Coh zeta functions for inert quadratic orders

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Motivation: from counting to zeta functions

- Many classic problems in enumerative algebra involve counting matrices over $\mathbb{F}_q$.
- A modern perspective unifies these problems by studying modules over specific rings.
- Central object: The **Coh zeta function**, $\widehat{\zeta}_R(s)$, is a generating series that provides a weighted count of all finite modules over a ring R:

$$\widehat{\zeta}_R(s) := \sum_{Q} \frac{1}{|\mathrm{Aut}_R(Q)|} |Q|^{-s}.$$

- It is directly connected to arithmetic statistics (Cohen–Lenstra), the stack of coherent sheaves, quiver varieties, and motivic degree 0 Donaldson–Thomas theory (Behrend, Bryan, Fantechi, Morrison, Ricolfi, Szendrői, and others).

Classical Examples

Throughout, let $t := q^{-s}$. The Coh zeta function connects module theory to matrix enumeration and classical product formulas.

Smooth curve ($R = \mathbb{F}_q[[X]]$)

Fine–Herstein '58: counting nilpotent matrices

$$\widehat{\zeta}_{\mathbb{F}_q[[x]]}(s) = \sum_{n=0}^{\infty} \frac{|\mathrm{Nilp}_n(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} t^n = \frac{1}{(tq^{-1}; q^{-1})_\infty}.$$

Smooth surface ($R = \mathbb{F}_q[X, Y]$)

Feit–Fine '60: counting commuting pairs

$$\widehat{\zeta}_{\mathbb{F}_q[x,y]}(s) = \sum_{n=0}^{\infty} \frac{|\{A, B \in \mathrm{Mat}_n(\mathbb{F}_q) : AB = BA\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|} t^n = \prod_{i,j \geq 1} \frac{1}{1 - t^i q^{2-j}}.$$

Singularities give q-series

Punchline (H. '23, H.–Jiang '23+)

When R is the ring of a singular curve, $\widehat{\zeta}_R(s)$ reveals a surprising source of new q-series.

We focused on the so-called A_k singularities, which correspond to some simplest torus knots and links. Throughout, $m \in \mathbb{Z}_{\geq 1}$.

$(2, 2m+1)$-torus knots ($R = \mathbb{F}_q[[X, Y]]/(Y^2 - X^{2m+1})$)

The formula for $\widehat{\zeta}_R(s)$ was identified as a t-deformed Andrews–Gordon m-fold sum. At $t = \pm 1$, this evaluates to a classic AG infinite product.

$(2, 2m)$-torus links ($R = \mathbb{F}_q[[X, Y]]/(Y(Y - X^m))$)

The initial formula for $\widehat{\zeta}_R(s)$ was a mysterious 2m-fold sum. It was proved (by geometric methods) to evaluate to $1/(q^{-1}; q^{-1})_\infty^2$ at $t = 1$. Based on numerical data, it was conjectured to evaluate to a Bressoud-type product at $t = -1$.

Resolving the torus link mystery

The conjecture for the torus link at $t = -1$ was equivalent to the following Rogers–Ramanujan type identity:

Identity (conjectured H.–Jiang '23+, proved Chern '24+)

$$(-q;q)_\infty \sum_{\substack{r_1,\ldots,r_m \\ s_1,\ldots,s_m}} \frac{(-1)^{\sum s_i} q^{\sum r_i^2 - r_i s_i + s_i^2}}{(q;q)_{r_1-r_2} \cdots (-q;q)_{r_m}^2 (q;q)_{s_m}} \times \begin{bmatrix} r_1 - s_2 \\ r_1 - s_1 \end{bmatrix}_q \cdots \begin{bmatrix} r_m \\ r_m - s_m \end{bmatrix}_q = \frac{(q^{m+1}, q^{m+1}, q^{2m+2}; q^{2m+2})_\infty}{(q;q)_\infty}$$

- This identity was recently proven by Shane Chern '24+.
- He proved that a finitization of the left-hand side equals a finitized Bressoud-type m-fold sum.
- But which finitization? Is it necessary to guess one?

Review: Rogers–Ramanujan type identities

(Central) Andrews–Gordon identity

$$\sum_{n_1,\dots,n_m} \frac{q^{n_1^2+\cdots+n_m^2}}{(q)_{n_1-n_2}\cdots(q)_{n_{m-1}-n_m}(q)_{n_m}} = \frac{(q^{m+1},q^{m+2},q^{2m+3};q^{2m+3})_\infty}{(q;q)_\infty}$$

(Central) Bressoud identity

$$\sum_{n_1,\dots,n_m} \frac{q^{n_1^2+\cdots+n_m^2}}{(q)_{n_1-n_2}\cdots(q)_{n_{m-1}-n_m}(q^2;q^2)_{n_m}} = \frac{(q^{m+1},q^{m+1},q^{2m+2};q^{2m+2})_\infty}{(q)_\infty}$$

Review: Finitization

- A classical proof strategy for such identities is to find and prove a finitization.
- For example, here is a '(finite multi-sum) = (finite single sum)' finitized AG identity (Paule '85, Andrews–Schilling–Warnaar '99):

$$(q)_n \sum_{n_1,\dots,n_m} \frac{q^{\sum n_i^2}}{(q)_{n-n_1}(q)_{n_1-n_2}\cdots(q)_{n_m}} = (q)_n \sum_r \frac{(-1)^r q^{\binom{r}{2}+(m+1)r^2}}{(q)_{n-r}(q)_{n+r}}$$

- As $n \to \infty$, applying the Jacobi triple product formula to the single sum gives the required infinite product.
- There are more tools to prove finite identities, such as the Bailey lemma.

A natural finitization for Coh zeta function

- The computation of the Coh zeta function in H.–Jiang '23+ is achieved by first computing the **lattice zeta function** (Solomon '77, Bushnell–Reiner '80), defined for any R-module M as $\zeta_M^R(s) := \sum_{L \subseteq_R M} (M : L)^{-s}$.
- The key observation is $\lim_{n\to\infty} \widehat{\zeta}_{R,n}(s) = \widehat{\zeta}_R(s)$ coefficient-wise (H.–Jiang '23+), where

$$\widehat{\zeta}_{R,n}(s) := \zeta_{R^n}^R(s+n).$$

- This inspires us to call $\widehat{\zeta}_{R,n}(s)$ the **finitized Coh zeta function**. A natural candidate to finitize $\widehat{\zeta}_R(s)$ without guesswork!

The t-deformation

The following t-deformations of the AG and Bressoud sums appear naturally in our framework.

t-deformed Andrews–Gordon Sum

$$\mathbf{AG}_n(q,t;2m+3) := (q)_n \sum_{n_1,\ldots,n_m} \frac{q^{\sum n_i^2} t^{2\sum n_i}}{(q)_{n-n_1}(q)_{n_1-n_2}\cdots(q)_{n_m}}.$$

t-deformed Bressoud Sum

$$\mathbf{Br}_n(q,t;2m+2) := (q)_n \sum_{n_1,\ldots,n_m} \frac{q^{\sum n_i^2} t^{2\sum n_i}}{(q)_{n-n_1}(q)_{n_1-n_2}\cdots(q)_{n_m}(-tq;q)_{n_m}}.$$

The full picture of the known cases

Thanks to Chern's work, the finitized Coh zeta functions for the $(2,k)$-torus knots/links can be described in a unified way:

Theorem (combining H.–Jiang '23+, Chern '24+)

$$\begin{aligned}\widehat{\zeta}_{R_{2,2m+1},n}(s) &= \frac{1}{(tq^{-1};q^{-1})_n}\mathbf{AG}_n(q^{-1},t;2m+3),\\ \widehat{\zeta}_{R_{2,2m},n}(s) &= \frac{1}{(tq^{-1};q^{-1})_n}\mathbf{Br}_n(q^{-1},-t;2m+2).\end{aligned}$$

Two puzzles arising from the picture

Question 1

What geometric object corresponds to the Bressoud sum *without* the $t \mapsto -t$ substitution?

Question 2

Why is our t-deformed Andrews–Gordon sum a polynomial in t^2, which makes $\mathbf{AG}_n(q, t; 2m+3) = \mathbf{AG}_n(q, -t; 2m+3)$? Is there a structural reason for this symmetry?

A new viewpoint via quadratic orders

- Our proposal: The puzzles can be resolved by shifting from a geometric to an algebraic viewpoint. The singularities correspond to **quadratic orders**.
- Torus knot $R_{2,2m+1}$: Orders in the **ramified** quadratic extension $\mathbb{F}_q((\sqrt{X}))/\mathbb{F}_q((X))$.
- Torus link $R_{2,2m}$: Orders in the **split** quadratic extension $\mathbb{F}_q((X)) \times \mathbb{F}_q((X))/\mathbb{F}_q((X))$.
- The missing piece: The order $R'_{2,2m} := \mathbb{F}_q[[T]] + T^m\mathbb{F}_{q^2}[[T]]$ in the **inert** quadratic extension $\mathbb{F}_{q^2}((X))/\mathbb{F}_q((X))$, which does not have a counterpart in the geometry of curves (over algebraically closed fields).

Main conjecture & deeper structure

Conjecture (H.)

The inert orders correspond to the direct t-deformed Bressoud sum (answering Question 1).

$$\widehat{\zeta}_{R'_{2,2m},n}(s) = \frac{1}{(tq^{-1};q^{-1})_n}\mathbf{Br}_n(q^{-1},t;2m+2).$$

This structure also strongly hints at a deeper explanation via **quadratic twists**, because

- Inert is a twist of Split.
- Ramified is its own twist (addressing Question 2).

See below.

More on quadratic twists

If q is odd, let $\alpha \in \mathbb{F}_q$ be a non-square.

- The ramified order is its own twist:

$$\begin{aligned} R_{2,2m+1} &:= \mathbb{F}_q[[X,Y]]/(Y^2 - X^{2m+1}) \\ &\simeq \mathbb{F}_q[[X,Y]]/(Y^2 - \alpha X^{2m+1}) =: R'_{2,2m+1}. \end{aligned}$$

- The split and inert orders form a twist pair:

$$\begin{aligned} R_{2,2m} &\simeq \mathbb{F}_q[[X,Y]]/(Y^2 - X^{2m}), \\ R'_{2,2m} &\simeq \mathbb{F}_q[[X,Y]]/(Y^2 - \alpha X^{2m}). \end{aligned}$$

The completed trilogy

A unified picture

The conjecture suggests a complete 2×2 picture relating geometry, algebra, and q, t-series:

$$\widehat{\zeta}_{R_{2,2m+1},n}(s) = \frac{1}{(tq^{-1};q^{-1})_n}\mathbf{AG}_n(q^{-1}, -t; 2m+3),$$

$$\widehat{\zeta}_{R'_{2,2m+1},n}(s) = \frac{1}{(tq^{-1};q^{-1})_n}\mathbf{AG}_n(q^{-1}, t; 2m+3),$$

$$\widehat{\zeta}_{R_{2,2m},n}(s) = \frac{1}{(tq^{-1};q^{-1})_n}\mathbf{Br}_n(q^{-1}, -t; 2m+2),$$

$$\widehat{\zeta}_{R'_{2,2m},n}(s) \stackrel{?}{=} \frac{1}{(tq^{-1};q^{-1})_n}\mathbf{Br}_n(q^{-1}, t; 2m+2).$$

Our contribution: Evidence for the conjecture

- We provide the first explicit formulas for the Coh zeta function of inert quadratic orders in two key settings:
  1. The $t = 1$ (i.e., $s = 0$) specialization for all m.
  2. The full function for the base case $m = 1$.
- Our method: A new technique using Möbius inversion on posets, on top of techniques of H.–Jiang '23+.

The $t = 1$ $(s = 0)$ case

Theorem (H. '25+)

For any $m \geq 1$ *and* $n \geq 0$, $\widehat{\zeta}_{R'_{2,2m},n}(0)$ *is given by the explicit multi-sum:*

$$\widehat{\zeta}_{R'_{2,2m},n}(0) = \frac{1}{(q^{-2};q^{-2})_n} \sum_{r,s_1,\ldots,s_m \in \mathbb{Z}} \frac{(-1)^r q^{-r-(s_1-r)^2-\sum_{k=2}^{m} s_k^2}}{\prod_{k=1}^{m-1}(q^{-1};q^{-1})_{s_k-s_{k+1}}} \times \begin{bmatrix} n \\ r \end{bmatrix}_{q^{-2}} \begin{bmatrix} 2n-2r \\ n-s_1 \end{bmatrix}_{q^{-1}} \frac{(q^{-1};q^{-1})_{n+s_1}}{(q^{-1};q^{-1})_{n+s_m}}.$$

Connection to Conjecture

Equating our formula with the conjecture's prediction, $\widehat{\zeta}_{R'_{2,2m},n}(0) = \frac{1}{(q^{-1};q^{-1})_n}\mathbf{Br}_n(q^{-1},1;2m+2)$, gives a new conjectural multi-sum identity. This is numerically checked; a full proof will appear in forthcoming joint work with Chern.

The $m = 1$ case

Theorem (H. '25+)

The finitized Coh zeta function $\widehat{\zeta}_{R'_{2,2},n}(s)$ is given by the explicit double sum:

$$\widehat{\zeta}_{R'_{2,2},n}(s) = \sum_{i,j} (-1)^j q^{-(i^2+ij+j)} \begin{bmatrix} n \\ i \end{bmatrix}_{q^{-1}} \frac{(q^{2i};q^{-2})_j}{(q^{-1};q^{-1})_j (t^2q^{-2};q^{-2})_i} t^{i+j}.$$

Conclusion

Our conjecture for $m = 1$ is equivalent to a new two-variable '(double sum) = (single sum)' identity.

We note that it is, in principle, machine verifiable by the q-WZ algorithm.

Summary

- The zeta functions for A_k singular curves appear to give AG and Bressoud q-series, but the picture was not complete.
- A new viewpoint of quadratic orders and twists proposes a framework to resolve these puzzles and suggests a complete, unified picture.
- We provide the first explicit formulas for the missing inert case, giving strong evidence for this picture.
- For $t = 1$, we reduce the conjecture to an $(m+1)$-fold sum equals m-fold sum identity.
- For $m = 1$, we reduce the conjecture to a double sum equals single sum identity.

Outline of proof techniques

How do we compute the zeta function?

- The starting point is the Reflection Principle (proved in H.–Jiang '23+ using a Tate's thesis style argument), which relates $\widehat{\zeta}_{R,n}(s)$ at $s=0$ to a simpler object: $\zeta^{R}_{\widetilde{R}^n}(0)$, where $\widetilde{R}$ is the normalization of R. (Another observation is also needed here: $\zeta^{R}_{\widetilde{R}^n}(0) = \zeta^{R}_{R^n}(0)$.)
- We compute $\zeta^{R}_{\widetilde{R}^n}(s)$ using Möbius inversion on the poset of submodules. The terms in the sum involve counting submodules over certain DVR quotients, which are given by Hall polynomials. A key observation is that while R is singular, a quotient of R is a DVR quotient.
- For the $m=1$ case (all s), the Reflection Principle no longer helps. Instead, we use a special property that the maximal ideal $\mathfrak{m} \simeq \widetilde{R}$ as an R-module. This allows us to relate $\zeta^{R}_{R^n}(s)$ back to the now-known $\zeta^{R}_{\widetilde{R}^n}(s)$ via a formula proved in H.–Jiang '23+ using Nakayama's Lemma.

Merci!

See https://arxiv.org/abs/2507.21966

Questions?