Counting 0-dimensional sheaves on singular curves

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Counting invariants

- Zeta function $\rightsquigarrow$ point-counting invariant
- Gromov–Witten invariant $\rightsquigarrow$ curve-counting invariant

What is counting?

Count = an enumerative invariant of the corresponding moduli space
The enumerative invariant could be:

- A fundamental class
- The integral of such
- Betti numbers, Hodge numbers
- Euler characteristic
- Point-count over finite fields

Motives

Let k be a field.

Definition

(Informal) The motive of a k-variety is itself "up to cut-and-paste".
(Formal) The motive of X is the class $[X]$ in the Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$ defined as the abelian group generated by k-varieties with the cut-and-paste relation $[X] = [Z] + [X \setminus Z]$.

The motive recovers the point-count over finite fields and the Euler characteristic, but not the Betti numbers. The motive remembers the cell-counts in a cell decomposition when there is one.
Example: X has a 0-cell, two 1-cells and a 2-cell $\rightsquigarrow [X] = 1 + 2\mathbb{L} + \mathbb{L}^2$.

0-dimensional sheaves

Definition

A 0-dimensional sheaf on a variety X is a coherent sheaf on X supported on finitely many points. The length of a 0-dimensional sheaf M is defined as $\dim_k H^0(X; M)$ (the same as the degree, or Euler characteristic).

Intuition

A 0-dimensional sheaf of length n on X can be thought of as an "n-point configuration" on X, together with some extra information remembered at points of collision.

Examples of 0-dimensional sheaves

$X = \mathbb{A}^1$

A 0-dimensional sheaf on X can be encoded by a module over $k[t]$ that is finite-dimensional as a k-vector space.

$M_1 = \dfrac{k[t]}{(t^2)} \oplus \dfrac{k[t]}{(t-1)}$ is a 0-dimensional sheaf of length 3.

$M_2 = \dfrac{k[t]}{(t)} \oplus \dfrac{k[t]}{(t)} \oplus \dfrac{k[t]}{(t-1)}$ is a different 0-dimensional sheaf of length 3.

Moduli spaces of 0-dimensional sheaves

- The Hilbert scheme of points $\mathrm{Hilb}_n(X) = \{\mathcal{O}_X \twoheadrightarrow M : \ell(M) = n\}$
- The ($\mathcal{E}$-framed) Quot scheme of points $\mathrm{Quot}_{\mathcal{E},n}(X) = \{\mathcal{E} \twoheadrightarrow M : \ell(M) = n\}$ for any given coherent sheaf $\mathcal{E}$
- The stack of 0-dimensional sheaves $\mathrm{Coh}_n(X) = \{M : \ell(M) = n\}$

Counting functions

- Hilbert zeta function $Z_X^{\mathrm{Hilb}}(t) = \sum_{n\geq 0}[\mathrm{Hilb}_n(X)]\, t^n$
- Quot zeta function $Z_{\mathcal{E}}(t) = \sum_{n\geq 0}[\mathrm{Quot}_{\mathcal{E},n}(X)]\, t^n$
- $\widehat{Z}_X(t) = \sum_{n\geq 0}[\mathrm{Coh}_n(X)]\, t^n$.

Hilbert zeta function

Facts

- X smooth curve: $\mathrm{Hilb}_n(X) = \mathrm{Sym}^n(X) \implies Z_X^{\mathrm{Hilb}}(t)$ is the motivic zeta function.
- X smooth surface: $\mathrm{Hilb}_n(X)$ is smooth and resolves the singularity of $\mathrm{Sym}^n(X)$. But what is $Z_X^{\mathrm{Hilb}}(t)$?
- (Ellingsrud–Strømme '87) Found a cell decomposition for $\mathrm{Hilb}_n(\mathbb{P}^2)$.
- (Göttsche '01) Computed $Z_X^{\mathrm{Hilb}}(t)$ in terms of the motivic zeta function for X smooth surface.

Consequences

- X smooth curve: $Z_X^{\mathrm{Hilb}}(t)$ is rational in t (Kapranov '00)
- X smooth surface: $Z_X^{\mathrm{Hilb}}(t)$ is rational in t whenever the motivic zeta function for X is.

Rationality

Question

Do we have rationality of $Z_X^{\mathrm{Hilb}}(t)$ for other X?

Theorem (Bejleri–Ranganathan–Vakil, '20)

If X is a reduced curve, then $Z_X^{\mathrm{Hilb}}(t)$ is rational in t with a known denominator.

Remark

The Hilbert scheme is sensitive to the singularities, so $Z_X^{\mathrm{Hilb}}(t)$ is different from the motivic zeta function.

Knot theory?

Remarkable fact

The exact formula of the numerator is also quite interesting – it says a lot about the singularities!

For planar singularities

- The numerator seems to always be a polynomial in $\mathbb{L}, t$.
- The numerator satisfies a functional equation. (PT '07, ...)
- The numerators give interesting combinatorial polynomials, such as generalized q, t-Catalan. (Gorsky–Mazin, '13)
- (Oblomkov–Rasmussen–Shende conjecture, '18) The numerator should read some knot-theoretic invariants about the singularities. More precisely, the triply-graded link homology of the algebraic link associated to the singularity.

How about other counting functions?

Recall

$\mathrm{Hilb}_n(X) = \{\mathcal{O}_X \twoheadrightarrow M : \ell(M) = n\} \rightsquigarrow Z_X^{\mathrm{Hilb}}(t)$
$\mathrm{Quot}_{\mathcal{E},n}(X) = \{\mathcal{E} \twoheadrightarrow M : \ell(M) = n\} \rightsquigarrow Z_{\mathcal{E}}(t)$
So $Z_X^{\mathrm{Hilb}}(t) = Z_{\mathcal{O}_X}(t)$.

Questions

- Is the Quot zeta function $Z_{\mathcal{E}}(t)$ as nice as the Hilbert zeta function?
- By varying $\mathcal{E}$, how much does $Z_{\mathcal{E}}(t)$ enrich $Z_X^{\mathrm{Hilb}}(t)$?

Short answers

- 99% yes;
- A lot!

Main results about $Z_{\mathcal{E}}(t)$

Settings

X reduced curve over $k = \overline{k}$; $\mathcal{E}$ a rank-d torsion-free bundle over X.
Typical example: $\mathcal{E} = \mathcal{O}_X^d, d \geq 0$.

Theorem (H.–Jiang)

$Z_{\mathcal{E}}(t)$ is rational in t "with known denominator". More precisely, $Z_{\mathcal{E}}(t)/Z_{\mathcal{O}^d_{\widetilde{X}}}(t)$ is a polynomial, where $\widetilde{X}$ is the normalization of X.

Remark

For smooth curve $\widetilde{X}$, $Z_{\mathcal{O}^d_{\widetilde{X}}}(t)$ is rational. (Bifet '89, BFP '20)

Theorem (H.–Jiang)

If X has only planar singularities, and $\mathcal{E} = \mathcal{O}_X^d$, then $Z_{\mathcal{E}}(t)$ satisfies a functional equation when specialized to point-counts over finite fields.

Relation to combinatorics

Theorem (H.–Jiang)

Let X be the curve $\{y^2 = x^n\}$ (when $n = 2m$ is even and $\operatorname{char} k = 2$, replace by $y(y - x^m) = 0$), and $\mathcal{E} = \mathcal{O}_X^d$. Then there are explicit polynomials in $\mathbb{L}$ and t that compute $Z_{\mathcal{O}_X^d}(t)$. The formulas

- *depend on whether n is odd or even;*
- *involve partitions, Hall polynomials and q-hypergeometric series.*

Consequences

The functional equation implies a nontrivial identity about Hall polynomials. A direct proof can be given if $n = 2$ or n is odd. A direct proof is so far unknown if $n \geq 4$ is even.

Open question

Do these complicated polynomials recover extra info about the associated links?

Break

Stack of 0-dimensional sheaves

- $\mathrm{Coh}_n(X)$ parametrizes 0-dimensional sheaves of length n up to isomorphism.
- It is a stack.
- Its motive is still defined. (Behrend–Dhillon '07)

Question

How to make sense of the motive of $\mathrm{Coh}_n(X)$, or a stack in general?

Orbit-stabilizer theorem

Example from counting

- Let a finite group G act on a finite set X.
- The orbit space X/G can be viewed as a "quotient stack" $[X/G]$ by counting each element with a fractional weight: $1/|\mathrm{Stabilizer}|$.
- By the orbit-stabilizer theorem, the weighted cardinality of $[X/G]$ is precisely $|X|/|G|$. (Not necessarily an integer)

Motive of a quotient stack

- Let the algebraic group GL_n act on a variety X.
- One can form the quotient stack $[X/\mathrm{GL}_n]$.
- The motive of $[X/\mathrm{GL}_n]$ is defined formally as $[X]/[\mathrm{GL}_n]$.
- $[[X/\mathrm{GL}_n]]$ lives in the localization $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}, (\mathbb{L}^b - 1)^{-1} : b \geq 1]$ because $[\mathrm{GL}_n] = \mathbb{L}^{\binom{n}{2}}(\mathbb{L} - 1)\dots(\mathbb{L}^n - 1)$.

So, is $\mathrm{Coh}_n(X)$ a quotient stack?

Yes — using the variety of "matrix points".

Matrix points

For $n \geq 0$ and a variety X/k, we can define a variety $C_n(X)$ of $n \times n$-matrix points on X. As a moduli space, $C_n(X)$ parametrizes length-n sheaves together with an ordered basis on the global sections: $C_n(X)(k) = \{(M, \iota) : \ell(M) = n, \iota \in \mathrm{Isom}_{\mathrm{Vect}_k}(k^n, H^0(X; M))\}$. Concretely, if X is an affine variety cut out by $f_1(T_1, \ldots, T_m) = \cdots = f_r(T_1, \ldots, T_m) = 0$, then $C_n(X)(k)$ is the set of pairwise commuting matrices $A_1, \ldots, A_m \in \mathrm{Mat}_n(k)$ satisfying $f_j(A_1, \ldots, A_m) = 0$ for all j. (We say $(A_1, \ldots, A_m)$ is a matrix point on X.)

Key fact

$\mathrm{Coh}_n(X) = [C_n(X)/\,\mathrm{GL}_n]$, so $[\mathrm{Coh}_n(X)] = [C_n(X)]/[\mathrm{GL}_n]$.

The generating function

Recall $\widehat{Z}_X(t) = \sum_n [\mathrm{Coh}_n(X)]\, t^n = \sum_n [C_n(X)]/[\mathrm{GL}_n]\, t^n$. View it in $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}, (\mathbb{L}^b - 1)^{-1}] \subseteq K_0(\mathrm{Var}_k)[[\mathbb{L}^{-1}]]$.

Facts

- (Euler's identity) $\widehat{Z}_{\mathbb{A}^1}(t) = 1/[(1-t)(1-\mathbb{L}^{-1}t)(1-\mathbb{L}^{-2}t)\dots]$.
- (Feit–Fine '60) $\widehat{Z}_{\mathbb{A}^2}(t)$ is also of the form 1/(infinite product).
- When $X = \mathbb{A}^2$, $C_n(X)$ is the commuting variety, as well as an example of unframed quiver variety. Feit–Fine formula played a role in the Donaldson–Thomas theory of 3-folds. (Behrend–Bryan–Szendrői, '13)
- (H.) Explicitly computed $\widehat{Z}_X(t)$ in terms of the zeta function for X smooth of dim ≤ 2.
- The formulas played a role in providing matrix-point models for Sato–Tate type distributions in arithmetic geometry. (H.–Ono–Saad, BBRX)

Singular curves?

Question

If X is a reduced singular curve, then what does $\widehat{Z}_X(t)$ look like?

Theorem (H.)

If $X = \{xy = 0\}$ (same as $y^2 = x^2$ when $\operatorname{char} k \neq 2$), then $\widehat{Z}_X(t)$ has an explicit formula of the form (interesting inf sum)/(easy inf product). The infinite sum involves partitions and basic hypergeometric functions.

Conjecture

In general, $\widehat{Z}_X(t)$ should be of the form (some numerator) / (well-understood denominator). More precisely, a series can be called a "numerator" if its specialization to finite-field point-count gives an entire function in t.

New results

Theorem (H.–Jiang)

"Locally speaking", for any variety X (not necessarily a curve), $\widehat{Z}_X(t)$ can be explicitly computed by a formula in terms of the Quot zeta function $Z_{\mathcal{O}_X^d}(t)$ for all $d \geq 0$.

As a consequence of this and our formulas for $Z_{\mathcal{O}_X^d}(t)$, we get

Theorem (H.–Jiang)

Let X be the curve $\{y^2 = x^n\}$ (when $n = 2m$ is even and $\operatorname{char} k = 2$, replace by $y(y - x^m) = 0$), then the "numerator" for $\widehat{Z}_X(t)$ is an explicit power series in $\mathbb{L}^{-1}$ and t involving partitions, Hall polynomials and q-hypergeometric series.

Modular forms and Ramanujan?

Some cases of $\{y^2 = x^n\}$ are actually simpler, e.g., when $n = 3$, the numerator (called $H(t;q)$) is $\sum_{n\geq 0} q^{n^2}/((1-q)\dots(1-q^n))\, t^{2n}$, where $q = \mathbb{L}^{-1}$.

Special values at $t = \pm 1$

When $t = \pm 1$ and $n = 3$, this series is the Fourier expansion of a modular form by Rogers–Ramanujan identity. Similar for $n \geq 5$ odd, except we need Gordon–Rogers–Ramanujan identity. For $n = 2m$ even, the $\mathbb{L}^{-1}, t$-series is far from understood (except $n = 2$), but it appears that $H(1;q) = 1$ and $H(-1;q)$ is an explicit Dedekind η-quotient that gives a modular form.

Far-reaching open question

Why should modular forms even appear? What does the modular form say about the singularity? For example, when X has planar singularities, what does the modular form say about the associated links?

Summary

In this talk, I have talked about

- Two moduli spaces of 0-dimensional sheaves: Quot scheme and the stack of 0-dimensional sheaves;
- Two counting functions they produce: $Z_{\mathcal{E}}(t)$ and $\widehat{Z}_X(t)$;
- A result that relate them explicitly;
- Some general theorems and exact formulas about them, when X is a singular curve.
- Open questions suggested by the exact formulas in relation to combinatorics, modular forms, knot theory...

Thank you for listening!