Zeta functions on orders

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joint with Ruofan Jiang

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- Has meromorphic continuation and functional equation $s \mapsto 1 - s$.

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- (Oblomkov–Rasmussen–Shende conjecture '18) For any curve with planar singularities, $\zeta_R(s)$ should encode knot invariants!!!

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- Thus: the functional equation implies combinatorial identities; we have found a nontrivial direct proof for the case $n = 2$ or n odd.

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- Example: $n = 3$, get $\sum_{n\geq 0} Q^{n^2}/((1-Q)\dots(1-Q^n))\, t^{2n}$, where $Q = q^{-1}$. At $t = \pm 1$, get modular form by Rogers–Ramanujan.

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Further-reaching questions

- How do these richer formulas, modular forms, etc. say about the knot theory associated to the singularities?
- Any hope of exact formulas in the number field case?

Thank you for listening!