From counting square-free polynomials to configuration spaces

(joint work with G. Cheong)

Motivation

Q1. How many integers in $[X, 2X)$ are square-free?

A1. The number is asymptotically $\frac{6}{\pi^2}X + O(X^{1/2})$

In other words, the probability of an integer being square free is $\frac{6}{\pi^2} = \zeta_{\mathbb{Z}}(2)^{-1}$

Q2. How many monic polynomials of degree n in $\mathbb{F}_q[t]$ are square-free?

A2. The number is precisely (!) $q^n - q^{n-1}$ for $n \geq 2$.

So among the q^n monic deg-n polynomials in $\mathbb{F}_q[t]$, the probability of being square-free is $1 - q^{-1} = \zeta_{\mathbb{F}_q[t]}(2)^{-1}$ $(n \geq 2)$

Q3. Instead of $\mathbb{A}^1/\mathbb{F}_q$, can we make an analog for $\mathbb{A}^1 - \{P_1, \cdots, P_h\}$, where $P_1, \cdots, P_h$ are $\mathbb{F}_q$-points of $\mathbb{A}^1$?

A3. Yes!

Kurlberg-Rudnick 2009 gives asymptotic count for

$\#\{f \in \mathbb{F}_q[t] \text{ monic, square-free} : f(P_1) = a_1, \cdots, f(P_h) = a_h\}$

where $a_1, \cdots, a_h \in \mathbb{F}_q - \{0\}$ are given.

It in particular implies

$$\lim_{n\to\infty} \frac{\#\{f\in\mathbb{F}_q[t] \text{ monic, square-free} : f(P_1)\neq 0, \cdots, f(P_h)\neq 0\}}{\#\{f\in\mathbb{F}_q[t] \text{ monic} : f(P_1)\neq 0, \cdots, f(P_h)\neq 0\}}$$

$$= \zeta_{\mathbb{A}^1-\{P_1,\cdots,P_h\}/\mathbb{F}_q}(2)^{-1} = \frac{1-q^{-1}}{(1-q^{-2})^h}$$

Geometric Picture: Configuration Spaces

Given arbitrary field k and a quasi-proj X/k, define

<u>Symmetric product</u> $\operatorname{Sym}^n X = X^n/S_n$

S_n is the symmetric gp acting by coordinate permutation. The quotient is the scheme theoretic quotient.

and <u>(unordered) configuration space</u> as an open subset of $\operatorname{Sym}^n X$ whose $\bar{k}$-point is described by

$$(\operatorname{Conf}^n X)(\bar{k}) = \frac{\{(x_1,\cdots,x_n)\in X^n(\bar{k}) : x_i\neq x_j \ \forall i\neq j\}}{S_n}$$

The bijection

$$(\operatorname{Sym}^n \mathbb{A}^1)(\bar{k}) \longrightarrow \{\text{monic deg-}n \text{ poly in } \bar{k}[t]\}$$

$$\{x_1,\cdots,x_n\} \longmapsto (t-x_1)\cdots(t-x_n)$$

(unordered, repetition allowed)

induces bijection

$$(\operatorname{Sym}^n \mathbb{A}^1)(k) \xrightarrow{\simeq} \{\text{monic deg-}n \text{ poly in } k[t]\}$$

Restricted to the open subset $\operatorname{Conf}^n \mathbb{A}^1$, we get bijection

$(\mathrm{Conf}^n \mathbb{A}^1)(k) \xrightarrow{\simeq} \{\text{monic deg-}n \text{ square-free poly in } k[t]\}$

$\therefore$ Counting square free polys is counting $\mathbb{F}_q$-pts of varieties $\mathrm{Conf}^n \mathbb{A}^1$!

How to count points? Cut and Paste.

e.g. $\mathrm{Conf}^2 X = \mathrm{Sym}^2 X - \Delta$ "=" $\mathrm{Sym}^2 X - X$

$\mathrm{Conf}^3 X = \mathrm{Sym}^3 X - \{\{a,a,b\} : a, b \in X\}$

"=" $\mathrm{Sym}^3 X - X^2$

$\mathrm{Conf}^4 X$

$= \mathrm{Sym}^4 X - \{$elts of big diagonal <u>not</u> of the form $\{a,a,b,b\}\} - \{\{a,a,b,b\} : \{a,b\} \in \mathrm{Sym}^2 X\}$

"=" $\mathrm{Sym}^4 X - \{\{a,a,b,c\} : \{b,c\} \in \mathrm{Conf}^2 X\} - \mathrm{Sym}^2 X$

"=" $\mathrm{Sym}^4 X - X \times (\mathrm{Sym}^2 X - X) - \mathrm{Sym}^2 X$

$\vdots$

where "=" is modulo the relation $V = U + Z$ for $Z = V \setminus U$, U open in V.

In fact, we have a combinatorial formula to Vakil and Wood 2015:

let $[Z_X(t)] := \sum_{n=0}^{\infty} [\mathrm{Sym}^n X] t^n$

$[K_X(t)] := \sum_{n=0}^{\infty} [\mathrm{Conf}^n X] t^n$

where $[V]$ is the equivalence class in $K_0(\mathrm{Var}_k)$, the Grothendieck ring of varieties. It is the abelian gp gen by k-varieties with relation "=". The ring structure is given by

$[X][Y]=[X\times Y]$

Then $[K_X(t)] = \dfrac{[Z_X(t)]}{[Z_X(t^2)]}$

Back to point count. For $\mathbb{F}_q$-scheme X,

$$\# Z_X(t) := \sum_{n=0}^{\infty} |(\mathrm{Sym}^n X)(\mathbb{F}_q)|\, t^n$$ is the Hasse-Weil zeta function of X.

So $$\# K_{\mathbb{A}^1}(t) = \frac{\# Z_{\mathbb{A}^1}(t)}{\# Z_{\mathbb{A}^1}(t^2)}$$

$$= \frac{\frac{1}{1-qt}}{\frac{1}{1-qt^2}} = \frac{1-qt^2}{1-qt}$$

$$= 1 + qt + (q^2-q)t^2 + (q^3-q^2)t^3 + \cdots + (q^n - q^{n-1})t^n + \cdots$$

$\therefore \#\{\text{monic deg-}n \text{ square-free poly in } \mathbb{F}_q[t]\}$

$= |(\mathrm{Conf}^n \mathbb{A}^1)(\mathbb{F}_q)| = q^n - q^{n-1}$ $(n \geq 2)$,

answering <u>Q2</u>.

Using similar techniques for $\mathbb{A}^1 - \{P_1, \cdots, P_h\}$, one can find the asymptotic for

$\#\{\text{monic deg-}n$ square-free $\text{poly in } \mathbb{F}_q[t] \text{ not vanishing at } P_1, \cdots, P_h\}$

Topology of Configuration Spaces

A classical theorem of Arnold 1969:

$$H^i(\mathrm{Conf}^n \mathbb{C}; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & i=0 \\ \mathbb{Q}, & i=1,\ n \geq 2 \\ 0, & \text{otherwise} \end{cases}$$

Compare

$$|(\mathrm{Conf}^n \mathbb{A}^1)(\mathbb{F}_q)| = \begin{cases} q^n, & n=0,1 \\ q^n - q^{n-1}, & n \geq 2 \end{cases}$$

This is no coincidence!

In fact, for $X = \mathbb{A}^1_{\mathbb{Z}} - \{P_1, \cdots, P_h\}$, we have

singular Betti number

$$(\mathrm{Conf}^n X)(\mathbb{F}_q) = \sum_i (-1)^i h^i(\mathrm{Conf}^n X(\mathbb{C}))\, q^{n-i}$$

[We know a generating function for LHS, so this info $\forall q$ determines all $h^i(\mathrm{Conf}^n X)$.]

Proof. The special property of $\mathbb{A}^1 - \{P_1, \cdots, P_h\}$ is the following consequence of Kim 1994:

Frob$_q$ acts on $H^i((\mathrm{Conf}^n X)(\overline{\mathbb{F}}_q); \mathbb{Q}_\ell)$ as multiplication by q^i.

By Poincaré duality, Frob_q acts on $H^i_c(\cdots)$ by q^{i-n}. Thus Grothendieck-Lefschetz gives

$$|\mathrm{Conf}^n X(\mathbb{F}_q)| = \sum_{i=0}^{2n} (-1)^i \, tr(\mathrm{Frob}_q \mid H^i_c(\mathrm{Conf}^n X(\overline{\mathbb{F}_q}); \mathbb{Q}_\ell))$$

$$= \sum_i (-1)^i q^{i-n} h^i_c(\mathrm{Conf}^n X(\mathbb{C}))$$

$$= \sum_i (-1)^i q^{i-n} h^{2n-i}(\mathrm{Conf}^n X(\mathbb{C}))$$

$$\overset{j=2n-i}{=} \sum_j (-1)^j q^{n-j} h^j(\mathrm{Conf}^n X(\mathbb{C})) \quad \square$$

<u>Goal Question</u>

Since we understand the class of $\mathrm{Conf}^n X$ in $K_0(\mathrm{Var}_{\mathbb{C}})$ well, can we use it to understand the singular cohomology of $\mathrm{Conf}^n X$ for other complex varieties X?

<u>Negative Answer</u>

In general, class in $K_0(\mathrm{Var}_{\mathbb{C}})$ and Betti numbers don't determine each other.

- Betti numbers are not <u>motivic</u>, unlike point count. This means, if $U \subseteq X$ open, $Z = X \setminus U$, we may <u>not</u> have

$$h^i_c(X) = h^i_c(U) + h^i_c(Z)$$

- All elliptic curves have the same analytic topology, but they can have different point counts.

However, there are still cases where the goal can be achieved.

Joint work with Gilyoung Cheong

Let E be an elliptic curve$/\mathbb{C}$, $P \in E(\mathbb{C})$, consider $X = E - \{P\}$.

We record Betti numbers using compactly supported Poincaré polynomial

$$P_c(V)(u) := \sum_i h_c^i(V)(-u)^i$$

Napolitano 2003 has a table that gives the first few terms of

$$P_c(K_{E-P})(u,t) := \sum_{i,t \geq 0} h_c^i(\mathrm{Conf}^n X)(-u)^i t^n$$

$$\begin{aligned}
= 1 &+ (-2u + u^2)t \\
&+ (2u^2 - 2u^3 + u^4)t^2 \\
&+ (-4u^3 + 4u^4 - 2u^5 + u^6)t^3 \\
&+ (3u^4 - 5u^5 + 4u^6 - 2u^7 + u^8)t^4 \\
&+ (-6u^5 + 7u^6 - 5u^7 + 4u^8 - 2u^9 + u^{10})t^5 \\
&+ (4u^6 - 8u^7 + 7u^8 - 5u^9 + 4u^{10} - 2u^{11} + u^{12})t^6 \\
&+ \cdots
\end{aligned}$$

It can be obtained from a rational function via a curious rule:

$$\frac{(1-ut)^2(1-u^2t^2)}{(1-u^2t)(1-ut^2)^2} = 1 + (-2u + u^2)t + (2u^1 - 2u^3 + u^4)t^2$$
$$+ (-4u^2 + 4u^3 - 2u^5 + u^6)t^3$$
$$+ (3u^2 - 5u^4 + 4u^5 - 2u^7 + u^8)t^4$$
$$+ (-6u^3 + 7u^4 - 5u^6 + 4u^7 - 2u^9 + u^{10})t^5$$
$$+ (4u^3 - 8u^5 + 7u^6 - 5u^8 + 4u^9 - 2u^{11} + u^{12})t^6$$
$$+ \cdots$$

To summarize:

Thm 1. (Cheong-H. 2020)

$$\sum_{i,n \geq 0} (-1)^i h^i(\mathrm{Conf}^n(E-P))\, u^{2n-w(i)} t^n = \frac{(1-ut)^2(1-u^2t^2)}{(1-u^2t)(1-ut^2)^2}$$

where $w(i) = \begin{cases} 3i/2, & i \text{ even} \\ (3i-1)/2, & i \text{ odd} \end{cases}$

i	0	1	2	3	4	5	6	7
$w(i)$	0	1	3	4	6	7	9	10

It turns out that the rational function above is the <u>virtual Poincaré polynomial</u> of K_{E-p}.

The virtual Poincaré polynomial is defined by the following:

<u>Fact.</u> There is a unique way to assign to each complex variety V a polynomial $P_c^{vir}(V) \in \mathbb{Z}[u]$ s.t.

① $P_c^{vir}(V) = P_c(V)$ whenever V is smooth and projective

② If $U \subseteq V$ open, $Z = V - U$, then $P_c^{vir}(V) = P_c^{vir}(U) + P_c^{vir}(Z)$

③ $P_c^{vir}(V_1 \times V_2) = P_c^{vir}(V_1)\, P_c^{vir}(V_2)$

②③ $\Leftrightarrow$ P_c^{vir} well-defined on $K_0(Var_{\mathbb{C}})$

We say P_c^{vir} is <u>motivic</u>

We have

- $P_c(E) = 1 - 2u + u^2$ from

i	0	1	2
$h^i(E)$	1	2	1

- $P_c(P) = 1$

So
- $P_c^{vir}(E - P) = P_c(E) - P_c(P) = u^2 - 2u$
- By a special case of a formula of Cheah 1994, which says

$$P_c^{vir}(Z_V) = \left(\frac{1}{1-t}\right)^{P_c^{vir}(V)},$$

where the power is computed with the convention $\left(\frac{1}{1-t}\right)^{u^i} = \frac{1}{1-u^i t}$, we see that

$$P_c^{vir}(Z_{E-p}) = \left(\frac{1}{1-t}\right)^{u^2-2u}$$

$$:= \left(\frac{1}{1-t}\right)^{u^2} \left(\left(\frac{1}{1-t}\right)^{u}\right)^{-2}$$

$$:= \frac{1}{1-u^2 t} \cdot (1-ut)^2$$

Formulas like this first appeared in Macdonald 1975, in which P_c^{vir} is replaced by P_c.

Since P_c^{vir} is well-defined on $K_0(Var_{\mathbb{C}})$,

$$P_c^{vir}(K_{E-p})(u,t) = \frac{P_c^{vir}(Z_{E-p})(u,t)}{P_c^{vir}(Z_{E-p})(u,t^2)}$$

$$= \frac{(1-ut)^2}{1-u^2t} \cdot \frac{1-u^2t^2}{(1-ut^2)^2}$$, as appears above

[Can't do the same for $P_c(K_{E-p})$ because P_c is not motivic!]

Hence Thm 1 has the same theme as the goal question: motivic quanity (in this case P_c^{vir}) records the information about Betti numbers.

But why it happens? Again, in general, neither of P_c and P_c^{vir} determines the other.

Key: a finer invariant that determines both P_c and P_c^{vir}!

Mixed Hodge Theory

Recall: If V is a smooth projective var$/\mathbb{C}$, then we have a Hodge decomposition

$$H^i(V;\mathbb{C}) = \bigoplus_{\substack{p+q=i \\ p,q\geq 0}} H^{p,q}(V)$$

Write $h^{p,q;i}(V)$ = dim of (p,q)-piece of $H^i(V;\mathbb{C})$.
So far, i is redundant because $H^i(V;\mathbb{C})$ only has (p,q)-piece for $p+q=i$.

However, Deligne has an extension to all var$/\mathbb{C}$, called mixed Hodge structure (MHS):

Thm (Deligne) There is a natural filtration of $H^i(V;\mathbb{Q})$:

$$0 = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_{2i} = H^i(V;\mathbb{Q})$$

and a natural decomposition

$$\mathrm{Gr}^W_m H^i(V;\mathbb{C}) := \frac{W_m}{W_{m-1}} \otimes \mathbb{C} = \bigoplus_{p+q=m} H^{p,q;i}(V)$$

These data are called MHS of $H^i(V;\mathbb{Q})$

Similarly for $H^i_c(V;\mathbb{Q})$.

Fact. Most of "basic" exact sequences you can think of strictly preserve MHS. For example, X smooth proj, $Z \subseteq X$ smooth closed, $U = X - Z$ open, then the "cut & paste" LES

$$\cdots \to H^i_c(U) \to H^i_c(X) \to H^i_c(Z) \to H^{i+1}_c(U) \to \cdots$$

induces LES

$$\cdots \to H^{p,q;i}_c(U) \to H^{p,q;i}_c(X) \to H^{p,q;i}_c(Z) \to H^{p,q;i+1}_c(U) \to \cdots$$

$\forall$ fixed $p,q \geq 0$.

Rmk. We have $h^i(V) = \sum_{p,q\geq 0} h^{p,q;i}(V)$. When V is smooth proj, $h^{p,q;i} = 0$ unless $p+q = i$.

Def. Compactly supported mixed Hodge polynomial

$$H_c(V)(x,y,u) := \sum_{p,q,i\geq 0} h_c^{p,q;i}(V)\, x^p y^q (-u)^i$$

Hodge-Euler polynomial or E-polynomial:

$$E(V)(x,y) := H_c(V)(x,y,1)$$

$$= \sum_{p,q} \sum_i (-1)^i h_c^{p,q;i} \, x^p y^q$$

Important fact:

Just like Euler characteristic, E-poly is motivic.

Pf. Cut & Paste LES strictly preserves MHS.

Note that

$$P_c(V)(u) = H_c(V)(1,1,u)$$

$$P_c^{vir}(V)(u) = H_c(V)(u,u,1)$$

$$= E(V)(u,u)$$

[This is how we construct P_c^{vir} and prove it exists!]

Dependence Graph:

$$\begin{tikzcd}
 & & \chi(\cdot) \text{ (Euler char)} \\
 & P_c^{vir}(\cdot)(u) \arrow[ur, "u=1"] & \\
 & E(\cdot)(x,y) \text{ or } ``H_c^{vir}" \arrow[u, "x=y=u"] \arrow[r, phantom, "\star"] & P_c(\cdot)(u) \arrow[uu, "u=1"'] \arrow[ul, "\times"] \\
\text{Point count in } \mathbb{F}_q \arrow[ur, "\text{Katz}"] & & H_c(\cdot)(x,y,u) \arrow[ul, "u=1"] \arrow[u, "x=y=1"']
\end{tikzcd}$$

Class in $K_0(\mathrm{Var}_{\mathbb{C}})$

[In fact, to make the parallelogram ☆ more obvious, one could consider $H_c^{vir}(\cdot)(x,y,u) = E(\cdot)(xu, yu)$ instead of $E(\cdot)$. Then $H_c^{vir}(\cdot)$ is the unique motivic invariant that agrees with $H_c(\cdot)$ on smooth proj var. But H_c^{vir} and $E(\cdot)$ determine each other, so we may as well use $E(\cdot)$.]

One can make a *natural* guess about $H_c(K_{E-p})$ that specializes to both the conjectured form of $P_c(K_{E-p})$ and the known $E(K_{E-p})$, $P_c^{vir}(K_{E-p})$. It turns out to be equiv. to

__Thm 2.__ (Cheong-H. 2020) $H^i(\mathrm{Conf}^n(E-p))$ is __pure of degree__ $w(i) = \begin{cases} 3i/2, & i \text{ even} \\ (3i-1)/2, & i \text{ odd} \end{cases}$, which means

$$h^{p,q;i}(\mathrm{Conf}^n(E-p)) = 0 \quad \text{unless } p+q = w(i)$$

The rest of the talk is a sketch of proof of Thm 2.

__Leray Spectral Seq__

Consider __ordered conf. space__

$$F(X,n) = \{(x_1, \cdots, x_n) \in X^n : x_i \neq x_j \ \forall i \neq j\} \overset{\text{open}}{\subseteq} X^n.$$

The embedding $f: F(X,n) \to X^n$ induces a Leray S.S. $E_2^{p,q} = H^p(X^n; R^q f_* \mathbb{Q}) \Rightarrow H^{p+q}(F(X,n); \mathbb{Q})$

We will study $H^*(F(X,n))$ and then take S_n-invariant:

$$H^*(F(X,n))^{S_n} = H^*(\mathrm{Conf}^n X)$$

Totaro 1996 has an explicit description of $E_2^{p,q}$ above for any variety X and $n \geq 0$ [also in Bibby 2016 but in different setting]

$E_2(X,n) := \bigoplus_{p,q\geq 0} E_2^{p,q}$ is bigraded-comm. alg. generated by $g_{\{i,j\}}$ ($\{i,j\} \subseteq \{1,\dots,n\}$, $i \neq j$) (write $g_{ij} = g_{ji} = g_{\{i,j\}}$)

and $H^*(X^n)$, where

- $E_2^{p,q}$ is the bidegree (p,q)-part [Not to confuse with Hodge type.]
- Bigraded commutativity means
$$\alpha\beta = (-1)^{\deg\alpha \deg\beta}\beta\alpha, \text{ where deg means } p+q$$
for an element of bidegree (p,q).
- $H^p(X^n)$ has bidegree $(p,0)$
- g_{ij} has bidegree $(0,1)$

with relations

- $g_{ij}g_{jk} + g_{jk}g_{ki} + g_{ki}g_{ij} = 0$ [Ref: look for "Orlik-Solomon Algebra"]
- $g_{ij}\alpha_i = g_{ij}\alpha_j$, where $\alpha \in H^*(X)$ and $\alpha_i := (p_i : X^n \twoheadrightarrow X)^*\alpha$. p_i: i-th coord proj.

The differential of E_2 page is given by

$dg_{ij} = [\Delta_{ij}] \in H^2(X^n)$ the class asso. to the

$d|_{H^*(X^n)} = 0$

and graded Leibniz rule.

big diagonal divisor

$\Delta_{ij} = \{x_i = x_j\}$ [The association is the "alg. cycle map" that appears in Hodge conj.]

The important properties are:

Also for MHS, as other literature points out

(1) The S.S. can compute the weight filtration of $H^*(F(X,n))$:

One can give weight filtration to all pages of Leray S.S., and on E_2 it is given by letting g_{ij} have weight 2, $H^*(X^n)$ have wt from MHS.

In particular E_∞ has a weight filtration, which gives a weight filtration on $H^*(F(X,n))$. The assertion is that it is the one from MHS.

(2) S_n action is kept track of. On E_2 page, it is given by permuting all indices.

(3) If $H^i(X)$ is pure of wt i $\forall i$, then the S.S. degenerates at E_3, i.e. $E_3 = E_\infty$. So we can compute E_∞ just from E_2 and d_2.

<u>Special Feature</u> $H^i(E-P)$ is pure of wt i.

In fact this property holds for smooth proj var and singly punctured smooth proj var, but not (smooth proj curve) $-\{\geq 2\text{ pts}\}$.

In the case of $X = E - P$, we have

$H^*(X)$ is generated by $x, y \in H^1(X)$ with relation $xy = 0$ [and of course $x^2 = y^2 = 0$ from graded commutativity]

So $E_2(X,n)$ is gen by g_{ij}, x_i, y_i w/ relations and

$$dg_{ij} = [\Delta_{ij}] = -x_i y_j - x_j y_i$$

Write $\alpha_{ij} := g_{ij}\alpha_i = g_{ij}\alpha_j$ (recall the relations) for $\alpha \in H^*(X)$.

Observe $dx_{ij} = dg_{ij} \cdot x_i - g_{ij} \underbrace{dx_i}_{=0}$

$$= (-x_i y_j - x_j y_i) x_i$$

$$= 0 \text{ because } x_i y_i = p_i^*(\underbrace{xy}_{=0}) = 0$$

Same for $dy_{ij} = 0$.

☆ This is not true for smooth proj. curve w/o punctures.

Next, we find a vector space generator for $E_2(X,n)$, and use the fact standard in rep. theory of S_n:

- For a $\mathbb{Q}$-vector space M with S_n-action, we have

$$M^{S_n} = e_n M$$

where $e_n = \frac{1}{|S_n|} \sum_{\sigma \in S_n} \sigma$

So we find a vector space generator for $E_2(X,n)^{S_n}$ of the form $e_n(\cdots)$. It turns out many such elements are 0,

e.g.
- $e_n(x_1 x_2) = 0$ because $\sigma = (12)$ acts on $x_1 x_2$ as -1

  so $e_n(x_1 x_2) = e\sigma(x_1 x_2) = e(-x_1 x_2) = -e(x_1 x_2)$
- $e_n(g_{12} g_{34}) = 0$ because $\sigma = (13)(24)$ acts on $g_{12}g_{34}$ as -1
- (Harder, needs Orlik-Solomon relation & combinatoric trick)

$$e_n(g_{12}g_{23}\cdots g_{(r-1)r}) = 0$$

The conclusion is:

$E_2(X,n)^{S_n}$ has a basis given by

$$e_n(g_{\bullet\bullet}^{r}\, x_{\bullet}^{s_1}\, y_{\bullet}^{s_2}\, \underbrace{x_{\bullet\bullet}\cdots x_{\bullet\bullet}}_{t_1}\, \underbrace{y_{\bullet\bullet}\cdots y_{\bullet\bullet}}_{t_2})$$

where "$\bullet$" are distinct indices, $r, s_1, s_2 \in \{0,1\}$,

$t_1, t_2 \geq 0$, $r+s_1+s_2+2t_1+2t_2 \leq n$

Thanks to $d\alpha_i = 0$, $d\alpha_{ij} = 0$, the differential on $E_2(X,n)^{S_n}$ is easy to describe.

Using $E_3(X,n)^{S_n} = \left(\mathrm{Ker}(d)/\mathrm{Im}(d)\right)^{S_n}$

$$= \frac{\mathrm{Ker}(d|_{E_2(X,n)^{S_n}})}{\mathrm{Im}(d|_{E_2(X,n)^{S_n}})}$$

since taking S_n invariant is exact in characteristic zero representation

we get an explicit basis for $E_3(X,n)^{S_n}$, thus for $H^*(\mathrm{Conf}^n X)$.

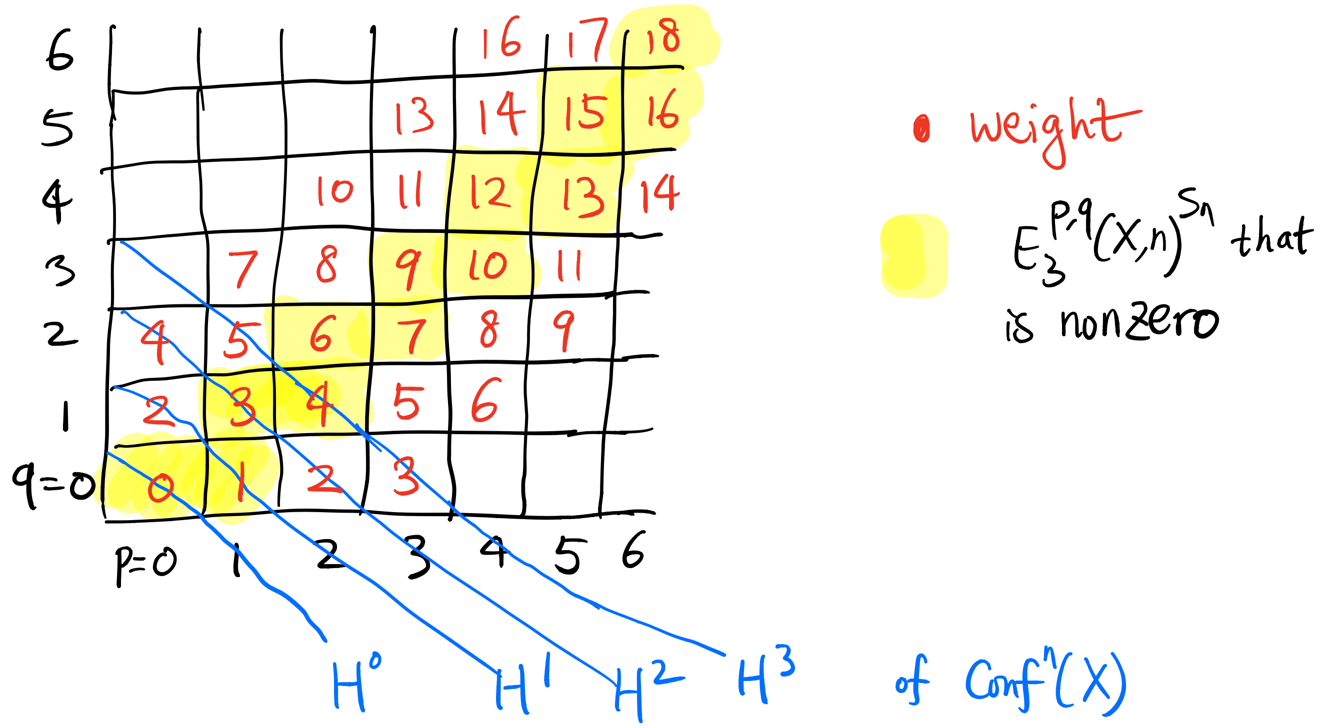
How do the weights distribute?

It turns out $E_3(X,n)^{S_n}$ is concentrated at bidegree (p,q) where $p-q=1$ by observing this basis.

But the weight of bidegree (p,q) piece is $p+2q$.

(g_{ij} has bideg $(0,1)$ and weight 2, and $H^p(X^n)$ has bideg

$(p,0)$ and weight p — this relies on purity of $H^*(X)$ and Künneth.) Also recall (p,q) piece contributes to H^{p+q}.



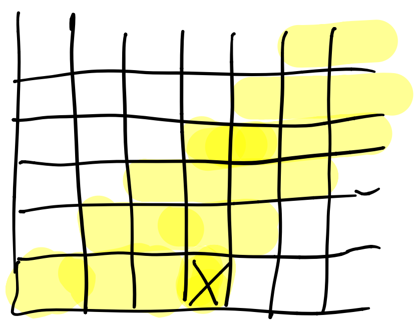
We see that H^i is pure of weight $w(i)$, where

$$i = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \ldots$$
$$w(i) = 0,\ 1,\ 3,\ 4,\ 6,\ 7,\ \ldots$$

Further Remarks

1. One can do similar things for $X = C - P$, C a smooth proj curve of genus $g > 1$, except that we no longer have purity.

   In fact, for $g=2$, we have



Each H^i has two weights.

There should be a combinatorial description of mixed Hodge numbers $h^{p,q;i}(\mathrm{Conf}^n(\mathbb{C}-P)) = \dim H^{p,q;i}(\mathrm{Conf}^n(\mathbb{C}-P))$, but will not be as simple as "rational generating func + degree shifting rule".

In particular, stabilization phenomenon is true:

Fix p,q,i. Then $h^{p,q;i}(\mathrm{Conf}^n(\mathbb{C}-P))$ stabilizes as $n \to \infty$.

2. Once we understand $\mathbb{C}-P$, we can understand $\mathbb{C}-\{P_1, \cdots, P_h\}$ because of "splitting theorem" due to Napolitano:

X: smooth non projective curve$/\mathbb{C}$, $P \in X(\mathbb{C})$,

Then

$$P_c(K_{X-P}) = \frac{1}{1+ut} P_c(K_X)$$

In fact, I can refine it to the level of MHS (Paper in preparation).

3. The case for smooth proj curve C is a mess.

Δ

Though:

- Napolitano computes first few $h^i(\mathrm{Conf}^n E)$
- Maybe one can find combinatorial formula
- It is unclear if anything can be said about mixed Hodge numbers.