Notes

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Calculus

Differentiation and Integration

Integration By Substitution: $\int f(g(x))g'(x)dx = \int f(u)du$, where $u = g(x)$. (Note: Remember to change domain for definite integrals.)

Integration By Parts: $\int fg'dx = fg - \int f'gdx$.

Multiple Integration

Double Integral

Polar Coordinates: Let $x = r\cos(\theta), y = r\sin(\theta)$, such that $r^2 = x^2 + y^2$.

$$\iint_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))rdrd\theta$$

Triple Integral

Spherical Coordinates: Let $x = p\sin(\phi)\cos(\theta), y = p\sin(\phi)\sin(\theta), z = p\cos(\phi)$, such that $x^2 + y^2 + z^2 = p^2$.

$$\iiint_E f(x,y,z)dV = \int_\delta^\gamma \int_\alpha^\beta \int_a^b p^2\sin(\phi)f(p\sin(\phi)\cos(\theta), p\sin(\phi)\sin(\theta), p\cos(\phi))dpd\theta d\phi$$

Cylindrical Coordinates: Let $x = r\cos(\theta), y = r\sin(\theta), z = z$.

$$\iiint_E f(x,y,z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos(\theta), r\sin(\theta))}^{u_2(r\cos(\theta), r\sin(\theta))} rf(r\cos(\theta), r\sin(\theta), z)dzdrd\theta$$

Multivariable

Let $f : \mathbb{R}^n \to \mathbb{R}^m$. The **Jacobian Matrix** is the $m \times n$ matrix J, where $J_{ij} = \frac{\partial f_i}{\partial x_j}$. The **Jacobian (Determinant)** is the determiant of J, denoted as $|J|$.

To integrate $f(x,y)$ over the region R. Let $x = g(u,v), y = h(u,v)$ be the transformation where the region becomes S. Then the integral becomes: $\iint_R f(x,y)dA = \iint_S f(g(u,v), h(u,v))\,|J|\,\overline{dA}$.

Lagrange Multipliers

For finding the Absolute Extrema of a $f(x,y,z)$ given $g(x,y,z) = c, h(x,y,z) = k$ constraints, solve

$$\begin{aligned}\nabla f(x,y,z) &= \lambda_1 \nabla g(x,y,z) + \lambda_2 \nabla h(x,y,z)\\ g(x,y,z) &= c\\ h(x,y,z) &= k\end{aligned}$$

where the constant λ_1, λ_2 are the **Lagrange Multipliers**. This should yield 5 equations with 5 variables.

Dimensions for f other than 3 works similarly. Number of constraints other than 2 works similarly.

(Note: using $\nabla f(x,y,z) + \lambda_1 \nabla g(x,y,z) + \lambda_2 \nabla h(x,y,z) = 0$ works aswell.)(?)

Vector

Let $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ be a vector function.

$\boldsymbol{\vec{r}'(t)}$ is the **Tangent Vector**.

$\boldsymbol{\vec{T}(t)}$ is the **Unit Tangent Vector** where $\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$.

$\boldsymbol{\vec{N}(t)}$ is the **Unit Normal Vector** where $\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$.

$\boldsymbol{\vec{B}(t)}$ is the **Binormal Vector** where $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$.

Define the operator $\boldsymbol{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ where $\nabla f(x,y,z) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$.

∇f is a vector with the gradient vectors of f.

Let $\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ be a vector field.

$\vec{F}$ is **Conservative** $\iff \exists f$ such that $\vec{F} = \nabla f$, where f is the **Potential Function** of $\vec{F}$.

Line Integral

Let $\vec{F}$ be a continuous vector field with domain D. Let C be a curve. Let f be a function of $x_1, x_2, \ldots, x_n$.

The **Line Integral** along C of $\begin{cases} f \text{ is } \int_C f(x_1,\ldots,x_n)ds = \int_a^b f(x_1(t),\ldots,x_n(t))\|\vec{r}'(t)\|dt \\ f \text{ respect to } x_k \text{ is } \int_C f(x_1,\ldots,x_n)dx_k = \int_a^b f(x_1(t),\ldots,x_n(t))x_k'(t)dt \\ \vec{F} \text{ is } \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)dt \end{cases}$

Fundamental Theorem of Calculus (for Line Integrals):
If C is a Smooth curve $\vec{r}(t)$ and ∇f is continuous on C, then $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

Green's Theorem
Let C positively oriented smooth simple closed curve, and D be the region enclosed by the curve (TODO).

$$\int_C Pdx + Qdy = \iint_D (Q_x - P_y)dA$$

$$A = \oint_C xdy = -\oint_C ydx = \frac{1}{2}\oint_C xdy - ydx$$

Surface Integral

Define $\mathbf{curl}(\boldsymbol{\vec{F}}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k}$.

If $\vec{F}$ is Conservative, then $\operatorname{curl}(\vec{F}) = \vec{0}$.
If $\operatorname{curl}(\vec{F}) = \vec{0}$ and $TODO$, then $\vec{F}$ is Conservative.

Define $\mathbf{div}(\boldsymbol{\vec{F}}) = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

$\operatorname{div}(\operatorname{curl}(\vec{F})) = 0$.

Fora surface S: Let $z = g(x,y)$. Let $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$. Depending on which exists (TODO $dS = \sqrt{g_x^2 + g_y^2 + 1}dA$):

To Integrate $f(x, y, z)$ on a surface S,

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^2 + g_y^2 + 1}dA = \iint_D f(\vec{r}(u,v))\|\vec{r}_u \times \vec{r}_v\|dA$$

To Integrate $\vec{F}$ on the surface S (TODO ($\vec{n}$ is the unit normal of S)),

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS = \iint_D \vec{F} \cdot \nabla h dA = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA = \iint_D -Pg_x - Qg_y + RdA$$

where $h(x, y, z) = z - g(x, y)$.

Divergence Theorem

Let E be a bounded solid with S being the surface. Let $\vec{F}$ be a vector field with continuous first partial derivatives on E. (TODO)

$$\iint_S F \cdot d\vec{S} = \iiint_E \mathrm{div}\vec{F} dV$$

Stokes' Theorem

Let S be an oriented smooth surface that is bounded by a Smooth Simple Closed curve C with positive orientation. Let $\vec{F}$ be a vector field with continuous first partial derivatives on S.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \mathrm{curl}(\vec{F}) \cdot d\vec{S}$$

Differential Equation TODO

First Order

Integrating Factor: $y' + p(t)y = q(t)$

$$y(t) = \frac{\int \mu(t)q(t)dt + c}{\mu(t)}, \text{ where } \mu(t) = e^{\int p(t)dt}$$

Bernoulli: $y' + p(t)y = q(t)y^n$ Substitute with $v = y^{1-n}$, then solve $v(t)$ with a Integrating Factor.

Second Order

We define a $n \times n$ matrix W, such that $W_{ij} = f_j^{i-1}(x)$. The **Wronskian** is the determiant of W, denoted as $W(f_1, f_2, \dots, f_n)$.

Example: The Wronskian of two functions f, g is $W(f, g) = fg' - gf'$.

If $y_1(t)$ and $y_2(t)$ are two solutions to $p(t)y'' + q(t)y' + r(t)y = 0$ and $W(y_1, y_2)(t) \neq 0$, then the two solutions are a **Fundamental Set of Solutions**, where the General solution is $y(t) = c_1y_1(t) + c_2y_2(t)$.

Derived Formulas:

Radioactive decay: $Q(t) = Q(0)2^{-t/H}$, where H is the half-life.

Cooling: $T(t) = T_m + (T(0) - T_m)e^{-kt}$, where k is the cooling constant. And $T'(t) = k(T_m - T(t))$.

Concentration: At time t. Let $Q(t) =$ amount of salt. Let $V(t) =$ volume of solution in the tank. Let $C_i(t), C_o(t)$ be the concentrations of solution flowing into/out the tank. Let $R_i(t), R_o(t)$ be the volumes of solution flowing into/out the tank. Let $C_o = Q/V$. $Q' = C_iR_i - C_oR_o$. Solve with Integrating Factor.

Finance

Market Model DRAFT

Let time $t \geq 0, t \in \mathbb{Q}$. Let $t = n\tau$ where $n \in \mathbb{N} \cup \{0\}$ (Discrete Time).

Let $m \in \mathbb{N}, \forall i \in \mathbb{N}, 1 \leq i \leq m$: $\boldsymbol{S_i(n)}$ is the price of the ith **Risky Asset** at time n, with a total of m risky assets.

$\boldsymbol{A(n)}$ is the price of the **Risk-Free Asset** at time n.

Unless stated otherwise, $A(0)$ is used as a reference, with $A(0) = 1$ or 100.

A **Portfolio** is a vector $\vec{v}(n) = \langle x_1(n), \ldots, x_m(n), y(n) \rangle$ representing the amount of assets held between time $n-1$ and time n, where $\vec{v}(0)$ is undefined. Let the **Value** of the Portfolio at time n being

$$\boldsymbol{V(n)} = \begin{cases} y(1)A(0) + \sum_{j=1}^{m} x_j(1)S_j(0) & \text{for } n = 0 \text{ (Initial Value/Wealth)} \\ y(n)A(n) + \sum_{j=1}^{m} x_j(n)S_j(n) & \text{for } n > 0 \end{cases}$$

Rate of Return TODO**** K_i =return of S_i is $\boldsymbol{K_F} = \frac{F(1)-F(0)}{F(0)}$.

Six Assumptions:

1. $\forall n > 0, S(0), A(0), A(n)$ are known, while $S(n)$ is a random variable (Randomness).
2. $\forall n, S(n) > 0 \wedge A(n) > 0$ (Positivity of Prices).
3. $\forall i, \forall n, x_i(n) \in \mathbb{R}, y(n) \in \mathbb{R}$ (Divisibility, Liquidity and Short Selling).
4. $\forall n, V(n) \geq 0$ (Solvency).
5. $\forall n, S(n)$ can only have a finite number of values (Discrete Unit Prices).
6. $\forall n > 0, V(0) = 0 \rightarrow V(n) = 0$ (No-Arbitrage Principle).

An Investment Strategy is **Self-Financing** $\iff \forall n \geq 1, V(n) = y(n+1)A(n) + \sum_{j=1}^{m} x_j(n+1)S_j(n)$

An Investment Strategy is **Predictable** $\iff \forall n,$ the Portfolio $\vec{v}(n+1)$ constructed at time n only depends on previous Portfolios.

An Investment Strategy is **Admissible** $\iff$ it is Predictable, Self-Financing and Assumption 6 (?).

TODO (Fundamental Theorem of Asset Pricing, Page 83) (Extended Model with Derivative securities, very simular)

A **Forward Contract** is an agreement to exchange a Risky Asset at a future **Delivery Date** for a **Forward Price $\boldsymbol{F}$**. To sell is to take a **Short Forward Position**. To buy is to take a **Long Forward Position**.

A **Call Option** is a contract that gives the holder a right/option to purchase a Risky Asset at a furture **Exercise Time** for a **Strike/Exercise Price**. Let $\boldsymbol{C(t)}$ denote the price of the option at time t.

The prices for Options and Forward Contracts are determined by the No-Arbitrage Principle.

Options are a derivative asset.

Risk-Free Asset DRAFT

Let $V(s) = \boldsymbol{P}$ be the **Principle** investment at time s.

Let $\boldsymbol{K(s,t)}$ denote the return on an investment from time s to time t, where $K(s,t) = \frac{V(t)-V(s)}{V(s)}$.

The **Growth Factor** is $V(t)/V(s)$, while **Discount Factor** is the multiplicative inverse.

Simple Interest

Let $\boldsymbol{r}$ be the annual intrest for exactly 365 days.

For **Simple Intrest**: $V(t) = (1+(t-s)r)V(s)$, $K(s,t) = (t-s)r$.

Periodic Compound Interest

Let $\boldsymbol{r}$ be the annual intrest for exactly 365 days.

For **Compound Intrest**: $V(t) = \left(1+\frac{r}{m}\right)^{(t-s)m} V(s)$, $K(s,t) = \left(1+\frac{r}{m}\right)^{(t-s)m} - 1$, where m is the number of intrest payments per annum/year.

A **Perpetuity** is a constant infinite sequence of payments made at equal time intervals.
A **Annuity** is a constant finite sequence of payments made at equal time intervals.
A **Amortised Loan** is a Annuity from the point of view of the borrower.
Let $\boldsymbol{C}$ be the constant.

Let $\mathbf{PA}(\boldsymbol{r},\boldsymbol{n}) = \sum_{i=1}^{n}(1+r)^{-i} = (\frac{1}{r}) - (\frac{1}{r}\frac{1}{(1+r)^n}) = \frac{1-(1+r)^{-n}}{r}$, where $\mathrm{PA}(r,n)$ returns the **Present Value Factor** for an Annuity.

$C \times \mathrm{PA}(r,n)$ is the present value of an Annuity that produces n annual payments of C.

$\lim_{n\to\infty} C \times \mathrm{PA}(r,n) = \frac{C}{r}$ is the present value of a Perpetuity that produces annual payments of C.

The above formulas are adjusted accordingly for $m \neq 1$.

Continuous Compound Interest

Let $\boldsymbol{r}$ be the interest rate for the continuous compounding method.

The **Effective Rate** $\boldsymbol{r_e} = e^r$. And r_e is defined similarly for periodic compounding aswell.

For **Continuous Compounding**: $V(t) = e^{(t-s)r}V(s)$.

Define $\boldsymbol{k(s,t)}$ to be the **Logarithmic Return**, where $k(s,t) = \ln(\frac{V(t)}{V(s)}$.

For $a \le b \le c, k(a,c) = k(a,b) + k(b,c)$ (Not true for $K(a,c)$).

Coupon Bond

A **Zero-Coupon Bond** promises one payment for a **Face Value** $\boldsymbol{F}$ on a **Maturity Date** $\boldsymbol{T}$.

$\boldsymbol{B(t,T)}F$ denotes the price of the Zero-Coupon Bond at time t, where $B(t,T) = e^{-r(T-t)}$ depending on the compounding method.

A **Coupon Bond** promises a sequence of payments. The value of a Coupon Bond at time t can be calculated by the values and times of the payments discounted by a constant interest rate.

The values for the payments are of the form $\{C, C, \ldots, C, C+F\}$.

Risky Asset TODO

TODO, Binomial Tree, u, d.

Let $\boldsymbol{r}$ be the one-step return for a risk-free investment, where $d < r < u$.

Define the **Risk-Neutral Probability** $\boldsymbol{p_*} = \frac{r-d}{u-d}$, where $0 < p_* < 1$ and $p_*(u-r) + (1-p_*)(d-r) = 0$.

$E_*(S(t+1)|S(t)) = S(t)(1+r)$

TODO r_*, q_*.

TODO Continuous-Time Limit

$\ln(1+u) = m\tau + \sigma\sqrt{\tau}, \ln(1+d) = m\tau - \sigma\sqrt{\tau},$

Portfolio Management TODO

Let K be the return on a risky investment, where K is a random variable.

Let $\sigma_K^2 = \mathrm{Var}(K)$.

σ_K and σ_K^2 are both used to measure **Risk**.

Risk-Free assets have no Risk, therefore is ignored for the following ($\sigma = 0$ messes up Pearson's Correlation Coefficient):

Weight $\boldsymbol{w_i} = \frac{x_i S_i(0)}{V(0)}$, where $\sum w_i = 1$ and $K_V = \sum w_i K_i$.

$\mathrm{E}(K_V)$ and $\mathrm{Var}(K_V)$ can be calculated using the properties of Var and E.

TODO Logarithmic return for Chapter 3

TODO Page 101

Forward and Futures Contracts TODO

Options TODO

Financial Engineering TODO

Real Analysis

Sets, Relations and Functions TODO

Sequence

A **Sequence** is a function $f : \mathbb{N} \to \mathbb{R}$. Notations are $\boldsymbol{a_n} = f(n), \boldsymbol{\{a_n\}}, \boldsymbol{\{a_n\}_{n=1}^{\infty}}$.

A sequence is **Bounded** $\iff \exists M \in R$ such that $\forall n \in \mathbb{N}, |a_n| \leq M$.
Bounded Above/Below are defined similarly.

A sequence is **(Strictly) Increasing** if $\forall n \in \mathbb{N}, a_{n+1} \geq a_n (a_{n+1} > a_n)$. **(Strictly) Decreasing** is defined similarly.

A sequence is **Monotone** if it's either Decreasing or Increasing.

A sequence is **Convergent** to $L \iff \forall \epsilon \in \mathbb{R}, \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$, if $n \geq N$ then $|a_n - L| < \epsilon$.

A sequence is **Divergent** to $\infty \iff \forall M \in \mathbb{R}, M > 0, \exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$, if $n \geq N$ then $a_n > M$.
Divergent to $-\infty$ is defined similarly.

A sequence is **Cauchy** $\iff \forall \epsilon \in \mathbb{R}, \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall m, n \in \mathbb{N}$, if $m, n \geq N$ then $|a_n - a_m| < \epsilon$.

A sequence is Convergent $\iff$ it's Cauchy.

If a sequence is Convergent, then it is Bounded.

Monotone Convergence Theorem:
If a sequence is Bounded Above and Increasing, then it converges.
If a sequence is Bounded Below and Decreasing, then it converges.
If a sequence is Bounded and Monotone, then it converges.

Every sequence has a Monotone subsequence.

Define $\mathbf{sub}(\boldsymbol{a_n})$ to be a set containing the limits of all subsequences of $\{a_n\}$.

The **Limit Superior** of $\{a_n\}$ is $\sup(\mathrm{sub}(a_n)) = \max(\mathrm{sub}(a_n))$, denoted as $\boldsymbol{\limsup_{n\to\infty}(a_n)}$.

The **Limit Inferior** of $\{a_n\}$ is $\inf(\mathrm{sub}(a_n)) = \min(\mathrm{sub}(a_n))$, denoted as $\boldsymbol{\liminf_{n\to\infty}(a_n)}$.

Let $\{q_n\}$ be any denumeration of $\mathbb{Q}$, then $\mathrm{sub}(q_n) = \mathbb{R}$.

Topology of the Real Line

Let $x \in \mathbb{R}$ and $S \subseteq \mathbb{R}$.

The **Neighbourhood (nhd)** of x is the set $\boldsymbol{N_\epsilon(x)} = \{y \in \mathbb{R} : |x - y| < \epsilon\} = (x - \epsilon, x + \epsilon)$, for some $\epsilon > 0$.

The **Deleted Neighbourhood** of x is the set $\boldsymbol{N^*_\epsilon(x)} = N_\epsilon(x) \setminus \{x\}$.

An **Interior Point** of S is an element $x \in S$ such that $N_\epsilon(x) \subseteq S$ for some $\epsilon > 0$.

The set of all Interior points of S is the **Interior** of S, denoted as $\mathbf{int}(\boldsymbol{S})$.

x is a **Boundary Point** for $S \iff \forall \epsilon > 0, N_\epsilon(x) \cap S \neq \varnothing \land N_\epsilon(x) \cap (\mathbb{R} \setminus S) \neq \varnothing$.

The set of all Boundary Points of S is the **Boundary** of S, denoted as $\mathbf{bd}(\boldsymbol{S})$.

x is a **Accumulation (Limit) Point** for $S \iff \forall \epsilon > 0, N^*_\epsilon(x) \cap S \neq \varnothing \iff \forall \epsilon > 0, \exists y \in S$ such that $0 < |x - y| < \epsilon$.

The set of all Accumulation Points of S is denoted as $\boldsymbol{S'}$.

The **Closure** of S is the set $\overline{\boldsymbol{S}} = S \cup S' = S \cup \mathrm{bd}(S)$.

x is a **Isolated Point** for $S \iff x \in S \setminus S' \iff x \in S \wedge \exists \epsilon > 0$ such that $N^*_\epsilon(x) \cap S = \varnothing$.

$S \subseteq \mathrm{int}(S) \cup \mathrm{bd}(S)$ and $\mathrm{int}(S) \cap \mathrm{bd}(S) = \varnothing$.

S is **Open** $\iff S = \mathrm{int}(S) \iff \mathrm{bd}(S) \subseteq \mathbb{R} \setminus S$.

S is **Closed** $\iff \mathbb{R} \setminus S$ is Open $\iff \mathrm{bd}(S) \subseteq S \iff S = \overline{S}$.

S is **Clopen** $\iff S$ is both Open and Closed. The only Clopen sets are $\varnothing$ and $\mathbb{R}$.

If $S \neq \varnothing$, then S is **Compact** $\iff$ every sequence in S has a Convergent subsequence with it's limit in S.

Bolzano–Weierstrass Theorem (BW):
If $S \subseteq \mathbb{R}$ is infinite and bounded, then $S' \neq \varnothing$.
Every bounded sequence has a convergent subsequence.

Heine-Borel Theorem:
$S \neq \varnothing$. S is Compact $\iff S$ is Closed and Bounded.

Cantor's Intersection Theorem:
If $K_1 \supseteq K_2 \supseteq K_3 \dots$ are Compact sets, then $\bigcap_{n=1}^\infty K_n \neq \varnothing$ is Compact.

Limit and Continuity

Let $f : D \to \mathbb{R}$ such that $D \subseteq \mathbb{R}$. Let $a \in D' \subseteq \mathbb{R}$. Let $a_n \in D \setminus \{a\}, a_n \to a$.

$\boldsymbol{\lim_{x \to a^+} f(x) = L} \iff \forall \epsilon > 0, \exists \delta > 0, \forall x \in D, 0 < x - a < \delta \to |f(x) - L| < \epsilon$

$\boldsymbol{\lim_{x \to a^-} f(x) = L}$ is defined similarly with $a - x$ instead.

$\boldsymbol{\lim_{x \to a} f(x) = L} \iff \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L \iff \lim_{n \to \infty} f(a_n) = L$

$\boldsymbol{\lim_{x \to a} f(x) = \infty} \iff \forall M > 0, \exists \delta > 0, \forall x \in D, 0 < |x - a| < \delta \to f(x) \geq M$

$\boldsymbol{\lim_{x \to a} f(x) = -\infty}$ is defined similarly with $M < 0$ instead.

$\boldsymbol{\lim_{x \to a^\pm} f(x) = \pm\infty}$ are defined similarly (all 4 cases).

Let $b, x \in D$. Let $S \subseteq D$. Let $b_n \in D, b_n \to b$.

f is **Continuous at** $b \iff \forall \epsilon > 0, \exists \delta > 0, |x - b| < \delta \to |f(x) - f(b)| < \epsilon \iff \lim_{n \to \infty} f(b_n) = f(b) \iff \lim_{x \to b} f(x) = L = f(b)$

f is **Discontinuous at** $b \iff f$ is not Continuous at b.

f is **Right (Left) Continuous at** b is defined similarly with $x - b$ ($b - x$) instead.

f is **Continuous on** $S \iff f$ is Continuous $\forall s \in S$.

f is **Uniformly Continuous (UC) on** $S \iff \forall \epsilon > 0, \exists \delta > 0, \forall x, y \in S, |x - y| < \delta \to |f(x) - f(y)| < \epsilon$.

f is **Continuous** $\iff f$ is Continuous on D.

(Strictly) Increasing/Decreasing/Monotone for f is defined similarly to sequences.

Let $g : S \to \mathbb{R}, S \subseteq D \cap \mathbb{R}$. g has a **Continuous Extension** to $D \iff \exists \tilde{g} : D \to \mathbb{R}$ such that $\tilde{g}$ is Continuous and $\tilde{g}(x) = g(x), \forall x \in S$. Notation is $\tilde{g}|_S = g$.

If f is UC and $\{c_n\} \subseteq D$ is Cauchy, then $\{f(c_n)\}$ is Cauchy.

If f is Continuous and D is Compact, then f is UC on D.

If f is UC, then f has a Continuous Extension $\tilde{f} : \bar{D} \to \mathbb{R}$.

Squeeze Theorem:
If $f, g, h : D \to \mathbb{R}, a \in D'$, and $\exists \epsilon > 0$ such that $\forall x \in (a - \epsilon, a + \epsilon), f(x) \leq g(x) \leq h(x)$, and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.

Extreme Value Theorem:
If $f : D \to \mathbb{R}$ is continuous and D is compact, then $f(D)$ has a $\min$ and $\max$.

Intermediate Value Theorem (IVT):
Suppose $f : I \to \mathbb{R}$ is continuous and I is an interval. If $a, b \in I, a < b, f(a) \neq f(b)$, then $\exists c \in (a, b)$ such that $f(c) = k$, where k is any value between $f(a)$ and $f(b)$.

Types of Discontinuity:
Removable Discontinuity: $\lim_{x \to a} f(x) = L \neq f(a)$
Jump Discontinuity: $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ and $f(a)$ exists.
Essential Discontinuity: $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a} f(x)$ has no limit L.
Infinite Discontinuity: TODO (LEC OCT 24)

Trig Functions:
$\forall x$:
$|\sin(x)| \leq |x|$,
$cos(x) = 1 - 2\sin^2(\frac{x}{2})$,
$\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$.
$\cos(\arcsin(x)) = \sqrt{1 - x^2}$.

Differentiation

Let $f : D \to \mathbb{R}$, $a \in \text{int}(D)$, $S \subseteq \text{int}(D) \neq \varnothing$.

f is **Differentiable (diff.) at** $a \iff$ The Limit $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.
$f'(a)$ is the **Derivative** of f at a.

f is **Differentiable (diff.) on** $S \iff f$ is Differentiable $\forall x \in S$.

The **Derivative** of f on S is the function $\boldsymbol{f'} : S \to \mathbb{R}$ or $\boldsymbol{f^{(1)}}$.

If f is Differentiable at a, then f is Continuous at a.

f has a **Local Maximum at** $c \iff \exists \delta > 0$ such that $\forall x \in N_\delta(c), f(c) \geq f(x)$.

f has a **Local Minimum at** $c \iff \exists \delta > 0$ such that $\forall x \in N_\delta(c), f(c) \leq f(x)$.

Rolle's Theorem
Suppose $f : [a, b] \to \mathbb{R}$ is Continuous on $[a, b]$ and Differentiable on (a, b).
If $f(a) = f(b)$, then $\exists c \in (a, b)$ such that $f'(c) = 0$.

Mean Value Theorem
Suppose $f : [a, b] \to \mathbb{R}$ is Continuous on $[a, b]$ and Differentiable on (a, b).
$\exists c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Let I be an interval and $f : I \to \mathbb{R}$. If $\exists M > 0$ such that $\forall x \in I, |f'(x)| \leq M$, then f is UC on I.

Intermediate Value Theorem (IVT) (for Derivatives)
Let f is Differentiable on $[a, b]$ and $f'(a) \neq f'(b)$. $\forall k$ Strictly between $f'(a)$ and $f'(b)$, $\exists c \in (a, b)$ such that $f'(c) = k$.

Inverse Function Theorem (for One Variable)
Let f be diff. on an open interval I and f' is non-zero on I (f' is either Strictly positive or Strictly negative).

1. $f : I \to f(I)$ is invertible.
2. $f^{-1} : f(I) \to I$ is diff.
3. $(f^{-1})'(f(x)) = \frac{1}{f'(x)}$.

Cauchy's Mean Value Theorem (Cauchy's MVT)
Let $f, g : [a, b] \to \mathbb{R}$ be Continuous and diff. on (a, b). $\exists c \in (a, b)$ such that $[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$.

L'Hôpital's Rule
Let f, g be diff. on an open interval containing $[a, b]$ and $c \in [a, b]$ with $f(c) = g(c) = 0$. If g' is non-zero in some deleted nhd of c and $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$ exists, then $\lim_{x \to c} \frac{f(x)}{g(x)} = L$.
Let f, g be diff. on (a, ∞) and $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty$. If $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L$ exists, then $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$.

Let f be n times diff. at $x = x_0$.

The n^{th} **degree Taylor Polynomial for f at** x_0 is $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x - x_0)^k}{k!}$.

$f^{(k)}(x_0) = T_n^{(k)}(x_0)$ for $0 \leq k \leq n$. T_n is unique.

Taylor's Theorem
Let f be n times diff. on an open set (?) containing $[a, b]$, where $f^{(n)}$ is continuous on $[a, b]$ and diff. on (a, b). Let $x_0 \in (a, b)$. $\forall x \in (a, b) \setminus \{x_0\}, \exists c$ between x and x_0, such that $f(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$.

Taylor's Inequality
If $f^{(n+1)}$ is bounded on (a, b) by M, then $\forall x \in (a, b), |f(x) - T_n(x)| \leq \frac{M}{(n+1)!} |x - x_0|^{n+1} \leq \frac{M}{(n+1)!}(b - a)^{n+1}$.

Integration

Let f be a Bounded function on $[a, b]$.

A **Partition of** $[a, b]$ is $\boldsymbol{P} = \{x_0, x_1, \dots, x_n\}$ where $a = x_0, b = x_n$ and $\forall k, 0 \leq k < n \to x_k < x_{k+1}$.

Let Q, P be Partitions of $[a, b]$.

Q is a **Refinement of** $P \iff P \subseteq Q$.

Defined $\boldsymbol{M_i(f, P)} = \sup\{f(x) | x_{i-1} \leq x \leq x_i\}$.
Defined $\boldsymbol{m_i(f, P)} = \inf\{f(x) | x_{i-1} \leq x \leq x_i\}$.

The **Upper Sum for f on a partition** $[a, b]$ is $\boldsymbol{U(f, P)} = \sum_{i=1}^{n} M_i(f, P)\Delta x_i$ where $\Delta x_i = x_i - x_{i-1}$.
The **Lower Sum for f on a partition** $[a, b]$ is $\boldsymbol{L(f, P)} = \sum_{i=1}^{n} m_i(f, P)\Delta x_i$.

The **Upper Integral for f on** $[a, b]$ is $U(f) = \inf_P\{U(f, P)\}$.
The **Lower Integral for f on** $[a, b]$ is $L(f) = \sup_P\{L(f, P)\}$.

f is **(Riemann) Integrable on** $[a, b]$ if $L(f) = U(f) = \boldsymbol{\int_a^b f(x)dx = \int_a^b f}$.

$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$.

f is Integrable on $[a, b] \iff \forall \epsilon > 0, \exists$ a partition P such that $U(f, P) - L(f, P) < \epsilon$.

Fundamental Theorem of Calculus

1. Let f be Bounded and Integrable on $[a, b]$. Define $F : [a, b] \to \mathbb{R}$, so that $F(x) = \int_a^x f(t)dt$. F is UC on $[a, b]$. If f is Continuous at c, then F is diff. at c and $F'(c) = f(c)$.

2. Let f' be Bounded and Integrable on $[a, b]$. $\int_a^b f' = f(b) - f(a)$.

Statistics

Introduction TODO

Define Independent, E, $P(A|B)$, etc.

Let X, Y be random variables.

$\boldsymbol{\sigma_X^2} = \mathbf{Var}(X) = \mathrm{E}[(X-\mu_X)^2] = \mathrm{E}(X)^2 - \mathrm{E}(X^2)$,
where σ_X^2 is the **Variance** of X, and σ_X is the **Standard Deviation**.

The **Covariance** of X is $\mathbf{Cov}(X,Y) = \mathrm{E}[(X-\mu_X)(Y-\mu_Y)]$.

Pearson Correlation Coefficient $\boldsymbol{\rho_{X,Y}} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$.

$\forall X, \forall Y, -1 \leq \rho(X,Y) \leq 1$.

Let $\{X_1, \ldots, X_n\}$ be any set of random variables. Let $\{a_1, \ldots, a_n\}$ be constants.

$\mathrm{Var}(\sum_{i=1}^n a_i X_i) = \sum_{i=1}^n a_i \mathrm{Var}(x_i) + 2\sum_{i<j} a_i a_j \mathrm{Cov}(X_i, X_j)$.

If X_i and X_j are Independent, then $\mathrm{Cov}(X_i, X_j) = 0$.

Law of Total Variance
$\mathrm{Var}(Y) = \mathrm{E}(\mathrm{Var}(Y|X)) + \mathrm{Var}(\mathrm{E}(Y|X))$.

Markov Chain (MC)

State j is **Accessible** from state $i \iff P_{ij}^n > 0$ for some $n \geq 0$.

If two states i and j are accessible to each other, then they are **Communicate**, denoted as $i \longleftrightarrow j$.

A Markov Chain is **irreducible** $\iff$ there is only one class.

Let $\boldsymbol{f_i}$ denote thet probability that, starting in state i, the process will ever reenter state i.

State i is **Recurrent** if $f_i = 1$, and **Transient** if $f_i < 1$.

Recurrent and Transient are class properties.

State i has a **Period** of $d \iff n \not\equiv 0 \pmod d \to P_{ii}^n = 0$, where d is the largest integer with this property.

A state is **Aperiodic** $\iff$ it has a peroid of 1.

Periodicity is a class property.

If state i is reccurent, and starting in i, the expected time until the process returns to state i is finite, then it is **Positive Reccurent**.

Positive Reccurency is a class property.

A state is **Ergodic** $\iff$ it is positive recurrent and aperiodic.

If a Markov Chain is irreducible and ergodic, then $\lim_{n\to+\infty} P_{ij}^n$ exists and is independent of i. Let $\boldsymbol{\pi_j}$ be the **Erodic/Limiting Probabilities**, where $\pi_j = \lim_{n\to+\infty} P_{ij}^n$.

TODO:
$S = (I - P_T)^{-1}$ (Lec 10).

Continuous-time Markov Chain (CTMC)

A stochastic process is a **CTMC** $\iff$ When the process enters state i,

1. The amount of time it spends in i before making a transition into a different state is Exponentially Distributed (where mean is $1/\boldsymbol{v_i}$).
2. The probability of entering state j from i is P_{ij}.
3. $\forall i, P_{ii} = 0$ and $\sum_j Pij = 1$.

Birth and Death Processes (BDP) are a CTMC with states $(0, 1, 2, \dots)$ and state n can go to either state $n-1$ or $n+1$. The transition probabilities are such that:
$v_0 = \lambda_0, P_{0,1} = 1$, and for $i > 0$, $v_i = \lambda_i + \mu_i$. $P_{i,i+1} = \frac{\lambda_i}{\lambda_i+\mu_i}$, $P_{i,i-1} = \frac{\mu_i}{\lambda_i+\mu_i}$

The **Transition Probability Function (TPF) $\boldsymbol{P_{ij}(t)}$** for a CTMC is defined as $P_{ij}(t) = P(X(t+s) = j | X(s) = i)$.

The **Instantaneous Transition Rates $\boldsymbol{v_{ij}}$** is defined as $v_{ij} = v_i P_{ij} = \lim_{h\to 0} \frac{P_{ij}(h)}{h}$.

v_i is the **Rate of Transition**, where $v_i = \lim_{h\to 0} \frac{1-P_{ii}(h)}{h}$.

$$v_i = \sum_j v_i P_{ij} = \sum_j v_{ij} \text{ and } P_{ij} = \frac{v_{ij}}{v_i} = \frac{v_{ij}}{\sum_j v_{ij}}$$

Defined the matrix $\boldsymbol{R}$ such that $\boldsymbol{r_{ij}} = \begin{cases} v_{ij} & \text{for } i \neq j. \\ -v_j & \text{for } i = j. \end{cases}$ So $\forall i, \sum_j r_{ij} = 0$.

Kolmogorov's Backward Equation: $\boldsymbol{P'_{ij}(t)}$ $= \left(\sum_{k\neq i} v_{ik} P_{kj}(t)\right) - v_i P_{ij}(t) = \sum_k r_{ij} P_{kj}(t)$

Kolmogorov's Forward Equation: $\boldsymbol{P'_{ij}(t)}$ $= \left(\sum_{k\neq j} P_{ik}(t) v_{kj}\right) - P_{ij}(t) v_i = \sum_k P_{ik}(t) r_{kj}$

Define the matrices $\boldsymbol{P(t), P'(t)}$ using $P_{ij}(t)$ and $P'_{ij}(t)$.
The Kolmogorov equations become $P'(t) = RP(t) = P(t)R$, and

$$P(t) = \boldsymbol{e^{Rt} = \sum_{n=0}^{\infty} R^n \frac{t^n}{n!} = \lim_{n\to\infty} (I + R\frac{t}{n})^n = \lim_{n\to\infty} (I - R\frac{t}{n})^{-n}}$$

.

If R is a $n \times n$ matrix with n Eigenvectors ($R = VDV^{-1}$), then e^{Rt} can be solved using a diagonal matrix ($e^{Rt} = Ve^{Dt}V^{-1}$), where e^{Dt} has the elements $e^{\lambda_i t}$, and λ_i is the ith Eigenvalue.

The **Limiting/Stationary Probability $\boldsymbol{P_j = \pi_j}$** $= \lim_{t\to\infty} P_{ij}(t)$ exists $\iff$ the CTMC is Ergodic.

$v_j P_j = \sum_{k\neq j} v_{kj} P_k, \sum_j P_j = 1$ and $\vec{\pi} \times R = \vec{0}$

Poisson Process (PP)

$N(t)$ for $t \geq 0$ is a **Counting Process (CP)** if

1. $N(t) \geq 0$.
2. $N(t) \in \mathbb{Z}$.
3. If $s < t$, then $N(s) \leq N(t)$.
4. For $s < t$, $N(t) - N(s)$ equals the number of events that occur in the interval $(s, t]$.

A CP has **Independent Increments** if the number of events that occur in disjoint intervals are independent.

A CP has **Stationary Increments** if the distribution of the number of events that occur in any interval depends only on the length of the interval.

Let $\boldsymbol{o(h)}$ be any function of h that satisfies the condition: $\lim_{h\to 0}(o(h)/h) = 0$.

$N(t)$ is a **Poisson Process (PP)** with rate $\lambda > 0$, If

1. $N(0) = 0$.
2. $N(t)$ has Independent and Stationary increments.
3. $P(N(t+s) - N(s) = n) = e^{-\lambda t}(\lambda t)^n n!^{-1}, n \geq 0, n \in \mathbb{Z}$.
4. $P(N(h) = 1) = \lambda h + o(h)$.
5. $P(N(h) \geq 2) = o(h)$.

Where $4 \wedge 5 \leftrightarrow 3$.

Let $\boldsymbol{\{T_n\}}$ be the sequence of **Interarrival Times** of $N(t)$. $P(T_n > t) = e^{-\lambda t}$.

T_n has an Exponential Distribution with a mean of $1/\lambda$ for all n.

Let $\boldsymbol{S_n}$ be the **Arrival/Waiting Time** of the nth event. Where $S_n = \sum_{i=1}^{n} T_i, n \geq 1$.

S_n has a Gamma Distribution with parameters n and λ. So $f_{S_n}(t) = \lambda e^{-\lambda t}(\lambda t)^{n-1}(n-1)!^{-1}, t \geq 0$.

$S_n \leq t = N(t) \geq n$. The formula for probability density can also be obtained using this fact.

$P(T_1 < s | N(t) = 1) = s/t$.

Two Poisson Processes

Suppose a PP with rate λ is split into two types, $\boldsymbol{N_1(t)}$ with probability p or $\boldsymbol{N_2(t)}$ with probability $1-p$. Where $N(t) = N_1(t) + N_2(t)$.

$N_1(t)$ has a rate of λp. $N_2(t)$ has a rate of $\lambda(1-p)$.

Let $\boldsymbol{S_n^1, S_m^2}$ denote the time of the nth/mth event of $N_1(t)$ and $N_2(t)$ respectively.

$P(S_1^1 < S_1^2) = \lambda_1(\lambda_1 + \lambda_2)^{-1}$.

Each event that occurs is going to be $N_1(t)$ with probability $\lambda_1(\lambda_1+\lambda_2)^{-1}$, and $N_2(t)$ with probability $\lambda_2(\lambda_1 + \lambda_2)^{-1}$. So $P(S_n^1 < S_m^2)$ is based on the Binomial Distribution with $p = \lambda_1(\lambda_1+\lambda_2)^{-1}$.

$\boldsymbol{n}$ Poisson Processes

Defined n processes similarly to two processes.

$\sum_{i=1}^{n} P_i(y) = 1$.

$\mathrm{E}(N_i(t)) = \lambda \int_0^t P_i(s)ds$.

Non-homogeneous Poisson Process (NHPP)

Every event that occurs has a $p(t)$ chance of being recorded. Let $\boldsymbol{N_c(t)}$ be the NHPP, then it has a rate of $\boldsymbol{\lambda(t)} = \lambda p(t)$.

A NHPP with $\lambda(t) = \lambda$ is a regular PP.

Let $\boldsymbol{m(t)}$ be the mean value function = $\mathrm{E}(N_c(t)) = \lambda \int_0^t p(s)ds$.

The increments of a NHPP are Independent, but not necessarily stationary.

$\forall s, t, 0 \leq s < t, N(t) - N(s)$ has a Poisson Distribution with mean $m(t) - m(s) = \int_s^t \lambda(x)dx$.

Compound Poisson Process (CPP)

$\boldsymbol{X(t)}$ is a CPP if $X(t) = \sum_{i=1}^{N(t)} Y_i$. where $N(i)$ is a PP, and $Y_i, i \geq 1$ is some i.i.d.r.v.'s that are also independent of $N(t)$.

$\mathrm{E}(X(t)) = \lambda t \mathrm{E}(Y_1)$.

$\mathrm{Var}(X(t)) = \lambda t \mathrm{E}(Y_1^2)$ (Derived from the Law of Total Variance) (?).

$M_X(s) = \exp(\lambda t(M_Y(s) - 1))$ (Derived from the Law of Total Probability) (?).

If $Y_i = 1$ then $X(t)$ is a normal PP.

TODO:
$f(y_1, y_2 \ldots y_n) = f(s_1, s_2 \ldots s_n | n) = n!/t^n$ (lec 15).

Renewal Process (RP) DRAFT

Let $N(t)$ be a CP. Let X_n be the time between any two consecutive events.

If X_n are i.i.d.r.v.'s, then $N(t)$ is a **Renewal Process**.

$m(t) = \mathrm{E}[N(t)]$ is the **Renewal/Mean-value Function**.

Let $\boldsymbol{F(x)} = P(X_n < x)$.

The PP is an example of a RP with $F(x) = 1 - e^{-\lambda x}, x \geq 0$ and $m(t) = \lambda t$.

$m(t) = F(t) + \int_0^t m(t-x) f(x) dx$ is the **Renewal Equation**.

Let $\{X_n\}$ be some sequence of r.v.'s. This stochastic process is **Martingale** $\iff \forall n$,

1. $\mathrm{E}(|X_n|) < \infty$.
2. $\mathrm{E}(X_{n+1} | X_u, 0 \leq u \leq n) = X_n$.

Notations and Tables TODO

$x \in [a, b]$ means $a \leq x \leq b$.

$x \in (a, b)$ means $a < x < b$.

$\{a$ such that $a = b\} = \{a | a = b\} = \{a : a = b\}$.

n choose $k = \binom{n}{k} = \frac{n!}{k!(n-k)!} = C_n^k = {}_nC_k = {}^nC_k = C(n, k)$.

For Trigonometric Functions, $\sin^2(x) = (\sin(x))^2$, not $\sin(\sin(x))$.
Unless the exponent is -1 in which case, $y = \sin^{-1}(x) = \arcsin(x)$, is defined $x = \sin(y)$.

i.i.d.r.v.'s means "Idependent Identically Distributed Random Variables".