

My main result about counting $A, B \in \mathrm{Mat}_n(\mathbb{F}_q)$ with $AB = BA = 0$ precisely says $\widehat{Z}_X(x) = \big((1-x)(1-q^{-1}x)(1-q^{-2}x)\dots\big)^{-2} H_q(x)$ for $X = \mathrm{Spec}\, \frac{k[u,v]}{(uv)} = \{u\text{- and } v\text{-axes}\}$. It can be elementarily proven that $H_q(x) > 0$ if $x \leq q$ or $x = q^n$, $n \in \mathbb{Z}$. Therefore $\widehat{Z}_X(x)$ has double poles at $1, q, q^2, \dots$.

I will later propose a heuristics to explain the poles in terms of the geometry of X.

Zeros of $H_q(x)$

This slide contains numerical observations that are not proven.
It appears that $H_q(x)$ has zeros, and there are infinitely many, distributing in an interesting pattern.

1. All zeros are strictly in the first and fourth quadrants.
2. The zeros in the first quadrant are $z_1, z_2, \ldots$, sorted by $|z_1| < |z_2| < \ldots$. The zeros in the fourth quadrants are $\bar{z}_1, \bar{z}_2, \ldots$.
3. The norm $|z_j|$ is "slightly" less than $q^{-1/2}q^j$.
4. The zeros $z_1, z_2, \ldots$ are almost on a straigt line, in that $z_{j+1} \approx qz_j$.

Here are the data for $q = 4$:

$$
\begin{aligned}
z_1 &= 0.41614 + 1.72467i, & |z_1| &= 1.77288 < 2 = q^{1/2} \\
z_2 &= 1.65483 + 7.60611i, & |z_2| &= 7.78405 < 8 = q^{3/2} \\
z_3 &= 6.62192 + 31.08907i, & |z_3| &= 31.7865 < 32 = q^{5/2} \\
z_4 &= 26.4883 + 125.0116i, & |z_4| &= 127.787 < 128 = q^{7/2}
\end{aligned}
$$

Zeros of $H_q(x)$

The reason to study the zeros of $H_q(x)$ is to look for a further factorization and a functional equation. If we exactly had $z_{j+1} = qz_j$, then it would suggest a functional equation about $x \mapsto qx$.

Such things are not likely to exist in an exact sense, but one can probably get some ideas about these by attempting to prove any feature of the zero distribution of $H_q(x)$. For example, the following problems are all open:

1. $H_q(x) > 0$ for $x \in \mathbb{R}$ and $H_q(x) \neq 0$ for $\mathrm{Im}(x) \leq 0$.
2. $H_q(x)$ has exactly $2j$ simple zeros in $|x| < q^{-1/2}q^j$.
3. Does $\lim_{j\to\infty} z_j/q^j$ exist?

$H_q(x)$ is given by the convergent sum:

$$H_q(x) = \sum_{k=0}^{\infty} \frac{q^{-k^2}x^{2k}}{(1-q^{-1})\dots(1-q^{-k})}(1-q^{-k-1}x)(1-q^{-k-2}x)\dots$$

Homework 2. Can you say anything about $H_q(x)$?

Poles of $\widehat{Z}_X(x)$

My main result can be rephrased this way. Let X be $\{uv = 0\}$ again, and consider its resolution of singularity $\widetilde{X}$, which is the disjoint union of two affine lines. Then the main result is equivalent to

$$\frac{\widehat{Z}_X(x)}{\widehat{Z}_{\widetilde{X}}(x)} = H_q(x).$$

This motivates the conjectural geometric explanation of the double poles of $\widehat{Z}_X(x)$: the poles are the same as those of $\widehat{Z}_{\widetilde{X}}(x)$.

Conjecture (H.) Let X be a singular curve over $\mathbb{F}_q$ with a resolution of singularity $\widetilde{X}$. Then $\widehat{Z}_X(x)$ has a meromorphic continuation to $\mathbb{C}$, and

$$H_X(x) := \frac{\widehat{Z}_X(x)}{\widehat{Z}_{\widetilde{X}}(x)}$$

is an entire function in $x \in \mathbb{C}$.

Heuristics: Hilbert schemes

I propose another heuristical reason behind the conjecture.

The $\widehat{Z}$ function shares some behaviors with another widely studied generating function, temporarily denoted by

$$Z_X^H(x) := \sum_{n \geq 0} |\mathrm{Hilb}^n(X)(\mathbb{F}_q)| x^n,$$

where $\mathrm{Hilb}^n(X)$ is the Hilbert scheme of n points on X. More concretely, for an $\mathbb{F}_q$-algebra R,

$$Z_R^H(x) = \sum_I x^{\dim_{\mathbb{F}_q}(R/I)},$$

where I ranges over all ideals of R whose quotient is finite.

In other words, $Z_R^H(x) = \sum_{R \twoheadrightarrow M} x^{\deg M}$, where M ranges over Artinian modules over R with a marked surjection $R \twoheadrightarrow M$.

Heuristics: Hilbert schemes

	$Z_X^H(x)$	$\widehat{Z}_X(x)$
Defn.	$Z_R^H(x) = \sum_{R \twoheadrightarrow M} x^{\deg M}$	$\widehat{Z}_R(x) = \sum_M \lvert \mathrm{Aut}\, M\rvert^{-1} x^{\deg M}$
$X = \mathbb{A}^1$	$Z_X^H(x/q) = \frac{1}{1-x}$	$\widehat{Z}_X(x) = \prod_{j \geq 0} \frac{1}{1-q^{-j}x}$
$X = \mathbb{A}^2$	$Z_X^H(x/q) = \prod_{i \geq 1} \frac{1}{1-qx^i}$ [1]	$\widehat{Z}_X(x) = \prod_{i \geq 1} \prod_{j \geq 0} \frac{1}{1-q^{-j}qx^i}$
X nodal curve [2]	$\frac{Z_X^H(x)}{Z_{\widetilde{X}}^H(x)} = 1 - x + qx^2$ [3]	$\frac{\widehat{Z}_X(x)}{\widehat{Z}_{\widetilde{X}}(x)} = H_q(x)$ (H.)
Gorenstein curve [4]	$\frac{Z_X^H(x)}{Z_{\widetilde{X}}^H(x)}$ is poly. [3]	$\frac{\widehat{Z}_X(x)}{\widehat{Z}_{\widetilde{X}}(x)}$ is holo. (Conj.)

[1]: Cheah '94

[2]: a curve with one nodal singularity

[3]: Göttsche–Shende '14

[4]: a condition on singularities. All planar singularities are Gorenstein.

Heuristics: Hilbert schemes

Let's also look at functional equations when X is a projective curve.

	$Z_X^H(x)$	$\widehat{Z}_X(x)$
$X = \mathbb{P}^1$	$Z_X^H(\frac{1}{qx}) = qx^2 Z_X^H(x)$	$\Lambda_X(x) := \widehat{Z}_X(x)\widehat{Z}_X(x^{-1})$ $\Lambda_X(qx) = (qx^2)^{-1}\Lambda_X(x)$ [5]
smooth projective curve	$Z_X^H(x) = Z_X(x)$ Func. equ. for $Z_X^H(\frac{1}{qx})$	Func. equ. for $\Lambda_X(qx)$ [5]
Gorenstein curve [4]	Func. equ. for $Z_X^H(\frac{1}{qx})$ [3]	??

This suggests that $\Lambda_q(x) := H_q(x)H_q(x^{-1})$ may have something for $x \mapsto qx$. Note that the zeros of $\Lambda_q(x)$ have norms approximately $\ldots, q^{-3/2}, q^{-1/2}, q^{1/2}, q^{3/2}, \ldots$, all apart by a multiple of q.

[3]: Göttsche–Shende '14
[4]: a condition on singularities. All planar singularities are Gorenstein.
[5]: the function-field version of the functional equation considered by Cohen–Lenstra '84

Heuristics: Hilbert schemes

There are algebro-geometric connections between $Z_X^H(x)$ and $\widehat{Z}_X(x)$. First, Hilbert schemes are moduli spaces of zero-dimensional closed subschemes, while the x^n coefficient of $\widehat{Z}_X(x)$ is the (stacky) point count of the moduli stack of length-n coherent sheaves on X. See [6][7] for a connection about ADHM data and [8] for a story about wall crossing.

However, we emphasize that the analogy in the tables above are purely heuristical. None of these connections are known to have any contribution to the proof. The main theorem is proved using pure combinatorics. It would be very instructive if one found an alternative proof.

[6]: Nakajima '99, Lectures on Hilbert schemes of points on surfaces
[7]: Henni–Jardim '18, Commuting matrices and the Hilbert schemes ...
[8]: Behrend–Bryan '12, Motivic degree 0 Donaldson–Thomas invariants

Summary

1. We have investigated one problem from three perspectives: matrix enumeration, Artinian modules over a ring, and the stack of finite-length coherent sheaves.
2. The nature and the answer of this problem are determined by the geometry. The problem is well-understood if the underlying variety is smooth of dimension ≤ 2, and the new result is about a curve with nodal singularities.
3. We have discussed a geometric heuristics that generate new surprising conjectures that can be tested elementarily.

Proof of the main result

Let $X = \operatorname{Spec} \mathbb{F}_q[u,v]/(uv)$. Recall the main theorem

$$\widehat{Z}_X(x) := \sum_{n=0}^{\infty} \frac{|\{A, B \in \mathrm{Mat}_n(\mathbb{F}_q) : AB = BA = 0\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n$$
$$= \big((1-x)(1-q^{-1}x)(1-q^{-2}x)\dots\big)^{-2} H_q(x) \quad (3)$$

for some entire function $H_q(x)$.

The following notation for a formal variable t is widely used in q-series:

$$(x;t)_n := (1-x)(1-tx)\dots(1-t^{n-1}x)$$
$$(x;t)_\infty := (1-x)(1-tx)(1-t^2x)\dots$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_t := \frac{(t;t)_n}{(t;t)_k (t;t)_{n-k}}$$

The last one is called the q-binomial coefficient. We have $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the number of k-dimensional subspaces of ${\mathbb{F}_q}^n$.

Proof of the main result

From now on, fix $t = q^{-1}$. It is not hard to find that

$$\widehat{Z}_X(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{{n \brack k}_t}{(t;t)_k} x^n.$$

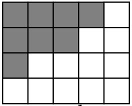
The key observation is that if $A \in \mathrm{Mat}_n(\mathbb{F}_q)$ has rank $n-k$, then there are q^{k^2} matrices B such that $AB = BA = 0$. This explains the summation in k.

To count the number of matrices A of any rank r, we first pick an r-dimensional subspace V (there are ${n \brack r}_q$ choices), and then pick a surjection ${\mathbb{F}_q}^n \twoheadrightarrow V$ (there are choices as many as there are full-rank $r \times n$ matrices; elementary to count).
The hard part, however, is to find a factorization, without which $\widehat{Z}_X(x)$ would not have been known to have a meromorphic continuation beyond $|x| < 1$!

Proof of the main result

In order to factorize $\widehat{Z}_X(x)$, we first express $\widehat{Z}_X(x)$ in terms of a summation of partitions, using formulas such as

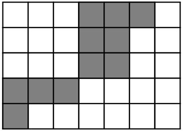
$$\begin{bmatrix} n \\ k \end{bmatrix}_t = \sum_{\lambda \subseteq k \times (n-k)} t^{|\lambda|} =: \sum$$



The shorthand notation means to sum over $t^{\text{number of shaded boxes}}$ for each (Young) diagram that fits.

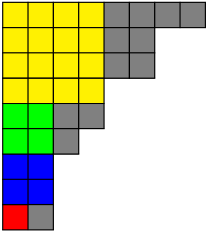
We end up with summing up diagrams with at most n rows with the largest square starting from the top-left (the Durfee square) removed.

$$\widehat{Z}_X(x) = \sum_{n \geq k \geq 0} x^n \sum$$



Proof of the main result

We can take Durfee squares repetitively. Here are the (first) Durfee square (in yellow), second Durfee square (in green), etc.



We classify our Young diagrams by the sizes of the first two Durfee squares, denoted k, l. A Young diagram can be reconstructed from the first two Durfee squares and the (three) subdiagrams $\lambda_1, \lambda_2, \lambda'$ separated by them. So we get a big sum involving $k, l, \lambda_1, \lambda_2, \lambda'$. Via a magical cancellation, we can decouple many of them, getting a factorization

$$\widehat{Z}_X(x) = (x;t)_\infty^{-2} \sum_{l=0}^{\infty} \frac{t^{l^2} x^{2l}}{(1-t)\dots(1-t^l)} (1-t^{l+1}x)(1-t^{l+2}x)\dots$$

Proof of the main result

More precise details need diagrams that are too painful to TeXify, so please refer to the handwritten notes on my website. The take-home message for the proof is that in the computation

$$\begin{aligned}\widehat{Z}_X(x) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\begin{bmatrix}n\\k\end{bmatrix}_t}{(t;t)_k}x^n \\ &= (x;t)_\infty^{-2}\sum_{l=0}^{\infty}\frac{t^{l^2}x^{2l}}{(1-t)\dots(1-t^l)}(1-t^{l+1}x)(1-t^{l+2}x)\dots\end{aligned}$$

where $t = q^{-1}$, the index k is really the sidelength of the first Durfee square, while l is the sidelength of the second Durfee square.

The last sum is our $H_q(x)$, which always converges due to the rapidly decaying factor t^{l^2}. (Note: this is a direct contribution of the l^2 boxes in the second Durfee square, which unlike the first, is not removed!)

Thank you!

For more details, please wait for my preprint. I will update its status on my website.

Main references:

[3] Göttsche–Shende '14, Refined curve counting on complex surfaces
[9] Bryan–Morrison '15, Motivic classes of commuting varieties via power structures