MATH 315 Assignment 2

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1 Let G be a group.

1.1 Prove that the relation $a \sim b$ if $b = gag^{-1}$ for some $g \in G$, is an equivalence relation on G.

Let $a, b, c \in G$, and $\sim$ be a relation on G.

Suppose $a \sim b$. So $\exists g \in G$ such that $b = gag^{-1}$. Therefore $g^{-1}bg = a$. Let $g' = g^{-1} \in G$, so $a = g'bg'^{-1}$. Therefore $b \sim a$.

Suppose $a \sim b, b \sim c$. So $\exists g$, and $h \in G$ such that $b = gag^{-1}$, and $c = hbh^{-1}$. So $c = hgag^{-1}h^{-1}$. Now $hgg^{-1}h^{-1} = h1h^{-1} = 1 = g^{-1}1g = g^{-1}h^{-1}hg$. Therefore $(hg)^{-1} = g^{-1}h^{-1}$. Let $g' = hg \in G$, where $c = g'ag'^{-1}$. Therefore $a \sim c$.

Lastly, $a = 1a1^{-1}$. Therefore $a \sim a$. $\square$

1.2 Prove that $\forall u, v \in G, uv \sim vu$.

Let $u, v \in G$. Choose $g = v^{-1}u^{-1}v^{-1} \in G$. Now,

$$\begin{aligned} 1v^{-1} &= v^{-1}1 \\ \rightarrow (uvv^{-1}u^{-1})v^{-1} &= v^{-1}(u^{-1}v^{-1}vu) \\ \rightarrow uvg &= gvu \\ \rightarrow uv &= gvug^{-1} \end{aligned}$$

Therefore $uv \sim vu$. $\square$

1.3 Show that if there is a surjective group homomorphism $G' \rightarrow G$ for an abelian group G', then G is an abelian group.

Let $\phi : G' \rightarrow G$ be surjective. Let G' be an abelian group.

We need to prove G is commutative.

Suppose $a, b \in G$. Since ϕ is surjective, $\exists a', b' \in G'$ such that $a = \phi(a'), b = \phi(b')$. Since G' is abelian, $ab = \phi(a')\phi(b') = \phi(a'b') = \phi(b'a') = \phi(b')\phi(a') = ba$. $\square$

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2.1 Show that the map $\alpha : GL_2(\mathbb{R}) \to GL_2(\mathbb{R})$ sending a matrix A to the inverse of its transposed matrix $(A^t)^{-1}$ is an isomorphism of groups.

Let $\alpha : GL_2(\mathbb{R}) \to GL_2(\mathbb{R}), \alpha(A) = (A^t)^{-1}$.

We need to prove α is a homomorphism, surjective, and injective.

Suppose $A, B \in GL_2(\mathbb{R})$. Now, $\alpha(AB) = ((AB)^t)^{-1} = (B^t A^t)^{-1} = (A^t)^{-1}(B^t)^{-1} = \alpha(A)\alpha(B)$.

Let $C \in GL_2(\mathbb{R})$. Choose $C' = (C^{-1})^t \in GL_2(\mathbb{R})$. Now, $\alpha(C') = (((C^{-1})^t)^t)^{-1} = C$.

Let $A, B \in GL_2(\mathbb{R})$. Suppose $\alpha(A) = \alpha(B)$. So $(A^t)^{-1} = (B^t)^{-1}$. Since the inverse of B^t is unique, $A^t = B^t$, and $A = B$. $\square$

2.2 Let $U = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| a, b, d \in \mathbb{R} \text{ and } ad \neq 0 \right\} \subset GL_2(\mathbb{R})$ denote the subset of invertible upper-triangular matrices. Show that U is a subgroup of $GL_2(\mathbb{R})$.

U is a subset of $GL_2(\mathbb{R})$. We need to prove U is a group.

Let $A, A' \in U$, where $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, A' = \begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix}$. So $AA' = \begin{bmatrix} aa' & b'a + d'b \\ 0 & d'd \end{bmatrix} \in U$.

U inherits associativity from $GL_2(\mathbb{R})$.

$I_2 \in U$.

Let $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in U$. $A^{-1} = \begin{bmatrix} a^{-1} & -b(ad)^{-1} \\ 0 & d^{-1} \end{bmatrix} \in U$, which exists as $ad \neq 0$. $\square$

2.3 Show that the map $\phi : U \to \mathbb{R}^\times$ which sends $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ to a^2 is a group homomorphism and determine its image and kernel.

Let $A, A' \in U$, where $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, A' = \begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix}$, and $AA' = \begin{bmatrix} aa' & b'a + d'b \\ 0 & d'd \end{bmatrix} \in U$.

$\phi(A)\phi(A') = a^2 a'^2 = (aa')^2 = \phi(AA')$. $\square$

$\phi(U) = (0, \infty)$. $\ker(\phi) = \left\{ A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| A \in U, a \in \{1, -1\} \right\}$.

3 Let $\psi : G \to G'$ be a group homomorphism such that $\psi(G) \neq \{1\}$. Suppose that the orders $|G| = 18$ and $|G'| = 15$.

3.1 Show that $1 < |G : \ker(\psi)|$ and the index $|G : \ker(\psi)|$ divides the order $|G'|$.

Since $\psi(G) \neq \{1\}$, $\exists g \in G$ such that $\psi(g) \neq 1$, and $\psi(g^{-1}) = \psi(g)^{-1} \neq 1$. So $g \notin \ker(\psi)$.

Let, $g \notin \ker(\psi) = \{1s | s \in \ker(\psi)\} = A$ be a left coset of G. Since $1 \in \ker(\psi)$, let $g \in \{gs | s \in \ker(\psi)\} = B$ be a left coset of G. Since A and B are distinct, $|G : \ker(\psi)| \geq 2$. ☐

Since $\psi(G)$ is a subgroup of G' , $|G : \ker(\psi)| = |\psi(G)|$ divides G'. ☐

3.2 What is the order $|\ker(\psi)|$?

We know $|\ker(\psi)|\,|G : \ker\psi| = |G| = 18$.

From (3.1): $|G : \ker\psi| = |\psi(G)| \neq 1$ divides $|G'| = 15$. So $|\psi(G)| \in \{3, 5, 15\}$.

Solve $|\ker(\psi)| \in \mathbb{Z}$ to get $|\ker(\psi)| = 6$.

3.3 Let S be a subset of G such that the identity element $1 \in S$. Assume that the subsets $aS = \{as | s \in S\} \subset G$ for $a \in G$ form a partition of G. Prove that S is a subgroup of G.

Let S contain the identity.

Let C contain the subsets $aS = \{as | s \in S\} \subset G$ for same $a \in G$.

Since C is a partition of G, C defines the equivalence relation: $a, b \in G, a \sim b \iff a \in bS \iff \exists s \in S, a = bs$.

Let $a = 1$. Let $b \in S$. Since $b = 1b$, $b \sim a$. Since $a \sim b, \exists b^{-1} \in S, a = 1 = bb^{-1}$. Therefore every element of S has an inverse in S.

Let $x, y \in S$. Let $a, ax^{-1}, ax^{-1}y^{-1} \in G$. $a \sim ax^{-1}$, as $a = ax^{-1}x$. And $ax^{-1} \sim ax^{-1}y^{-1}$, as $ax^{-1} = ax^{-1}y^{-1}y$. Therefore $a \sim ax^{-1}y^{-1}$. Since $a \sim ax^{-1}y^{-1}$, $\exists s = yx \in S$ such that $a = ax^{-1}y^{-1}yx$. Therefore the binary operation on S is closed.

S inherits associativity from G.

Therefore S is a subgroup of G. ☐

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4.1 List the even permutations in the symmetric group of degree 4, i.e. the elements of the alternating group A_4. How many of them are of order 3?

$$\begin{aligned} p(\langle 1,2,3,4\rangle) \in \{ &\langle 1,2,3,4\rangle, \langle 1,3,4,2\rangle, \langle 1,4,2,3\rangle, \\ &\langle 2,1,4,3\rangle, \langle 2,3,1,4\rangle, \langle 2,4,3,1\rangle, \\ &\langle 3,1,2,4\rangle, \langle 3,2,4,1\rangle, \langle 3,4,1,2\rangle, \\ &\langle 4,1,3,2\rangle, \langle 4,2,1,3\rangle, \langle 4,3,2,1\rangle \} \end{aligned}$$

[1] There are 8 of order 3.

4.2 Let G be a group. Show that a subgroup H of G of index 2 is necessarily normal.

[2] Let H be a subgroup of G, and $|G : H| = 2$.

Since the index is 2, partition G into H, H', so $H \cup H' = G$, and $H \cap H' = \varnothing$.

Let $g \in G$.

If $g \in H$, then $gH = Hg$, and we're done.

Suppose $g \in H'$. Then $gH = H'$, otherwise $H = H'$. Similarly $Hg = H'$. Therefore $gH = H' = Hg$.

Therefore H is a normal subgroup of G. $\square$

4.3 Let K be a subgroup of A_4 of order 6. Show that for all $a \in A_4$, the cosets K, aK, and a^2K cannot all be distinct, and deduce that K must necessarily contain all elements of order 3 of A_4. Conclude that A_4 does not have a subgroup of order 6, even though 6 divides the order of A_4.

[3] Let K be a subgroup of A_4. Let $|K| = 6$.

If K, aK, a^2K are all distinct, then $|G : H| \geq 3$, but $|G : H| = 2$. Therefore K, aK, a^2K are not all distinct. $\square$

Since there are 8 elements of order 3 in A_4, and K only contains 6 elements, choose $a \in A_4, a \notin K$, and the order of $a = 3$.

Suppose $K = aK$. Since $1 \in K, a = a1$, and $a \in K$. Contradiction.

Suppose $K = a^2K$. Similarly $a^2 \in K$, but K is a group, so $(a^2)^{-1} = a \in K$. Contradiction.

Suppose $aK = a^2K$. Similarly $a(a) = a^2(1)$, so $a \in K$. Contradiction.

Therefore K must necessarily contain all $a \in A_4$ of order 3.
Therefore the subgroup K of A_4 such that $|K| = 6$ does not exist. $\square$

[1] Source: `https://groupprops.subwiki.org/wiki/Element_structure_of_alternating_group:A4`
[2] Source: `https://proofwiki.org/wiki/Subgroup_of_Index_2_is_Normal`
[3] Source: `https://math.stackexchange.com/questions/582658/a-4-has-no-subgroup-of-order-6`

5 Recall our notation for the dihedral group $D_n, n \geq 1$. We have $x, y \in D_n$ such that the orders $o(x) = n, o(y) = 2, yx = x^{-1}y$ and $D_n =< x, y >$.

5.1 Write down the element $x^2yx^{-1}y^{-1}x^3y^3$ of D_n in the form x^iy^i, for integers $i, j \geq 0$.

We have $x^n = 1, y^2 = 1, yx = x^{-1}y$. Now,

$$\begin{aligned}
yx &= x^{-1}y \\
y &= x^{-1}yx^{-1} \\
xy &= yx^{-1} \\
\rightarrow &x^2yx^{-1}y^{-1}x^3y^3 \\
= &x^2yx^{-1}y^{-1}x^3y \\
= &x(yx^{-1})x^{-1}y^{-1}x^2(yx^{-1}) \\
= &xyx^{-2}y^{-1}x^2yx^{-1} \\
= &yx^{-3}y^{-1}yx^{-3} \\
= &yx^{-3}x^{-3} \\
= &yx^{-6} \\
= &x^6y \\
\rightarrow &i = 6, j = 1
\end{aligned}$$

5.2 Let G be a group. Show that for all $a \in G$, if the subset $\{1, a\}$ is a normal subgroup of G then a is in the center of G. Prove that $N = \{1, x^5\}$ is a normal subgroup of D_{10}.

Suppose $a \in G$ such that $\{1, a\}$ is a normal subgroup of G.

Let $x \in G$. Since $\{1, a\}$ is a normal subgroup of $G, xax^{-1} \in \{1, a\}$. If $xax^{-1} = a$, then $xa = ax$, we're done. If $xax^{-1} = 1$, then $a = 1$, so $xa = ax$. □

We have $x, y \in D_{10}, x^{10} = 1, y^2 = 1, yx = x^{-1}y$.

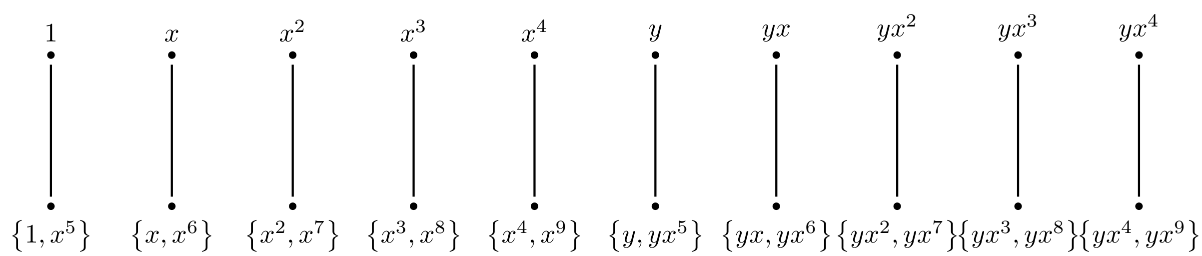
Let $h \in N = \{1, x^5\}$. $h = 1$ is the trivial case, so assume $h = x^5$.

Now, $xx^5x^{-1} = x^5 \in N$, and $yx^5y^{-1} = x^{-5}yy^{-1} = x^{-5} = x^5 \in N$. And any combination of x and y would yield the same result. Therefore N is a normal subgroup of D_{10}. □

5.3 Compute the left cosets of N in D_{10} and show that the quotient group $\frac{D_{10}}{N}$ is isomorphic to D_5.

$\{\{1, x^5\}, \{x, x^6\}, \{x^2, x^7\}, \{x^3, x^8\}, \{x^4, x^9\}, \{y, yx^5\}, \{yx, yx^6\}, \{yx^2, yx^7\}, \{yx^3, yx^8\}, \{yx^4, yx^9\}\}$.

We find a bijective group homomorphism from D_{10}/N to D_5.



□

Citations

"Algebra" by Michael Artin (ISBN 13: 9780132413770).

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