

# Notes

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# Abstract Algebra

## TODO Categories

### Groups

A **Law of Composition/(Closed) Binary Operation** on  $S$  is a function/map of the form  $S \times S \rightarrow S$ .  
(Common Notations:  $*$ ,  $\star$ ,  $\cdot$ ,  $\bullet$ ,  $\times$ ,  $+$ , or by concatenating the two elements.)

A Law of Composition on  $S$  is  $\begin{cases} \text{Associative} & \iff \forall a, b, c \in S, (ab)c = a(bc). \\ \text{Commutative} & \iff \forall a, b \in S, ab = ba. \end{cases}$

The **Identity/Unity** of a Law of Composition is an element  $e \in S$ , such that  $\forall a \in S, ea = ae = a$ .

A element  $a \in S$  is **Invertible**  $\iff \exists b \in S, ab = ba = e$ , where  $b$  is the inverse of  $a$ .

A **Group** is a set  $S$  together with a Law of Composition denoted by  $G = (S, *)$ , such that

1.  $*$  is Associative.
2.  $\exists$  an Identity  $e_G$ .
3. Every element of  $S$  is Invertible.

A **Abelian Group** is a Group that is Commutative.

The **Order** of a Group  $(S, *)$  is the number of elements in the group, where  $|G| = |(S, *)| = |S|$ .  
(Notation:  $(S, *)$  should not be treated as an ordered set, for instance:  $(\mathbb{R}, \times) = \mathbb{R}^\times$ , and  $(\mathbb{Z}, +) \subseteq (\mathbb{R}, +)$ .)

The **Center** of a group  $G$  is defined:  $Z = \{z \in G \mid \forall x \in G, zx = xz\}$ .

A **Partition**  $\Pi$  of a set  $S$  is a subdivision of  $S$  into nonoverlapping, nonempty subsets.

A Partition is logically equivalent to an **Equivalence Relation** on  $S$ .

### Subgroups

A Group  $H$  is a **Subgroup of a Group**  $G \iff \emptyset \subsetneq H \subseteq G \wedge \forall a, b \in H, ab^{-1} \in H$ .

Every Group  $G$  has two trivial Subgroups:  $G$  and  $\{e\}$  (The trivial subgroups are also Normal).

A **Cyclic Subgroup**  $H$  of  $G$  is generated by a element  $x \in G$ , such that  $H = \{x^n \mid n \in \mathbb{Z}\}$ .

$H$  is a **Normal Subgroup** of  $G \iff \forall h \in H, \forall g \in G, ghg^{-1} \in H$ .

### Homomorphisms

Let  $G, G'$  be groups.

A **Homomorphism**  $\phi$  is a function from  $G \rightarrow G'$ , such that  $\forall a, b \in G, \phi(ab) = \phi(a)\phi(b)$ .

The **Kernel**  $\ker(\phi)$  of a Homomorphism is the set of all elements in  $G$  such that  $a \in G, \phi(a) = \text{the identity in } G'$ .

An **Isomorphism**  $\psi : G \rightarrow G'$  is a bijective group Homomorphism.

### Cosets

Let  $H$  be a subgroup of  $G$ ,  $a \in G$ .

$aH = \{ah | h \in H\}$  is a **Left Coset**.

$Ha = \{ha | h \in H\}$  is a **Right Coset**.

All Left Cosets of a subgroup  $H$  of  $G$  have the same order, and form a Partition of  $G$ .

The **Index** of  $H$  in  $G$  is the number of left cosets of the subgroup. Denoted by  $|G : H|$ .

Right Cosets have the same properties, but may not equal to the Left Coset.

Let  $\psi : G \rightarrow G'$  be a group homomorphism.  $|G : \ker \psi| = |\psi(G)|$ .

### Quotient Group

Let  $C$  be the set of left cosets of a Normal subgroup  $N$  of  $G$ .

A **Quotient Group**  $G/N$  is defined on  $C$  such that:  $\forall aN, bN \in G/N, (aN)(bN) = abN$ .

### Symmetric Group

A **Symmetric group**  $S_n$  is the set of all permutations of  $\{1, 2, \dots, n\}$ .

Every permutation  $p$  has an associated **Permutation Matrix**  $P$  with a single 1 in each row and column, and 0s everywhere else.

$$P^{-1} = P^T.$$

A permutation  $p$  is **Even**  $\iff \det(P) = 1$ .

A permutation  $p$  is **Odd**  $\iff \det(P) = -1$ .

An **Alternating Group**  $A_n$  contains all even elements of  $S_n$ .

The order of  $p$  is the smallest positive integer  $k$  such that  $p^k = 1$ .

### Dihedral Group TODO

#### Action/Operation

Let  $G$  be a group,  $X$  be a set.

A **Left Group Action (Operation)** is a function  $\phi : G \times X \rightarrow X, (g, x) \mapsto \phi(g, x) = g \cdot x$  such that

1.  $\forall s \in S, \phi(e_G, s) = s$ .
2.  $\forall g, h \in G, \forall x \in X, \phi(gh, x) = \phi(g, \phi(h, x))$ .

A **Right Group Action** is defined similarly.

A **Orbit** of an element  $x \in X$  by the group  $G$  is  $G \cdot x = \{g \cdot x | g \in G\}$ .

The set of Orbits of  $X$  form a partition of  $X$ .

A **Stabilizer** of  $x$  by  $G$  is  $g \in G | g \cdot x = x$ .

The set of all Stabilizers is the **Stabilizer (Subgroup)** of  $x$  under  $G$ .

## Conjugation

A **Conjugation** is a action of  $G$  on itself such that  $(g, x) \mapsto gxg^{-1}$ .

The Stabilizer of this action is called the **Centralizer** denoted by  $Z(x)$ .

A Orbit of this action is called the **Conjugacy Class** denoted by  $C(x)$ .

$$|G| = |Z(x)| |C(x)|.$$

## Rings

A (**Commutative**) **Ring**  $R$  is a set with two binary operation  $+$  and  $\times$ , referred to as **Addition** and **Multiplication**, such that

1.  $(R, +)$  is an abelian group denoted as  $R^+$ , with the identity denoted as 0.
2.  $\times$  is Commutative and Associative, with the identity denoted as 1 ( $R$  doesn't necessarily contain 1).
3.  $\forall a, b, c \in R, (a + b)c = ac + bc$ . (Distributive Law)

A **Noncommutative Ring** is a Ring without the Commutative Law for Multiplication.

A **Subring** of  $R$  is a Ring that is a subset of  $R$ .

A **Ring Homomorphism**  $\phi$  is a function from  $R \rightarrow R'$  such that  $\forall a, b \in R$ ,

1.  $\phi(a + b) = \phi(a) + \phi(b)$ .
2.  $\phi(ab) = \phi(a)\phi(b)$ .
3.  $\phi(1) = 1$ .

$R$  and  $R'$  are **Isomorphic**, denoted by  $R \approx R' \iff$  There exists an Ring Isomorphism between them.

## Ideal

An **Ideal**  $I$  of a ring  $R$  is a subset of  $R$  such that

1.  $I \neq \emptyset$ .
2.  $I$  is closed under addition.
3.  $\forall s \in I, \forall r \in R, rs \in I$ .

The kernel of a ring homomorphism is an Ideal.

Ideals are subrings.

Let  $a \in \mathbb{R}$ . The multiples of  $a$  form an Ideal called the **Principal Ideal** generated by  $a$ . Denoted as  $(a) = aR = Ra = \{ra \mid r \in R\}$ .

$(1) = R$  is the **Unit Ideal**.

$(0) = \{0\}$  is the **Zero Ideal**.

A Ideal that is not  $(1)$  nor  $(0)$  is **Proper**.

A Ring with exactly two ideals is a **Field**.

An Ideal generated by a set of elements  $\{a_1, \dots, a_n\} \subseteq R$  is the smallest ideal that contains those elements. Denoted as  $(a_1, \dots, a_n) = \{r_1a_1 + \dots + r_na_n \mid r_i \in R\}$ .

### **Quotient Ring**

Let  $R$  be a Ring with an Ideal  $I$ .

Let  $C$  be the set of cosets of  $I^+$  of  $R^+$ .

A **Quotient/Factor Ring  $R/I$**  is defined on  $C$  such that:

1.  $\forall aI, bI \in R/I, (a + I) + (b + I) = (a + b) + I.$
2.  $\forall aI, bI \in R/I, (a + I)(b + I) = (ab) + I.$

# Real Analysis

## Sequences

A **Sequence** is a function  $f : \mathbb{N} \rightarrow \mathbb{R}$ . Notations are  $a_n = f(n)$ ,  $\{a_n\}$ ,  $\{a_n\}_{n=1}^\infty$ .

A sequence is **Bounded**  $\iff \exists M \in \mathbb{R}$  such that  $\forall n \in \mathbb{N}, |a_n| \leq M$ .

**Bounded Above/Below** are defined similarly.

A sequence is **(Strictly) Increasing** if  $\forall n \in \mathbb{N}, a_{n+1} \geq a_n$  ( $a_{n+1} > a_n$ ). **(Strictly) Decreasing** is defined similarly.

A sequence is **Monotone** if it's either Decreasing or Increasing.

A sequence is **Convergent** to  $L \iff \forall \epsilon \in \mathbb{R}, \epsilon > 0, \exists N \in \mathbb{N}$  such that  $\forall n \in \mathbb{N}$ , if  $n \geq N$  then  $|a_n - L| < \epsilon$ .

A sequence is **Divergent** to  $\infty \iff \forall M \in \mathbb{R}, M > 0, \exists N \in \mathbb{N}$  such that  $\forall n \in \mathbb{N}$ , if  $n \geq N$  then  $a_n > M$ .  
Divergent to  $-\infty$  is defined similarly.

A sequence is **Cauchy**  $\iff \forall \epsilon \in \mathbb{R}, \epsilon > 0, \exists N \in \mathbb{N}$  such that  $\forall m, n \in \mathbb{N}$ , if  $m, n \geq N$  then  $|a_n - a_m| < \epsilon$ .

A sequence is Convergent  $\iff$  it's Cauchy.

If a sequence is Convergent, then it is Bounded.

### Monotone Convergence Theorem:

If a sequence is Bounded Above and Increasing, then it converges.

If a sequence is Bounded Below and Decreasing, then it converges.

If a sequence is Bounded and Monotone, then it converges.

Every sequence has a Monotone subsequence.

Define **sub**( $a_n$ ) to be a set containing the limits of all subsequences of  $\{a_n\}$ .

The **Limit Superior** of  $\{a_n\}$  is  $\sup(\text{sub}(a_n)) = \max(\text{sub}(a_n))$ , denoted as  $\limsup_{n \rightarrow \infty} (a_n)$ .

The **Limit Inferior** of  $\{a_n\}$  is  $\inf(\text{sub}(a_n)) = \min(\text{sub}(a_n))$ , denoted as  $\liminf_{n \rightarrow \infty} (a_n)$ .

Let  $\{q_n\}$  be any denumeration of  $\mathbb{Q}$ , then  $\text{sub}(q_n) = \mathbb{R}$ .

## Topology of the Real Line

Let  $x \in \mathbb{R}$  and  $S \subseteq \mathbb{R}$ .

The **Neighbourhood (nhd)** of  $x$  is the set  $N_\epsilon(x) = \{y \in \mathbb{R} : |x - y| < \epsilon\} = (x - \epsilon, x + \epsilon)$ , for some  $\epsilon > 0$ .

The **Deleted Neighbourhood** of  $x$  is the set  $N_\epsilon^*(x) = N_\epsilon(x) \setminus \{x\}$ .

An **Interior Point** of  $S$  is an element  $x \in S$  such that  $N_\epsilon(x) \subseteq S$  for some  $\epsilon > 0$ .

The set of all Interior points of  $S$  is the **Interior** of  $S$ , denoted as  $\text{int}(S)$ .

$x$  is a **Boundary Point** for  $S \iff \forall \epsilon > 0, N_\epsilon(x) \cap S \neq \emptyset \wedge N_\epsilon(x) \cap (\mathbb{R} \setminus S) \neq \emptyset$ .

The set of all Boundary Points of  $S$  is the **Boundary** of  $S$ , denoted as  $\text{bd}(S)$ .

$x$  is a **Accumulation (Limit) Point** for  $S \iff \forall \epsilon > 0, N_\epsilon^*(x) \cap S \neq \emptyset \iff \forall \epsilon > 0, \exists y \in S \text{ such that } 0 < |x - y| < \epsilon.$

The set of all Accumulation Points of  $S$  is denoted as  $S'$ .

The **Closure** of  $S$  is the set  $\bar{S} = S \cup S' = S \cup \text{bd}(S)$ .

$x$  is a **Isolated Point** for  $S \iff x \in S \setminus S' \iff x \in S \wedge \exists \epsilon > 0 \text{ such that } N_\epsilon^*(x) \cap S = \emptyset.$

$S \subseteq \text{int}(S) \cup \text{bd}(S)$  and  $\text{int}(S) \cap \text{bd}(S) = \emptyset$ .

$S$  is **Open**  $\iff S = \text{int}(S) \iff \text{bd}(S) \subseteq \mathbb{R} \setminus S$ .

$S$  is **Closed**  $\iff \mathbb{R} \setminus S$  is Open  $\iff \text{bd}(S) \subseteq S \iff S = \bar{S}$ .

$S$  is **Clopen**  $\iff S$  is both Open and Closed. The only Clopen sets are  $\emptyset$  and  $\mathbb{R}$ .

If  $S \neq \emptyset$ , then (  $S$  is **Compact**  $\iff$  every sequence in  $S$  has a Convergent subsequence with it's limit in  $S$ . )

**Bolzano–Weierstrass Theorem (BW):**

If  $S \subseteq \mathbb{R}$  is infinite and bounded, then  $S' \neq \emptyset$ .

Every bounded sequence has a convergent subsequence.

**Heine–Borel Theorem:**

$S \neq \emptyset, S$  is Compact  $\iff S$  is Closed and Bounded.

**Cantor's Intersection Theorem:**

If  $K_1 \supseteq K_2 \supseteq K_3 \dots$  are Compact sets, then  $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$  is Compact.

## Limits and Continuity

Let  $f : D \rightarrow \mathbb{R}$  such that  $D \subseteq \mathbb{R}$ . Let  $a \in D' \subseteq \mathbb{R}$ . Let  $a_n \in D \setminus \{a\}, a_n \rightarrow a$ .

$\lim_{x \rightarrow a^+} f(x) = L \iff \forall \epsilon > 0, \exists \delta > 0, \forall x \in D, 0 < x - a < \delta \rightarrow |f(x) - L| < \epsilon$

$\lim_{x \rightarrow a^-} f(x) = L$  is defined similarly with  $a - x$  instead.

$\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L \iff \lim_{n \rightarrow \infty} f(a_n) = L$

$\lim_{x \rightarrow a} f(x) = \infty \iff \forall M > 0, \exists \delta > 0, \forall x \in D, 0 < |x - a| < \delta \rightarrow f(x) \geq M$

$\lim_{x \rightarrow a} f(x) = -\infty$  is defined similarly with  $M < 0$  instead.

$\lim_{x \rightarrow a^\pm} f(x) = \pm\infty$  are defined similarly (all 4 cases).

Let  $b, x \in D$ . Let  $S \subseteq D$ . Let  $b_n \in D, b_n \rightarrow b$ .

$f$  is **Continuous at  $b$**   $\iff \forall \epsilon > 0, \exists \delta > 0, |x - b| < \delta \rightarrow |f(x) - f(b)| < \epsilon \iff \lim_{n \rightarrow \infty} f(b_n) = f(b) \iff \lim_{x \rightarrow b} f(x) = f(b)$

$f$  is **Discontinuous at  $b$**   $\iff f$  is not Continuous at  $b$ .

$f$  is **Right (Left) Continuous at  $b$**  is defined similarly with  $x - b$  ( $b - x$ ) instead.

$f$  is **Continuous on  $S$**   $\iff f$  is Continuous  $\forall s \in S$ .

$f$  is **Uniformly Continuous (UC) on  $S$**   $\iff \forall \epsilon > 0, \exists \delta > 0, \forall x, y \in S, |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon$ .

$f$  is **Continuous**  $\iff f$  is Continuous on  $D$ .

**(Strictly) Increasing/Decreasing/Monotone** for  $f$  is defined similarly to sequences.

Let  $g : S \rightarrow \mathbb{R}, S \subseteq D \cap \mathbb{R}$ .  $g$  has a **Continuous Extension** to  $D \iff \exists \tilde{g} : D \rightarrow \mathbb{R}$  such that  $\tilde{g}$  is Continuous and  $\tilde{g}(x) = g(x), \forall x \in S$ . Notation is  $\tilde{g}|_S = g$ .

If  $f$  is UC and  $\{c_n\} \subseteq D$  is Cauchy, then  $\{f(c_n)\}$  is Cauchy.

If  $f$  is Continuous and  $D$  is Compact, then  $f$  is UC on  $D$ .

If  $f$  is UC, then  $f$  has a Continuous Extension  $\tilde{f} : \bar{D} \rightarrow \mathbb{R}$ .

### Squeeze Theorem:

If  $f, g, h : D \rightarrow \mathbb{R}$ ,  $a \in D'$ , and  $\exists \epsilon > 0$  such that  $\forall x \in (a - \epsilon, a + \epsilon)$ ,  $f(x) \leq g(x) \leq h(x)$ , and  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$ , then  $\lim_{x \rightarrow a} g(x) = L$ .

### Extreme Value Theorem:

If  $f : D \rightarrow \mathbb{R}$  is continuous and  $D$  is compact, then  $f(D)$  has a min and max.

### Intermediate Value Theorem (IVT):

Suppose  $f : I \rightarrow \mathbb{R}$  is continuous and  $I$  is an interval. If  $a, b \in I$ ,  $a < b$ ,  $f(a) \neq f(b)$ , then  $\exists c \in (a, b)$  such that  $f(c) = k$ , where  $k$  is any value between  $f(a)$  and  $f(b)$ .

### Types of Discontinuity:

**Removable Discontinuity:**  $\lim_{x \rightarrow a} f(x) = L \neq f(a)$

**Jump Discontinuity:**  $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$  and  $f(a)$  exists.

**Essential Discontinuity:**  $\lim_{x \rightarrow a^-} f(x)$  or  $\lim_{x \rightarrow a^+} f(x)$  or  $\lim_{x \rightarrow a} f(x)$  has no limit  $L$ .

**Infinite Discontinuity:** TODO (LEC OCT 24)

### Trig Functions:

$\forall x$ :

$$|\sin(x)| \leq |x|,$$

$$\cos(x) = 1 - 2 \sin^2\left(\frac{x}{2}\right),$$

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a).$$

$$\cos(\arcsin(x)) = \sqrt{1-x^2}.$$

## Differentiation

Let  $f : D \rightarrow \mathbb{R}$ ,  $a \in \text{int}(D)$ ,  $S \subseteq \text{int}(D) \neq \emptyset$ .

$f$  is **Differentiable (diff.) at  $a$**   $\iff$  The Limit  $f'(a) = \lim_{x \rightarrow a} \frac{f(x)-f(a)}{x-a} = \lim_{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}$  exists.  
 $f'(a)$  is the **Derivative** of  $f$  at  $a$ .

$f$  is **Differentiable (diff.) on  $S$**   $\iff$   $f$  is Differentiable  $\forall x \in S$ .

The **Derivative** of  $f$  on  $S$  is the function  $f' : S \rightarrow \mathbb{R}$  or  $f^{(1)}$ .

If  $f$  is Differentiable at  $a$ , then  $f$  is Continuous at  $a$ .

$f$  has a **Local Maximum at  $c$**   $\iff \exists \delta > 0$  such that  $\forall x \in N_\delta(c)$ ,  $f(c) \geq f(x)$ .

$f$  has a **Local Minimum at  $c$**   $\iff \exists \delta > 0$  such that  $\forall x \in N_\delta(c)$ ,  $f(c) \leq f(x)$ .

### Rolle's Theorem

Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is Continuous on  $[a, b]$  and Differentiable on  $(a, b)$ .

If  $f(a) = f(b)$ , then  $\exists c \in (a, b)$  such that  $f'(c) = 0$ .

### Mean Value Theorem

Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is Continuous on  $[a, b]$  and Differentiable on  $(a, b)$ .

$\exists c \in (a, b)$  such that  $f'(c) = \frac{f(b)-f(a)}{b-a}$ .

Let  $I$  be an interval and  $f : I \rightarrow \mathbb{R}$ . If  $\exists M > 0$  such that  $\forall x \in I$ ,  $|f'(x)| \leq M$ , then  $f$  is UC on  $I$ .

### Intermediate Value Theorem (IVT) (for Derivatives)

Let  $f$  be Differentiable on  $[a, b]$  and  $f'(a) \neq f'(b)$ .  $\forall k$  Strictly between  $f'(a)$  and  $f'(b)$ ,  $\exists c \in (a, b)$  such that  $f'(c) = k$ .

### Inverse Function Theorem (for One Variable)

Let  $f$  be diff. on an open interval  $I$  and  $f'$  is non-zero on  $I$  ( $f'$  is either Strictly positive or Strictly negative).

1.  $f : I \rightarrow f(I)$  is invertible.

2.  $f^{-1} : f(I) \rightarrow I$  is diff.

$$3. (f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

### Cauchy's Mean Value Theorem (Cauchy's MVT)

Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be Continuous and diff. on  $(a, b)$ .  $\exists c \in (a, b)$  such that  $[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$ .

### L'Hôpital's Rule

Let  $f, g$  be diff. on an open interval containing  $[a, b]$  and  $c \in [a, b]$  with  $f(c) = g(c) = 0$ . If  $g'$  is non-zero in some deleted nhd of  $c$  and  $\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} = L$  exists, then  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = L$ .

Let  $f, g$  be diff. on  $(a, \infty)$  and  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = \infty$ . If  $\lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)} = L$  exists, then  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$ .

Let  $f$  be  $n$  times diff. at  $x = x_0$ .

The  $n^{\text{th}}$  degree Taylor Polynomial for  $f$  at  $x_0$  is  $T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}$ .

$f^{(k)}(x_0) = T_n^{(k)}(x_0)$  for  $0 \leq k \leq n$ .  $T_n$  is unique.

### Taylor's Theorem

Let  $f$  be  $n$  times diff. on an open set (?) containing  $[a, b]$ , where  $f^{(n)}$  is continuous on  $[a, b]$  and diff. on  $(a, b)$ .

Let  $x_0 \in (a, b)$ .  $\forall x \in (a, b) \setminus \{x_0\}$ ,  $\exists c$  between  $x$  and  $x_0$ , such that  $f(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ .

### Taylor's Inequality

If  $f^{(n+1)}$  is bounded on  $(a, b)$  by  $M$ , then  $\forall x \in (a, b)$ ,  $|f(x) - T_n(x)| \leq \frac{M}{(n+1)!} |x - x_0|^{n+1} \leq \frac{M}{(n+1)!} (b - a)^{n+1}$ .

## Integration

Let  $f$  be a Bounded function on  $[a, b]$ .

A Partition of  $[a, b]$  is  $P = \{x_0, x_1, \dots, x_n\}$  where  $a = x_0, b = x_n$  and  $\forall k, 0 \leq k < n \rightarrow x_k < x_{k+1}$ .

Let  $Q, P$  be Partitions of  $[a, b]$ .

$Q$  is a Refinement of  $P \iff P \subseteq Q$ .

Defined  $M_i(f, P) = \sup \{f(x) | x_{i-1} \leq x \leq x_i\}$ .

Defined  $m_i(f, P) = \inf \{f(x) | x_{i-1} \leq x \leq x_i\}$ .

The Upper Sum for  $f$  on a partition  $[a, b]$  is  $U(f, P) = \sum_{i=1}^n M_i(f, P) \Delta x_i$  where  $\Delta x_i = x_i - x_{i-1}$ .

The Lower Sum for  $f$  on a partition  $[a, b]$  is  $L(f, P) = \sum_{i=1}^n m_i(f, P) \Delta x_i$ .

The Upper Integral for  $f$  on  $[a, b]$  is  $U(f) = \inf_P \{U(f, P)\}$ .

The Lower Integral for  $f$  on  $[a, b]$  is  $L(f) = \sup_P \{L(f, P)\}$ .

$f$  is (Riemann) Integrable on  $[a, b]$  if  $L(f) = U(f) = \int_a^b f(x) dx = \int_a^b f$ .

$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$ .

$f$  is Integrable on  $[a, b] \iff \forall \epsilon > 0, \exists$  a partition  $P$  such that  $U(f, P) - L(f, P) < \epsilon$ .

### Fundamental Theorem of Calculus

1. Let  $f$  be Bounded and Integrable on  $[a, b]$ . Define  $F : [a, b] \rightarrow \mathbb{R}$ , so that  $F(x) = \int_a^x f(t) dt$ .  $F$  is UC on  $[a, b]$ . If  $f$  is Continuous at  $c$ , then  $F$  is diff. at  $c$  and  $F'(c) = f(c)$ .

2. Let  $f'$  be Bounded and Integrable on  $[a, b]$ .  $\int_a^b f' = f(b) - f(a)$ .

## Infinite Series

Let  $\{a_n\}$  be a sequence.

Define the **Sequence of Partial Sums**  $\sum a_n = s_n = \sum_{i=1}^n a_i$ .

If  $\{s_n\}$  is convergent, then  $\{a_n\}$  is convergent to 0.

### Cauchy's Convergence Test

$\{s_n\}$  is convergent  $\iff \forall \epsilon > 0, \exists N \in \mathbb{N}$  so that  $\forall n, m \in \mathbb{N}, N < m < n \rightarrow \sum_{i=m}^n a_i < \epsilon$ .

### Comparison Test

Suppose  $\forall n, 0 \leq a_n \leq b_n$ .

If  $\sum b_n$  is convergent, then  $\sum a_n$  is convergent.

### Limit Comparison Test

Suppose  $\forall n, a_n$  and  $b_n$  are non-negative. Suppose  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \in [0, \infty]$ .

1. If  $L \in [0, \infty)$  and  $\sum b_n$  is convergent, then  $\sum a_n$  is convergent.
2. If  $L \in (0, \infty]$  and  $\sum b_n$  is divergent, then  $\sum a_n$  is divergent.
3. If  $L \in (0, \infty)$ , then ( $\sum a_n$  is convergent  $\iff \sum b_n$  is convergent).

## Sequences and Series of Functions

Let  $C \subseteq D \subseteq \mathbb{R}$ .

$\{f_n\}$  is a sequence of functions each from  $D \rightarrow \mathbb{R}$ .

$\{f_n\}$  **Converges Pointwise** to  $f : C \rightarrow \mathbb{R} \iff \forall x \in C, \lim_{n \rightarrow \infty} f_n(x) = f(x)$ . Denoted as  $f_n \xrightarrow[C]{\text{c.p.}} f$ , and  $f$  is the **Pointwise Limit Function**.

$\{f_n\}$  **Converges Uniformly** to  $f : C \rightarrow \mathbb{R} \iff \forall \epsilon > 0, \exists N = N \in \mathbb{N}, \forall n \geq N, \forall x \in C, |f_n(x) - f(x)| < \epsilon$ . Denoted as  $f_n \xrightarrow[C]{\text{c.u.}} f$ .

If  $\{f_n\}$  Converges Uniformly to  $f$ , then it Converges Pointwise to  $f$ .

Suppose  $f_n \xrightarrow[C]{\text{c.p.}} f$ . If  $\exists \{u_n\}$  such that  $\lim_{n \rightarrow \infty} u_n = 0$  and  $\forall x \in C, \forall n \in \mathbb{N}, |f_n(x) - f(x)| \leq u_n$ , then  $f_n \xrightarrow[C]{\text{c.u.}} f$ .

Suppose  $\{f_n\}$  is a sequence of functions Integrable on  $[a, b]$ . If  $f_n \xrightarrow[C]{\text{c.u.}} f$ , then  $f$  is Integrable on  $[a, b]$ , and  $\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx$ .

## Differentiation in $\mathbb{R}^n$

Let  $A \subseteq \mathbb{R}^n$ . Let  $f : A \rightarrow \mathbb{R}^m$ . Let  $x_0 \in \text{int } A$ . Let  $x'_0 \in A$ . (Does it matter? for derivative.) Let  $v \in \mathbb{R}^n$  be a vector. Let  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation.

$f$  is **Differentiable at  $x_0$  with derivative  $L$**   $\iff \lim_{x \rightarrow x_0, x \in A \setminus \{x_0\}} \frac{\|f(x) - f(x_0) - L(x - x_0)\|}{\|x - x_0\|} = 0$ .

$f$  is **Differentiable in the direction  $v$  at  $x_0$**   $\iff \lim_{t \rightarrow 0^+, x_0 + tv \in A} \frac{f(x_0 + tv) - f(x_0)}{t}$  exists.

Let  $1 \leq i \leq n$ . Let  $e_i$  be a unit vector in the direction of  $x_i$ .

The **Partial Derivative of  $f$  with respect to  $x_i$  at  $x_0$**  is defined as  $\partial_{x_i} f(x_0) := \lim_{t \rightarrow 0, x_0 + te_i \in E} \frac{f(x_0 + te_i) - f(x_0)}{t}$

## Integration in $\mathbb{R}^n$

Let  $A \subset \mathbb{R}^n$  be bounded. Let  $f : A \rightarrow \mathbb{R}$  be bounded. Let  $D \subset A$  be the set of all points where  $f$  is discontinuous. If  $\text{bd}A$  and  $D$  both have measure zero, then  $f$  is integrable on  $A$ .

Let  $R$  be any compact rectangle that contains  $A$ . ( $R$  exists since  $A$  is bounded.)

Extend  $f$  to  $\bar{f} : R \rightarrow \mathbb{R}$ , such that

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{if } x \in R \setminus A \end{cases}$$

Let  $\bar{D} \subset D \cup \text{bd}A$  be the set of all points where  $\bar{f}$  is discontinuous.

By the Riemann–Lebesgue Theorem in  $\mathbb{R}^n$ :  $\bar{f}$  is integrable over  $R \iff \bar{D}$  has measure zero.

By the definition of the integral of  $f$  over  $A$ :  $f$  is integrable over  $A \iff \bar{f}$  is integrable over  $R$ .

# Calculus

## Differentiation and Integration TODO

**Integration By Substitution:**  $\int f(g(x))g'(x)dx = \int f(u)du$ , where  $u = g(x)$ . (Note: Remember to change domain for definite integrals.)

**Integration By Parts:**  $\int fg'dx = fg - \int f'gdx$ .

## Multiple Integration

### Double Integral

**Polar Coordinates:** Let  $x = r \cos(\theta), y = r \sin(\theta)$ , such that  $r^2 = x^2 + y^2$ .

$$\iint_D f(x, y)dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos(\theta), r \sin(\theta))rdrd\theta$$

### Triple Integral

**Spherical Coordinates:** Let  $x = p \sin(\phi) \cos(\theta), y = p \sin(\phi) \sin(\theta), z = p \cos(\phi)$ , such that  $x^2 + y^2 + z^2 = p^2$ .

$$\iiint_E f(x, y, z)dV = \int_{\delta}^{\gamma} \int_{\alpha}^{\beta} \int_a^b p^2 \sin(\phi) f(p \sin(\phi) \cos(\theta), p \sin(\phi) \sin(\theta), p \cos(\phi))dpd\theta d\phi$$

**Cylindrical Coordinates:** Let  $x = r \cos(\theta), y = r \sin(\theta), z = z$ .

$$\iiint_E f(x, y, z)dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos(\theta), r \sin(\theta))}^{u_2(r \cos(\theta), r \sin(\theta))} rf(r \cos(\theta), r \sin(\theta), z)dzdrd\theta$$

## Multivariable TODO

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ . The **Jacobian Matrix** is the  $m \times n$  matrix  $J$ , where  $J_{ij} = \frac{\partial f_i}{\partial x_j}$ . The **Jacobian (Determinant)** is the determinant of  $J$ , denoted as  $|J|$ .

To integrate  $f(x, y)$  over the region  $R$ . Let  $x = g(u, v), y = h(u, v)$  be the transformation where the region becomes  $S$ . Then the integral becomes:  $\iint_R f(x, y)dA = \iint_S f(g(u, v), h(u, v)) |J| dA$ .

### Lagrange Multipliers

For finding the Absolute Extrema of a  $f(x, y, z)$  given  $g(x, y, z) = c, h(x, y, z) = k$  constraints, solve

$$\nabla f(x, y, z) = \lambda_1 \nabla g(x, y, z) + \lambda_2 \nabla h(x, y, z)$$

$$g(x, y, z) = c$$

$$h(x, y, z) = k$$

where the constant  $\lambda_1, \lambda_2$  are the **Lagrange Multipliers**. This should yield 5 equations with 5 variables.  
Dimensions for  $f$  other than 3 works similarly. Number of constraints other than 2 works similarly.  
(Note: using  $\nabla f(x, y, z) + \lambda_1 \nabla g(x, y, z) + \lambda_2 \nabla h(x, y, z) = 0$  works aswell.) (?)

## Vector

Let  $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$  be a vector function.

$\vec{r}'(t)$  is the **Tangent Vector**.  $\vec{N}(t)$  is the **Unit Normal Vector** where  $\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$ .  
 $\vec{T}(t)$  is the **Unit Tangent Vector** where  $\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$ .  $\vec{B}(t)$  is the **Binormal Vector** where  $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ .

Define the operator  $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$  where  $\nabla f(x, y, z) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$ .

$\nabla f$  is a vector with the gradient vectors of  $f$ .

Let  $\vec{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$  be a vector field.

$\vec{F}$  is **Conservative**  $\iff \exists f$  such that  $\vec{F} = \nabla f$ , where  $f$  is the **Potential Function** of  $\vec{F}$ .

## Line Integral

Let  $\vec{F}$  be a continuous vector field with domain  $D$ . Let  $C$  be a curve. Let  $f$  be a function of  $x_1, x_2, \dots, x_n$ .

The **Line Integral** along  $C$  of  $\int_C f(x_1, \dots, x_n) ds = \int_a^b f(x_1(t), \dots, x_n(t)) \|\vec{r}'(t)\| dt$   
 $f$  respect to  $x_k$  is  $\int_C f(x_1, \dots, x_n) dx_k = \int_a^b f(x_1(t), \dots, x_n(t)) x'_k(t) dt$   
 $\vec{F}$  is  $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$

### Fundamental Theorem of Calculus (for Line Integrals):

If  $C$  is a Smooth curve  $\vec{r}(t)$  and  $\nabla f$  is continuous on  $C$ , then  $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

### Green's Theorem

Let  $C$  positively oriented smooth simple closed curve, and  $D$  be the region enclosed by the curve (TODO).

$$\begin{aligned} \int_C P dx + Q dy &= \iint_D (Q_x - P_y) dA \\ A &= \oint_C x dy - y dx = \frac{1}{2} \oint_C x dy + y dx \end{aligned}$$

## Surface Integral

Define  $\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k}$ .

If  $\vec{F}$  is Conservative, then  $\text{curl}(\vec{F}) = \vec{0}$ .

If  $\text{curl}(\vec{F}) = \vec{0}$  and TODO, then  $\vec{F}$  is Conservative.

Define  $\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ .

$\text{div}(\text{curl}(\vec{F})) = 0$ .

For a surface  $S$ : Let  $z = g(x, y)$ . Let  $\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$ . Depending on which exists (TODO  $dS = \sqrt{g_x^2 + g_y^2 + 1} dA$ ):

To Integrate  $f(x, y, z)$  on a surface  $S$ ,

$$\iint_S f(x, y, z) dS = \iint_D f(x, y, g(x, y)) \sqrt{g_x^2 + g_y^2 + 1} dA = \iint_D f(\vec{r}(u, v)) \|\vec{r}_u \times \vec{r}_v\| dA$$

To Integrate  $\vec{F}$  on the surface  $S$  (TODO ( $\vec{n}$  is the unit normal of  $S$ )),

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS = \iint_D \vec{F} \cdot \nabla h dA = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA = \iint_D -Pg_x - Qg_y + R dA$$

where  $h(x, y, z) = z - g(x, y)$ .

### Divergence Theorem

Let  $E$  be a bounded solid with  $S$  being the surface. Let  $\vec{F}$  be a vector field with continuous first partial derivatives on  $E$ . (TODO)

$$\iint_S F \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} dV$$

### Stokes' Theorem

Let  $S$  be an oriented smooth surface that is bounded by a Smooth Simple Closed curve  $C$  with positive orientation. Let  $\vec{F}$  be a vector field with continuous first partial derivatives on  $S$ .

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S}$$

# Linear Algebra TODO

TODO Define determinant, vector magnitude, etc.

Let  $u, v$  be vectors of size  $n$ . Let  $\theta$  be the angle between the  $u$  and  $v$ .

**Dot Product** of  $u$  and  $v$ ,  $u \cdot v = \sum_{i=1}^n u_i v_i = \|u\| \|v\| \cos \theta$ .

If  $u, v$  are of size 3, then the **Cross Product**  $u \times v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$  where  $u \times v$  is orthogonal to both  $u$  and  $v$ , and

$$\|u \times v\| = \|u\| \|v\| \sin \theta.$$

$$\text{proj}_v(u) = \frac{u \cdot v}{\|v\|^2} v.$$

## Unorganized

$$(AC)^T = C^T A^T (AC)^{-1} = C^{-1} A^{-1}$$

$$(pA)^{-1} = 1/p * A^{-1}$$

A is mxn b is nxp  $ab_i, j = \text{sum of } a_i, k * b_k, j \text{ from } k = 1 \text{ to } n$

$$Qm \text{ on } y = mx \quad Qm = 1/(1 + m^2) 1 - m^2, 2m, 2m, m^2 - 1$$

$$QmQn = R\theta \text{ where } \theta \text{ is } 2x \text{ the angle between lines } y=mx \text{ and } y=nx.?$$

Effects of the 3 elementary row operations on the determinant of triangular matrices

Co-factors is the determinant after deleting the col and row. That's it

Cramer's rule on invertible matrices. Unique solution

The Vandermonde Determinant  $\prod_{1 \leq j < i \leq n} (a_i - a_j)$ .

Cofactor expansion theorem

Orthogonal matrix  $Q^T = Q^{-1}$

Adjudicate  $A(\text{adj}A) = (\det A)I$

Elementary matrices

dynamic systems

Proof that if A is invertable it's RREF is I

Interpolating polynomials

cofactor expansion theorem

quadrilateral parallelograms

fundamental theorem

$$\text{law of cosine } c^2 = a^2 + b^2 - 2ab * \cos(C)$$

# Statistics DRAFT

## Introduction TODO

Define Independent, E,  $P(A|B)$ , etc.

Let  $X, Y$  be random variables.

$\sigma_X^2 = \text{Var}(X) = E[(X - \mu_X)^2] = E(X)^2 - E(X^2)$ ,  
where  $\sigma_X^2$  is the **Variance** of  $X$ , and  $\sigma_X$  is the **Standard Deviation**.

The **Covariance** of  $X$  is  $\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$ .

**Pearson Correlation Coefficient**  $\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$ .

$\forall X, \forall Y, -1 \leq \rho(X, Y) \leq 1$ .

Let  $\{X_1, \dots, X_n\}$  be any set of random variables. Let  $\{a_1, \dots, a_n\}$  be constants.

$\text{Var}(\sum_{i=1}^n a_i X_i) = \sum_{i=1}^n a_i \text{Var}(x_i) + 2 \sum_{i < j} a_i a_j \text{Cov}(X_i, X_j)$ .

If  $X_i$  and  $X_j$  are Independent, then  $\text{Cov}(X_i, X_j) = 0$ .

## Law of Total Variance

$\text{Var}(Y) = E(\text{Var}(Y|X)) + \text{Var}(E(Y|X))$ .

## Markov Chain (MC)

State  $j$  is **Accessible** from state  $i \iff P_{ij}^n > 0$  for some  $n \geq 0$ .

If two states  $i$  and  $j$  are accessible to each other, then they are **Communicate**, denoted as  $i \longleftrightarrow j$ .

A Markov Chain is **irreducible**  $\iff$  there is only one class.

Let  $f_i$  denote the probability that, starting in state  $i$ , the process will ever reenter state  $i$ .

State  $i$  is **Recurrent** if  $f_i = 1$ , and **Transient** if  $f_i < 1$ .

Recurrent and Transient are class properties.

State  $i$  has a **Period** of  $d \iff n \not\equiv 0 \pmod{d} \rightarrow P_{ii}^n = 0$ , where  $d$  is the largest integer with this property.

A state is **Aperiodic**  $\iff$  it has a period of 1.

Periodicity is a class property.

If state  $i$  is recurrent, and starting in  $i$ , the expected time until the process returns to state  $i$  is finite, then it is **Positive Recurrent**.

Positive Recurrence is a class property.

A state is **Ergodic**  $\iff$  it is positive recurrent and aperiodic.

If a Markov Chain is irreducible and ergodic, then  $\lim_{n \rightarrow +\infty} P_{ij}^n$  exists and is independent of  $i$ . Let  $\pi_j$  be the **Ergodic/Limiting Probabilities**, where  $\pi_j = \lim_{n \rightarrow +\infty} P_{ij}^n$ .

TODO:

$$S = (I - P_T)^{-1} \text{ (Lec 10).}$$

### Continuous-time Markov Chain (CTMC)

A stochastic process is a **CTMC**  $\iff$  When the process enters state  $i$ ,

1. The amount of time it spends in  $i$  before making a transition into a different state is Exponentially Distributed (where mean is  $1/v_i$ ).
2. The probability of entering state  $j$  from  $i$  is  $P_{ij}$ .
3.  $\forall i, P_{ii} = 0$  and  $\sum_j P_{ij} = 1$ .

**Birth and Death Processes (BDP)** are a CTMC with states  $(0, 1, 2, \dots)$  and state  $n$  can go to either state  $n - 1$  or  $n + 1$ . The transition probabilities are such that:

$$v_0 = \lambda_0, P_{0,1} = 1, \text{ and for } i > 0, v_i = \lambda_i + \mu_i, P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$$

The **Transition Probability Function (TPF)**  $P_{ij}(t)$  for a CTMC is defined as  $P_{ij}(t) = P(X(t+s) = j | X(s) = i)$ .

The **Instantaneous Transition Rates**  $v_{ij}$  is defined as  $v_{ij} = v_i P_{ij} = \lim_{h \rightarrow 0} \frac{P_{ij}(h)}{h}$ .

$v_i$  is the **Rate of Transition**, where  $v_i = \lim_{h \rightarrow 0} \frac{1 - P_{ii}(h)}{h}$ .

$$v_i = \sum_j v_i P_{ij} = \sum_j v_{ij} \text{ and } P_{ij} = \frac{v_{ij}}{v_i} = \frac{v_{ij}}{\sum_j v_{ij}}$$

Defined the matrix  $R$  such that  $r_{ij} = \begin{cases} v_{ij} & \text{for } i \neq j. \\ -v_j & \text{for } i = j. \end{cases}$  So  $\forall i, \sum_j r_{ij} = 0$ .

**Kolmogorov's Backward Equation:**  $P'_{ij}(t) = \left( \sum_{k \neq i} v_{ik} P_{kj}(t) \right) - v_i P_{ij}(t) = \sum_k r_{ij} P_{kj}(t)$

**Kolmogorov's Forward Equation:**  $P'_{ij}(t) = \left( \sum_{k \neq j} P_{ik}(t) v_{kj} \right) - P_{ij}(t) v_i = \sum_k P_{ik}(t) r_{kj}$

Define the matrices  $P(t), P'(t)$  using  $P_{ij}(t)$  and  $P'_{ij}(t)$ .

The Kolmogorov equations become  $P'(t) = RP(t) = P(t)R$ , and

$$P(t) = e^{Rt} = \sum_{n=0}^{\infty} R^n \frac{t^n}{n!} = \lim_{n \rightarrow \infty} (I + R \frac{t}{n})^n = \lim_{n \rightarrow \infty} (I - R \frac{t}{n})^{-n}$$

If  $R$  is a  $n \times n$  matrix with  $n$  Eigenvectors ( $R = VDV^{-1}$ ), then  $e^{Rt}$  can be solved using a diagonal matrix ( $e^{Rt} = Ve^{Dt}V^{-1}$ ), where  $e^{Dt}$  has the elements  $e^{\lambda_i t}$ , and  $\lambda_i$  is the  $i$ th Eigenvalue.

The **Limiting/Stationary Probability**  $P_j = \pi_j = \lim_{t \rightarrow \infty} P_{ij}(t)$  exists  $\iff$  the CTMC is Ergodic.

$$v_j P_j = \sum_{k \neq j} v_{kj} P_k, \sum_j P_j = 1 \text{ and } \vec{\pi} \times R = \vec{0}$$

### Poisson Process (PP)

$N(t)$  for  $t \geq 0$  is a **Counting Process (CP)** if

1.  $N(t) \geq 0$ .
2.  $N(t) \in \mathbb{Z}$ .
3. If  $s < t$ , then  $N(s) \leq N(t)$ .
4. For  $s < t$ ,  $N(t) - N(s)$  equals the number of events that occur in the interval  $(s, t]$ .

A CP has **Independent Increments** if the number of events that occur in disjoint intervals are independent.

A CP has **Stationary Increments** if the distribution of the number of events that occur in any interval depends only on the length of the interval.

Let  $o(h)$  be any function of  $h$  that satisfies the condition:  $\lim_{h \rightarrow 0} (o(h)/h) = 0$ .

$N(t)$  is a **Poisson Process (PP)** with rate  $\lambda > 0$ , If

1.  $N(0) = 0$ .
2.  $N(t)$  has Independent and Stationary increments.
3.  $P(N(t+s) - N(s) = n) = e^{-\lambda t} (\lambda t)^n n!^{-1}, n \geq 0, n \in \mathbb{Z}$ .
4.  $P(N(h) = 1) = \lambda h + o(h)$ .
5.  $P(N(h) \geq 2) = o(h)$ .

Where 4  $\wedge$  5  $\leftrightarrow$  3.

Let  $\{T_n\}$  be the sequence of **Interarrival Times** of  $N(t)$ .  $P(T_n > t) = e^{-\lambda t}$ .

$T_n$  has an Exponential Distribution with a mean of  $1/\lambda$  for all  $n$ .

Let  $S_n$  be the **Arrival/Waiting Time** of the  $n$ th event. Where  $S_n = \sum_{i=1}^n T_i, n \geq 1$ .

$S_n$  has a Gamma Distribution with parameters  $n$  and  $\lambda$ . So  $f_{S_n}(t) = \lambda e^{-\lambda t} (\lambda t)^{n-1} (n-1)!^{-1}, t \geq 0$ .

$S_n \leq t = N(t) \geq n$ . The formula for probability density can also be obtained using this fact.

$$P(T_1 < s | N(t) = 1) = s/t.$$

### Two Poisson Processes

Suppose a PP with rate  $\lambda$  is split into two types,  $N_1(t)$  with probability  $p$  or  $N_2(t)$  with probability  $1-p$ . Where  $N(t) = N_1(t) + N_2(t)$ .

$N_1(t)$  has a rate of  $\lambda p$ .  $N_2(t)$  has a rate of  $\lambda(1-p)$ .

Let  $S_n^1, S_m^2$  denote the time of the  $n$ th/ $m$ th event of  $N_1(t)$  and  $N_2(t)$  respectively.

$$P(S_1^1 < S_2^2) = \lambda_1(\lambda_1 + \lambda_2)^{-1}.$$

Each event that occurs is going to be  $N_1(t)$  with probability  $\lambda_1(\lambda_1 + \lambda_2)^{-1}$ , and  $N_2(t)$  with probability  $\lambda_2(\lambda_1 + \lambda_2)^{-1}$ . So  $P(S_n^1 < S_m^2)$  is based on the Binomial Distribution with  $p = \lambda_1(\lambda_1 + \lambda_2)^{-1}$ .

### $n$ Poisson Processes

Defined  $n$  processes similarly to two processes.

$$\sum_{i=1}^n P_i(y) = 1.$$

$$E(N_i(t)) = \lambda \int_0^t P_i(s) ds.$$

### Non-homogeneous Poisson Process (NHPP)

Every event that occurs has a  $p(t)$  chance of being recorded. Let  $N_c(t)$  be the NHPP, then it has a rate of  $\lambda(t) = \lambda p(t)$ .

A NHPP with  $\lambda(t) = \lambda$  is a regular PP.

Let  $m(t)$  be the mean value function =  $E(N_c(t)) = \lambda \int_0^t p(s) ds$ .

The increments of a NHPP are Independent, but not necessarily stationary.

$\forall s, t, 0 \leq s < t, N(t) - N(s)$  has a Poisson Distribution with mean  $m(t) - m(s) = \int_s^t \lambda(x) dx$ .

### Compound Poisson Process (CPP)

$X(t)$  is a CPP if  $X(t) = \sum_{i=1}^{N(t)} Y_i$ , where  $N(i)$  is a PP, and  $Y_i, i \geq 1$  is some i.i.d.r.v's that are also independent of  $N(t)$ .

$$E(X(t)) = \lambda t E(Y_1).$$

$$\text{Var}(X(t)) = \lambda t E(Y_1^2) \text{ (Derived from the Law of Total Variance) (?).}$$

$$M_X(s) = \exp(\lambda t(M_Y(s) - 1)) \text{ (Derived from the Law of Total Probability) (?).}$$

If  $Y_i = 1$  then  $X(t)$  is a normal PP.

TODO:

$$f(y_1, y_2 \dots y_n) = f(s_1, s_2 \dots s_n | n) = n! / t^n \text{ (lec 15).}$$

### Renewal Process (RP) DRAFT

Let  $N(t)$  be a CP. Let  $X_n$  be the time between any two consecutive events.

If  $X_n$  are i.i.d.r.v's, then  $N(t)$  is a **Renewal Process**.

$m(t) = E[N(t)]$  is the **Renewal/Mean-value Function**.

Let  $F(x) = P(X_n < x)$ .

The PP is an example of a RP with  $F(x) = 1 - e^{-\lambda x}, x \geq 0$  and  $m(t) = \lambda t$ .

$m(t) = F(t) + \int_0^t m(t-x)f(x)dx$  is the **Renewal Equation**.

Let  $\{X_n\}$  be some sequence of r.v.'s. This stochastic process is **Martingale**  $\iff \forall n,$

1.  $E(|X_n|) < \infty$ .
2.  $E(X_{n+1}|X_u, 0 \leq u \leq n) = X_n$ .

# Finance

## Market Model DRAFT

Let time  $t \geq 0, t \in \mathbb{Q}$ . Let  $t = n\tau$  where  $n \in \mathbb{N} \cup \{0\}$  (Discrete Time).

Let  $m \in \mathbb{N}, \forall i \in \mathbb{N}, 1 \leq i \leq m$ :  $S_i(n)$  is the price of the  $i$ th **Risky Asset** at time  $n$ , with a total of  $m$  risky assets.

$A(n)$  is the price of the **Risk-Free Asset** at time  $n$ .

Unless stated otherwise,  $A(0)$  is used as a reference, with  $A(0) = 1$  or 100.

A **Portfolio** is a vector  $\vec{v}(n) = \langle x_1(n), \dots, x_m(n), y(n) \rangle$  representing the amount of assets held between time  $n - 1$  and time  $n$ , where  $\vec{v}(0)$  is undefined. Let the **Value** of the Portfolio at time  $n$  being

$$V(n) = \begin{cases} y(1)A(0) + \sum_{j=1}^m x_j(1)S_j(0) & \text{for } n = 0 \text{ (Initial Value/Wealth)} \\ y(n)A(n) + \sum_{j=1}^m x_j(n)S_j(n) & \text{for } n > 0 \end{cases}$$

**Rate of Return** TODO\*\*\*\*  $K_i$  = return of  $S_i$  is  $K_F = \frac{F(1) - F(0)}{F(0)}$ .

Six Assumptions:

1.  $\forall n > 0, S(0), A(0), A(n)$  are known, while  $S(n)$  is a random variable (Randomness).
2.  $\forall n, S(n) > 0 \wedge A(n) > 0$  (Positivity of Prices).
3.  $\forall i, \forall n, x_i(n) \in \mathbb{R}, y(n) \in \mathbb{R}$  (Divisibility, Liquidity and Short Selling).
4.  $\forall n, V(n) \geq 0$  (Solvency).
5.  $\forall n, S(n)$  can only have a finite number of values (Discrete Unit Prices).
6.  $\forall n > 0, V(0) = 0 \rightarrow V(n) = 0$  (No-Arbitrage Principle).

An Investment Strategy is **Self-Financing**  $\iff \forall n \geq 1, V(n) = y(n+1)A(n) + \sum_{j=1}^m x_j(n+1)S_j(n)$

An Investment Strategy is **Predictable**  $\iff \forall n$ , the Portfolio  $\vec{v}(n+1)$  constructed at time  $n$  only depends on previous Portfolios.

An Investment Strategy is **Admissible**  $\iff$  it is Predictable, Self-Financing and Assumption 6 (?).

TODO (Fundamental Theorem of Asset Pricing, Page 83) (Extended Model with Derivative securities, very similar)

A **Forward Contract** is an agreement to exchange a Risky Asset at a future **Delivery Date** for a **Forward Price**  $F$ . To sell is to take a **Short Forward Position**. To buy is to take a **Long Forward Position**.

A **Call Option** is a contract that gives the holder a right/option to purchase a Risky Asset at a future **Exercise Time** for a **Strike/Exercise Price**. Let  $C(t)$  denote the price of the option at time  $t$ .

The prices for Options and Forward Contracts are determined by the No-Arbitrage Principle.

Options are a derivative asset.

## Risk-Free Asset DRAFT

Let  $V(s) = P$  be the **Principle** investment at time  $s$ .

Let  $K(s, t)$  denote the return on an investment from time  $s$  to time  $t$ , where  $K(s, t) = \frac{V(t) - V(s)}{V(s)}$ .

The **Growth Factor** is  $V(t)/V(s)$ , while **Discount Factor** is the multiplicative inverse.

### Simple Interest

Let  $r$  be the annual interest for exactly 365 days.

For **Simple Interest**:  $V(t) = (1 + (t - s)r)V(s)$ ,  $K(s, t) = (t - s)r$ .

### Periodic Compound Interest

Let  $r$  be the annual interest for exactly 365 days.

For **Compound Interest**:  $V(t) = \left(1 + \frac{r}{m}\right)^{(t-s)m} V(s)$ ,  $K(s, t) = \left(1 + \frac{r}{m}\right)^{(t-s)m} - 1$ , where  $m$  is the number of interest payments per annum/year.

A **Perpetuity** is a constant infinite sequence of payments made at equal time intervals.

A **Annuity** is a constant finite sequence of payments made at equal time intervals.

A **Amortised Loan** is an Annuity from the point of view of the borrower.

Let  $C$  be the constant.

Let  $\text{PA}(r, n) = \sum_{i=1}^n (1 + r)^{-i} = \left(\frac{1}{r}\right) - \left(\frac{1}{r} \frac{1}{(1+r)^n}\right) = \frac{1 - (1+r)^{-n}}{r}$ , where  $\text{PA}(r, n)$  returns the **Present Value Factor** for an Annuity.

$C \times \text{PA}(r, n)$  is the present value of an Annuity that produces  $n$  annual payments of  $C$ .

$\lim_{n \rightarrow \infty} C \times \text{PA}(r, n) = \frac{C}{r}$  is the present value of a Perpetuity that produces annual payments of  $C$ .

The above formulas are adjusted accordingly for  $m \neq 1$ .

### Continuous Compound Interest

Let  $r$  be the interest rate for the continuous compounding method.

The **Effective Rate**  $r_e = e^r$ . And  $r_e$  is defined similarly for periodic compounding as well.

For **Continuous Compounding**:  $V(t) = e^{(t-s)r} V(s)$ .

Define  $k(s, t)$  to be the **Logarithmic Return**, where  $k(s, t) = \ln\left(\frac{V(t)}{V(s)}\right)$ .

For  $a \leq b \leq c$ ,  $k(a, c) = k(a, b) + k(b, c)$  (Not true for  $K(a, c)$ ).

### Coupon Bond

A **Zero-Coupon Bond** promises one payment for a **Face Value**  $F$  on a **Maturity Date**  $T$ .

$B(t, T)F$  denotes the price of the Zero-Coupon Bond at time  $t$ , where  $B(t, T) = e^{-r(T-t)}$  depending on the compounding method.

A **Coupon Bond** promises a sequence of payments. The value of a Coupon Bond at time  $t$  can be calculated by the values and times of the payments discounted by a constant interest rate.

The values for the payments are of the form  $\{C, C, \dots, C, C + F\}$ .

## Risky Asset TODO

TODO, Binomial Tree,  $u, d$ .

Let  $r$  be the one-step return for a risk-free investment, where  $d < r < u$ .

Define the **Risk-Neutral Probability**  $p_* = \frac{r-d}{u-d}$ , where  $0 < p_* < 1$  and  $p_*(u-r) + (1-p_*)(d-r) = 0$ .

$$E_*(S(t+1)|S(t)) = S(t)(1+r)$$

TODO  $r_*, q_*$ .

TODO Continuous-Time Limit

$$\ln(1+u) = m\tau + \sigma\sqrt{\tau}, \ln(1+d) = m\tau - \sigma\sqrt{\tau},$$

## Portfolio Management TODO

Let  $K$  be the return on a risky investment, where  $K$  is a random variable.

Let  $\sigma_K^2 = \text{Var}(K)$ .

$\sigma_K$  and  $\sigma_K^2$  are both used to measure **Risk**.

Risk-Free assets have no Risk, therefore is ignored for the following ( $\sigma = 0$  messes up Pearson's Correlation Coefficient):

**Weight**  $w_i = \frac{x_i S_i(0)}{V(0)}$ , where  $\sum w_i = 1$  and  $K_V = \sum w_i K_i$ .

$E(K_V)$  and  $\text{Var}(K_V)$  can be calculated using the properties of Var and E.

TODO Logarithmic return for Chapter 3

TODO Page 101

## Forward and Futures Contracts TODO

## Options TODO

## Financial Engineering TODO

## Notations and Tables TODO

$x \in [a, b]$  means  $a \leq x \leq b$ .

For definitions:  $\coloneqq$  means  $=$ .

$x \in (a, b)$  means  $a < x < b$ .

$\{a \text{ such that } a = b\} = \{a | a = b\} = \{a : a = b\}$ .

$n$  choose  $k = \binom{n}{k} = \frac{n!}{k!(n-k)!} = C_n^k = {}_nC_k = {}^nC_k = C(n, k)$ .

For Trigonometric Functions,  $\sin^2(x) = (\sin(x))^2$ , not  $\sin(\sin(x))$ .

Unless the exponent is  $-1$  in which case,  $y = \sin^{-1}(x) = \arcsin(x)$ , is defined  $x = \sin(y)$ .

i.i.d.r.v.'s means "Independent Identically Distributed Random Variables".

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