RESEARCH STATEMENT: YIFENG HUANG

My research programs span algebraic geometry, combinatorics, number theory, and topology, unified by a central theme: the *exact enumeration* of *particle configurations* on a space. More precisely, I study generating functions encoding the point counts, cohomology, or motives (i.e., classes in the Grothendieck ring of varieties) for moduli spaces that parametrize zero-dimensional objects on a variety X. These moduli spaces of interest include the Hilbert and Quot schemes of points ($\mathrm{Hilb}_n(X)$, $\mathrm{Quot}_n(\mathcal{O}_X^{\oplus d})$), the stack of zero-dimensional sheaves ($\mathrm{Coh}_n(X)$), the variety of "matrix points" ($C_n(X)$), and the configuration spaces of non-colliding points ($\mathrm{Conf}_n(X)$). Apart from $\mathrm{Conf}_n(X)$, each space has its own nuance in handling the points that collide.

My investigations are organized into several programs. The study of matrix points and their cousins ($C_n(X)$, $\mathrm{Coh}_n(X)$, $\mathrm{Hilb}_n(X)$, and $\mathrm{Quot}_n(\mathcal{O}_X^{\oplus d})$) forms the core of my recent work, revealing surprising connections to number theory, q-series, and mathematical physics (Section 1). The study of the cohomology of $\mathrm{Conf}_n(X)$ reveals combinatorial structures in the generating function (Section 2). Combinatorics of a moduli space is often related to statistics of "random" points in it; this philosophy naturally leads to a related program in discrete random matrix theory (Section 3). Finally, a distinct program (Section 4) in arithmetic geometry and Diophantine equations explores the Mordell–Lang problem in the unprecedented setting of non-commutative algebraic groups.

1. Matrix points on varieties and their cousins

Given a variety X over a field k, the variety of $n \times n$ matrix points on X, denoted by $C_n(X)$, is the scheme parametrizing tuples of pairwise commuting $n \times n$ matrices that satisfy the defining equations for X. Equivalently, $C_n(X)$ parametrizes zero-dimensionally supported coherent sheaves on X of length n, together with an ordered k-basis of global sections. This perspective reveals a fundamental relationship between $C_n(X)$ and the stack of such sheaves, $\mathrm{Coh}_n(X)$, via the global quotient construction $\mathrm{Coh}_n(X) = [C_n(X)/\mathrm{GL}_n]$.

The study of $C_n(X)$ is notoriously challenging, even for the most classical case $X = \mathbb{A}^d$. This gives the celebrated "commuting variety" of d-tuples of matrices, which is of central importance in algebraic geometry and linear algebra. For $d = 3$, fundamental properties such as the irreducibility and dimension of $C_n(\mathbb{A}^d)$ remain open (the Gerstenhaber problem), indicating that the behavior of these spaces can be quite "wild". My research has found two regimes of study that have proven to be particularly rewarding:

(a) The enumeration of matrix points over finite fields, especially when X is a reduced planar singular curve.
(b) The study of the cohomology of $C_n(X)$ for general varieties.

1.1. Matrix points on singular curves.

The study of matrix point counts naturally motivates a finer investigation of their motives in the Grothendieck ring of varieties. This perspective fits within, and significantly extends, the framework for studying the Hilbert scheme of points $\mathrm{Hilb}_n(X)$, a cornerstone of degree-zero Donaldson–Thomas theory on X. The extension arises from the observation ([23]) that as far as motives are concerned, $\mathrm{Coh}_n(X)$ (and equivalently, $C_n(X)$) can be realized as an infinite-rank limit of the rank-r Quot schemes of points on X, whereas $\mathrm{Hilb}_n(X)$ is precisely the rank-1 case. From the point of view of Nakajima's constructions [31, Chapter 1], if the Hilbert schemes are viewed as the framing-rank-one story, then $C_n(X)$ and $\mathrm{Coh}_n(X)$ can be viewed as the framing-rank-infinity or unframed story.

This high-rank viewpoint has proven to be remarkably powerful. For planar curve singularities, the rank-1 story is already exceptionally rich, featuring the celebrated Oblomkov–Rasmussen–Shende (ORS) conjecture [33], which connects the geometry of Hilbert schemes to link homology, and works [14–17, 28] that connect Catalan combinatorics to the starting family of examples, the (a, b) **toric singularity** defined by $y^a = x^b$. My program aims to extend this entire story to higher ranks, and a key discovery is that new phenomena, exclusive to the high-rank setting, emerge—most notably, a surprising connection to *Rogers–Ramanujan (RR) type q-series*.

This connection is already striking for toric singularities. In joint work with Jiang [23], we established the aforementioned "rank $\to \infty$ limit" theorem for a general reduced curve singularity, rigorously showing that the motivic generating function for matrix points (the Coh zeta function) can be recovered from the motivic generating functions for rank-r Quot schemes (the Quot zeta function). The Quot zeta function fits in the framework of lattice zeta functions developed by Solomon, Bushnell, and Reiner [4, 34], and its connection to p-adic integration and Fourier transform implies a Weil-conjecture-type functional equation (rank 1 case proved by Yun [37] and higher rank case proved in [23]). We developed the first recipe for computing the rank-r Quot zeta function, proving its rationality. Applying our computational framework to toric singularities reveals formulas with remarkable structure. For the $(2, 2m + 1)$ cusp singularity, the Coh zeta function (computed by taking the $r \to \infty$ limit) is precisely the sum side of the Andrews–Gordon identities, a classical generalization of the RR identities [23]. Ongoing work with Jiang and Oblomkov [24] on general unibranched (a,b) singularities (which means $\gcd(a,b)=1$) has produced a wealth of new, conjectural RR-type identities, whose sum sides are indexed by a *infinite-rank generalization of* (a,b)*-rational Dyck paths*. This suggests an amalgamation of RR-identities and Catalan combinatorics in an unprecedented way; its form is summarized as follows. Let $(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$ denote the q-Pochhammer symbol.

Conjecture 1 ([24])**.** For each $a,b>1$, $\gcd(a,b)=1$, there is an identity of the form "explicit sum = a RR-type infinite product with modulus $a+b$". More precisely,

$$\sum_{\mathbf{v}=(v_i)_{i\in\Gamma}} q^{Q(\mathbf{v})}F_{\mathbf{v}}(q) = \prod_{i=1}^{\lfloor\frac{a+b}{2}\rfloor}\Big((q^i;q^{a+b})_\infty(q^{a+b-i};q^{a+b})_\infty\Big)^{-m_i} \tag{1}$$

for some choices of $m_i\in\mathbb{Z}_{\geq 0}$, where $\Gamma=\mathbb{Z}_{\geq 0}\setminus\langle a,b\rangle$ is the complement of the semigroup generated by a,b, $Q(\mathbf{v})$ is an explicit quadratic form, and $F_{\mathbf{v}}(q)$ is an explicit product/quotient of q-Pochhammers.

The sum side of (1) arises from the Birula-Białynicki (BB) decomposition of the Quot scheme under the $\mathbb{G}_m$ action on the toric (a,b) singularity. Despite the wild global geometry, we show that the BB decomposition is as tame as possible (Theorem 2), a key contribution of our work. The sum side then results from the motivic Quot zeta function: $\mathbf{v}$ indexes the connected components of the fixed point locus, $F_{\mathbf{v}}(q)$ computes their motives, and $Q(\mathbf{v})$ represents the rank of the affine bundle arising from the BB map.

Theorem 2 ([24])**.** *Fix $a,b>1$, $\gcd(a,b)=1$, $r,n\geq 0$, and let X be the Quot scheme classifying n-codimensional $k[[T^a,T^b]]$-submodules of $k[[T]]^r$. Let $\mathbb{G}_m$ act on X by the natural action $t\cdot T=tT$. Let*

$$X^{\mathbb{G}_m}=\coprod_i Y_i$$

be the decomposition of the fixed point locus into connected components, and let

$$X_i=\{x\in X : \lim_{t\to 0} t\cdot x\in Y_i\}$$

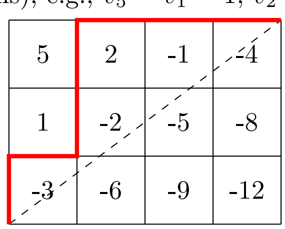
be the corresponding BB stratum. Then

(a) *Each Y_i is a smooth projective variety with an explicit iterated Grassmannian bundle structure;*
(b) *The BB map $X_i\to Y_i$ is an affine bundle.*

Example 3 ([24])**.** If $(a,b)=(3,4)$, then $\Gamma=\{1,2,5\}$, and

$$Q(\mathbf{v})=v_1^2+v_2^2+v_5^2+v_1v_2-v_1v_5,\quad F_{\mathbf{v}}(q)=\frac{(q;q)_{v_5}}{(q;q)_{v_1}(q;q)_{v_2}(q;q)_{v_5-v_2}(q;q)_{v_5-v_1}}.$$

The conjectured product is a modulus 7 infinite product with $(m_1,m_2,m_3)=(2,0,1)$. The sum is over all v_1,v_2,v_5 such that $v_5\geq v_1\geq 0, v_5\geq v_2\geq 0$ (since otherwise $F_{\mathbf{v}}(q)=0$). This index set is an infinite-rank generalization of $(3,4)$-Dyck paths, in which the rank-n version means that the index set is further restricted by $v_i\leq n$. Indeed, if $n=1$, then $(0,1)$-assignments to v_1,v_2,v_5 satisfying the inequalities above are in bijection with lattice paths in a 3×4 rectangle above the diagonal (Dyck paths), e.g., $v_5=v_1=1$, $v_2=0$ corresponds to



The quadratic form $Q(\mathbf{v})$ is a generalization of the celebrated dinv statistics: if $v_i\leq 1$ for all $i\in\Gamma$, then $Q(\mathbf{v})$ is precisely the dinv of the Dyck path corresponding to $\mathbf{v}$. This formulation of dinv appears to be new.

The picture for multibranched singularities (where $\gcd(a,b)>1$) is even more subtle and reveals the necessity of an arithmetic perspective. For the $(2,2m)$ node singularity, the initial formula was an enigmatic $2m$-fold summation in two variables q,t [23] that was later proved by Chern [8] to be an m-fold sum that deforms the sum side of the Bressoud identity (a sibling of the Andrews–Gordon identity). Puzzles remain, as the t-deformation requires a minus sign. The key insight, developed in [21], is that even to understand the geometry over $\mathbb{C}$, one must consider two distinct arithmetic models over $\mathbb{F}_q$: the usual, **split** model $(y-\alpha_1x^m)(y-\alpha_2x^m)=0$ where $\alpha_1\neq\alpha_2\in\mathbb{F}_q$, and the twisted, **inert** model where $\alpha_1,\alpha_2\in\mathbb{F}_{q^2}$ are Galois conjugates over $\mathbb{F}_q$. The Quot zeta functions for these two models are proposed to differ by a sign change $t \mapsto -t$, so the inert model gives the previously missing t-deformed Bressoud sum *without* the minus sign. Joint work with Chern [9] verifies this claim and furthermore proves a unifying a-deformed identity that interpolates between the split and inert cases, solidifying this arithmetic-geometric dictionary.

This program also extends in other directions. For non-reduced singularities, such as $k[x,y]/(x^a)$ ("fat line") and $k[x,y]/(x^a y)$, initial results on matrix point counts have been obtained in [20]. On the global arithmetic side, joint work with Ono and Saad establishes a "vertical" analogue of the Sato–Tate conjecture for the distribution of matrix point counts over families of elliptic curves and K3 surfaces [26]; this is later extended to the "horizontal" regime by our REU students [3].

1.2. **Matrix points on general varieties.** The second major direction of this program studies $X \mapsto C_n(X)$ as a general construction, not restricted to one example of X at a time. As a first thing we can say about this construction, $X \mapsto C_n(X)$ is a *functor* and it can be viewed as a non-commutative Weil restriction from the matrix algebra (see [12]). The guiding question in this direction is:

Can we make general connections between properties of $C_n(X)$ and those of X?

In recent joint work [12], we study the rational singular cohomology of $C_n(X)$ for general varieties X over $k = \mathbb{C}$. In general, Betti numbers do not determine and are not determined by the point count or the motive, but somewhat surprisingly, we reveal that the Betti numbers of $C_n(X)$ are determined by the Betti numbers of X via a uniform formula of Macdonald type [29]. The key to our argument is to show that the cohomology of $C_n(X)$ agrees with that of a related but simpler variety $S_n(X)$ that can be understood as a "Fermionic partner" of $C_n(X)$, as well as the variety of "diagonalized matrix points". To explain the analogy, consider the support map $C_n(X) \to \mathrm{Sym}^n(X)$ obtained by taking simultaneous eigenvalues of commuting matrices. The fibers of this map allow for nontrivial Jordan structure, so "quantum states", or eigenspaces, can fuse together to form generalized eigenspaces. If we view $\mathrm{Sym}^n(X)$ as the space of n-particle configuration on X, then $C_n(X)$ can be viewed as a *Bosonic* quantization of it. On the other hand, $S_n(X)$ is defined as

$$S_n(X) = \{((x_1, L_1), \dots, (x_n, L_n)) : x_i \in X, L_1 \oplus \cdots \oplus L_n \text{ is a direct sum decomposition of } k^n \text{ into lines}\}/\mathfrak{S}_n,$$

where $\mathfrak{S}_n$ is the permutation group. The fibers of the natural map $S_n(X) \to \mathrm{Sym}^n(X)$ record the one-dimensional eigenspace L_i attached to each point of support, and they are not allowed to fuse together; hence $S_n(X)$ is a *Fermionic* quantization of $\mathrm{Sym}^n(X)$. There is a natural map $\sigma : S_n(X) \to C_n(X)$ whose image is "semisimple matrix points" on X. If $X = \mathbb{A}^1$, this map is simply the diagonalization map: given a point $x := \{(x_1, L_1), \dots, (x_n, L_n)\}$ in $S_n(X)$, the image $\sigma(x)$ is defined as the unique matrix that has eigenspace L_i with eigenvalue x_i, for each i. In other words, $S_n(\mathbb{A}^1)$ can also be viewed as the space of "diagonalized matrices", i.e., it remembers not just a diagonalizable matrix, but also how it is diagonalized.

The key theorem is:

Theorem 4 ([12])**.** *For any complex variety X, $\sigma : S_n(X) \to C_n(X)$ induces isomorphisms on singular cohomologies.*

By studying the cohomology on $S_n(X)$ by combinatorial methods, we obtain the Macdonald formula. Let $P_u(X) := \sum_i (-u)^i h^i(X)$ denote the Poincaré polynomial of X, where $h^i(X) = \dim H^i(X; \mathbb{Q})$ is the i-th Betti number.

Corollary 5 ([12])**.** *For any complex variety X, we have*

$$\sum_{n=0}^{\infty} P_u(\mathrm{Coh}_n(X))\, t^n = \sum_{n=0}^{\infty} \frac{P_u(C_n(X))}{(u^2; u^2)_n}\, t^n = \prod_{i \geq 0, j \geq 1} \left(\frac{1}{1 - u^{i+2j} t} \right)^{(-1)^i h^i(X)}.$$

For comparison, the original Macdonald formula [29] says the analogous generating function for $\mathrm{Sym}^n(X)$ is the above product over $i \geq 0, j = 0$.

As another direction of the guiding question, I seek explicit global descriptions of $C_n(X)$ when X is not an affine variety (so the easy description "commuting matrices satisfying equations" fails). In ongoing joint project with Anderson, Jiang, and Oblomkov, we found explicit descriptions, in terms of GIT quotients and quiver representations, for $C_n(X)$ with X a projective space or certain toric variety.

2. Cohomology of configuration spaces of points

This program studies the topology of (unordered) configuration spaces of points on varieties, a classical topic that has recently seen renewed interest through the perspective of representation stability. For example, the fundamental group of $\mathrm{Conf}_n(\mathbb{C})$ is the braid group, and its rational cohomology is well-known (Arnol'd [1]) and is directly connected to counting square-free polynomials over finite fields [10]. The central guiding question is whether a "finite" algebraic structure governs the cohomology groups $H^i(\mathrm{Conf}_n(X); \mathbb{Q})$ for all n and i simultaneously. In particular, research in homological and representation stability has sought stability phenomena for fixed i, or along linear families of pairs (n,i) [10, 11, 30]. Although this question could have several interpretations, it motivates a concrete conjecture asserting the rationality of the full generating function of mixed Hodge numbers for these configuration spaces. The search for precise formulas for generating functions is, in some sense, equivalent to the goal of capturing all unstable behaviors at once. Mixed Hodge numbers refine Betti numbers, and often serve as links between topology and arithmetic of a variety (e.g., via purity). The ability to compute them typically indicates that the topology of the space can be accessed by algebro-geometric methods rather than only by more flexible topological manipulations.

In recent joint work with Ramos [27], we prove this conjecture for $\Sigma_{g,r}$, the genus-g Riemann surface with $r \geq 1$ punctures. The main result is an explicit rational formula for the generating function

$$f_X(x,y,u,t) := \sum_{p,q,i,n \geq 0} (-1)^i h^{p,q;i}(\mathrm{Conf}_n(X))\, x^p y^q u^i t^n$$

in the case $X = \Sigma_{g,r}$. Here, $h^{p,q;i}(M)$ denotes the (p,q)-mixed Hodge number of $H^i(M;\mathbb{Q})$.

Theorem 6 ([27])**.** *For $g \geq 0$ and $r \geq 1$, we have*

$$f_X(x,y,u,t) = \frac{1}{(1+xyut)^{r-1}} \frac{\Phi_g\{(1-xyz^2)(1-xz)^g(1-yz)^g\}}{(1-t)(1-x^2yu^2t^2)^g(1-xy^2u^2t^2)^g}, \tag{2}$$

where Φ_g is the piecewise shift operator *that is $\mathbb{Z}[x,y]$-linear with*

$$\Phi_g(z^j) = \begin{cases} u^j t^j, & 0 \leq j \leq g; \\ u^{j-1}t^j, & g+2 \leq j \leq 2g+2. \end{cases} \tag{3}$$

The proof strategy involves several key steps. First, a general "further puncturing" theorem that I established in [18] reduces the problem to the once-punctured case $r = 1$. For these surfaces, Totaro's spectral sequence provides a powerful algebraic model for the cohomology. The crucial insight is that the differential in this model can be identified with the Hard Lefschetz map for any g-dimensional abelian variety. The injectivity/surjectivity properties of the hard Lefschetz map precisely explain the mysterious "piecewise shift" phenomenon observed in the final formula. This result extends earlier work with Cheong that established the case $g = r = 1$ using purity of the mixed Hodge structure [6]; however, the same shortcut cannot be applied for more general g, r due to the lack of purity.

Future directions.

- In my previous work [18], I demonstrated that the generating function $f_X(x,y,u,t)$ behaves well under further puncturing, provided that X is smooth and non-compact, along with some additional purity assumptions:
$$f_{X \setminus P}(x,y,u,t) = \frac{1}{1+x^d y^d u^{2d-1} t} f_X(x,y,u,t),$$
where $d = \dim_{\mathbb{C}} X$. Can we eliminate these additional assumptions? The key may lie in the degeneracy of spectral sequences arising from arrangements of subvarieties.
- How "rational" is the *equivariant* generating function
$$F_X(x,y,u,t;\mathbf{z}) := \sum_{p,q,i,n \geq 0} (-1)^i \mathbf{ch}(H^{p,q;i}(\mathrm{PConf}_n(X));\mathbf{z})\, x^p y^q u^i t^n?$$
Here, $\mathrm{PConf}_n(X)$ denotes the ordered configuration space with an action of $\mathfrak{S}_n$, and $\mathbf{ch}(-;\mathbf{z})$ represents the Frobenius character of an $\mathfrak{S}_n$-representation, interpreted as a symmetric function in $\mathbf{z} = (z_1, z_2, \dots)$. Even in the simplest case where $X = \mathbb{C}$, determining this generating function remains an open problem; related conjectures have been formulated (see [11, p. 1843]).

3. Discrete random matrices and Cohen–Lenstra distributions

This program explores universal distributions for cokernels of random integral matrices, a topic with deep connections to the Cohen–Lenstra heuristics in arithmetic statistics and the classical random matrix theory. A fundamental philosophy in the classical random matrix is that taking the *spectrum* (of eigenvalues or singular values) of a random complex matrix gives a random point-configuration model that is often of physical and mathematical significance. In discrete random matrices, the corresponding operation is taking the *cokernel* of a random matrix over $\mathbb{Z}$ or the p-adic integers, $\mathbb{Z}_p$. The cokernel of a generic random matrix follows the "Cohen–Lenstra distribution": the probability of obtaining a particular object is inversely proportional to the size of its automorphism group [36]. However, certain algebraic operations on matrices can lead to new distributions, and often, these new distributions are best explained by identifying *additional algebraic structures* on the cokernels. My research in this area is guided by two central questions:

1. Given a sort of algebraic structure (formulated as a category or a moduli space), can we find interesting distributions on it and realize them through random matrix models?
2. Given a random matrix model, what algebraic structures best explain the resulting distribution, ideally as a Cohen–Lenstra type distribution?

One direction of my work investigates polynomials $P(X)$ of a single random matrix X. In joint work with Cheong [5, 7], I studied the cokernel distribution of $P(X)$ where X is a Haar-random matrix over $\mathbb{Z}_p$, as well as for matrices with a fixed residue modulo p. These settings are novel: the polynomial structure creates a matrix with intricately dependent entries, while fixing a residue violates the "ϵ-balancedness" condition essential to previous universality results. A key insight is that the cokernel of $P(X)$ is not merely an abelian p-group but is canonically a module over the ring $R := \mathbb{Z}_p[t]/P(t)$. This reframes the problem from the study of finite abelian groups (classified by partitions) to modules over more general rings. Our main result in [7] provides an explicit formula for the distribution of $\operatorname{coker}(P(X))$ as a finite R-module, demonstrating that the R-module structure is precisely the extra algebraic data that explains the distribution (addressing Question 2). This, in turn, feeds back to Question 1 by producing new families of Cohen–Lenstra type distributions on finite modules over general Noetherian local rings, where the probability of a module M depends on its homological invariants (the zeroth and first Betti numbers).

Another important model is a product $M_1 \cdots M_k$ of independent random $\mathbb{Z}_p$-matrices, initially investigated by Nguyen and Van Peski [32]. They observed that its cokernel distribution could be explained by the Cohen–Lenstra heuristics applied to a richer structure: a *flag* of cokernels, $\mathbf{cok}(M_1, \dots, M_k)$, given by the chain of natural surjections

$$\operatorname{cok}(M_1 \cdots M_k) \twoheadrightarrow \dots \twoheadrightarrow \operatorname{cok}(M_1).$$

In [22], I answered their question by proving that for independent Haar-random matrices over $\mathbb{Z}_p$, this random flag does follow the Cohen–Lenstra distribution. Building on this, my joint work with Nguyen and Van Peski [25] establishes that this distribution is, in fact, universal. We show that for any independent ϵ-balanced matrices (a broad class not restricted to the Haar measure), the limiting distribution (as the matrix size approaches infinity) is the same. As a corollary, we obtain universality for the "conditional convolution" of cokernels: the limiting distribution of $\operatorname{cok}(M_1 M_2)$ given the isomorphism types of $\operatorname{cok}(M_1)$ and $\operatorname{cok}(M_2)$ is governed by a fixed formula expressed in terms of Hall–Littlewood polynomials, as long as M_1, M_2 are independent and ϵ-balanced. The conditional convolution law was previously known only in the case where the distributions for M_1 and M_2 are $\mathrm{GL}_n(\mathbb{Z}_p)$-invariant [35].

4. Non-commutative Mordell–Lang problem

This research program charts new territory in Diophantine geometry by extending the framework of the Mordell–Lang conjecture to non-commutative algebraic groups. The classical Mordell–Lang conjecture (now Faltings' Theorem) describes the intersection of a finitely generated subgroup of a semiabelian variety with a subvariety. My initial work in this direction [19] established a first effective non-commutative analogue by proving a finiteness theorem for the unit equation $axb + cyd = 1$ in the algebra of Hamilton quaternions, $\mathbb{H}$, where a, b, c, d are fixed and x, y vary in finitely generated semigroups satisfying certain conditions. This result has direct applications to arithmetic dynamics, leading to an "infinite intersection implies common iterate" theorem for orbits of self-maps on genus-one curves.

This direction has proven fruitful, leading to a systematic development of a non-abelian Mordell–Lang theory in broader settings. In joint work with Ghioca [13], we establish a direct analogue of the classical Mordell–Lang theorem for division algebras.

Theorem 7 ([13])**.** *Let D be a finite-dimensional division algebra over a field K of characteristic 0, let $f_1, \dots, f_r \in D^\times$, and let V be a K-subvariety of D. The set of exponents*

$$\mathcal{S} := \{(n_1, \dots, n_r) \in \mathbb{Z}^r : f_1^{n_1} f_2^{n_2} \cdots f_r^{n_r} \in V\}$$

is a finite union of cosets of subgroups of $\mathbb{Z}^r$.

This result shows that a *dynamical* Mordell–Lang question in this specific non-commutative setup, where the dynamics are given by left-multiplication maps $\Phi_i(x) := f_i x$ starting from $v = 1$, the solution set of exponents for the dynamical Mordell–Lang equation $(\Phi_1^{n_1} \circ \cdots \circ \Phi_r^{n_r})(v) \in V$ retains the same well-behaved structure seen in the classical commutative case. Furthermore, we prove a finiteness result when the elements f_i have "multiplicatively independent norms" and V does not contain the origin. A semigroup version of this result provides a direct extension to the finiteness theorem for the unit equation in $\mathbb{H}$ from my earlier work [19].

While division algebras are a natural first step, the central simple algebra $\mathrm{Mat}_m(K)$ presents greater challenges due to zero divisors and non-diagonalizable elements. The problem of describing intersections with arbitrary matrix products is known to be undecidable in general. However, in joint work with Bell and Ghioca [2], we establish a powerful finiteness result by imposing a strong hypothesis on the eigenvalues. This condition is carefully chosen to be strong enough to avoid the known undecidability of the general problem, yet it remains flexible enough to accommodate essential challenges of the matrix setting, such as repeated eigenvalues and non-trivial Jordan structures.

Theorem 8 ([2]). *Let K be an algebraically closed field of characteristic zero, let $B_1, \ldots, B_r \in \mathrm{GL}_m(K)$ be matrices with "multiplicatively independent eigenvalues", and let V be a closed subvariety of $\mathrm{GL}_m(K)$ not passing through the origin. Then the intersection*

$$V(K) \cap \{B_1^{n_1} \cdot \cdots \cdot B_r^{n_r} : n_1, \ldots, n_r \in \mathbb{Z}\}$$

is finite.

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