RESEARCH STATEMENT: YIFENG HUANG

My primary interests are algebraic geometry and number theory, and I am interested in various questions on the interface of arithmetic statistics, combinatorics and algebraic topology. I have studied problems involving moduli spaces, Diophantine equations, random matrices, elliptic curves, singular curves, zeta functions, partitions and q-series. Among other problems, I have initiated the investigation of "matrix points on varieties", which amounts to the study of certain matrix equations naturally arising from arithmetic geometry. This topic has exhibited rich connections with a broad territory of mathematics, ranging from modular forms, hypergeometric functions to knot theory and mathematical physics.

1. Counting matrix points on varieties

For millenia, mathematicians aspire to understand Diophantine equations and their solutions. An important aspect of this immortal subject is counting solutions in finite fields. The investigation of it has led to a number of deep findings in the last century. One celebrated example is the Sato–Tate conjecture in 1960s, which concerns $\mathbb{F}_p$-point counts on elliptic curves and their distribution as the prime p varies. Half a century later, it was famously proved by a series of works [8, 22, 35, 63] which helped Richard Taylor win the Breakthrough Prize in Mathematics in 2015.

On the other hand, matrix equations play an important role in algebraic geometry, combinatorics, representation theory, topology and mathematical physics. They give rise to geometric objects such as quiver varieties, commuting varieties and character varieties that have rich connections to multiple branches of mathematics. Counting solutions to matrix equations over finite fields has been a fruitful method to study the relevant varieties, see for example [9, 36]. However, solution counting is much harder in the matrix setting and most of the successful instances rely on specific structures of the equation.

In a series of works [39, 42, 43], I introduced a robust approach to matrix equations by considering matrix solutions to a general Diophantine equation. More precisely, given a Diophantine equation $f_1(T_1, \dots, T_m) = \cdots = f_r(T_1, \dots, T_m) = 0$ over any field k and given an integer $n \geq 1$, we consider the matrix equation

$$A_iA_j = A_jA_i \text{ and } f_j(A_1, \dots, A_m) = 0, \text{ for } A_i \in \mathrm{Mat}_n(k), \tag{1}$$

where we note that the commuting condition is naturally required so that substituting matrices into a polynomial is well-defined. In a geometric language, if X is the affine variety cut out by the Diophantine equation $f_1 = \cdots = f_r = 0$, then we denote the solution set of (1) by $X(\mathrm{Mat}_n(k))$ and call it the set of $n \times n$**-matrix points** on X. This construction has led to many examples of matrix equations whose solution counting can be understood using arithmetic insights about its associated polynomial equation. My study of matrix points on varieties has also resulted in a number of theorems and open problems showing surprising connections to other branches of mathematics.

I now survey some of my results on matrix point counts on X, organized by the geometry of X.

1.1. Sato–Tate type distributions for special families of varieties.

We start with a motivating example. Over a finite field $\mathbb{F}_q$ with $q = p^r$ elements, consider the Legendre elliptic curve E_λ with parameter $\lambda \neq 0, 1$. Then the set of $n \times n$-matrix points on E_λ is defined as

$$E_\lambda(\mathrm{Mat}_n(\mathbb{F}_q)) := \{(A, B) \in C_{n,2}(\mathbb{F}_q) : B^2 = A(A - I_n)(A - \lambda I_n)\},$$

where $C_{n,m}(\mathbb{F}_q)$ denotes the set of pairwise commuting m-tuples of $n\times n$-matrices over $\mathbb{F}_q$. It is natural to ask whether $\#E_\lambda(\mathrm{Mat}_n(\mathbb{F}_q))$ can be computed, and how it distributes as λ varies. In joint work with Ono and Saad [43], we found a formula for $\#E_\lambda(\mathrm{Mat}_n(\mathbb{F}_q))$ in terms of certain combinatorial polynomials depending on n, and certain arithmetic functions [33] depending on λ. Moreover, the limiting distribution of $\#E_\lambda(\mathrm{Mat}_n(\mathbb{F}_q))$ is described as follows. For any $n \geq 1$, let $a_n^*(\lambda; q)$ denote a suitable normalization of $\#E_\lambda(\mathrm{Mat}_n(\mathbb{F}_q))$, then we have the following result.

Theorem 1 (H.–Ono–Saad [43]). *The histogram of $\{a_n^*(\lambda; q) : \lambda \in \mathbb{F}_q\}$ approaches the semicircular distribution that has famously appeared in the Sato–Tate conjecture. More precisely, for $-2 \leq a < b \leq 2$ and $n, r \geq 1$, we have*

$$\lim_{p\to\infty} \frac{\#\{\lambda \in \mathbb{F}_{p^r} : a_n^*(\lambda; p^r) \in [a, b]\}}{p^r} = \frac{1}{2\pi}\int_a^b \sqrt{4 - t^2}\, dt.$$

Theorem 1 is a distributional result about matrix point counts on elliptic curves. I have also obtained similar theorems in higher dimensions. In joint work [43] with Ono and Saad, I proved analogous results for a special family of $K3$ surfaces discovered by Ahlgren, Ono and Penniston [2]. More precisely, for $\lambda \neq 0, -1$, consider

$$X_\lambda(\mathrm{Mat}_n(\mathbb{F}_q)) := \{(A, B, C) \in C_{n,3}(\mathbb{F}_q) : C^2 = AB(A + I_n)(B + I_n)(A + \lambda B), C \in \mathrm{GL}_n(\mathbb{F}_q)\}.$$

We found an exact formula for $\#X_\lambda(\mathrm{Mat}_n(\mathbb{F}_q))$ and its limiting distribution as λ varies [43]. The distribution has an interesting and nontrivial shape, which is very different from the Sato–Tate semicircular distribution.

In an ongoing project with Ono, I determined the exact values and the distribution for $n \times n$ matrix point counts of some special Calabi–Yau 3-folds for $n \leq 3$. For example, for an odd prime p and an integer $n \leq 3$, let

$$Y(\mathrm{Mat}_n(\mathbb{F}_p)) := \{(A_1, \dots, A_4) \in C_{n,4}(\mathbb{F}_p) : A_i \in \mathrm{GL}_n(\mathbb{F}_p), \sum_{i=1}^{4}(A_i + A_i^{-1}) = 0\}.$$

Then I showed that as p varies, the normalized $\#Y(\mathrm{Mat}_n(\mathbb{F}_p))$ follows the semicircular distribution.

The proof of the theorems in this section is based on known arithmetic results [1, 59, 66] about special varieties, and my previous work on matrix point counts of a variety in general, see below.

1.2. **Varieties in general.** Matrix point counts for a variety as simple as the affine plane are already interesting: $\mathbb{A}^2(\mathrm{Mat}_n(\mathbb{F}_q))$ corresponds to the set of commuting pairs of $n \times n$-matrices over $\mathbb{F}_q$. Feit and Fine proved a remarkable theorem [26] in 1960 that provides the count of these matrix points. Their formula involves integer partitions.

For X in general, I established a relation [39] between $\#X(\mathrm{Mat}_n(\mathbb{F}_q))$ and a zeta function defined by Cohen and Lenstra in their seminal work [24]. When X is a smooth variety of dimension d, its associated Cohen–Lenstra zeta function can further be expressed in terms of counting d-tuples of pairwise commuting matrices over $\mathbb{F}_q$. This allows us to prove general formulas about matrix point counts on smooth varieties using Feit–Fine's theorem and its potential generalizations. For example, if $d \leq 2$, I explicitly determined $\#X(\mathrm{Mat}_n(\mathbb{F}_q))$ in terms of the ordinary point counts $\#X(\mathbb{F}_{q^r})$ for $1 \leq r \leq n$ [39] as an application of Feit–Fine. If $d \geq 3$, I gave analogous formulas for $\#X(\mathrm{Mat}_n(\mathbb{F}_q))$ for $n \leq 3$ in an ongoing project with Ono. This is achieved by counting commuting tuples of small matrices using observations in [28, 34]. The case $n \geq 4$ remains open even when $d = 3$.

1.3. **Reduced singular curves.** The next nontrivial case, namely, the case where X is a reduced singular curve, turns out to exhibits unexpected phenomena that open the gate to new connections. In my series of works [39, 42] and my current joint work with Jiang [41], I computed a generating function associated with the matrix point counts on curves with some simplest examples of singularities, and showed some surprising properties. I now define the generating function precisely.

Definition 2 (H. [39])**.** For any variety $X/\mathbb{F}_q$, define $\widehat{Z}_X(t) := 1 + \sum_{n=1}^{\infty} \frac{\#X(\mathrm{Mat}_n(\mathbb{F}_q))}{\#\mathrm{GL}_n(\mathbb{F}_q)} t^n \in \mathbb{Q}[[t]]$.

If X is a reduced singular curve, define its normalization by $\widetilde{X}$ and consider a normalized generating series $N\widehat{Z}_X(t) := \widehat{Z}_X(t)/\widehat{Z}_{\widetilde{X}}(t)$. Since $\widehat{Z}_{\widetilde{X}}(t)$ is known [24], counting matrix points on X amounts to computing $N\widehat{Z}_X(t)$. The reason why we introduce $N\widehat{Z}_X(t)$ is that if X has exactly one singular point p, then $N\widehat{Z}_X(t)$ is determined by the completed local ring $R := \widehat{\mathcal{O}}_{X,p}$. We may thus treat $N\widehat{Z}_X(t)$ as an invariant for curve singularities, and denote it by $N\widehat{Z}_R(t)$.

I explicitly determined the matrix point counts on reduced curves with singularities of the form $y^2 = x^n$, where $n \geq 2$. This amounts to computing $N\widehat{Z}_R(t)$ for rings R associated with these singularities. The explicit formulas are of combinatorial interest. We use the standard notation $(a;q)_n := (1-a)(1-aq)\dots(1-aq^{n-1})$.

- If $n = 2$, then $N\widehat{Z}_R(t)$ is given by a ${}_0\phi_1$-basic hypergeometric series [39]. More precisely, $\widehat{Z}_R(t)|_{q\mapsto q^{-1}} = (tq;q)_\infty \cdot {}_0\phi_1(-;tq;q,t^2q)$. This series has a combinatorial interpretation in terms of partitions and Durfee squares [39]. The proof involves explicitly enumerating solutions to $AB = BA = 0$ with $A, B \in \mathrm{Mat}_n(\mathbb{F}_q)$.
- If n is odd, then $N\widehat{Z}_R(t)$ is given by a Ramanujan-type partition sum [41]; see Theorem 3 below.
- If $n \geq 4$ is even, then $N\widehat{Z}_R(t)$ can be given by a partition sum involving Hall polynomials [41].

In the theorem below, $a_q(\lambda)$ is an explicit polynomial in q associated with a partition λ that plays a role in symmetric functions [50, p. 181] and the celebrated Gordon–Rogers–Ramanujan identities [4, p. 111].

Theorem 3 (H.–Jiang [41])**.** *Let $m \geq 1$ and $R := \mathbb{F}_q[[x,y]]/(y^2 - x^{2m+1})$, then we have*

$$N\widehat{Z}_R(t) = \sum_{\lambda_1 \leq m} \frac{1}{a_q(\lambda)} t^{2|\lambda|},$$

where $\lambda_1 \leq m$ means the sum extends over partitions λ whose parts are at most m.

For these singularities, I also derived product formulas for the special values $N\widehat{Z}_R(\pm 1)$.

Theorem 4 ([39] for $n = 2$, [41] for $n \geq 3$)**.** *Let $n \geq 2$ and $R = \mathbb{F}_q[[x, y]]/(y^2 - x^n)$. If $n = 2m + 1$, then*

$$N\widehat{Z}_R(\pm 1)|_{q \mapsto q^{-1}} = \prod_{\substack{i=1 \\ i \not\equiv 0, \pm(m+1) \bmod 2m+3}}^{\infty} (1 - q^i)^{-1}.$$

On the other hand, if $n = 2m$, then $N\widehat{Z}_R(1) = 1$ and

$$N\widehat{Z}_R(-1)|_{q \mapsto q^{-1}} = \frac{(q^2; q^2)_\infty (q^{m+1}; q^{m+1})^2_\infty}{(q; q)^2_\infty (q^{2m+2}; q^{2m+2})_\infty}.$$

The computation of $N\widehat{Z}_R(t)$ for $n \geq 3$ requires my work in enumerative geometry, see §2.2.2. Based on these explicit formulas, I formulated two general conjectures. First, $N\widehat{Z}_R(t)$ should converge for all $t \in \mathbb{C}$ [39]. Second, the special values $N\widehat{Z}_R(\pm 1)$ should give rise to modular functions [41]. These conjectures are verified for the $y^2 = x^n$ singularities on a case-by-case basis. It is not known why $t = \pm 1$ should be special. An important goal of my current and future research is to understand why these phenomena occur. This investigation will potentially lead to new connections between matrix equations, modular forms, combinatorics, and other fields.

2. Enumerative geometry of moduli spaces

Many important invariants in geometry and physics are counting invariants. For example, Weil's zeta function is a point-counting invariant, and the Gromov–Witten invariant in physics is a curve-counting invariant. Often, the configurations we are counting form a continuum, so naïve counting does not make sense. Instead, we shall consider a suitable *enumerative invariant* of the *moduli space* parametrizing all possible configurations. The enumerative invariant could be a dimension, a (co)homology class, or a Betti number, etc.

I am interested in "counting n-point configurations" on a variety. There are several moduli spaces of this flavor.

2.1. **Configuration spaces.** Let X be a manifold and let $H^i(\mathrm{Conf}_n(X))$ denote the i-th cohomology group of the configuration space $\mathrm{Conf}_n(X)$ parametrizing n unlabeled *distinct* points on X. A fundamental problem in topology is to understand how $H^i(\mathrm{Conf}_n(X))$ behaves as i and n vary. This problem has motivated a rich theory of representation stability introduced in a series of works by Church, Ellenberg and Farb [18–20]. A remarkable consequence of the representation stability is that for a "nice" X and any fixed i, the Betti numbers $h^i(\mathrm{Conf}_n(X))$ stabilizes as $n \to \infty$.

My work is aimed at understanding $h^i(\mathrm{Conf}_n(X))$ for complex varieties X by making connections to point counting over finite fields. Such connections, if made, would lead to a rational generating function for $h^i(\mathrm{Conf}_n(X))$ due to a rationality result on the point count side [65, Prop. 5.9]. The rationality would in turn strongly constrain how $h^i(\mathrm{Conf}_n(X))$ behaves as both i and n vary. For example, the aforementioned stabilization can be captured by one factor in the denominator, while any other factor amounts to a different stability behavior.

Deligne's mixed Hodge theory is naturally involved in making connections between Betti numbers and point counts over finite fields. The motivating example $X = \mathbb{A}^1$ is as follows [19]:

$$\begin{array}{lll} n = 0, 1, \quad \#\,\mathrm{Conf}_n(\mathbb{F}_q) = q^n & \longleftrightarrow & h^0(\mathrm{Conf}_n(\mathbb{C})) = 1; \\ n \geq 2, \quad \#\,\mathrm{Conf}_n(\mathbb{F}_q) = q^n - q^{n-1} & \longleftrightarrow & h^0(\mathrm{Conf}_n(\mathbb{C})) = 1, h^1(\mathrm{Conf}_n(\mathbb{C})) = 1, \end{array}$$

where a proof was given using the *purity* of the mixed Hodge structure of $\mathrm{Conf}_n(\mathbb{C})$. Discoveries like this have led to surprising connections between configuration spaces and factorization statistics of polynomials over finite fields, which shed light to both sides [19, 44, 45, 61]. In the example above, the knowledge of $h^i(\mathrm{Conf}_n(\mathbb{C}))$ is a famous theorem of Arnol'd [5], and $\mathrm{Conf}_n(\mathbb{F}_q)$ corresponds to the set of monic square-free polynomials of degree n in $\mathbb{F}_q[T]$.

Joint with Cheong [13], I obtained a high-genus analogue of the above. For $g, r \geq 0$, let $\Sigma_{g,r}$ be an r-punctured algebraic curve of genus g; note that $\mathbb{A}^1 = \Sigma_{0,1}$ belongs to the genus 0 case. Our result connects $h^i(\mathrm{Conf}_n(\Sigma_{1,1}))$, via a curious degree shift, to a rational generating function closely related to the $\mathbb{F}_q$-point count of $\mathrm{Conf}_n(\Sigma_{1,1})$.

Theorem 5 (Cheong–H. [13])**.** $\displaystyle\sum_{n,i \geq 0} (-1)^i h^i(\mathrm{Conf}_n(\Sigma_{1,1})) u^{2n - w(i)} t^n = \frac{(1 - ut)^2 (1 - u^2 t^2)}{(1 - u^2 t)(1 - ut^2)^2}$, *where* $w(i) = \lfloor \frac{3i}{2} \rfloor$.

The proof of Theorem 5 is based on the following purity statement, which explains why $w(i)$ appears.

Theorem 6 (Cheong–H. [13])**.** *For $i, n \geq 1$, the mixed Hodge structure of $H^i(\mathrm{Conf}_n(\Sigma_{1,1}))$ is pure of weight $w(i)$.*

We have seen that mixed Hodge theory is instrumental to relate Betti numbers and point counts of $\mathrm{Conf}_n(X)$. In view of the above, it is natural to investigate the mixed Hodge numbers of $H^i(\mathrm{Conf}_n(X))$ for X in general. In [13] we gave a complete answer for $X = \Sigma_{1,1}$ as a consequence of Theorem 6 (cf. Corollary 8). However, analogues of Theorem 6 no longer hold in general [13, §1.2]. Yet I was able to give a partial answer. We denote the (p, q)-th mixed Hodge number of $H^i(\mathrm{Conf}_n(X))$ by $h^{p,q;i}(\mathrm{Conf}_n(X))$ for $p, q \geq 0$.

Theorem 7 (H. [37])**.** *Let X be a smooth compact complex variety of dimension d, and let X_r denote the r-punctured version of X. Then for $r \geq 1$, we have*

$$\sum_{p,q,i,n\geq 0} h^{p,q;i}(\mathrm{Conf}_n(X_r))x^p y^q (-u)^i t^n = \frac{1}{(1+x^d y^d u^{2d-1} t)^{r-1}} \sum_{p,q,i,n\geq 0} h^{p,q;i}(\mathrm{Conf}_n(X_1))x^p y^q (-u)^i t^n.$$

This allows us to reduce all cases with $r \geq 1$ to the case $r = 1$, which implies the following.

Corollary 8 ([13, 37])**.** *For $r \geq 1$, we have*

$$\sum_{p,q,i,n\geq 0} h^{p,q;i}(\mathrm{Conf}_n(\Sigma_{1,r}))x^p y^q (-u)^i t^n = \frac{1}{(1+xyut)^{r-1}} \frac{1-(x+y)ut+(x+y)xyu^2t^3 - x^2y^2u^3t^4}{(1-t)(1-x^2yu^2t^2)(1-xy^2u^2t^2)}.$$

Investigating the analogue of Corollary 8 for $\Sigma_{g,r}$ with $g \geq 2$ and $r \geq 1$ is a project in progress, in which an explicit generating function is conjectured. It is related to point counts over finite fields via a novel degree shift.

Theorem 7 is an instance of the "splitting" phenomenon. It is first discovered in 1981 [29] and has been intensively studied [23, 55, 56], though not from the Hodge-theoretic perspective. Theorem 7 is a Hodge-theoretic refinement of a result of Kallel [46]. To prove Theorem 7, I first established an S_n-equivariant strengthening by examining the spectral sequence [64] for a certain hyperplane-like arrangement, and using mixed-Hodge purity of $H^i(X \setminus \{\mathrm{pt}\})$. The proof suggests that hypotheses about X can be relaxed, and "categorifications" of Theorem 7 exist.

2.2. **Hilbert schemes of points and beyond.** We work over an algebraically closed field k. The **Hilbert scheme** of n points on a variety X (denoted by $\mathrm{Hilb}_n(X)$) parametrizes quotients of the structure sheaf $\mathcal{O}_X$ that are supported in 0 dimension and have degree n. Hilbert schemes of points play a prominent role in algebraic combinatorics, geometric representation theory and mathematical physics. Exciting developments in the recent decade by Gorsky, Maulik, Oblomkov, Rasmussen, Shende and many others reveal surprising connections between Hilbert schemes of points on singular curves, knot theory, and q,t-Catalan combinatorics [30, 31, 51, 57, 58]. Joint with Jiang [41, 42], we investigated two analogous moduli spaces on singular curves through explicit computations. This would potentially lead to finer models for link invariants and generalizations of the q,t-Catalan numbers.

For $n \geq 0$ and a variety X/k, let $\mathrm{Coh}_n(X)$ be the moduli stack parametrizing coherent sheaves on X of degree n with zero-dimensional support. We call $\mathrm{Coh}_n(X)$ the **stack of length-n sheaves** on X. Given a coherent sheaf $\mathcal{E}$ on X, we consider the moduli space $\mathrm{Quot}_{\mathcal{E},n} := \{\mathcal{E} \twoheadrightarrow M : M \in \mathrm{Coh}_n(X)\}$ parametrizing quotients of $\mathcal{E}$ in $\mathrm{Coh}_n(X)$. This is a scheme, which we call the **Quot scheme** of $\mathcal{E}$. The enumerative invariants considered here are their motivic generating functions in appropriate Grothendieck rings:

$$\mathcal{Z}_{\mathcal{E}}(t) := \sum_{n\geq 0} [\mathrm{Quot}_{\mathcal{E},n}]\, t^n \in K_0(\mathrm{Var}_k)[[t]], \qquad \widehat{\mathcal{Z}}_X(t) := \sum_{n\geq 0} [\mathrm{Coh}_n(X)]\, t^n \in K_0(\mathrm{Stck}_k)[[t]].$$

2.2.1. *Quot schemes.* If $\mathcal{E} = \mathcal{O}_X$, then $\mathrm{Quot}_{\mathcal{E},n}$ is $\mathrm{Hilb}_n(X)$ by definition. The notion of the Quot scheme thus generalizes the Hilbert scheme. It was recently shown that when X is a reduced curve (possibly singular), then $\mathcal{Z}_{\mathcal{O}_X}(t)$ is rational in t [10]. On the other hand, if X is a smooth curve and $\mathcal{E}$ is a vector bundle on X, then $\mathcal{Z}_{\mathcal{E}}(t)$ is rational in t [6]. Our first result is a common generalization of both.

Theorem 9 (H.–Jiang [41])**.** *If X is a reduced curve and $\mathcal{E}$ is a torsion-free bundle of rank d on X, then $\mathcal{Z}_{\mathcal{E}}(t)$ is rational in t. Moreover, if $\widetilde{X}$ is the normalization of X, then $\mathcal{NZ}_{\mathcal{E}}(t) := \mathcal{Z}_{\mathcal{E}}(t)/\mathcal{Z}_{\mathcal{O}_{\widetilde{X}}^{\oplus d}}(t)$ is a polynomial in t.*

The second general property we proved about $\mathcal{Z}_{\mathcal{E}}(t)$ is a functional equation. Let $\mathbb{L} := [\mathbb{A}^1]$.

Theorem 10 (H.–Jiang [41])**.** *If X is a reduced projective curve whose all singularities are planar, g_a is its arithmetic genus, $d \geq 1$, and $\mathcal{E} = \mathcal{O}_X^{\oplus d}$, then the identity of rational functions*

$$\mathcal{Z}_{\mathcal{E}}(t) \approx (\mathbb{L}^{d^2} t^{2d})^{g_a - 1} \mathcal{Z}_{\mathcal{E}}(\mathbb{L}^{-d} t^{-1}) \tag{2}$$

holds in a weaker sense of point counting over finite fields.

We conjecture that (2) holds in $K_0(\mathrm{Var}_k)(t)$ in the original sense. If $d = 1$, this is a consequence of the Serre duality and the compactified Jacobian [32, 52, 60]. Attacking this for $d \geq 2$ would likely require a deep geometric understanding of the high-rank analogue of the compactified Jacobian, which is no longer a scheme, but a stack.

Our next result connects to combinatorics by considering the first explicit examples of $\mathcal{Z}_{\mathcal{E}}(t)$. Let X be the singular curve $y^2 = x^n$ over $\mathbb{C}$, where $n \geq 2$. Let $\mathcal{E} = \mathcal{O}_X^{\oplus d}$. Consider $\mathcal{NZ}_{\mathcal{E}}(t)$ defined in Theorem 9, where it is proved to be in $K_0(\mathrm{Var}_{\mathbb{C}})[t]$. The following theorem computes $\mathcal{NZ}_{\mathcal{E}}(t)$ for all d, n in terms of summations over partitions. Here, (m^d) is the partition associated with the $d \times m$ box, λ'_i denotes the i-th column of a partition λ, and $g^{\lambda}_{\mu,\nu}(q)$ is the Hall polynomial [50].

Theorem 11 (H.–Jiang [41]). *Let $n \geq 2$, $d \geq 0$ and assume the notation above. Then there is a polynomial $F_{d,n}(t;q) \in \mathbb{Z}[t,q]$ such that $\mathcal{NZ}_{\mathcal{E}}(t) = F_{d,n}(t;\mathbb{L})$. Moreover, for $m \geq 1$, we have*

$$F_{d,2m+1}(t;q) = \sum_{\mu \subseteq (m^d)} g^{(m^d)}_{\mu,(m^d)-\mu}(q)\,(q^d t^2)^{|\mu|},$$

$$F_{d,2m}(t;q) = \sum_{\lambda,\mu,\nu \subseteq (m^d)} g^{(m^d)}_{\lambda,(m^d)-\lambda}(q)\, g^{\lambda}_{\mu,\nu}(q)\, t^{|\lambda|}(q^d t)^{|\lambda|-|\mu|}(t;q)^2_{d-\lambda'_m}\frac{(q^{-1};q^{-1})_{\lambda'_m}}{(q^{-1};q^{-1})_{\mu'_m}}.$$

One combinatorial consequence of Theorems 2 and 11 is the formal identity $F_{d,n}(t;q) = (q^d t^2)^{dm} F_{d,n}(q^{-d}t^{-1};q)$ where $n = 2m$ or $n = 2m+1$. This has no direct proof so far if $n \geq 4$ is even and $d \geq 2$. In general, an important reason to consider $\mathcal{NZ}_{\mathcal{O}_X^{\oplus d}}(t)$ is that it generalizes $\mathcal{NZ}_{\mathcal{O}_X^{\oplus 1}}(t)$, an algebro-geometric model that is conjectured to recover link invariants and rational q,t-Catalan numbers [57]. For example, say n is odd, then these conjectures predict how $F_{1,n}(t;q)$ is related to the Khovanov–Rozansky homology of the $(2,n)$ torus knot and statistics of $2 \times n$ Dyck paths; they are verified in [53] among other cases. The following questions are natural from the perspectives above. For a planar singular curve X, is $\mathcal{NZ}_{\mathcal{O}_X^{\oplus d}}(t)$ determined by the algebraic link associated to the singularity? If yes, what link invariants does it encode, and can we categorify it? If X is the curve $y^m = x^n$, can one give a combinatorial interpretation for $\mathcal{NZ}_{\mathcal{O}_X^{\oplus d}}(t)$, which should specialize to the rational q,t-Catalan numbers when $d = 1$?

Our generalization from $\mathcal{Z}_{\mathcal{O}_X^{\oplus 1}}(t)$ to $\mathcal{Z}_{\mathcal{O}_X^{\oplus d}}(t)$ also plays a key role in the study of matrix points, see below.

2.2.2. *Stack of length-n sheaves.* In [42], Jiang and I studied the stack of length-n sheaves by making the following connections to matrix points (§1) and Quot schemes. First, we constructed a scheme $C_n(X)$ for *any* variety X, such that whenever X is affine, the k-points of $C_n(X)$ are $\mathrm{Mat}_n(k)$-points of X. Second, we showed that $\mathrm{Coh}_n(X)$ can be realized as the stack quotient $[C_n(X)/\operatorname{GL}_n(X)]$, which implies a motivic relation $[C_n(X)] = [\mathrm{Coh}_n(X)][\mathrm{GL}_n]$. Third, we proved a formula that computes $\widehat{\mathcal{Z}}_X(t)$ (recall that this is the motivic generating function for $\mathrm{Coh}_n(X)$) in terms of $\mathcal{Z}_{\mathcal{O}_X^{\oplus d}}(t)$ for all $d \geq 0$. In [41], we combined these general connections and our explicit computation of $\mathcal{Z}_{\mathcal{O}_X^{\oplus d}}(t)$ in Theorem 11 to prove formulas about matrix point counts such as Theorem 3.

The connections above also suggest that invariants arising from matrix point counts of singular curves, such as the modular forms in Theorem 4, may be link invariants. It would be interesting to find out what the modular form says about the algebraic link associated to the singularity. We remark that $\mathrm{Coh}_n(X)$ and $C_n(X)$ are also of importance in mathematical physics. For example, when X is an affine plane, $C_n(X)$ is a quiver variety, and $[C_n(X)]$ and $[\mathrm{Coh}_n(X)]$ played a crucial role in Donaldson–Thomas theory [9].

2.2.3. *Methods.* Our key approach in [41] to the study of $\mathcal{Z}_{\mathcal{E}}(t)$ is a new method to compute the **lattice zeta function**, a generalization of the Dedekind zeta function introduced in [62] that has had impacts in discrete mathematics, such as subgroup growth [49]. For an arithmetic order R and a rank-d lattice L over R, the lattice zeta function $\zeta_L(s)$ counts R-sublattices of L of each index. More precisely, $\zeta_L(s) := \sum_{M \subseteq L}(L:M)^{-s}$. If X is a curve with singularity p, the motivic generating function $\mathcal{Z}_{\mathcal{E}}(t)$ is closely related to a "geometrization" of a lattice zeta function over $R := \widehat{\mathcal{O}}_{X,p}$, which is an order in a function field. The classical treatment of $\zeta_L(s)$ involves orbital integrals [11], but doing explicit calculations with this requires classifying lattices over R up to isomorphism, which is hard. Our new method involves considering for each sublattice $M \subseteq L$ an $\widetilde{R}$-lattice defined by $\widetilde{R}M$, where $\widetilde{R}$ is the integral closure of R. Since lattices over the nicer ring $\widetilde{R}$ have a well-known parametrization (by the affine Grassmannian), we get an explicit parametrization of R-sublattices of L. This explicitness helps in two ways. First, we are able to obtain concrete formulas in Theorem 11. Second, it allows us to draw *motivic* conclusions of $\zeta_L(s)$ that lead to Theorem 9.

To prove Theorem 10, we obtained an analogous functional equation about the lattice zeta function, which is a $d \geq 1$ generalization of a result by Yun [70]. Our proof of this, however, is based on the classical method (orbital integrals and Fourier transforms), which is why we could only prove (2) in a weaker sense of point counts.

3. Noncommutative Diophantine equations

A central question in number theory and arithmetic geometry is to find all integer or rational solutions of a polynomial equation and *prove* they are all. An important situation where the goal can be achieved (in finite time) is when one shows that there are only finitely many solutions, and moreover, their "heights" are bounded by an effectively computable number. We call it an *effective finiteness* theorem for this equation.

The unit equations have played an important role in the theory of Diophantine equations in general. A unit equation, in a very generic language, is an equation $x+y=1$, where x,y are in certain sets multiplicatively generated by finitely many elements. For example, we may require x,y to be in certain finitely generated subgroups of $\mathbb{C}^\times$. A toy example is $2^n - 3^m = 1$, where $n,m \in \mathbb{Z}$. A celebrated theorem [47] of Lang in 1960 states that any unit equation on $\mathbb{C}^\times$ has only finitely many solutions. As an application of the unit equations to other Diophantine equations, Lang used this result to show that a curve over $\mathbb{Q}$ of genus ≥ 1 cannot have infinitely many integral points. Using Baker's method [7], both finiteness theorems above were made effective by Evertse and Győry; see [25].

In view of the philosophy that "addition and multiplication should not be compatible", it is natural to expect analogues of the finiteness theorem in noncommutative settings. However, my result below is the first of such. Moreover, the finiteness is effective. To state it, let $\mathbb{H}_a$ be the quaternion algebra over the real algebraic numbers $\mathbb{R} \cap \overline{\mathbb{Q}}$, and consider the Euclidean norm.

Theorem 12 (H. [38]). *Let Γ_1, Γ_2 be semigroups of $\mathbb{H}_a^\times$ generated by finitely many elements of norms > 1, and fix $a, a', b, b' \in \mathbb{H}_a^\times$. If Γ_1 is commutative, then the equation*

$$afa' + bgb' = 1$$

has only finitely many solutions with $f \in \Gamma_1$ and $g \in \Gamma_2$. Moreover, there are effective bounds for the solutions (f, g).

The proof is based on Baker's method. To establish Theorem 12, I proved that a commutative subsemigroup Γ of $\mathbb{H}_a^\times$ generated by finitely many elements of norms > 1 cannot infinitely intersect an $(\mathbb{R} \cap \overline{\mathbb{Q}})$-hyperplane of $\mathbb{H}_a$ not passing through the origin. This can be viewed a noncommutative Mordell–Lang theorem in the real algebraic group $\mathbb{H}^\times$. The classical Mordell–Lang theorem essentially says that in a semiabelian variety (which is necessarily a commutative algebraic group), subgroups and subvarieties are not compatible. The analogue of Mordell–Lang do not hold in any noncommutative algebraic group over an algebraically closed field, but my work suggests its possibility in noncommutative groups such as the multiplicative group of a division algebra over a field that is not algebraically closed. This is potential future research.

Unit equations have applications to arithmetic dynamics. In [38], by considering a unit equation on the endomorphism ring of a simple abelian variety A, I proved an "orbit intersection theorem" for two self-maps on A, which fits in the general framework of dynamical Mordell–Lang theorems.

4. Discrete random matrix theory

Random matrix theory, originated in 1950s from modeling distributions in physics, is nowadays a vast and ever-growing field that is deeply connected with many areas of mathematics, such as number theory and representation theory. Classical random matrix theory concerns the distribution of spectral invariants (eigenvalues, singular values, etc.) of a random $n \times n$ real or complex matrices, as $n \to \infty$. A result of Wigner (see [3, Chap. 2]) that marks the birth of this subject reveals the ubiquity of the "universality" phenomenon: it states that a random $n \times n$ Hermitian matrix with independent entries (subject to the Hermitian constraint) whose distributions are mean 0 and variance 1 has eigenvalue distribution approaching the semicircular distribution as $n \to \infty$, after a suitable normalization. Here, the limiting distribution of eigenvalues does not depend on the distributions of individual entries.

Discrete random matrix theory, on the other hand, concerns integer matrices or p-adic integer matrices. An important spectral invariant is the Smith normal form, or equivalently, the structure of the cokernel as an abelian group. Since the proposal of the Cohen–Lenstra heuristics [24], the cokernels of random (p-adic) integer matrices have been used to model the distributions of many finite abelian groups arising in arithmetic statistics. The universality phenomenon is also ubiquitous in discrete random matrix theory. An exemplary result by Wood [69] states that if a random $n \times n$ matrix X over $\mathbb{Z}_p$ has independent entries whose residue classes modulo p are not too concentrated, then the limiting distribution of $\operatorname{cok}(X)$ (as $n \to \infty$) is always given by the "Cohen–Lenstra distribution". Universality results with analogous distributions are also found in other models, such as cokernels of random symmetric integer matrices [68], which have significance in spectral graph theory.

Widely studied random matrix models generally have independent entries, possibly subject to symmetry constraints. Instead, I have studied a model where entries are intricately dependent. More precisely, we fix a monic polynomial $P(t) \in \mathbb{Z}_p[t]$, and consider the random matrix $P(X)$, where $X \in \operatorname{Mat}_n(\mathbb{Z}_p)$ has independent and random entires. The central problem is how $\operatorname{cok}(P(X))$ distributes, and whether this model gives rise to new distributions.

This model was proposed in my joint work with Cheong [12], where we proved Cohen–Lenstra type distributions in the first few nontrivial cases. More general cases were studied in subsequent works [15–17, 48]. These results show that it is inevitable to consider an *additional* structure on $\operatorname{cok}(P(X))$, namely, its structure as a module over the ring $R := \mathbb{Z}_p[t]/P(t)$. In [17], the limiting distribution of $\operatorname{cok}(P(X))$ as an R-module is determined. The distribution depends on how $P(t)$ factorizes modulo p and involves certain algebraic invariants of R-modules. My recent work with Cheong [14] considers two further questions: (i) What is the distribution of $\operatorname{cok}(P(X))$ for each fixed n? (ii) What if the entries of X have concentrated modulo classes modulo p, which is opposite to the setting in which universality has been known? The following theorem answers both at the same time.

Theorem 13 (Cheong–H. [14])**.** *Let $n \geq 1$ and recall $R = \mathbb{Z}_p[t]/P(t)$. Fix $A \in \mathrm{Mat}_n(\mathbb{F}_p)$ and a finite-sized R-module G. If G satisfies $G/pG \cong_R \mathrm{cok}_{\mathbb{F}_p}(P(A))$ and a "Hom=Ext" condition, then*

$$\operatorname*{Prob}_{X \in \mathrm{Mat}_n(\mathbb{Z}_p)} \left(\mathrm{cok}(P(X)) \cong_R G \,\middle|\, X \equiv A \bmod p \right) = \frac{|\mathrm{Aut}_R(G/pG)| \prod_{j=1}^{l} \prod_{i=1}^{u_j(G/pG)} (1 - p^{-d_j i})}{|\mathrm{Aut}_R(G)|},$$

where $d_1, \dots, d_l$ are degrees of irreducible factors of $P(t)$ mod p, and $u_j(G/pG)$ is the dimension of a certain submodule of G/pG. Otherwise, the conditional probability is 0.

The proof requires two novel ingredients. First, we proved a *noncommutative* Weierstrass preparation theorem for $\mathrm{Mat}_n(\mathbb{Z}_p)$, which, together with a linearization trick by Lee [48], connects Theorem 13 to the cokernel distribution of random R-matrices. Second, we established their distribution using tools from commutative algebra. Our formula involves zeroth and first Betti numbers of R-modules, which is where the "Hom=Ext" condition comes from.

Miscellany and future research

Apart from the works described above, I have also worked on random permutations [21], enumerative combinatorics [40] and machine learning [67]. An important objective of my future research is to combine my wide range of expertise to discover new connections that advance the understanding of both sides. This often involves developing methods of explicit computations, finding suitable organizations of the computational results using insights in other fields, and exploring "geometrization" and "categorification" that realize the concrete observation as a combinatorial "shadow".

To this end, I would like to explore potential applications of geometric representation theory and toric geometry to my research. Beautiful connections have been made between these topics and Hilbert schemes of points, and my research suggests possibility to further these already rich stories. For example, in light of [27, 54], it is reasonable to investigate how "nested Quot schemes" on singular curves reflect the geometric representation theory of affine flag varieties; we may ask whether the cell decomposition and the perverse filtration in [52, 57] has analogues in $\mathrm{Quot}_{\mathcal{E},n}$.

References

[1] S. Ahlgren and K. Ono. Modularity of a certain Calabi-Yau threefold. *Monatsh. Math.*, 129(3):177–190, 2000.

[2] S. Ahlgren, K. Ono, and D. Penniston. Zeta functions of an infinite family of $K3$ surfaces. *Amer. J. Math.*, 124(2):353–368, 2002.

[3] G. W. Anderson, A. Guionnet, and O. Zeitouni. *An introduction to random matrices*, volume 118 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[4] G. E. Andrews. *The theory of partitions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

[5] V. I. Arnol'd. The cohomology ring of the group of dyed braids. *Mat. Zametki*, 5:227–231, 1969.

[6] M. Bagnarol, B. Fantechi, and F. Perroni. On the motive of Quot schemes of zero-dimensional quotients on a curve. *New York J. Math.*, 26:138–148, 2020.

[7] A. Baker. Linear forms in the logarithms of algebraic numbers. I. *Mathematika*, 13:204–216, 1966.

[8] T. Barnet-Lamb, D. Geraghty, M. Harris, and R. Taylor. A family of Calabi–Yau varieties and potential automorphy II. *Publ. Res. Inst. Math. Sci.*, 47(1):29–98, 2011.

[9] K. Behrend, J. Bryan, and B. Szendrői. Motivic degree zero Donaldson-Thomas invariants. *Invent. Math.*, 192(1):111–160, 2013.

[10] D. Bejleri, D. Ranganathan, and R. Vakil. Motivic Hilbert zeta functions of curves are rational. *J. Inst. Math. Jussieu*, 19(3):947–964, 2020.

[11] C. J. Bushnell and I. Reiner. Zeta functions of arithmetic orders and Solomon's conjectures. *Math. Z.*, 173(2):135–161, 1980.

[12] G. Cheong and Y. Huang. Cohen–Lenstra distributions via random matrices over complete discrete valuation rings with finite residue fields. *Illinois Journal of Mathematics*, 65(2):385–415, 2021.

[13] G. Cheong and Y. Huang. Betti and Hodge numbers of configuration spaces of a punctured elliptic curve from its zeta functions. *Trans. Amer. Math. Soc.*, 375(9):6363–6383, 2022.

[14] G. Cheong and Y. Huang. The cokernel of a polynomial push-forward of a random integral matrix with concentrated residue. Preprint `https://arxiv.org/abs/2310.09491`, 2023.

[15] G. Cheong and N. Kaplan. Generalizations of results of Friedman and Washington on cokernels of random p-adic matrices. *J. Algebra*, 604:636–663, 2022.

[16] G. Cheong, Y. Liang, and M. Strand. Polynomial equations for matrices over integers modulo a prime power and the cokernel of a random matrix. *Linear Algebra Appl.*, 677:1–30, 2023.

[17] G. Cheong and M. Yu. The distribution of the cokernel of a polynomial evaluated at a random integral matrix. Preprint `https://arxiv.org/abs/2303.09125`.

[18] T. Church. Homological stability for configuration spaces of manifolds. *Invent. Math.*, 188(2):465–504, 2012.

[19] T. Church, J. S. Ellenberg, and B. Farb. Representation stability in cohomology and asymptotics for families of varieties over finite fields. In *Algebraic topology: applications and new directions*, volume 620 of *Contemp. Math.*, pages 1–54. Amer. Math. Soc., Providence, RI, 2014.

[20] T. Church and B. Farb. Representation theory and homological stability. *Adv. Math.*, 245:250–314, 2013.

[21] A. Clifton, B. Deb, Y. Huang, S. Spiro, and S. Yoo. Continuously increasing subsequences of random multiset permutations. *European J. Combin.*, 110:103708, 2023.

[22] L. Clozel, M. Harris, and R. Taylor. Automorphy for some ℓ-adic lifts of automorphic mod ℓ Galois representations. *Publ. Math. Inst. Hautes Études Sci.*, 108:1–181, 2008. With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras.

[23] F. R. Cohen. On the mapping class groups for punctured spheres, the hyperelliptic mapping class groups, SO(3), and $\mathrm{Spin}^c(3)$. *Amer. J. Math.*, 115(2):389–434, 1993.

[24] H. Cohen and H. W. Lenstra, Jr. Heuristics on class groups of number fields. In *Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983)*, volume 1068 of *Lecture Notes in Math.*, pages 33–62. Springer, Berlin, 1984.

[25] J.-H. Evertse and K. Győry. *Unit Equations in Diophantine Number Theory*. Cambridge University Press, Cambridge, 2015.

[26] W. Feit and N. J. Fine. Pairs of commuting matrices over a finite field. *Duke Math. J.*, 27:91–94, 1960.

[27] N. Garner and O. Kivinen. Generalized affine Springer theory and Hilbert schemes on planar curves. *Int. Math. Res. Not. IMRN*, (8):6402–6460, 2023.

[28] M. Gerstenhaber. On dominance and varieties of commuting matrices. *Ann. of Math. (2)*, 73:324–348, 1961.

[29] V. V. Gorjunov. Cohomology of braid groups of series C and D. *Trudy Moskov. Mat. Obshch.*, 42:234–242, 1981.

[30] E. Gorsky and M. Mazin. Compactified Jacobians and q,t-Catalan numbers, I. *J. Combin. Theory Ser. A*, 120(1):49–63, 2013.

[31] E. Gorsky, M. Mazin, and M. Vazirani. Affine permutations and rational slope parking functions. *Trans. Amer. Math. Soc.*, 368(12):8403–8445, 2016.

[32] L. Göttsche and V. Shende. Refined curve counting on complex surfaces. *Geom. Topol.*, 18(4):2245–2307, 2014.

[33] J. Greene. Hypergeometric functions over finite fields. *Trans. Amer. Math. Soc.*, 301(1):77–101, 1987.

[34] R. M. Guralnick. A note on commuting pairs of matrices. *Linear and Multilinear Algebra*, 31(1-4):71–75, 1992.

[35] M. Harris, N. Shepherd-Barron, and R. Taylor. A family of Calabi–Yau varieties and potential automorphy. *Ann. of Math. (2)*, 171(2):779–813, 2010.

[36] T. Hausel and F. Rodriguez-Villegas. Mixed Hodge polynomials of character varieties. *Invent. Math.*, 174(3):555–624, 2008. With an appendix by Nicholas M. Katz.

[37] Y. Huang. Cohomology of configuration spaces on punctured varieties. Preprint `https://arxiv.org/abs/2011.07153`, 2020.

[38] Y. Huang. Unit equations on quaternions. *Q. J. Math.*, 71(4):1521–1534, 2020.

[39] Y. Huang. Mutually annihilating matrices, and a Cohen–Lenstra series for the nodal singularity. *J. Algebra*, 619:26–50, 2023.

[40] Y. Huang and R. Jiang. Spiral shifting operators from the enumeration of finite-index submodules of $\mathbb{F}_q[[T]]^d$. Preprint `https://arxiv.org/abs/2210.10215`, 2022.

[41] Y. Huang and R. Jiang. Generating series for torsion-free bundles over singular curves: rationality, duality and modularity. In preparation, 2023.

[42] Y. Huang and R. Jiang. Punctual Quot schemes and Cohen–Lenstra series of the cusp singularity. Preprint `https://arxiv.org/abs/2305.06411`, 2023.

[43] Y. Huang, K. Ono, and H. Saad. Counting matrix points on certain varieties over finite fields. *Contemp. Math., Amer. Math. Soc.*, accepted for publication, 2023. `https://arxiv.org/abs/2302.04830`.

[44] T. Hyde. Polynomial factorization statistics and point configurations in $\mathbb{R}^3$. *Int. Math. Res. Not. IMRN*, (24):10154–10179, 2020.

[45] T. Hyde and J. C. Lagarias. Polynomial splitting measures and cohomology of the pure braid group. *Arnold Math. J.*, 3(2):219–249, 2017.

[46] S. Kallel. Symmetric products, duality and homological dimension of configuration spaces. In *Groups, homotopy and configuration spaces*, volume 13 of *Geom. Topol. Monogr.*, pages 499–527. Geom. Topol. Publ., Coventry, 2008.

[47] S. Lang. Integral points on curves. *Publications Mathématiques de l'IHÉS*, 6:27–43, 1960.

[48] J. Lee. Joint distribution of the cokernels of random p-adic matrices. *Forum Math.*, 35(4):1005–1020, 2023.

[49] A. Lubotzky and D. Segal. *Subgroup growth*, volume 212 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2003.

[50] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, second edition, 2015.

[51] D. Maulik. Stable pairs and the HOMFLY polynomial. *Invent. Math.*, 204(3):787–831, 2016.

[52] D. Maulik and Z. Yun. Macdonald formula for curves with planar singularities. *J. Reine Angew. Math.*, 694:27–48, 2014.

[53] A. Mellit. Homology of torus knots. *Geom. Topol.*, 26(1):47–70, 2022.

[54] S. Monavari and A. T. Ricolfi. On the motive of the nested Quot scheme of points on a curve. *J. Algebra*, 610, 2022.

[55] F. Napolitano. Configuration spaces on surfaces. *C. R. Acad. Sci. Paris Sér. I Math.*, 327(10):887–892, 1998.

[56] F. Napolitano. On the cohomology of configuration spaces on surfaces. *J. London Math. Soc. (2)*, 68(2):477–492, 2003.

[57] A. Oblomkov, J. Rasmussen, and V. Shende. The Hilbert scheme of a plane curve singularity and the HOMFLY homology of its link. *Geom. Topol.*, 22(2):645–691, 2018. With an appendix by Eugene Gorsky.

[58] A. Oblomkov and V. Shende. The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link. *Duke Math. J.*, 161(7):1277–1303, 2012.

[59] K. Ono, H. Saad, and N. Saikia. Distribution of values of Gaussian hypergeometric functions. *Pure Appl. Math. Q.*, 19(1):371–407, 2023.

[60] R. Pandharipande and R. P. Thomas. Stable pairs and BPS invariants. *J. Amer. Math. Soc.*, 23(1):267–297, 2010.

[61] D. Petersen and P. Tosteson. Factorization statistics and bug-eyed configuration spaces. *Geom. Topol.*, 25(7):3691–3723, 2021.

[62] L. Solomon. Zeta functions and integral representation theory. *Advances in Math.*, 26(3):306–326, 1977.

[63] R. Taylor. Automorphy for some ℓ-adic lifts of automorphic mod ℓ Galois representations. II. *Publ. Math. Inst. Hautes Études Sci.*, 108:183–239, 2008.

[64] P. Tosteson. Lattice spectral sequences and cohomology of configuration spaces. 2016. arXiv: 1612.06034.

[65] R. Vakil and M. M. Wood. Discriminants in the Grothendieck ring. *Duke Math. J.*, 164(6):1139–1185, 2015.

[66] H. A. Verrill. The L-series of certain rigid Calabi-Yau threefolds. *J. Number Theory*, 81(2):310–334, 2000.

[67] T. Wang, Y. Huang, and D. Li. From the Greene–Wu convolution to gradient estimation over riemannian manifolds. Preprint `https://arxiv.org/abs/2108.07406`, 2021.

[68] M. M. Wood. The distribution of sandpile groups of random graphs. *J. Amer. Math. Soc.*, 30(4):915–958, 2017.

[69] M. M. Wood. Random integral matrices and the Cohen-Lenstra heuristics. *Amer. J. Math.*, 141(2):383–398, 2019.

[70] Z. Yun. Orbital integrals and Dedekind zeta functions. In *The legacy of Srinivasa Ramanujan*, volume 20 of *Ramanujan Math. Soc. Lect. Notes Ser.*, pages 399–420. Ramanujan Math. Soc., Mysore, 2013.