Mutually annihilating matrices, and a Cohen–Lenstra series for the nodal singularity

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Abstract

We give a generating function for the number of pairs of $n \times n$ matrices (A, B) over a finite field that are mutually annihilating, namely, $AB = BA = 0$. This generating function can be viewed as a singular analogue of a series considered by Cohen and Lenstra. We show that this generating function has a factorization that allows it to be meromorphically extended to the entire complex plane. We also use it to count pairs of mutually annihilating nilpotent matrices. This work is essentially a study of the motivic aspects about the variety of modules over $\mathbb{C}[u, v]/(uv)$ as well as the moduli stack of coherent sheaves over an algebraic curve with nodal singularities.

1 Introduction

1.1 History and Motivation

Let R be a commutative ring with only finite quotient fields. We define the *Cohen–Lenstra zeta function* of R as

$$\widehat{\zeta}_R(s) := \sum_M \frac{1}{|\operatorname{Aut} M|} |M|^{-s}, \tag{1.1}$$

where M ranges over all isomorphism classes of finite(-cardinality) R-modules.

When R is a Dedekind domain, this function agrees with a function defined in the important work of Cohen and Lenstra [8] about the statistics of finite R-modules, motivated by the distribution of class groups of imaginary quadratic fields. They defined $\widehat{\zeta}_R(s)$ and proved a simple formula for it in [8, p. 39], which was crucial in their work.

If R contains a finite field $\mathbb{F}_q$ with q elements, we define the *Cohen–Lenstra series* of R over $\mathbb{F}_q$ as

$$\widehat{Z}_{R/\mathbb{F}_q}(x) := \sum_M \frac{1}{|\operatorname{Aut} M|} x^{\dim_{\mathbb{F}_q} M}, \tag{1.2}$$

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where M ranges over all isomorphism classes of finite R-modules. Clearly $\widehat{\zeta}_R(s) = \widehat{Z}_{R/\mathbb{F}_q}(q^{-s})$. When the ground field is clear from the context, we may simply denote $\widehat{Z}_{R/\mathbb{F}_q}(x)$ by $\widehat{Z}_R(x)$.

Existing work in various areas of mathematics can be put in the context of the Cohen–Lenstra series:

- When $R = \mathbb{F}_q$, the series $\widehat{Z}_{\mathbb{F}_q}(x)$ is the subject of Rogers–Ramanujan identities [1, p. 104], which state that $\widehat{Z}_{\mathbb{F}_q}(1)$ and $\widehat{Z}_{\mathbb{F}_q}(q^{-1})$ each equals to an infinite product:

$$\widehat{Z}_{\mathbb{F}_q}(1) = \frac{1}{(q^{-1}; q^{-5})_\infty (q^{-4}; q^{-5})_\infty}; \tag{1.3}$$

$$\widehat{Z}_{\mathbb{F}_q}(q^{-1}) = \frac{1}{(q^{-2}; q^{-5})_\infty (q^{-3}; q^{-5})_\infty}. \tag{1.4}$$

- When R is the power series ring $\mathbb{F}_q[[t]]$, giving a formula of $\widehat{Z}_R(x)$ is equivalent to finding the number of nilpotent matrices over $\mathbb{F}_q$, which was given by Fine and Herstein [11].
- When $R = \mathbb{F}_q[u, v]$, the series $\widehat{Z}_R(x)$ is the generating function evaluated by Feit and Fine [10] to enumerate pairs of commuting matrices.
- When $R = \mathbb{F}_q[[u, v]]$, the series $\widehat{Z}_R(x)$ is the generating function evaluted by Fulman and Guralnick [12] to enumerate pairs of commuting nilpotent matrices.

For any variety[1] X over $\mathbb{F}_q$, we define the *Cohen–Lenstra series* of X over $\mathbb{F}_q$ as

$$\widehat{Z}_{X/\mathbb{F}_q}(x) := \sum_M \frac{1}{|\mathrm{Aut}\, M|} x^{\dim_{\mathbb{F}_q} H^0(X;M)}, \tag{1.5}$$

where M ranges over all isomorphism classes of finite-length coherent sheaves over X, and $H^0(X; M)$ denotes the space of global sections of M. This generalizes the Cohen–Lenstra series over a ring, since $\widehat{Z}_R(x) = \widehat{Z}_{\mathrm{Spec}\, R}(x)$. If p is a closed point of X, we define the *local Cohen–Lenstra series* of X over p as

$$\widehat{Z}_{X,p}(x) := \widehat{Z}_{\mathcal{O}_{X,p}}(x) = \widehat{Z}_{\widehat{\mathcal{O}}_{X,p}}(x), \tag{1.6}$$

where $\mathcal{O}_{X,p}$ is the local ring of X at p and $\widehat{\mathcal{O}}_{X,p}$ is its completion.

It is implicit in the work of Bryan and Morrison [4] (and references therein) that $\widehat{Z}_X(x)$ and $\widehat{Z}_{X,p}(x)$ are known if X is a smooth curve or a smooth surface (Proposition 4.5). The Cohen–Lenstra series satisfies an important "Euler product" property (Proposition 4.2): for any variety X, we have

$$\widehat{Z}_X(x) = \prod_{p \in \mathrm{cl}(X)} \widehat{Z}_{X,p}(x), \tag{1.7}$$

where $\mathrm{cl}(X)$ denotes the set of closed points of X. In light of this property, the study of $\widehat{Z}_X(x)$ is equivalent to the study of the local factors $\widehat{Z}_{X,p}(x)$. When X is a reduced curve or surface, the only unknown factors are those at singular points. We can view the local Cohen–Lenstra series as an invariant attached to the classification of singularities up to analytic isomorphism, so it is natural to wonder what $\widehat{Z}_{X,p}(x)$ reveals about the geometry of X at p.

[1] separated scheme of finite type

1.2 Main results

The goal of this paper is to determine the properties of $\widehat{Z}_{X,p}(x)$ where X is a singular curve over $\mathbb{F}_q$ and p is a *nodal singularity*, namely, a singularity whose completed local ring is isomorphic to $\mathbb{F}_q[[u,v]]/(uv)$. This is the first result about the local Cohen–Lenstra series of a singularity. We use the q-Pochhammer symbol

$$(a;q)_n := (1-a)(1-qa)\dots(1-q^{n-1}a), \tag{1.8}$$

$$(a;q)_\infty := (1-a)(1-qa)(1-q^2a)\dots. \tag{1.9}$$

Theorem 1.1. *Fix a prime power $q>1$ and let $R_q = \mathbb{F}_q[[u,v]]/(uv)$. Then*

(a) *The power series $\widehat{Z}_{R_q}(x)$ in x has a meromorphic continuation to all of $\mathbb{C}$.*

(b) *The poles of the meromorphic continuation of $\widehat{Z}_{R_q}(x)$ are precisely double poles at $x=q^i$, $i=1,2,\dots$. Moreover, the power series $\widehat{Z}_{R_q}(x)$ admits a factorization*

$$\widehat{Z}_{R_q}(x) = \frac{1}{(xq^{-1};q^{-1})_\infty^2} H_q(x) \tag{1.10}$$

where

$$H_q(x) := \sum_{k=0}^{\infty} \frac{q^{-k^2}x^{2k}}{(q^{-1};q^{-1})_k}(xq^{-k-1};q^{-1})_\infty. \tag{1.11}$$

We point out that $H_q(x)$ is an entire power series in x whenever $|q|>1$.

(c) $\widehat{Z}_{R_q}(1) = \dfrac{1}{(q^{-1};q^{-1})_\infty^2}$, *and* $\widehat{Z}_{R_q}(-1) = \dfrac{1}{(-q^{-2};q^{-2})_\infty}$.

Since $R_q = \mathbb{F}_q[[u,v]]/(uv)$ is the completed local ring of a curve at a nodal singularity, the series $\widehat{Z}_{R_q}(x)$ in Theorem 1.1 is precisely the local Cohen–Lenstra series of a nodal singularity. Theorem 1.1(b) implies that the series $\widehat{Z}_{R_q}(x)$ has a radius of convergence equal to q; the content of Theorem 1.1(a)(b) lies in the description of $\widehat{Z}_{R_q}(x)$ outside the domain of convergence. The specialization of $\widehat{Z}_{R_q}(x)$ to $x=\pm 1$ is especially important because of the statistical interpretations below. The value of $\widehat{Z}_{R_q}(1)$ is the weighted count of finite-cardinality R-modules up to isomorphism, each weighted inversely by the size of the automorphism group. The numbers $(\widehat{Z}_{R_q}(1) \pm \widehat{Z}_{R_q}(-1))/2$ give the weighted counts of even- and odd-dimensional R_q-modules up to isomorphism, respectively. Theorem 1.1(c) thus gives a clean formula for all of the aforementioned weighted counts, and the cleanness is surprising given that the classification of R_q-modules (see [3] and its references) is much more complicated than the classically well-known classification of $\mathbb{F}_q[[t]]$-modules. For comparison, let $S_q = \mathbb{F}_q[[t]]$, the completed local ring of a curve at a smooth $\mathbb{F}_q$-point. We have

$$\widehat{Z}_{R_q}(x) = \frac{1}{(xq^{-1};q^{-1})_\infty^2} H_q(x); \tag{1.12}$$

$$\widehat{Z}_{S_q}(x) = \frac{1}{(xq^{-1};q^{-1})_\infty}. \tag{1.13}$$

The series $\widehat{Z}_{R_q}(x)$ has a complicated factor $H_q(x)$ not present in $\widehat{Z}_{S_q}(x)$; we may interpret $H_q(x)$ as a factor accounting for the nodal singularity. However, specializing to $x = \pm 1$, the values of $\widehat{Z}_{R_q}(x)$ are no longer "much more complicated" than $\widehat{Z}_{S_q}(x)$, as

$$\widehat{Z}_{R_q}(1) = \frac{1}{(q^{-1}; q^{-1})_\infty^2}; \tag{1.14}$$

$$\widehat{Z}_{R_q}(-1) = \frac{1}{(-q^{-2}; q^{-2})_\infty}; \tag{1.15}$$

$$\widehat{Z}_{S_q}(\pm 1) = \frac{1}{(\pm q^{-1}; q^{-1})_\infty}. \tag{1.16}$$

Theorem 1.1 also implies that if X is a reduced curve with only nodal singularities, and $\widetilde{X}$ is its resolution of singularity, then $\widehat{Z}_X(x)/\widehat{Z}_{\widetilde{X}}(x)$ is entire. This can be interpreted as that the Cohen–Lenstra series of a nodal singular curve, while being mysterious, is "not too far" from its smooth version.

Our proof of Theorem 1.1 depends on the following combinatorial identity.

Theorem 1.2. *As power series in x, we have*

$$\sum_{n=0}^{\infty} \frac{|\{(A,B) \in \mathrm{Mat}_n(\mathbb{F}_q) \times \mathrm{Mat}_n(\mathbb{F}_q) : AB = BA = 0\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n = \frac{1}{(x; q^{-1})_\infty^2} H_q(x), \tag{1.17}$$

where $H_q(x)$ is defined in (1.11).

The content of Theorem 1.2 lies not only in the enumeration of pairs of mutually annihilating matrices, but also in the unusual factorization identity (1.17). We point out that the left-hand side of (1.17) is precisely $\widehat{Z}_{\mathbb{F}_q[u,v]/(uv)}(x)$ (note the single bracket). We also point out that obtaining the specific expression of $H_q(x)$ in (1.11) is the key to prove Theorem 1.1. It is currently unknown if any part of Theorem 1.1 has a geometric proof.

We point out that (1.10) gives a formula that counts pairs of mutually annihilating *nilpotent* matrices. In specific,

$$\sum_{n=0}^{\infty} \frac{|\{(A,B) \in \mathrm{Nilp}_n(\mathbb{F}_q) \times \mathrm{Nilp}_n(\mathbb{F}_q) : AB = BA = 0\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n = \frac{1}{(xq^{-1}; q^{-1})_\infty^2} H_q(x), \tag{1.18}$$

where $\mathrm{Nilp}_n(\mathbb{F}_q)$ denotes the set of n by n nilpotent matrices over $\mathbb{F}_q$, and $H_q(x)$ is defined in (1.11). Pairs of mutually annihilating nilpotent matrices are much harder to count than pairs of mutually annihilating matrices, so deriving (1.10) from Theorem 1.2 can be viewed as an application of the Cohen–Lenstra series and its Euler product property.

As an attempt to extend Theorem 1.1 to other curve singularities, we formulate the following questions:

Question 1.3. Let p be an $\mathbb{F}_q$-point of a reduced curve X.

(a) Is it always true that $\widehat{Z}_{X,p}(x)$ has a meromorphic continuation to all of $\mathbb{C}$?

(b) If the answer to (a) is yes, is it true that the poles of $\widehat{Z}_{X,p}(x)$ are given by the factor $(xq^{-1};q^{-1})_\infty^{-r(p)}$, where $r(p)$ is some numeric invariant attached to the pair (X,p)?

(c) Do the special values $\widehat{Z}_{X,p}(\pm 1)$ read the geometry of X at p in a meaningful way?

A possibility that is compatible to all the known cases (i.e., smooth point and nodal singularity) is that $r(p)$ is the branching number of X at p. This specific choice of $r(p)$ would imply an elegant global statement that whenever $\widetilde{X}$ is a resolution of singularity of a reduced curve X, the quotient power series $\widehat{Z}_X(x)/\widehat{Z}_{\widetilde{X}}(x)$ is entire. An analogous quotient about the Hilbert schemes of points on a curve with planar singularities was considered in [16, p. 2259], where it was shown to be a polynomial as opposed to a rational function with a nontrivial denominator.

1.3 Related work

The coefficients of the Cohen–Lenstra series encode the point count of a wide family of varieties that arise as variants of the commuting variety. The commuting variety is the variety of pairs of commuting matrices, whose geometry was studied by Motzkin and Taussky [22] and Gerstenhaber [13]. Generalizations and variants of the commuting variety have been introduced and their geometry has been studied in both characteristic zero and positive characteristic; see [2, 6, 7, 9, 25].

One of the variants of the commuting variety is the variety of modules [9]. Our Theorem 1.1 and 1.2, in particular, give the point count of the varieties of modules over $\mathbb{F}_q[[u,v]]/(uv)$ and $\mathbb{F}_q[u,v]/(uv)$, respectively. Their point count can be viewed as a statistical information (in the sense of Cohen–Lenstra [8]) about the classification of finite-dimensional modules over $\mathbb{F}_q[u,v]/(uv)$ up to isomorphism. See [3, 24, 27] for studies of the structure of these varieties and the aforementioned classification problem. See also [21] for similar work about $\mathbb{F}_q[u,v]$.

For an $\mathbb{F}_q$-variety X, the coefficients of $\widehat{Z}_X(x)$ are precisely the point count of the motivic class of the moduli stack of coherent sheaves over X. Bryan and Morrison [4] reproved and refined the result of Feit and Fine about $\widehat{Z}_{\mathbb{F}_q[u,v]}$ from this motivic point of view. Much more is known motivically about a related moduli space, namely, the Hilbert scheme of points on X, not only when X is a smooth surface [5, 14, 15], but also when X is a curve with planar singularities [16, 20], and even for some singular surfaces [18]. One of the motivations of our work is to find analogy between known motivic statements about the Hilbert scheme and about the stack of coherent sheaves. Despite some known geometric connections between the Hilbert scheme and the stack of coherent sheaves [19, 23], it is unknown if they are motivically related in some way.

1.4 Organization of the paper

In Section 2, we give some preliminaries about partitions and q-series that will be used in the proof of Theorem 1.2, given in Section 3. In Section 4, we give a self-contained introduction to known properties of the Cohen–Lenstra series, part of which will be used to prove Theorem 1.1. In Section 5, we make some elementary observations about the series $H_q(x)$ appearing in the main theorems, and use them to finish the proof of Theorem 1.1. In Section 6, we discuss a possible connection between $H_q(x)$ and the partial theta function.

2 Preliminaries

A *partition* λ is a finite nonincreasing sequence of positive integers $(\lambda_1, \dots, \lambda_\ell)$, each of which is called a *part* of λ. The *length* of λ is the number of parts in λ, denoted $\ell(\lambda)$. The *size* of λ is $|\lambda| := \sum_i \lambda_i$. We denote by $a_i(\lambda)$ the number of parts of λ of size i, so we can write down a partition as

$$\lambda = a_1(\lambda) \cdot [1] + a_2(\lambda) \cdot [2] + \dots \tag{2.1}$$

The *Young diagram* of λ follows the convention such that it has λ_1 boxes in the top row, and $\ell(\lambda)$ boxes in the leftmost column. We will often refer to a partition by its Young diagram. The *conjugate* (or *transpose*) partition of λ is the partition, denoted λ', whose Young diagram is the transpose of the Young diagram of λ.

The (first) *Durfee square* of λ is the largest square that fits the top-left corner of its Young diagram. For $i \geq 1$, the $(i+1)$-st Durfee square is the Durfee square of the part of λ below the i-th Durfee square. We denote the sidelength of the i-th Durfee square by $\sigma_i(\lambda)$, and define the *Durfee partition* as

$$\operatorname{Durf}(\lambda) := (\sigma_1(\lambda), \sigma_2(\lambda), \dots). \tag{2.2}$$

Recall the *q-Pochhammer symbol*

$$(a;q)_n = (1-a)(1-qa)\dots(1-q^{n-1}a); \tag{2.3}$$

$$(a;q)_\infty = (1-a)(1-qa)(1-q^2a)\dots \tag{2.4}$$

The *q-binomial coefficient* is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}. \tag{2.5}$$

3 Proof of Theorem 1.2

We define $\widehat{Z}(x) := \sum_{n=0}^\infty \frac{|\{A, B \in \operatorname{Mat}_n(\mathbb{F}_q) : AB = BA = 0\}|}{|\operatorname{GL}_n(\mathbb{F}_q)|} x^n$ and compute it in two steps.

3.1 Counting pairs of mutually annihilating matrices

Fix n. We count the number of pairs (A, B) of $n \times n$ matrices such that $AB = BA = 0$.

First, we fix A and let $0 \leq k \leq n$ be the nullity of A (so the rank of A is $n-k$). Let $\operatorname{im} A = V$, $\ker A = W$, then $\dim V = n-k$, $\dim W = k$. We have

$$AB = 0 \iff A(\operatorname{im} B) = 0 \iff \operatorname{im} B \subseteq W; \tag{3.1}$$

$$BA = 0 \iff B(\operatorname{im} A) = 0 \iff \ker B \supseteq V. \tag{3.2}$$

Hence, choosing B that mutually annihilates A is equivalent to picking a linear map from ${\mathbb{F}_q}^n/V \to W$. Since $\dim {\mathbb{F}_q}^n/V = \dim W = k$, there are q^{k^2} choices of B. Notice that this number depends only on the rank of A.

It is a standard fact that the number of $n \times n$ $\mathbb{F}_q$-matrices of nullity k is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q (q^n - 1)(q^n - q)\dots(q^n - q^{n-k-1}). \tag{3.3}$$

(As one of the proofs, first choose an $(n-k)$-dimensional subspace $V \subseteq \mathbb{F}_q{}^n$ as the image, and then choose a surjection $\mathbb{F}_q{}^n \to V$. The former has $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ choices, and the latter $(q^n-1)(q^n - q)\dots(q^n - q^{n-k-1})$ choices.)

Recalling that $|\mathrm{GL}_n(\mathbb{F}_q)| = (q^n-1)(q^n-q)\dots(q^n-q^{n-1})$, it follows (after simplication) that

$$\widehat{Z}(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{|\mathrm{GL}_n(\mathbb{F}_q)|} q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_q (q^n-1)(q^n-q)\dots(q^n-q^{n-k-1})x^n \tag{3.4}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_t}{(t;t)_k} x^n, \tag{3.5}$$

where $t = q^{-1}$.

3.2 Factorization of $\widehat{Z}(x)$

We collect some standard q-series identities that will be used in the factorization.

Proposition 3.1 ([1])**.** *As formal power series in t (and x if applicable), we have*

(a) $\sum_{\ell(\lambda)\le k} t^{|\lambda|} = \dfrac{1}{(t;t)_k}$. *Here the sum is over all partitions with at most k parts.*

(b) $\sum_{\lambda\subseteq(n-k)\times k} t^{|\lambda|} = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_t$. *The notation here means that the sum is over all partitions whose Young diagram fits inside a $(n-k)\times k$ rectangle.*

(c) $\sum_{n=0}^{\infty} \dfrac{x^n}{(t;t)_n} = \dfrac{1}{(x;t)_\infty}$.

(d) *A* partition with zeros *is a nonincreasing sequence of finitely many nonnegative integers $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$, whose length is defined as ℓ. Then we have the identity*

$$\sum_{\substack{\lambda \text{ w/ } [0] \\ \lambda_1 \le k}} t^{|\lambda|} x^{\ell(\lambda)} = \frac{1}{(1-x)(1-tx)\dots(1-t^k x)}, \tag{3.6}$$

where the sum is over all partitions with zeros whose parts have sizes at most k.

Lemma 3.2. *We have*

$$\widehat{Z}(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_t}{(t;t)_k} x^n = \sum_{\lambda \text{ w/ } [0]} t^{|\lambda| - \sigma_1(\lambda)^2} x^{\ell(\lambda)} \tag{3.7}$$

Proof. A partition λ whose Durfee square has sidelength k can be reconstructed uniquely with a partition $\lambda^{(1)}$ such that $\ell(\lambda^{(1)}) \leq k$ (to be put to the right of the Durfee square) and a partition λ' such that $\lambda'_1 \leq k$ (to be put below the Durfee square).

By Proposition 3.1(1)(2), we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\begin{bmatrix}n\\k\end{bmatrix}_t}{(t;t)_k}x^n = \sum_{n=0}^{\infty}x^n\sum_{k=0}^{n}\left(\sum_{\lambda^{(1)}\subseteq k\times\infty}t^{|\lambda^{(1)}|}\sum_{\lambda'_1\subseteq(n-k)\times k}t^{|\lambda'|}\right) \tag{3.8}$$

$$= \sum_{n=0}^{\infty}x^n\sum_{\lambda\subseteq n\times\infty}t^{|\lambda|-\sigma_1(\lambda)^2} \tag{3.9}$$

$$= \sum_{n=0}^{\infty}x^n\sum_{\substack{\lambda \text{ w/ } [0]\\ \ell(\lambda)=n}}t^{|\lambda|-\sigma_1(\lambda)^2} \tag{3.10}$$

$$= \sum_{\lambda \text{ w/ } [0]}t^{|\lambda|-\sigma_1(\lambda)^2}x^{\ell(\lambda)}. \tag{3.11}$$

Here, the line (3.9) is because $\lambda^{(1)}$ and λ' consist of the part of λ outside the Durfee square. Note that $\lambda\subseteq n\times\infty$ means that λ fits inside the n (rows) $\times$ ∞ (columns) rectangle, which is equivalent to saying $\ell(\lambda)\leq n$. The line (3.10) is because specifying a partition whose length is at most n is equivalent to specifying a partition with zeros whose length is exactly n. $\square$

To complete the factorization, we reconstruct λ with zeros using the first two Durfee squares. Pick $k\geq l\geq 0$. Let $\lambda^{(1)}\subseteq k\times\infty$ and $\lambda^{(2)}\subseteq l\times(k-l)$ be usual partitions, and $\lambda''\subseteq\infty\times l$ be a partition with zeros. Then we have a bijection

$$\{(\lambda'' \text{ w/ } [0], \lambda^{(1)}, \lambda^{(2)}) \text{ as above}\} \to \{\lambda \text{ w/ } [0] : \sigma_1(\lambda)=k, \sigma_2(\lambda)=l\} \tag{3.12}$$

by putting $\lambda^{(1)}$ to the right of the first Durfee square, $\lambda^{(2)}$ to the right of the second Durfee square, and λ'' below the second Durfee square.

We have

$$\widehat{Z}(x) = \sum_{\lambda \text{ w/ } [0]}t^{|\lambda|-\sigma_1(\lambda)^2}x^{\ell(\lambda)} \tag{3.13}$$

$$= \sum_{\substack{k\geq l\geq 0\\ \lambda'',\lambda^{(1)},\lambda^{(2)}}}t^{|\lambda^{(1)}|^2+|\lambda^{(2)}|^2+|\lambda''|^2+l^2}x^{k+l+\ell(\lambda'')} \tag{3.14}$$

$$= \sum_{k\geq l\geq 0}t^{l^2}x^{l+k}\left(\sum_{\lambda^{(1)}\subseteq k\times\infty}t^{|\lambda^{(1)}|}\sum_{\lambda^{(2)}\subseteq k\times\infty}t^{|\lambda^{(2)}|}\sum_{\substack{\lambda'' \text{ w/ } [0]\\ \lambda''\subseteq\infty\times l}}t^{|\lambda''|}x^{\ell(\lambda'')}\right) \tag{3.15}$$

$$= \sum_{k\geq l\geq 0}t^{l^2}x^{l+k}\frac{1}{(t;t)_k}\begin{bmatrix}k\\l\end{bmatrix}_t\frac{1}{(1-x)(1-tx)\dots(1-t^lx)}, \tag{3.16}$$

where the line (3.16) uses Proposition 3.1(1)(2)(4), in that order.

Observe that

$$\frac{1}{(t;t)_k}\begin{bmatrix}k\\ l\end{bmatrix}_t = \frac{1}{(t;t)_k}\frac{(t;t)_k}{(t;t)_l(t;t)_{k-l}} = \frac{1}{(t;t)_l(t;t)_{k-l}}. \tag{3.17}$$

Letting $b=k-l$, we have

$$\widehat{Z}(x) = \sum_{k\geq l\geq 0} t^{l^2}x^{l+k}\frac{1}{(t;t)_k}\begin{bmatrix}k\\ l\end{bmatrix}_t \frac{1}{(1-x)(1-tx)\dots(1-t^lx)} \tag{3.18}$$

$$= \sum_{b,l\geq 0} t^{l^2}x^{l+(b+l)}\frac{1}{(t;t)_l(t;t)_b}\frac{1}{(1-x)(1-tx)\dots(1-t^lx)} \tag{3.19}$$

$$= \left(\sum_{b=0}^{\infty}\frac{x^b}{(t;t)_b}\right)\left(\sum_{l=0}^{\infty} t^{l^2}x^{2l}\frac{1}{(t;t)_l}\frac{1}{(1-x)(1-tx)\dots(1-t^lx)}\right) \tag{3.20}$$

$$= \frac{1}{(x;t)_\infty}\sum_{l=0}^{\infty} t^{l^2}x^{2l}\frac{1}{(t;t)_l}\frac{1}{(1-x)(1-tx)\dots(1-t^lx)}, \tag{3.21}$$

where the line (3.21) follows from Proposition 3.1(3).

We remark that the key reason why this factorization works is that (3.17) does not depend explicitly on k after simplification.

Finally, recalling that $(x;t)_\infty = (1-x)(1-tx)(1-t^2x)\dots$, we get

$$\widehat{Z}(x) = \frac{1}{(x;t)_\infty}\sum_{l=0}^{\infty} t^{l^2}x^{2l}\frac{1}{(t;t)_l}\frac{1}{(1-x)(1-tx)\dots(1-t^lx)} \tag{3.22}$$

$$= \frac{1}{(x;t)_\infty^2}\sum_{l=0}^{\infty} t^{l^2}x^{2l}\frac{1}{(t;t)_l}(1-t^{l+1}x)(1-t^{l+2}x)\dots \tag{3.23}$$

$$=: \frac{1}{(x;t)_\infty^2}H_q(x), \tag{3.24}$$

which finishes the proof of Theorem 1.2.

4 General properties of the Cohen–Lenstra series

We give a self-contained introduction about the well-known properties of the Cohen–Lenstra series. These properties are implicit in [4, 17] from the motivic point of view, while in this introduction, we restrict our attention to counting over finite fields. We point out that the argument in Proposition 4.5 that uses Proposition 4.2 is essentially the use of "power structures" in [4, 17].

Let R be an algebra over $\mathbb{F}_q$ with only finite quotient fields, and let X be a variety over $\mathbb{F}_q$. Recall the definitions

$$\widehat{Z}_R(x) := \sum_M \frac{1}{|\mathrm{Aut}\, M|}x^{\dim_{\mathbb{F}_q} M}, \tag{4.1}$$

where M ranges over all isomorphism classes of finite-cardinality R-modules, and

$$\widehat{Z}_X(x) := \sum_M \frac{1}{|\mathrm{Aut}\, M|} x^{\dim_{\mathbb{F}_q} H^0(X;M)}, \tag{4.2}$$

where M ranges over all isomorphism classes of finite-length coherent sheaves over X, and $H^0(X;M)$ denotes the space of global sections of M. We denote both $\dim_{\mathbb{F}_q} M$ and $\dim_{\mathbb{F}_q} H^0(X;M)$ by $\deg M$. We also recall the local Cohen–Lenstra series for a closed point p of X:

$$\widehat{Z}_{X,p}(x) := \widehat{Z}_{\mathcal{O}_{X,p}}(x). \tag{4.3}$$

We state some basic properties.

Proposition 4.1.

(a) $\widehat{Z}_R(x) = \widehat{Z}_{\mathrm{Spec}\, R}(x)$.

(b) $\widehat{Z}_{\mathcal{O}_{X,p}}(x) = \widehat{Z}_{\widehat{\mathcal{O}}_{X,p}}(x)$.

(c) *We have*

$$\widehat{Z}_{X,p}(x) := \sum_{M_p} \frac{1}{|\mathrm{Aut}\, M_p|} x^{\deg M_p}, \tag{4.4}$$

where M_p ranges over all isomorphism classes of finite-length coherent sheaves over X that are supported at p.

Proof.

(a) This follows from the standard correspondence between modules over R and quasicoherent sheaves over $\mathrm{Spec}\, R$.

(b) This follows from the elementary fact that the classification of finite-length modules over a Noetherian local ring is the same as the classification of finite-length modules over its completion.

(c) A coherent sheaf is supported at p is determined by it stalk at p, thus corresponds to a module over $\mathcal{O}_{X,p}$.

□

Proposition 4.2 (Euler product)**.** *Let X be a variety over $\mathbb{F}_q$. Then*

$$\widehat{Z}_X(x) = \prod_{p\in \mathrm{cl}(X)} \widehat{Z}_{X,p}(x), \tag{4.5}$$

where $\mathrm{cl}(X)$ *is the set of closed points in X.*

Proof. For every finite-length coherent sheaf M over X, we have a unique decomposition $M = \bigoplus_{p\in \mathrm{cl}(X)} M_p$ into finite-length coherent sheaves M_p supported at p, with all but finitely many M_p's being zero. For closed points $p \neq q$ and sheaves M_p, M_q supported on p, q respectively, we have

$$\mathrm{Hom}_X(M_p, M_q) = 0. \tag{4.6}$$

It follows that

$$\mathrm{Aut}(M_{p_1} \oplus \cdots \oplus M_{p_r}) \cong \mathrm{Aut}(M_{p_1}) \times \cdots \times \mathrm{Aut}(M_{p_r}). \tag{4.7}$$

As a consequence,

$$\widehat{Z}_X(x) = \sum_M \frac{1}{|\mathrm{Aut}\, M|} x^{\deg M} \tag{4.8}$$

$$= \sum_{(M_p\colon p\in \mathrm{cl}(X))} \frac{1}{|\prod_p \mathrm{Aut}\, M_p|} x^{\sum_p \deg M_p} \tag{4.9}$$

$$= \prod_{p\in \mathrm{cl}(X)} \sum_{M_p} \frac{1}{|\mathrm{Aut}\, M_p|} x^{\deg M_p} \tag{4.10}$$

$$= \prod_{p\in \mathrm{cl}(X)} \widehat{Z}_{X,p}(x). \tag{4.11}$$

□

For any subvariety Y of X, if we set

$$\widehat{Z}_{X,Y}(x) := \prod_{p\in \mathrm{cl}(Y)} \widehat{Z}_{X,p}(x) = \sum_{\mathrm{supp} M\subseteq Y} \frac{1}{|\mathrm{Aut}\, M|} x^{\deg M}, \tag{4.12}$$

then the Euler product gives

$$\widehat{Z}_X(x) = \widehat{Z}_U(x) \cdot \widehat{Z}_{X,Z}(x) \tag{4.13}$$

for any open subvariety $U \subseteq X$ and closed subvariety $Z \subseteq X$ with $X \setminus Z = U$. This uses the fact that $\mathcal{O}_{U,p} = \mathcal{O}_{X,p}$ for all $p \in \mathrm{cl}(U)$, so that

$$\widehat{Z}_U(x) = \widehat{Z}_{X,U}(x) \text{ for open } U \subseteq X. \tag{4.14}$$

As a warning, $\widehat{Z}_{X,Z}(x)$ is not equal to $\widehat{Z}_Z(x)$, because $\mathcal{O}_{Z,p}$ and $\mathcal{O}_{X,p}$ are not isomorphic in general. Thus, the $\widehat{Z}$ construction is not motivic in the sense that $\widehat{Z}_X(x) \neq \widehat{Z}_U(x) \cdot \widehat{Z}_Z(x)$.

The following relates the Cohen–Lenstra zeta function to the variety of modules, or its nilpotent variant.

Proposition 4.3. *Let* $R = \dfrac{\mathbb{F}_q[t_1, \ldots, t_m]}{(f_1, ..., f_r)}$, *where* $t_1, \ldots, t_m$ *are indeterminates and* $f_1, \ldots, f_r$ *are polynomials in* $t_1, \ldots, t_m$. *Then*

(a) *We have*

$$\widehat{Z}_R(x) = \sum_{n=0}^{\infty} \frac{|M_n|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n, \tag{4.15}$$

where

$$M_n := \left\{ (A_1, \dots, A_m) \;\middle|\; \begin{array}{c} A_i \in \mathrm{Mat}_n(\mathbb{F}_q), A_iA_j = A_jA_i \\ f_s(A_1, \dots, A_m) = 0 \text{ for } 1 \le s \le r \end{array} \right\} \tag{4.16}$$

(b) *If $f_1, \dots, f_r$ all vanish at the origin 0, let $p = (t_1, \dots, t_m)R$ be the maximal ideal corresponding to $0 \in X := \operatorname{Spec} R$, then*

$$\widehat{Z}_{R_p}(x) = \widehat{Z}_{X,0}(x) = \sum_{n=0}^{\infty} \frac{|N_n|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n, \tag{4.17}$$

where

$$N_n := \left\{ (A_1, \dots, A_m) \;\middle|\; \begin{array}{c} A_i \in \mathrm{Nilp}_n(\mathbb{F}_q), A_iA_j = A_jA_i \\ f_s(A_1, \dots, A_m) = 0 \text{ for } 1 \le s \le r \end{array} \right\} \tag{4.18}$$

(c) *More generally, let $I \subseteq J \subseteq \mathbb{F}_q[t_1, \dots, t_m]$ be two ideals, and $Z = \operatorname{Spec} \mathbb{F}_q[t_1, \dots, t_m]/J \subseteq X = \operatorname{Spec} \mathbb{F}_q[t_1, \dots, t_m]/I$. Then*

$$\widehat{Z}_{X,Z}(x) = \sum_{n=0}^{\infty} \frac{|N_n|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n, \tag{4.19}$$

where

$$N_n := \left\{ A = (A_1, \dots, A_m) \;\middle|\; \begin{array}{c} A_i \in \mathrm{Mat}_n(\mathbb{F}_q), A_iA_j = A_jA_i \\ f(A) = 0 \text{ for } f \in I, g(A) \in \mathrm{Nilp}_n(\mathbb{F}_q) \text{ for } g \in J \end{array} \right\} \tag{4.20}$$

Proof.

(a) Fix n and consider an $\mathbb{F}_q$-vector space V of dimension n. Giving V a structure of an R-module is equivalent to specifying the actions of $t_1, \dots, t_m$ on V as linear endomorphisms $A_1, \dots, A_m$, under the constraints

$$A_iA_j = A_jA_i \tag{4.21}$$

and

$$f_s(A_1, \dots, A_m) = 0 \text{ for } 1 \le s \le r. \tag{4.22}$$

We denote by $(V; A_1, \dots, A_m)$ the R-module specified by the data above. Note that the constraints are satisfied if and only if $(A_1, \dots, A_m) \in M_n$.

Consider the action of $GL_n(\mathbb{F}_q)$ on the set M_n by simultaneous conjugation:

$$g \cdot (A_1, \dots, A_m) := (gA_1g^{-1}, \dots, gA_mg^{-1}). \tag{4.23}$$

Given two R-modules $M = (V; A_1, \dots, A_m)$ and $M' = (V'; A'_1, \dots, A'_m)$, an R-linear map $M \to M'$ is nothing but an $\mathbb{F}_q$-linear map $B : V \to V'$ such that $B \circ A_i = A'_i \circ B$

for all i. It follows that $(\mathbb{F}_q{}^n; A_1, \ldots, A_m)$ and $(\mathbb{F}_q{}^n; A'_1, \ldots, A'_m)$ are isomorphic as R-modules precisely if $(A_1, \ldots, A_m)$ and $(A'_1, \ldots, A'_m)$ are in the same orbit. Moreover, $g : \mathbb{F}_q{}^n \to \mathbb{F}_q{}^n$ gives an automorphism of $(\mathbb{F}_q{}^n; A_1, \ldots, A_m)$ as an R-module if and only if g fixes $(A_1, \ldots, A_m)$. Denoting by O_M the orbit corresponding to an R-module M with $\dim_{\mathbb{F}_q} M = n$, the orbit-stabilizer theorem gives

$$\sum_{\deg M = n} \frac{1}{|\mathrm{Aut}\, M|} = \sum_{M} \frac{|O_M|}{|\mathrm{GL}_n(\mathbb{F}_q)|} = \frac{|M_n|}{|\mathrm{GL}_n(\mathbb{F}_q)|}, \tag{4.24}$$

where the first and second sums are over the isomorphism classes of R-modules of degree M.

It follows that

$$\widehat{Z}_R(x) = \sum_{n=0}^{\infty} \sum_{\deg M = n} \frac{1}{|\mathrm{Aut}\, M|} x^n = \sum_{n=0}^{\infty} \frac{|M_n|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n. \tag{4.25}$$

(b) The proof is exactly the same, modulo the following observation:

The category of finite-length R_p-modules is a full subcatgory of finite-length R-modules consisting of those annihilated by some power of p.

An R-module $(V; A_1, \ldots, A_m)$ is annihilated by a power of $p = (t_1, \ldots, t_m)R$ if and only if a power of A_i annihilates V for all i. This is equivalent to requiring that all A_i are nilpotent.

(c) Let $R = \operatorname{Spec} \mathbb{F}_q[t_1, \ldots, t_m]/I$ and $S = \operatorname{Spec} \mathbb{F}_q[t_1, \ldots, t_m]/J$. It suffices to prove the following claim:

A finite-length R-module M is supported on $Z = \operatorname{Spec} S$ if and only if $J^n M = 0$ for some n.

To prove the claim, assume $J^n M = 0$. Consider any maximal ideal p of R that corresponds to a closed point in $X \setminus Z$, then p does not contain J. So there exists $u \in J$ such that $u \notin p$. Since $u^n M = 0$ and u is invertible in R_p, the localization M_p of M at p is zero. This shows that that M is supported on Z.

Conversely, assume M is supported on Z. Then there are maximal ideals $p_1, \ldots, p_h$ corresponding to closed points in Z such that

$$M = \bigoplus_{i=1}^{h} M_{p_i}, \tag{4.26}$$

where M_{p_i} is the localization of M at p_i. Note that $p_i \supseteq J$. Since M_{p_i} is a finite-length module over the local ring R_{p_i}, there is n_i such that $p_i^{n_i} M_{p_i} = 0$. It follows that $J^{n_i} M_{p_i} = 0$. Taking $n = \max\{n_1, \ldots, n_h\}$, we get $J^n M = \bigoplus_{i=1}^{h} J^n M_{p_i} = 0$, proving the claim.

$\square$

Corollary 4.4.

$$\widehat{Z}_{\mathbb{F}_q[[t]]}(x) = \prod_{i=1}^{\infty}(1 - q^{-i}x)^{-1}, \tag{4.27}$$

$$\widehat{Z}_{\mathbb{F}_q[[u,v]]}(x) = \prod_{i,j\geq 1}(1 - q^{-j}x^i)^{-1}. \tag{4.28}$$

Proof. Given Proposition 4.3(2), the two formulas follow from the matrix counting results of Fine and Herstein [11] and Fulman and Guralnick [12], respectively. □

Proposition 4.5. *Let $Z_X(x)$ be the Hasse–Weil zeta series of X. Then*

(a) *If X is a smooth curve over $\mathbb{F}_q$, then*

$$\widehat{Z}_X(x) = \prod_{i=1}^{\infty} Z_X(q^{-i}x) \in \mathbb{C}[[x]]. \tag{4.29}$$

(b) *If X is a smooth surface over $\mathbb{F}_q$, then*

$$\widehat{Z}_X(x) = \prod_{i,j\geq 1} Z_X(q^{-j}x^i) \in \mathbb{C}[[x]]. \tag{4.30}$$

Proof. (a) The Euler product (Proposition 4.2) gives

$$\widehat{Z}_X(x) = \prod_{p\in \mathrm{cl}(X)} \widehat{Z}_{X,p}(x) = \prod_{p\in \mathrm{cl}(X)} \widehat{Z}_{\widehat{\mathcal{O}}_{X,p}}(x). \tag{4.31}$$

Since X is a smooth curve, $\widehat{\mathcal{O}}_{X,p}$ is a complete regular local ring of dimension one. By the Cohen structure theorem, $\widehat{\mathcal{O}}_{X,p} \cong \kappa_p[[t]]$, where κ_p is the residue field of $\widehat{\mathcal{O}}_{X,p}$. By Corollary 4.4,

$$\widehat{Z}_{\kappa_p[[t]]/\kappa_p}(x) = \sum_{M/\kappa_p[[t]]} \frac{1}{|\mathrm{Aut}\, M|} x^{\dim_{\kappa_p} M} = \prod_{i=1}^{\infty}(1 - q^{-i\deg p}x)^{-1}, \tag{4.32}$$

where $\deg p$ is the degree of the field extension $[\kappa_p : \mathbb{F}_q]$, so that $q^{\deg p}$ is the cardinality of κ_p.

Noting that $\dim_{\mathbb{F}_q} M = (\deg p) \dim_{\kappa_p} M$ for any κ_p-vector space M, we have

$$\widehat{Z}_{\widehat{\mathcal{O}}_{X,p}}(x) = \sum_{M/\kappa_p[[t]]} \frac{1}{|\mathrm{Aut}\, M|} x^{\dim_{\mathbb{F}_q} M} \tag{4.33}$$

$$= \sum_{M/\kappa_p[[t]]} \frac{1}{|\mathrm{Aut}\, M|} (x^{\deg p})^{\dim_{\kappa_p} M} \tag{4.34}$$

$$= \prod_{i=1}^{\infty}(1 - q^{-i\deg p}x^{\deg p})^{-1}. \tag{4.35}$$

Recalling the Euler product of the Hasse–Weil zeta function

$$Z_X(x) = \prod_{p\in \mathrm{cl}(X)} (1 - x^{\deg p})^{-1}, \tag{4.36}$$

we get

$$\widehat{Z}_X(x) = \prod_{p\in \mathrm{cl}(X)} \prod_{i=1}^{\infty} (1 - q^{-i\deg p} x^{\deg p})^{-1} \tag{4.37}$$

$$= \prod_{i=1}^{\infty} \prod_{p\in \mathrm{cl}(X)} (1 - (q^{-i}x)^{\deg p})^{-1} \tag{4.38}$$

$$= \prod_{i=1}^{\infty} Z_X(q^{-i}x). \tag{4.39}$$

(b) By the Cohen structure theorem, for any closed point p on a smooth surface p, we have $\widehat{\mathcal{O}}_{X,p} \cong \kappa_p[[u,v]]$. Applying the same argument to the corresponding formula in Corollary 4.4, we have

$$\widehat{Z}_{\widehat{\mathcal{O}}_{X,p}}(x) = \widehat{Z}_{\kappa_p[[u,v]]/\kappa_p}(x^{\deg p}) \tag{4.40}$$

$$= \left(\prod_{i,j\geq 1} (1 - (q^{\deg p})^{-j} x^i)^{-1} \right)\Bigg|_{x\mapsto x^{\deg p}} \tag{4.41}$$

$$= \prod_{i,j\geq 1} (1 - (q^{-j}x^i)^{\deg p})^{-1}. \tag{4.42}$$

It follows that

$$\widehat{Z}_X(x) = \prod_{p\in \mathrm{cl}(X)} \widehat{Z}_{\widehat{\mathcal{O}}_{X,p}}(x) \tag{4.43}$$

$$= \prod_{i,j\geq 1} \prod_{p\in \mathrm{cl}(X)} (1 - (q^{-j}x^i)^{\deg p})^{-1} \tag{4.44}$$

$$= \prod_{i,j\geq 1} Z_X(q^{-j}x^i). \tag{4.45}$$

□

5 Properties of $H_q(x)$ and Proof of Theorem 1.1

Theorem 1.1(b) almost follow immediately from Theorem 1.2: let $X = \operatorname{Spec} \mathbb{F}_q[u,v]/(uv)$ be the union of x- and y-axes on a plane, and let p be the origin, then by (4.13),

$$\widehat{Z}_{X,p}(x) = \frac{\widehat{Z}_X(x)}{\widehat{Z}_{X\backslash p}(x)} \tag{5.1}$$

$$= \frac{(x;q^{-1})_\infty^{-2} H_q(x)}{(\widehat{Z}_{\mathbb{A}^1\backslash 0}(x))^2} \tag{5.2}$$

$$= \frac{(x;q^{-1})_\infty^{-2} H_q(x)}{(1-x)^{-2}} \tag{5.3}$$

$$= (xq^{-1};q^{-1})_\infty^{-2} H_q(x). \tag{5.4}$$

Here, $\mathbb{A}^1 \setminus 0$ denotes an affine line minus one point. Because it is a smooth curve, its Cohen–Lenstra series can be evaluated by Proposition 4.5. This finishes the proof of Theorem 1.1(b) except the claim that $q^i, i \geq 1$ are actually double poles of $\widehat{Z}_{\mathbb{F}_q[[u,v]]/(uv)}(x)$. This requires that $H_q(q^i) \neq 0$ for $i \geq 1$, which turns out to be elementary from the expression of $H_q(x)$, see Proposition 5.1 below.

The proof of Theorem 1.1 is complete given the observations about $H_q(x)$ in Proposition 5.1. Let $t = q^{-1}$ and we define

$$H(x;t) := H_q(x) = \sum_{k=0}^{\infty} t^{k^2} x^{2k} \frac{1}{(t;t)_k} (1-t^{k+1}x)(1-t^{k+2}x)\dots \tag{5.5}$$

$$= (tx;t)_\infty \sum_{k=0}^{\infty} t^{k^2} x^{2k} \frac{1}{(t;t)_k (tx;t)_k.} \text{ (if } x \neq t^{-1}, t^{-2}, \dots). \tag{5.6}$$

We note that the infinite sum (5.5) defines $H(x;t)$ in two possible ways. First, the infinite sum converges formally to a power series in x and t (due to the x^{2k} factor). Second, if $0 < t < 1$ is fixed, then each summand of (5.5) is an entire function in x, so (5.5) is an infinite sum of functions. We will show that the "formal" sum and the "analytic" sum are the same.

Proposition 5.1. *For any fixed real number $0 < t < 1$, we have*

(a) *The infinite sum (5.5) of entire functions in x converges uniformly on any bounded disc to an entire function whose Maclaurin series is the coefficient-wise limit of the sum (5.5) of formal power series in x.*

(b) $H(x;t) > 0$ *if $x < t^{-1}$ or $x = t^{-i}, i = 1, 2, \dots$.*

(c) $H(1;t) = 1$.

(d) $H(-1;t) = (-t;t)_\infty (-t;t^2)_\infty$

Proof.

(a) Fix a bounded disc $|x| \leq M$. Then as k goes to infinity, the factor $\frac{1}{(t;t)_k}(1-t^{k+1}x)(1-t^{k+2}x)\dots$ has a uniform bound for $|x| \leq M$ that only depends on M. The uniform convergence of the sum (5.5) follows from the convergence of $\sum t^{k^2} M^k$. Therefore, the sum (5.5) defines an entire function.

To find its Maclaurin series, consider the sequence of partial sums of (5.5). The assertion of (a) follows from the fact that if a sequence of holomorphic functions $f_k(x)$ converges uniformly to a holomorphic function $f(x)$ on a disc D centered at $x = 0$, then the Maclaurin series of $f_k(x)$ must converge to the Mauclaurin series of $f(x)$ coefficient-wise. For a proof, we recall that the n-th Maclaurin coefficient of $f(x)$ is given by $n! f^{(n)}(0)$. For any n, the sequence $f_k^{(n)}(x)$ converges uniformly to $f^{(n)}(x)$ on compact subsets of D (see for instance [26, Theorem 10.28]), so we have $n! f_k^{(n)}(0) \to n! f^{(n)}(0)$ as $k \to \infty$.

(b) If $x < t^{-1}$, then every term of (5.5) is positive. If $x = t^{-i}, i = 1, 2, \dots,$ then

$$H(t^{-i}; t) = \sum_{k=0}^{\infty} \frac{t^{k^2} (t^{-i})^{2k}}{(1-t)\dots(1-t^k)} (1 - t^{k+1} t^{-i})(1 - t^{k+2} t^{-i}) \dots \tag{5.7}$$

$$= \sum_{k=i}^{\infty} \frac{t^{k^2} (t^{-i})^{2k}}{(1-t)\dots(1-t^k)} (1 - t^{k+1-i})(1 - t^{k+2-i}) \dots \tag{5.8}$$

and every term in the last sum is positive.

(c) Using (5.6), we have

$$H(1; t) = (t; t)_\infty \sum_{k=0}^{\infty} t^{k^2} \frac{1}{(t; t)_k (t; t)_k}, \tag{5.9}$$

which is equal to 1 by the following standard identity due to Euler; see [1, p. 21, (2.2.9)].

$$\sum_{k=0}^{\infty} \frac{t^{k^2}}{(t; t)_k^2} = \frac{1}{(t; t)_\infty}. \tag{5.10}$$

(d) To compute $H(-1; t)$, we need the following identities:

$$(t^2; t^2)_n = (t; t)_n (-t; t)_n \tag{5.11}$$

$$\sum_{n=0}^{\infty} \frac{t^{\binom{n}{2}} x^n}{(t; t)_n} = (-x; t)_\infty, \tag{5.12}$$

where the first one is elementary:

$$(t^2; t^2)_n = \prod_{i=1}^{n} (1 - t^{2i}), \tag{5.13}$$

$$= \prod_{i=1}^{n} (1 - t^i)(1 + t^i) = (t; t)_n (-t; t)_n, \tag{5.14}$$

and the second one is due to Euler; see [1, p. 19, (2.2.6)]. Now we have

$$\begin{aligned}
H(-1;t) &= (-t;t)_\infty \sum_{k=0}^{\infty} \frac{t^{k^2}}{(t;t)_k(-t;t)_k} && (5.15)\\
&= (-t;t)_\infty \sum_{k=0}^{\infty} \frac{(t^2)^{\binom{k}{2}} t^k}{(t^2;t^2)_k} && \text{(by (5.11))}\\
&= (-t;t)_\infty (-t;t^2)_\infty. && \text{(by (5.12) with } x \mapsto t,\ t \mapsto t^2)
\end{aligned}$$

□

The proof of Theorem 1.1 now follows directly from Theorem 1.2 and Proposition 5.1. Here, we note that the proof of Theorem 1.1(c) requires the following computation:

$$\begin{aligned}
\widehat{Z}_{\mathbb{F}_q[[u,v]]/(uv)}(-1) &= (-t;t)_\infty^{-2} H(-1;t) && (5.16)\\
&= \frac{(-t;t^2)_\infty}{(-t;t)_\infty} && (5.17)\\
&= \frac{(1+t)(1+t^3)(1+t^5)\dots}{(1+t)(1+t^2)(1+t^3)\dots} && (5.18)\\
&= \frac{1}{(1+t^2)(1+t^4)(1+t^6)\dots} && (5.19)\\
&= \frac{1}{(-t^2;t^2)_\infty}. && (5.20)
\end{aligned}$$

6 Further discussion on $H_q(x)$

The factor $H_q(x)$ that appears in the factorization of $\widehat{Z}_{X,p}(x)$ (where (X,p) is a nodal singularity) is mostly mysterious. We note that if Question 1.3(b) has a positive answer, then we can attach a meaningful holomorphic function $H_{X,p}(x)$ to a curve singularity (X,p). In particular, $H_q(x)$ would be $H_{X,p}(x)$ associated to a nodal singularity (X,p). It is natural to ask about the properties of $H_{X,p}(x)$ for a general singularity (X,p), and what they reveal about the geometry of X at p. This section discusses possible analytic properties of $H_q(x)$, aiming at providing clues to the questions above.

The mysterious function $H_q(x) = H(x;t)$ (where $t = q^{-1} \in (0,1/2]$ is fixed) appears to share some analytical features with the *partial theta function* $\Theta_{\mathrm{p}}(x;t) := \sum_{n=0}^{\infty} t^{n^2} x^n$. We summarize several notable properties of the partial theta function; we refer the readers to an excellent survey paper [28] where many references are listed. The partial theta function satisfies the functional equation

$$\Theta_{\mathrm{p}}(x;t) - tx\Theta_{\mathrm{p}}(t^2x;t) = 1. \tag{6.1}$$

An important property of the partial theta function is having *smooth coefficients*. An entire function $f(x) = \sum a_n x^n$ is said to have smooth coefficients if $\lim_{n\to\infty} a_n^2/(a_{n-1}a_{n+1})$ converges. The partial theta function has smooth coefficients because $a_n^2/(a_{n-1}a_{n+1})$ is constant.

Having smooth coefficients is the reason behind many other analytic behaviors of $\Theta_{\mathrm{p}}(x;t)$, such as distribution of roots in an "almost geometric sequence", belong to Laguerre–Pólya class, etc.; see [28] for an excellent survey paper on this topic. Therefore, having smooth coefficients is a key feature to look for when comparing $H(x;t)$ to $\Theta_{\mathrm{p}}(x;t)$.

Based on numerical observation, the roots of $H(x;t)$ appear to be imaginary, and $H(x;t)$ does not appear to have smooth coefficients. However, the even-degree terms and odd-degree terms of $H(x;t)$ appear to have smooth coefficients and real roots. For any power series $f(x) = \sum a_n x^n$, denote

$$\ell_x f(x) := \lim_{n\to\infty} \frac{a_n^2}{a_{n-1}a_{n+1}}. \tag{6.2}$$

Our observations suggest the following conjecture.

Conjecture 6.1. The function $H(x;t)$ satisfies the following properties:

(a) As a power series in x and t, we have

$$H(x;t) = \sum_{n=0}^{\infty} (-1)^n t^{\lceil n^2/4 \rceil} (1 + O(t)) x^n. \tag{6.3}$$

(b) Let $F(x;t)$ and $G(x;t)$ be defined such that $H(x;t) = F(x^2;t) + xG(x^2;t)$. Then both $F(x;t)$ and $G(x;t)$ have smooth coefficients. Moreover, both $\ell_x F(x;t)$ and $\ell_x G(x;t)$ are equal to t^2.

Question 6.2. Does $F(x;t)$ (or $G(x;t)$) satisfy a functional equation, possibly similar to (6.1), the functional equation for $\Theta_{\mathrm{p}}(x;t)$?

We note $\ell_x \Theta_{\mathrm{p}}(x;t) = t^2$, the same as the conjectured value of $\ell_x F(x;t)$ and $\ell_x G(x;t)$.

Apart from the similarity to the partial theta function, another motivation why we look for a functional equation for $H(x;t)$ is an observation by Cohen and Lenstra [8, §7], where they find a functional equation for an entire function built from the Cohen–Lenstra zeta function of a Dedekind domain.

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