MATH 315 Assignment 3

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1 Let G be a group.

1.1 Show that $g \cdot x = xg^{-1}$ for $g, x \in G$ defines an action of G on itself.

Let $x \in G$. Now, $1 \cdot x = x1^{-1} = x$.

Let $g, h, x \in G$. Now $(gh) \cdot x = x(gh)^{-1} = xh^{-1}g^{-1} = g \cdot (h \cdot x)$. $\square$

1.2 The natural action of $GL_2(\mathbb{R})$ on $\mathbb{R}^2$ is given by $A \cdot v = Av$, for $A \in GL_2(\mathbb{R})$ and $v \in \mathbb{R}^2$. Compute the stabilizers and the orbits of $\binom{1}{0}$ and $\binom{1}{1}$.

For $v = \binom{1}{0}$:

The stabilizer subgroup is $\left\{ \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \middle| a, b \in \mathbb{R}, b \neq 0 \right\}$.

Since $\begin{bmatrix} a & 0 \\ b & a \end{bmatrix} v = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix} v = \begin{bmatrix} a \\ b \end{bmatrix}$, orbit $= \left\{ \binom{a}{b} | a, b \in \mathbb{R}, a \neq 0 \text{ or } b \neq 0 \right\}$.

For $v = \binom{1}{1}$.

The stabilizer subgroup is $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R}, a + b = 1, c + d = 1, ad - bc \neq 0, \right\}$.

Since $\binom{1}{1}$ is in the orbit of $\binom{1}{0}$, it has the same orbit as $\binom{1}{0}$.

1.3 Show that a group homomorphism for the alternating group A_5 to G is injective if its image contains an element different from the identity 1 of G.

Idea for the proof: [1]

Lemma: A_5 is a simple group. [2]

Let $\phi : A_5 \to G$, such that $\exists a \in A_5, \phi(a) \neq 1_G$.

We know the kernel of ϕ is a normal subgroup of A_5 ,and that the $\ker(\phi) \neq A_5$ from construction of ϕ.

Since A_5 is a simple group, and $\ker(\phi) \neq A_5$, we know $\ker(\phi) = \{1\}$.

Since $\ker(\phi) = \{1\}$, therefore ϕ is injective. [3]

$\square$

[1] From a since deleted question on StackExchange (probably from a student of this class): `https://web.archive.org/web/20200314122820/https://math.stackexchange.com/questions/3580662/prove-that-fa-5-to-g-is-injective`

[2] It's listed here: `https://groupprops.subwiki.org/wiki/Alternating_group#Group_properties`

[3] Second "Symbol-free definition" of: `https://groupprops.subwiki.org/wiki/Injective_homomorphism`

2

2.1 What is the class equation of the dihedral group D_4? Justify your answer.

Let $y, x \in D_4$, where x is basic rotation, y is basic reflection.

The identity forms a conjugacy class $\{\boldsymbol{x^0}\}$.

Let $x' = x^s$ be any rotation element.
We have $yx^n \cdot x' = yx^n x'(yx^n)^{-1} = yx^{n+s}x^{-n}y^{-1} = x'^{-1}, \forall n \in \mathbb{N}$.
And $x^n \cdot x' = x', \forall n \in \mathbb{N}$.
Therefore $\{\boldsymbol{x, x^3}\}$ and $\{\boldsymbol{x^2}\}$ are conjugacy classes.

Let $y' = x^s y$ be any reflection element.
We have $x^n \cdot y' = x^n x^s y x^{-n} = x^{2n+s} y$.
And $x^n y \cdot y' = x^n y x^s y y x^{-n} = x^{2n-s} y$.
Therefore $\{\boldsymbol{x^0 y, x^2 y}\}$ and $\{\boldsymbol{x^1 y, x^3 y}\}$ are conjugacy clasess.

Therefore $|D_4| = 1 + 1 + 2 + 2 + 2$. $\square$

2.2 The class equation of a group H is $1 + 4 + 5 + 5 + 5$. Does H have a subgroup of order 4? Could it be a normal subgroup? Justify your answers.

Let H be a group of order 20, with the class equation $1 + 4 + 5 + 5 + 5 = |C_1| + \ldots + |C_5|$.

H has a subgroup of order 4. Example:

The stabilizer/centralizer Z_5 is a subgroup with the order $|H| / |C_5| = 20/5 = 4$. $\square$

H does not have a normal subgroup of order 4. Proof:

Let I be a normal subgroup of order 4 of H.

Let $i \in I$, such that $i \neq 1$.

Since I is normal, the orbit of i is a subset of I.

But $i \neq 1$, so its orbit is at least 4.

Therefore $|I| \geq 4 + 1 = 5$. Therefore there exists no such I. $\square$

2.3 Let G be a group of order 12. Show that if G contains a conjugacy class of order 4, then the center of G is $\{1\}$.

Let G be a group with order 12 such that G have a conjugacy class of order 4.

So $\exists x$ such that $|C(x)| = 4$.

Now, $|Z(x)| = |G| / |C(x)| = 12/4 = 3$.

We know $|Z(G)| \leq |Z(x)| = 3$, where $Z(G) \subseteq Z(x)$.

If $x \in Z(G)$ then $|C(x)| = 1$, which would be a contradiction. Therefore $x \notin Z(G)$.

Since $x \in Z(x)$, we know $|Z(G)| \leq 2$.

Since $Z(x)$ and $Z(G)$ are both subgroups of G , and $Z(G) \subseteq Z(x)$, we know $Z(G)$ is a subgroup of $Z(x)$.

So $|Z(G)|$ divides $|Z(x)|$ Therefore $|Z(G)| \neq 2$.

Therefore $|Z(G)| \leq 1$.

Therefore $Z(G) = \{1\}$. $\square$

3

3.1 Let R, R' be rings. Show that the product of groups $R \times R'$ is a ring for the multiplication given by $(r_1, r_1') \cdot (r_2, r_2') = (r_1 r_2, r_1' r_2')$. It is called the (direct) product of the rings R and R'.

It's easy to verify that $(R \times R', +)$ is an abelian group with the identity $(0, 0)$.

It's also easy to verify that $\times$ is commutative and associative on $R \times R'$, with the identity being $(1, 1)$.

Let $(a, a'), (b, b'), (c, c') \in R \times R'$. Now,

$$\begin{aligned}((a, a') + (b, b'))(c, c') &= (a + b, a' + b')(c, c') \\ &= ((a + b)c, (a' + b')c') \\ &= (ac + bc, a'c' + b'c') \\ &= (a, a')(c, c') + (b, b')(c, c')\end{aligned}$$

Therefore the distributive law holds, and $R \times R'$ is a ring. □

3.2 Prove that a surjective map $f : R \to R'$ such that $f(r_1 r_2) = f(r_1)f(r_2)$ for all $r_1, r_2 \in R$ satisfies $f(1) = 1$.

Let $b \in R'$.

Since f is surjective, $\exists a \in R, f(a) = b$.

Now, $f(1)b = f(1)f(a) = f(1a) = f(a) = b$, and $bf(1) = b$.

Therefore $f(1)$ is the identity in R'. □

3.3 An element r of a ring R is nilpotent if $r^n = 0$ for some $n \geq 0$. Does the ring $F(\mathbb{R}, \mathbb{R})$ of functions from $\mathbb{R}$ to $\mathbb{R}$ have non-zero nilpotent elements? Does it have non-zero zero divisors? Prove or give an example.

Lemma: $F(\mathbb{R}, \mathbb{R})$ of all function from $\mathbb{R}$ to $\mathbb{R}$, is a ring.

There are no non-zero nilpotent elements. Proof:

Let $f \in F(\mathbb{R}, \mathbb{R})$, such that $f(x) \neq 0$.

So $\exists x \in \mathbb{R}$, such that $f(x) = c \neq 0$, for some $c \in \mathbb{R}$.

So $(f(x))^n = c^n \neq 0$ for all $n \in \mathbb{N}$. □

There are non-zero zero divisors. Proof:

Let $f(x) = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{if } x \neq 1 \end{cases}$.

Let $g(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases}$.

Where $f(x) \neq 0, g(x) \neq 0, f(x)g(x) = 0$. □

4

4.1 Let R be a ring. Show that if $a^2 = a$ for all $a \in R$ and $R \neq \{0\}$, then the characteristic of R is 2.

Choose $r \in R$ such that $r \neq 0$.

Now, $r + r = (r + r)^2 = r^2 + r^2 + r^2 + r^2 = r + r + r + r$. So $0 = r + r$. □

4.2 Show that in $\mathbb{Z}/6\mathbb{Z}$, we have $a^3 = a$ for all elements of a but the characteristic of $\mathbb{Z}/6\mathbb{Z}$ is not 3.

$\mathbb{Z}/6\mathbb{Z}$ has ring with 6 elements. Now,

$$\begin{aligned}
0^3 &= 0 \equiv 0 \pmod 6 \\
1^3 &= 1 \equiv 1 \pmod 6 \\
2^3 &= 8 \equiv 2 \pmod 6 \\
3^3 &= 27 \equiv 3 \pmod 6 \\
4^3 &= 64 \equiv 4 \pmod 6 \\
5^3 &= 125 \equiv 5 \pmod 6
\end{aligned}$$

. However, $1 + 1 + 1 = 3 \not\equiv 0 \pmod 6$. □

4.3 Show that if the characteristic of a commutative ring R' is 2, then the map $a \mapsto a^2$ is a ring homomorphism $R' \to R'$.

Let $\phi : a \to a^2$, for $a \in R'$.

Let $a, b \in R'$.

Now, $f(a+b) = (a+b)^2 = (a+b)(a+b) = a(a+b)+b(a+b) = aa+ab+ab+bb = f(a)+f(b)+ab+ab = f(a)+f(b)$.

And, $f(ab) = (ab)^2 = a^2b^2 = f(a)f(b)$.

Lastly, $f(1) = 1^2 = 1$. □

Citations

"Algebra" by Michael Artin (ISBN 13: 9780132413770).

Proofread by Devin Kwok (UCID: 10016484).