q-Series and Applications to counting matrices

Part I. q-Series

Ref: Andrews "Theory of Partitions"

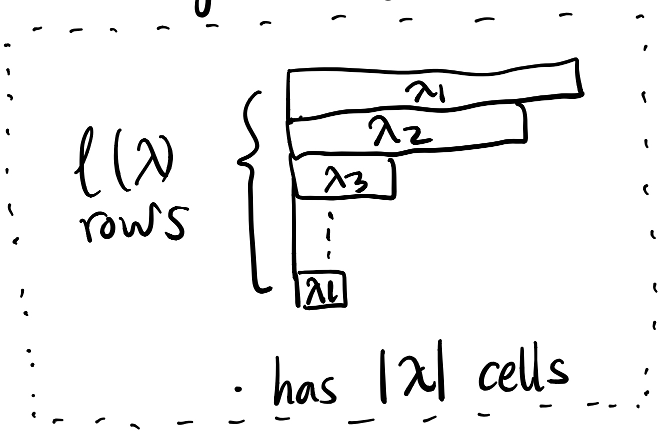
§1. Partitions

A *partition* is a sequence of integers

$$\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots)$$

that are eventually 0. Here λ_i is called a *part* of λ.

The *Young diagram* for a partition λ is a collection of cells given by



$l(\lambda)$ = length of λ
= last index l s.t. $\lambda_l \neq 0$

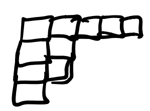
$|\lambda|$ = size of λ
$= \sum \lambda_i$

For $i > 0$, define $a_i(\lambda)$ = # of times that i occurs as a part of λ.

E.g. $10 = 5 + 2 + 2 + 1$ is a partition of 10, given by

$$\lambda = (5, 2, 2, 1)$$

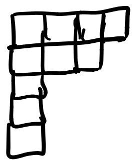
Young diagram:

$l(\lambda) = 4$ 

$|\lambda| = 10$, $a_1 = 1$, $a_2 = 2$, $a_5 = 1$

i.e. $10 = 1[1] + 2[2] + 1[5]$.

The *conjugate of λ* is defined by transposing the Young diagram.

$\lambda' =$  $= (4,3,1,1,1)$

§2. q-Series

Let $p(n) = \#$ of partitions of n (i.e. λ with $|\lambda| = n$)

It has no closed-form formula, but it can be read from a nice series.

Consider $\sum_{n=0}^{\infty} p(n) q^n = \sum_{\lambda} q^{|\lambda|} \in \mathbb{Z}[[q]]$

Thm (Euler).

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i} = \frac{1}{(1-q)(1-q^2)\cdots}$$

Pf. $LHS = \sum_{\lambda} q^{|\lambda|}$

Have bijection $\{\text{partition } \lambda\} \longleftrightarrow \left\{\begin{array}{l}\text{eventually} \\ \text{zero seq.} \\ a_1, a_2, \dots \geq 0\end{array}\right\}$

given by

$\lambda = a_1[1] + a_2[2] + \cdots$

$$= \sum_{a_1, a_2, \dots \geq 0} q^{a_1 + 2a_2 + 3a_3 + \cdots}$$

$$= \sum_{a_1=0}^{\infty} q^{a_1} \sum_{a_2=0}^{\infty} q^{2a_2} \cdots$$

$$= \frac{1}{1-q} \cdot \frac{1}{1-q^2} \cdots = RHS \qquad \square$$

The following notation will appear a lot when dealing with partitions:

Def. q-Pochhammer symbols

$$(x;q)_n = \underbrace{(1-x)(1-qx)\cdots(1-q^{n-1}x)}_{n \text{ factors}}$$

$$(x;q)_\infty = (1-x)(1-qx)(1-q^2x)\cdots \in \mathbb{Z}[[q,x]]$$

Then Euler's identity is $\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty}$

There are many combinatorial identities involving these symbols. Many do not explicitly involve partitions, but their proofs need the study of partitions.

Identities at a glance:

$$(-x;q)_\infty = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} x^n \qquad (1)$$

$$\frac{1}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n} x^n \qquad (2)$$

$$\left(\sum_{n=0}^{\infty} p(n) q^n =\right) \frac{1}{(q;q)_\infty} \overset{(2)}{=} \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} \overset{\text{but also}}{=} \sum_{n=0}^{\infty} \frac{q^{n^2}}{((q;q)_n)^2} \qquad (3)$$

$$\frac{(ax;q)_\infty}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n \qquad (4)$$

We will prove (2) and (3) as examples to demonstrate some tools for many such identities.

The word "q-series" can mean any series in this context. They are studied by methods of similar flavors. Let's not worry about the precise meaning of this term.

Pf of ②. $\dfrac{1}{(x;q)_\infty} = \sum_{n=0}^{\infty} \dfrac{1}{(q;q)_n} x^n$

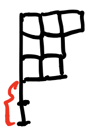
$$\text{LHS} = \frac{1}{1-x}\,\frac{1}{1-qx}\,\frac{1}{1-q^2x}\cdots$$

$$= \sum_{a_0=0}^{\infty} x^{a_0} \sum_{a_1=0}^{\infty} q^{a_1} x^{a_1} \sum_{a_2=0}^{\infty} q^{2a_2} x^{a_2} \cdots$$

$$= \sum_{\substack{a_0, a_1, \cdots \geq 0 \\ \boxed{\text{eventually } 0}}} q^{a_1 + 2a_2 + 3a_3 + \cdots} x^{a_0 + a_1 + a_2 + \cdots}$$

always omit writing this later

To each datum $(a_0, a_1, \cdots)$, we associate a "partition"

$\lambda = a_0 [0] + a_1 [1] + \cdots$ e.g. $a_0 \{$  $\lambda = (3,2,2,0,0)$

It is a "partition" whose parts are allowed to be 0.

Let's call this data structure a partition with zero.

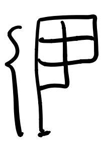
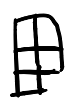
$$\left(\begin{matrix}\text{LHS,}\\ \text{Continue}\end{matrix}\right) = \sum_{\lambda \text{ with } [0]} q^{|\lambda|} x^{\ell(\lambda)}$$

Note: $[0]$'s count towards the length

$$= \sum_n \left[\sum_{\substack{\lambda \text{ w/ } [0] \\ \ell(\lambda) = n}} q^{|\lambda|} \right] x^n$$

[Recalling the goal is to find the coefficients of x^n]

$$= \sum_n \sum_{\substack{\lambda \\ \ell(\lambda) \leq n}} q^{|\lambda|} x^n$$

[$n \{$  $\longleftrightarrow$ 

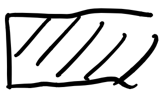
$\mapsto$: remove 0

$\leftarrow$: fill in 0s until length is n]

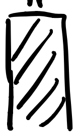
Claim: $\displaystyle\sum_{\substack{\lambda \\ \ell(\lambda)\le n}} q^{|\lambda|} = \frac{1}{(q;q)_n}$

- If you repeat the method in Euler's identity, there is a problem: it is hard to deal with the restriction $\ell(\lambda)\le n$.
- Think about conjugate!

$$\text{LHS} = \sum_{\substack{\lambda \\ \text{fits in} \\ n\times\infty}} q^{|\lambda|} = \sum_{\substack{\lambda \\ \text{fits in} \\ \infty\times n}} q^{|\lambda|} = \sum_{a_1=0}^{\infty} q^{a_1} \sum_{a_2=0}^{\infty} q^{2a_2} \cdots \sum_{a_n=0}^{\infty} q^{n a_n}$$

n 

$\Sigma a_i(\lambda)\le n$ ... ???

n 

← i.e. λ has no part of size $>n$.

i.e. $a_i(\lambda)=0 \ \forall i>n$

$$= \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)} \qquad \square$$

q-Binomial Coefficients

Let $\displaystyle\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \xleftrightarrow{\text{analog}} \binom{n}{k} = \frac{n!}{k!(n-k)!}$

It can be elementarily checked that

$\begin{bmatrix} n \\ k \end{bmatrix}_q = \#$ of k-dim subspaces of $\mathbb{F}_q^n$ $=$ Grassmannian $Gr(n,k)$ ←- - - - - → $\binom{n}{k} = \#$ of k-elt subset of $\{1,\cdots,n\}$

Important fact:

$\displaystyle\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\substack{\lambda \text{ fits} \\ (n-k)\times k}} q^{|\lambda|}$ [Not quite elementary!]

☆ Gateway to Schubert Calculus: $\#Gr(n,k)(\mathbb{F}_q) = \displaystyle\sum_{\substack{\lambda \text{ fits} \\ (n-k)\times k}} q^{|\lambda|}$ has a geometric reason: $Gr(n,k)$ is composed of Schubert cells indexed by λ fitting in $(n-k)\times k$, and the dim of each cell is $|\lambda|$.

Dictionary for sums over partitions

- $$\sum_{\lambda} q^{|\lambda|} = \frac{1}{(q;q)_\infty}$$

- $$\sum_{\substack{\lambda \text{ fits} \\ n\times\infty}} q^{|\lambda|} = \frac{1}{(q;q)_n}$$

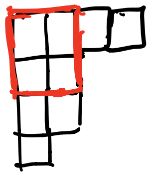
- $$\sum_{\substack{\lambda \text{ fits} \\ n\times k}} q^{|\lambda|} = \frac{(q;q)_{n+k}}{(q;q)_n (q;q)_k} = \begin{bmatrix} n+k \\ n \end{bmatrix}_q = \begin{bmatrix} n+k \\ k \end{bmatrix}_q$$

etc.

There are too many identities about q-series; it is hard even to search one that works. A practical strategy to think about q-series and recover many identities by yourself is,

- Rewrite a q-series in the form of
$$\sum_{\substack{\lambda \text{ ranging} \\ \text{over some set}}} (\text{some monomial associated to } \lambda)$$
- Break down λ into partitions ranging over sets that appear in the dictionary above.

Durfee Square : the largest square that fits the top-left corner

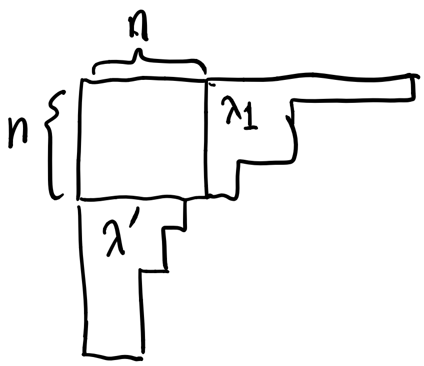
e.g. $\lambda =$ 

$\sigma_1(\lambda) :=$ side-length of Durfee square $= 2$

Proof of ③: $\dfrac{1}{(q;q)_\infty} = \displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^{\,2}}$

$$LHS = \sum_{\lambda} q^{|\lambda|} = \sum_{n=0}^{\infty} \sum_{\substack{\lambda \\ \sigma_1(\lambda)=n}} q^{|\lambda|}$$

A partition of Durfee size n is uniquely determined by a partition λ_1 that fits in $n \times \infty$ and a partition λ' that fits in $\infty \times n$



$\therefore$ (LHS, continue)

$$= \sum_{n=0}^{\infty} \sum_{\substack{\lambda_1 \text{ fits} \\ n\times\infty}} \sum_{\substack{\lambda' \text{ fits} \\ \infty\times n}} q^{n^2+|\lambda_1|+|\lambda'|}$$

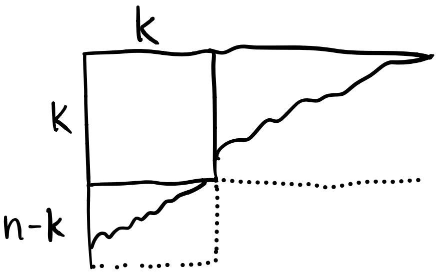
$$= \sum_{n=0}^{\infty} q^{n^2} \sum_{\substack{\lambda_1 \text{ fits} \\ n\times\infty}} q^{|\lambda_1|} \sum_{\substack{\lambda' \text{ fits} \\ \infty\times n}} q^{|\lambda'|}$$

$$= \sum_{n=0}^{\infty} q^{n^2} \cdot \frac{1}{(q;q)_n} \cdot \frac{1}{(q;q)_n}$$

Exercise.

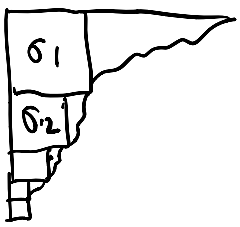
$$\frac{1}{(q;q)_n} = \sum_{k=0}^{n} q^{k^2} \frac{1}{(q;q)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

Hint:



Remark: One can take Durfee square repetitively.

$\lambda =$



Get a Durfee partition:

$\sigma(\lambda) = (\sigma_1 \geq \sigma_2 \geq \cdots)$

Have $|\sigma| = \ell(\lambda)$

Very useful!

Takeaway of §2. q-Series

- Many q-series can be expressed as a sum over partitions.
- To convert back, use the dictionary.
- When you get stuck, try conjugation and Durfee square.

§3. Enumeration in Linear Algebra

Fix finite field $\mathbb{F}_q$. Here is a dictionary of various matrix counting:

① $\# GL_n(\mathbb{F}_q) = \#$ of $n \times n$ full rank matrices

$$= \underbrace{(q^n - 1)}_{\substack{\text{choose} \\ 1^{\text{st}} \text{ column}}} \underbrace{(q^n - q)}_{\substack{\text{choose} \\ 2^{\text{nd}} \text{ column}}} \cdots\cdots \underbrace{(q^n - q^{n-1})}_{\substack{\text{choose} \\ n^{\text{th}} \text{ column}}}$$

② # of k-dim Subspace of $\mathbb{F}_q^n$ $= \begin{bmatrix} n \\ k \end{bmatrix}_q$

③ # of rank-k $n \times k$ matrices = choose k lin. indep columns

$$= \underbrace{(q^n-1)}_{\substack{\text{choose} \\ 1^{st} \text{ column}}} \underbrace{(q^n-q)}_{\substack{\text{choose} \\ 2^{nd} \text{ column}}} \cdots \underbrace{(q^n-q^{k-1})}_{\substack{\text{choose} \\ k\text{-th column}}}$$

In fact, ③ can prove ② because a rank-k $n \times k$ matrix determines a k-dim subspace with an ambiguity of $GL_k(\mathbb{F}_q)$.

④ # of rank-k $n \times n$ matrices

$$= \underbrace{\begin{bmatrix} n \\ k \end{bmatrix}_q}_{\substack{\text{choose a} \\ k\text{-dim subspace } V \\ \text{as its image}}} (q^n-1) \underbrace{(q^n-q) \cdots (q^n-q^{k-1})}_{\substack{\text{choose surjection } \mathbb{F}_q^n \to V, \\ \text{i.e., a full rank } k \times n \text{ matrix,} \\ \text{But this corresponds to a} \\ \text{full rank } n \times k \text{ matrix !}}}$$

Part II. My work: a factorization of generating function

§1. Motivation

Consider $\sum_{n=0}^{\infty} \frac{\#\mathrm{Mat}_n(\mathbb{F}_q)}{\# GL_n(\mathbb{F}_q)} x^n = \sum_{n=0}^{\infty} \frac{\#\{A : n\times n \text{ mat}/\mathbb{F}_q\}}{\# GL_n(\mathbb{F}_q)} x^n$

It has factorization

$$= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^n-1)\cdots(q^n-q^{n-1})} x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{(1-q^{-n})\cdots(1-q^{-1})} x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{(q^{-1};q^{-1})_n} x^n \overset{②}{=} \frac{1}{(x;q^{-1})_\infty}$$

We often need to work with q-series with q replaced by $t := q^{-1}$

$$= \frac{1}{(1-x)(1-q^{-1}x)\cdots}$$

In particular, it is a meromophic function in x (consider $q>1$ is fixed) with poles at $1, q, q^2, \ldots$.

Without the factorization, we wouldn't have known that the power series has a meromorphic extension beyond radius of conv, which is 1.

<u>Phenomenon</u>: Such a generating function about counting matrices often has a nice factorization

Examples

- $$\sum_{n=0}^{\infty} \frac{\#\{A \in \mathrm{Mat}_n(\mathbb{F}_q)\}}{\# GL_n(\mathbb{F}_q)} x^n = \frac{1}{(x;q^{-1})_\infty}$$

nxn nilpotent matrices

- $$\sum_{n=0}^{\infty} \frac{\#\{A \in \mathrm{Nilp}_n(\mathbb{F}_q)\}}{\# GL_n(\mathbb{F}_q)} x^n = \frac{1}{(xq^{-1};q^{-1})_\infty}$$ (Fine-Herstein)

- $$\sum_{n=0}^{\infty} \frac{\#\{A,B \in \mathrm{Mat}_n(\mathbb{F}_q) : AB=BA\}}{\# GL_n(\mathbb{F}_q)} x^n = \prod_{\substack{i \geq 1 \\ j \geq 0}} \frac{1}{1-x^i q^{1-j}}$$ (Feit-Fine)

- $$\sum_{n=0}^{\infty} \frac{\#\{A,B \in \mathrm{Nilp}_n(\mathbb{F}_q) : AB=BA\}}{\# GL_n(\mathbb{F}_q)} x^n = \prod_{\substack{i \geq 1 \\ j \geq 2}} \frac{1}{1-x^i q^{1-j}}$$ (Fulman)

Rmk. There is a systematic way to go between nilpotent version and usual version, so let's not worry about nilpotent matrices anymore.

Just from the four identities above, it is not hard to show:

Fact. Suppose $f_1, \cdots, f_r \in \mathbb{F}_q[x_1, \cdots, x_m]$ are polynomials that cut out the variety $X = \{x=(x_1,\cdots,x_m) \in \mathbb{A}^m : f_1(x) = \cdots = f_r(x) = 0\}$.

If X is a smooth curve or smooth surface, then

$$\sum_{n=0}^{\infty} \frac{\#\{A_1,\cdots,A_m \in \mathrm{Mat}_n(\mathbb{F}_q) : A_iA_j = A_jA_i,\ f_k(A_1,\cdots,A_m)=0\ \forall 1 \leq k \leq r\}}{\# GL_n(\mathbb{F}_q)} x^n$$

makes sense because A_i commute

has a nice factorization involving Hasse-Weil zeta function of X.

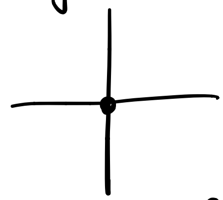
Q: What about other cases of X?

A: When $\dim X \geq 3$, this is hopeless.

So the simplest new case is when X is a singular curve.

It turns out that we can study this problem one singularity at a time.

The simplest singularity is a node, and has a local model

$\{xy = 0\}$ = 

§2 Main Result and Proof

Thm (H.) Fix finite field $\mathbb{F}_q$. Then

$$\sum_{n=0}^{\infty} \frac{\#\{A, B \in \mathrm{Mat}_n(\mathbb{F}_q) : AB = BA = 0\}}{\# GL_n(\mathbb{F}_q)} x^n = \frac{1}{(x; q^{-1})_\infty^2} \cdot \mathcal{H}(x)$$

for an explicit power series $\mathcal{H}(x)$ with infinite radius of conv.

Rmk. LHS is not hard to find. The hard part is the factorization, without which LHS is not known to have a meromorphic extension outside $|x| < 1$.

Pf. Step 1. Compute LHS.

Classify the pairs (A, B) according to $\mathrm{rk}\, A$.

Say $\mathrm{rk}\, A = n-k$. We know how many such A's are there.

Fix such A. Let's count how many B's satisfy $AB = BA = 0$.

$AB=0 \Leftrightarrow A$ kills $Bx\ \forall x \Leftrightarrow \operatorname{im}B \subseteq \ker A$ $\quad (*)$

$BA=0 \Leftrightarrow B$ kills $Ax\ \forall x \Leftrightarrow \operatorname{im}A \subseteq \ker B.$ $\quad (**)$

By $(*)$, just need to look for $B: \mathbb{F}_q^n \longrightarrow \ker A$

To find such B s.t. B vanishes on $\operatorname{im}A$ is the same as finding $\bar{B}: \underbrace{\mathbb{F}_q^n/\operatorname{im}A}_{\dim k} \longrightarrow \underbrace{\ker A}_{\dim k}.$

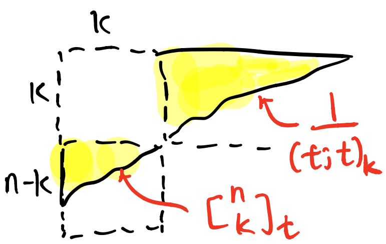
$\therefore$ There are q^{k^2} choices.

The end result:

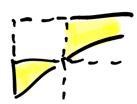
$$\mathrm{LHS} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{[{n \atop k}]_t}{(t;t)_k} x^n, \quad t := q^{-1}.$$

Step 2 Find a formula for $H(x)$

$$\mathrm{LHS} = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \sum$$



To read this,

$\sum$ 

means the sum over all such diagrams that fit. Each diagram contributes to a term $t^{\text{size of shaded region}}$

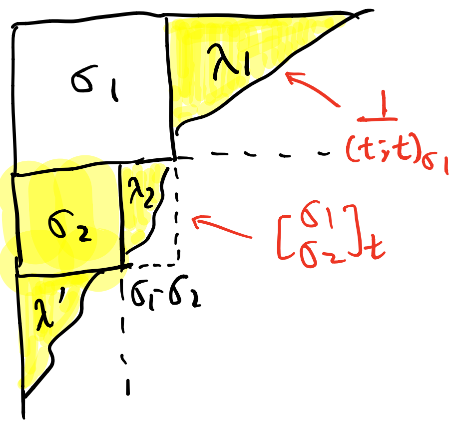
$$= \sum_{n=0}^{\infty} x^n \sum_{\ell(\lambda)\le n} t^{|\lambda| - \sigma_1(\lambda)^2}$$

$$= \sum_{n=0}^{\infty} x^n \sum_{\substack{\ell(\lambda)=n \\ \lambda \text{ w/ } [0]}} t^{|\lambda| - \sigma_1(\lambda)^2}$$

$\leftarrow$ recall the old trick of filling zeros.

$$= \sum_{\lambda \text{ w/ } [0]} t^{|\lambda| - \sigma_1(\lambda)^2} x^{\ell(\lambda)}$$

Now, classify λ w/[0] according to $\sigma_1(\lambda), \sigma_2(\lambda)$. It can be reconstructed by the data tuple $(\sigma_1, \sigma_2, \lambda_1, \lambda_2, \lambda')$ as shown below



- $\sigma_1 \geq \sigma_2$
- λ_1 fits in $\sigma_1 \times \infty$
- λ_2 fits in $\sigma_2 \times (\sigma_1 - \sigma_2)$
- λ' with zeros fits in $\infty \times \sigma_2$

(LHS continued)

$$= \sum_{\sigma_1, \sigma_2, \lambda_1, \lambda_2, \lambda' w/[0]} t^{|\lambda_1| + |\lambda_2| + |\lambda'| + \sigma_2^2} x^{\sigma_1 + \sigma_2 + \ell(\lambda')}$$

$$= \sum_{\sigma_1, \sigma_2} t^{\sigma_2^2} x^{\sigma_1 + \sigma_2} \sum_{\lambda_1} t^{|\lambda_1|} \sum_{\lambda_2} t^{|\lambda_2|} \sum_{\lambda' w/[0]} t^{|\lambda'|} x^{\ell(\lambda')}$$

$$\overset{\text{dictionary}}{=} \sum_{\sigma_1, \sigma_2} t^{\sigma_2^2} x^{\sigma_1 + \sigma_2} \underbrace{\frac{1}{(t;t)_{\sigma_1}} \begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix}_t} \frac{1}{(1-x)(1-tx)\cdots(1-t^{\sigma_2}x)}$$

The magic is here: since we have constraint $\sigma_1 \geq \sigma_2$, they are not free variables we can separate; rather, $\sigma_2 \geq 0$ and $\sigma_1 - \sigma_2 \geq 0$ are freely varying. The following eliminates σ_1 dependence:

$$= \frac{1}{\cancel{(t;t)_{\sigma_1}}} \frac{\cancel{(t;t)_{\sigma_1}}}{(t;t)_{\sigma_2} (t;t)_{\sigma_1 - \sigma_2}}$$

Denote $b = \sigma_1 - \sigma_2$, $k = \sigma_2$, we get

(LHS continued)

$$= \sum_{b,k\geq 0} t^{k^2} x^{b+2k} \frac{1}{(t;t)_k (t;t)_b} \frac{1}{(1-x)(1-tx)\cdots(1-t^k x)}$$

$$= \sum_{b=0}^{\infty} x^b \frac{1}{(t;t)_b} \sum_{k=0}^{\infty} t^{k^2} x^{2k} \frac{1}{(t;t)_k} \frac{1}{(1-x)(1-tx)\cdots(1-t^k x)}$$

$$\overset{\text{dict}}{=} \frac{1}{(x;t)_\infty} \sum_{k=0}^{\infty} t^{k^2} x^{2k} \frac{1}{(t;t)_k} \frac{1}{(1-x)(1-tx)\cdots(1-t^k x)}$$

This is a good enough factorization. For the sake of finding all poles,

$$= \frac{1}{(x;t)_\infty} \cdot \frac{1}{(1-x)(1-tx)\cdots} \sum_{k=0}^{\infty} t^{k^2} \frac{x^{2k}}{(t;t)_k} (1-t^{k+1}x)(1-t^{k+2}x)\cdots$$

$$= \frac{1}{(x;t)_\infty^2} \underbrace{\sum_{k=0}^{\infty} t^{k^2} \frac{x^{2k}}{(t;t)_k} (1-t^{k+1}x)(1-t^{k+2}x)\cdots}_{H(x)}$$

<u>Step 3.</u> Show $H(x)$ is entire.

Recall $0<t<1$. $t=q^{-1}$.

$$H(x) = \sum_{k=0}^{\infty} t^{k^2} x^{2k} \underbrace{\frac{1}{(t;t)_k} (1-t^{k+1}x)(1-t^{k+2}x)\cdots}_{\text{Fix } x \text{. can show this is bounded for large } k.}$$

$\therefore$ convergence follows from that of

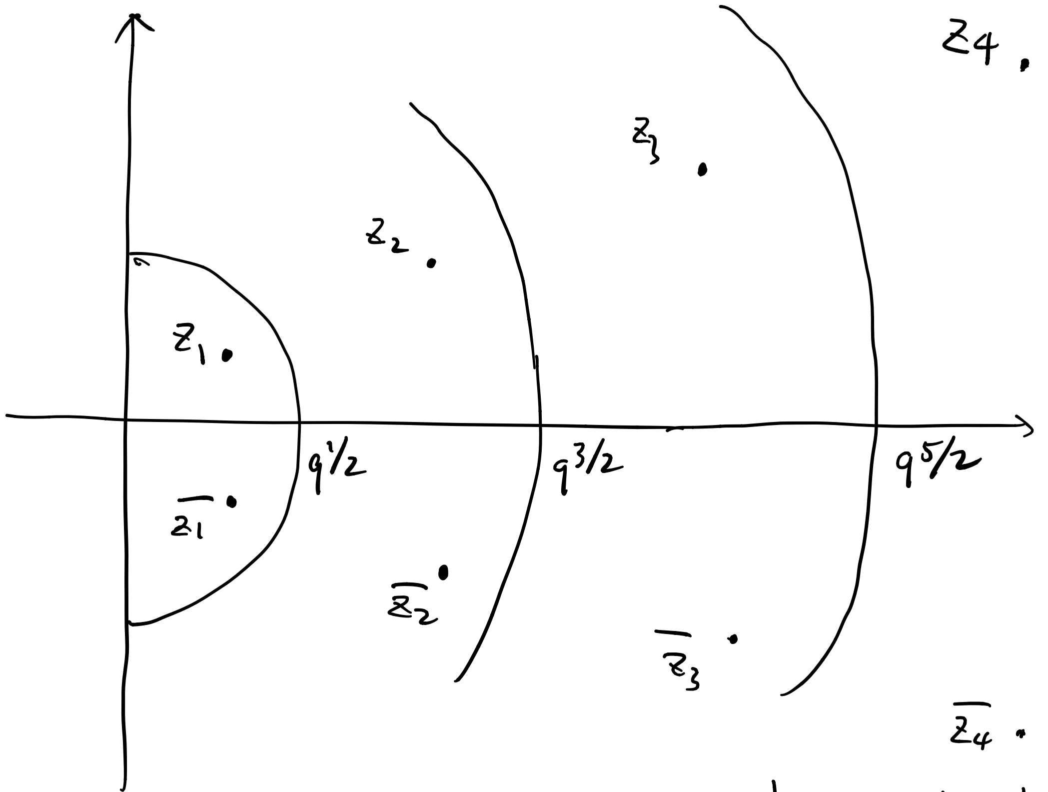
$$\sum t^{k^2} |x|^{2k}$$

$\square$

§3. Quirks about $\mathcal{H}(x) = \sum_{k=0}^{\infty} \frac{t^{k^2} x^{2k}}{(t;t)_k} (t^{k+1}x;t)_\infty$

It is not clear if $\mathcal{H}(x)$ can be factorized further.

But computer shows that $\mathcal{H}(x)$ seems to have infinitely many zeros that distribute like



But none of them seem to be understandable

Nor do we have $z_{n+1} = q z_n$ ... but "somewhat".

Any feature about this diagram is not proven.

You can try to show, for example,

- $H(x) > 0$ for $x \in \mathbb{R}$
- $H(x) \neq 0$ for $\mathrm{Im}(x) < 0$

(It is elementary to see that $H(x) > 0$ for $x \in (-\infty, q]$ or $x = q^n$, $n \in \mathbb{Z}$)

E.g. $q=4$

$z_1 = 0.41614 + 1.72467\, i$	$\lvert z_1 \rvert = 1.77288 < 2 = q^{1/2}$
$z_2 = 1.65483 + 7.60611\, i$	$\lvert z_2 \rvert = 7.78405 < 8 = q^{3/2}$
$z_3 = 6.62192 + 31.08907\, i$	$\lvert z_3 \rvert = 31.7865 < 32 = q^{5/2}$
$z_4 = 26.4883 + 125.0116\, i$	$\lvert z_4 \rvert = 127.787 < 128 = q^{7/2}$

§4. Algebraic Geometry Connection.

Recall the generating function

$$\sum_{n=0}^{\infty} \frac{\#\{A_1,\cdots,A_m \in \mathrm{Mat}_n(\mathbb{F}_q) : A_iA_j = A_jA_i,\ f_k(A_1,\cdots,A_m)=0\ \forall 1\le k\le r\}}{\# GL_n(\mathbb{F}_q)} x^n$$

associated to the variety

$$X = \{f_1 = \cdots = f_r = 0\} \subseteq \mathbb{A}^m$$

Now denote the function by $\hat{Z}_X(x)$.

- In fact, it can be defined for any quasi-projective variety $X/\mathbb{F}_q$, and it behaves like a "zeta function". E.g. There is Euler product $\hat{Z}_X(x) = \prod_{\substack{p\in X \\ \text{closed} \\ \text{pt}}} \hat{Z}_{X,p}(x)$ for a local version $\hat{Z}_{X,p}(x)$.

- $\hat{Z}_X(x) = \sum_M \frac{1}{\mathrm{Aut}M} x^{l(M)}$, where M ranges over isomorphism classes of torsion coherent sheaves of X, and $l(M)$ is its length.

The main result is equivalent to

Thm (H.) If X is a nodal curve$/\mathbb{F}_q$, and $\tilde{X}$ is its resolution of singularity, then

$$\frac{\hat{Z}_X(x)}{\hat{Z}_{\tilde{X}}(x)} \text{ is an entire function of } x.$$

Conj (H.) This works for singular curves with other examples of singularities, too.

One reason to believe it is an analogous result:

Thm (Göttsche–Shende 2016)

If X is a curve with Gorenstein singularities, consider

$$f_X(x) = \sum_{n=0}^{\infty} \# \mathrm{Hilb}^n(X)(\mathbb{F}_q)\, x^n$$

Then $\dfrac{f_X(x)}{f_{\tilde{X}}(x)}$ is a polynomial of x, where

$\tilde{X}$ is as above.

There have been several discussions of the connection between Hilbert schemes and spaces of commuting matrices (both are moduli spaces of something). To list some,

- $\mathrm{Hilb}^n(X)$ is a GIT quotient of (the latter) $\times \mathbb{A}^n$.
- The latter is the space of representation of a quiver, while the Hilbert scheme is the "framed" version of it.
- To be more concrete, these connections come from the following

observation:

Say $X = \mathrm{Spec}\, A$. Then Hilbert scheme classifies ring quotients $A \twoheadrightarrow M$ where $\dim_{\mathbb{F}_q} M = n$. The commuting matrix variety classifies A-mod M where $\dim_{\mathbb{F}_q} M = n$.

Then note that a quotient $A \twoheadrightarrow M$ is determined by an A-mod M and an element $\mathbb{1} \in M$ s.t. $A \cdot \mathbb{1} = M$.

Summary:

- We use tools about partitions and q-series to factorize a generating function counting $\#\{A, B \in \mathrm{Mat}_n(\mathbb{F}_q) : AB = BA = 0\}$, thus giving a meromorphic continuation on all of $\mathbb{C}$.
- This leads to a result about a moduli space on nodal curves.
- Analogy suggests that similar thing should work for other singular curves.