

Notes

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Real Analysis

Sets, Relations and Functions TODO

Sequence

A **Sequence** is a function $f : \mathbb{N} \rightarrow \mathbb{R}$. Notations are $a_n = f(n)$, $\{a_n\}$, $\{a_n\}_{n=1}^{\infty}$.

A sequence is **Bounded** $\iff \exists M \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$, $|a_n| \leq M$.

Bounded Above/Below are defined similarly.

A sequence is **(Strictly) Increasing** if $\forall n \in \mathbb{N}$, $a_{n+1} \geq a_n$ ($a_{n+1} > a_n$). **(Strictly) Decreasing** is defined similarly.

A sequence is **Monotone** if it's either Decreasing or Increasing.

A sequence is **Convergent** to L $\iff \forall \epsilon \in \mathbb{R}, \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$, if $n \geq N$ then $|a_n - L| < \epsilon$.

A sequence is **Divergent** to ∞ $\iff \forall M \in \mathbb{R}, M > 0, \exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$, if $n \geq N$ then $a_n > M$.
Divergent to $-\infty$ is defined similarly.

A sequence is **Cauchy** $\iff \forall \epsilon \in \mathbb{R}, \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall m, n \in \mathbb{N}$, if $m, n \geq N$ then $|a_n - a_m| < \epsilon$.

A sequence is Convergent \iff it's Cauchy.

If a sequence is Convergent, then it is Bounded.

Monotone Convergence Theorem:

If a sequence is Bounded Above and Increasing, then it converges.

If a sequence is Bounded Below and Decreasing, then it converges.

If a sequence is Bounded and Monotone, then it converges.

Every sequence has a Monotone subsequence.

Define $\text{sub}(a_n)$ to be a set containing the limits of all subsequences of $\{a_n\}$.

The **Limit Superior** of $\{a_n\}$ is $\sup(\text{sub}(a_n)) = \max(\text{sub}(a_n))$, denoted as $\limsup_{n \rightarrow \infty} (a_n)$.

The **Limit Inferior** of $\{a_n\}$ is $\inf(\text{sub}(a_n)) = \min(\text{sub}(a_n))$, denoted as $\liminf_{n \rightarrow \infty} (a_n)$.

Let $\{q_n\}$ be any denumeration of \mathbb{Q} , then $\text{sub}(q_n) = \mathbb{R}$.

Topology of the Real Line

Let $x \in \mathbb{R}$ and $S \subseteq \mathbb{R}$.

The **Neighbourhood (nhd)** of x is the set $N_\epsilon(x) = \{y \in \mathbb{R} : |x - y| < \epsilon\} = (x - \epsilon, x + \epsilon)$, for some $\epsilon > 0$.

The **Deleted Neighbourhood** of x is the set $N_\epsilon^*(x) = N_\epsilon(x) \setminus \{x\}$.

An **Interior Point** of S is an element $x \in S$ such that $N_\epsilon(x) \subseteq S$ for some $\epsilon > 0$.

The set of all Interior points of S is the **Interior** of S , denoted as $\text{int}(S)$.

x is a **Boundary Point** for S $\iff \forall \epsilon > 0, N_\epsilon(x) \cap S \neq \emptyset \wedge N_\epsilon(x) \cap (\mathbb{R} \setminus S) \neq \emptyset$.

The set of all Boundary Points of S is the **Boundary** of S , denoted as $\text{bd}(S)$.

x is a **Accumulation (Limit) Point** for $S \iff \forall \epsilon > 0, N_\epsilon^*(x) \cap S \neq \emptyset \iff \forall \epsilon > 0, \exists y \in S$ such that $0 < |x - y| < \epsilon$.

The set of all Accumulation Points of S is denoted as S' .

The **Closure** of S is the set $\bar{S} = S \cup S' = S \cup \text{bd}(S)$.

x is a **Isolated Point** for $S \iff x \in S \setminus S' \iff x \in S \wedge \exists \epsilon > 0$ such that $N_\epsilon^*(x) \cap S = \emptyset$.

$S \subseteq \text{int}(S) \cup \text{bd}(S)$ and $\text{int}(S) \cap \text{bd}(S) = \emptyset$.

S is **Open** $\iff S = \text{int}(S) \iff \text{bd}(S) \subseteq \mathbb{R} \setminus S$.

S is **Closed** $\iff \mathbb{R} \setminus S$ is Open $\iff \text{bd}(S) \subseteq S \iff S = \bar{S}$.

S is **Clopen** $\iff S$ is both Open and Closed. The only Clopen sets are \emptyset and \mathbb{R} .

If $S \neq \emptyset$, then (S is **Compact** \iff every sequence in S has a Convergent subsequence with it's limit in S .)

Bolzano–Weierstrass Theorem (BW):

If $S \subseteq \mathbb{R}$ is infinite and bounded, then $S' \neq \emptyset$.

Every bounded sequence has a convergent subsequence.

Heine–Borel Theorem:

$S \neq \emptyset$. S is Compact $\iff S$ is Closed and Bounded.

Cantor's Intersection Theorem:

If $K_1 \supseteq K_2 \supseteq K_3 \dots$ are Compact sets, then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$ is Compact.

Limit and Continuity

Let $f : D \rightarrow \mathbb{R}$ such that $D \subseteq \mathbb{R}$. Let $a \in D' \subseteq \mathbb{R}$. Let $a_n \in D \setminus \{a\}, a_n \rightarrow a$.

$\lim_{x \rightarrow a^+} f(x) = L \iff \forall \epsilon > 0, \exists \delta > 0, \forall x \in D, 0 < x - a < \delta \rightarrow |f(x) - L| < \epsilon$

$\lim_{x \rightarrow a^-} f(x) = L$ is defined similarly with $a - x$ instead.

$\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L \iff \lim_{n \rightarrow \infty} f(a_n) = L$

$\lim_{x \rightarrow a} f(x) = \infty \iff \forall M > 0, \exists \delta > 0, \forall x \in D, 0 < |x - a| < \delta \rightarrow f(x) \geq M$

$\lim_{x \rightarrow a} f(x) = -\infty$ is defined similarly with $M < 0$ instead.

$\lim_{x \rightarrow a^\pm} f(x) = \pm\infty$ are defined similarly (all 4 cases).

Let $b, x \in D$. Let $S \subseteq D$. Let $b_n \in D, b_n \rightarrow b$.

f is **Continuous at b** $\iff \forall \epsilon > 0, \exists \delta > 0, |x - b| < \delta \rightarrow |f(x) - f(b)| < \epsilon \iff \lim_{n \rightarrow \infty} f(b_n) = f(b) \iff \lim_{x \rightarrow b} f(x) = L = f(b)$

f is **Discontinuous at b** $\iff f$ is not Continuous at b .

f is **Right (Left) Continuous at b** is defined similarly with $x - b$ ($b - x$) instead.

f is **Continuous on S** $\iff f$ is Continuous $\forall s \in S$.

f is **Uniformly Continuous (UC) on S** $\iff \forall \epsilon > 0, \exists \delta > 0, \forall x, y \in S, |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon$.

f is **Continuous** $\iff f$ is Continuous on D .

(Strictly) Increasing/Decreasing/Monotone for f is defined similarly to sequences.

Let $g : S \rightarrow \mathbb{R}, S \subseteq D \cap \mathbb{R}$. g has a **Continuous Extension** to $D \iff \exists \tilde{g} : D \rightarrow \mathbb{R}$ such that \tilde{g} is Continuous and $\tilde{g}(x) = g(x), \forall x \in S$. Notation is $\tilde{g}|_S = g$.

If f is UC and $\{c_n\} \subseteq D$ is Cauchy, then $\{f(c_n)\}$ is Cauchy.

If f is Continuous and D is Compact, then f is UC on D .

If f is UC, then f has a Continuous Extension $\tilde{f} : \bar{D} \rightarrow \mathbb{R}$.

Squeeze Theorem:

If $f, g, h : D \rightarrow \mathbb{R}$, $a \in D'$, and $\exists \epsilon > 0$ such that $\forall x \in (a - \epsilon, a + \epsilon)$, $f(x) \leq g(x) \leq h(x)$, and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$, then $\lim_{x \rightarrow a} g(x) = L$.

Extreme Value Theorem:

If $f : D \rightarrow \mathbb{R}$ is continuous and D is compact, then $f(D)$ has a min and max.

Intermediate Value Theorem (IVT):

Suppose $f : I \rightarrow \mathbb{R}$ is continuous and I is an interval. If $a, b \in I$, $a < b$, $f(a) \neq f(b)$, then $\exists c \in (a, b)$ such that $f(c) = k$, where k is any value between $f(a)$ and $f(b)$.

Types of Discontinuity:

Removable Discontinuity: $\lim_{x \rightarrow a} f(x) = L \neq f(a)$

Jump Discontinuity: $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$ and $f(a)$ exists.

Essential Discontinuity: $\lim_{x \rightarrow a^-} f(x)$ or $\lim_{x \rightarrow a^+} f(x)$ or $\lim_{x \rightarrow a} f(x)$ has no limit L .

Infinite Discontinuity: TODO (LEC OCT 24)

Trig Functions:

$\forall x$:

$$|\sin(x)| \leq |x|,$$

$$\cos(x) = 1 - 2\sin^2\left(\frac{x}{2}\right),$$

$$\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a).$$

$$\cos(\arcsin(x)) = \sqrt{1 - x^2}.$$

Differentiation

Let $f : D \rightarrow \mathbb{R}$, $a \in \text{int}(D)$, $S \subseteq \text{int}(D) \neq \emptyset$.

f is **Differentiable (diff.)** at $a \iff$ The Limit $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ exists.
 $f'(a)$ is the **Derivative** of f at a .

f is **Differentiable (diff.)** on $S \iff f$ is Differentiable $\forall x \in S$.

The **Derivative** of f on S is the function $f' : S \rightarrow \mathbb{R}$ or $f^{(1)}$.

If f is Differentiable at a , then f is Continuous at a .

f has a **Local Maximum** at $c \iff \exists \delta > 0$ such that $\forall x \in N_\delta(c)$, $f(c) \geq f(x)$.

f has a **Local Minimum** at $c \iff \exists \delta > 0$ such that $\forall x \in N_\delta(c)$, $f(c) \leq f(x)$.

Rolle's Theorem

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is Continuous on $[a, b]$ and Differentiable on (a, b) .

If $f(a) = f(b)$, then $\exists c \in (a, b)$ such that $f'(c) = 0$.

Mean Value Theorem

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is Continuous on $[a, b]$ and Differentiable on (a, b) .

$\exists c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Let I be an interval and $f : I \rightarrow \mathbb{R}$. If $\exists M > 0$ such that $\forall x \in I$, $|f'(x)| \leq M$, then f is UC on I .

Intermediate Value Theorem (IVT) (for Derivatives)

Let f is Differentiable on $[a, b]$ and $f'(a) \neq f'(b)$. $\forall k$ Strictly between $f'(a)$ and $f'(b)$, $\exists c \in (a, b)$ such that $f'(c) = k$.

Inverse Function Theorem (for One Variable)

Let f be diff. on an open interval I and f' is non-zero on I (f' is either Strictly positive or Strictly negative).

1. $f : I \rightarrow f(I)$ is invertible.
2. $f^{-1} : f(I) \rightarrow I$ is diff.
3. $(f^{-1})'(f(x)) = \frac{1}{f'(x)}$.

Cauchy's Mean Value Theorem (Cauchy's MVT)

Let $f, g : [a, b] \rightarrow \mathbb{R}$ be Continuous and diff. on (a, b) . $\exists c \in (a, b)$ such that $[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$.

L'Hôpital's Rule

Let f, g be diff. on an open interval containing $[a, b]$ and $c \in [a, b]$ with $f(c) = g(c) = 0$. If g' is non-zero in some deleted nhd of c and $\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} = L$ exists, then $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = L$.

Let f, g be diff. on (a, ∞) and $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = \infty$. If $\lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)} = L$ exists, then $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$.

Let f be n times diff. at $x = x_0$.

The n^{th} degree Taylor Polynomial for f at x_0 is $T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}$.

$f^{(k)}(x_0) = T_n^{(k)}(x_0)$ for $0 \leq k \leq n$. T_n is unique.

Taylor's Theorem

Let f be n times diff. on an open set (?) containing $[a, b]$, where $f^{(n)}$ is continuous on $[a, b]$ and diff. on (a, b) .

Let $x_0 \in (a, b)$. $\forall x \in (a, b) \setminus \{x_0\}$, $\exists c$ between x and x_0 , such that $f(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$.

Taylor's Inequality

If $f^{(n+1)}$ is bounded on (a, b) by M , then $\forall x \in (a, b)$, $|f(x) - T_n(x)| \leq \frac{M}{(n+1)!} |x - x_0|^{n+1} \leq \frac{M}{(n+1)!} (b - a)^{n+1}$.

Integration

Let f be a Bounded function on $[a, b]$.

A Partition of $[a, b]$ is $P = \{x_0, x_1, \dots, x_n\}$ where $a = x_0, b = x_n$ and $\forall k, 0 \leq k < n \rightarrow x_k < x_{k+1}$.

Let Q, P be Partitions of $[a, b]$.

Q is a Refinement of $P \iff P \subseteq Q$.

Defined $M_i(f, P) = \sup\{f(x) | x_{i-1} \leq x \leq x_i\}$.

Defined $m_i(f, P) = \inf\{f(x) | x_{i-1} \leq x \leq x_i\}$.

The Upper Sum for f on a partition $[a, b]$ is $U(f, P) = \sum_{i=1}^n M_i(f, P) \Delta x_i$ where $\Delta x_i = x_i - x_{i-1}$.

The Lower Sum for f on a partition $[a, b]$ is $L(f, P) = \sum_{i=1}^n m_i(f, P) \Delta x_i$.

The Upper Integral for f on $[a, b]$ is $U(f) = \inf_P \{U(f, P)\}$.

The Lower Integral for f on $[a, b]$ is $L(f) = \sup_P \{L(f, P)\}$.

f is (Riemann) Integrable on $[a, b]$ if $L(f) = U(f) = \int_a^b f(x) dx = \int_a^b f$.

$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$.

f is Integrable on $[a, b] \iff \forall \epsilon > 0, \exists$ a partition P such that $U(f, P) - L(f, P) < \epsilon$.

Fundamental Theorem of Calculus

1. Let f be Bounded and Integrable on $[a, b]$. Define $F : [a, b] \rightarrow \mathbb{R}$, so that $F(x) = \int_a^x f(t) dt$. F is UC on $[a, b]$. If f is Continuous at c , then F is diff. at c and $F'(c) = f(c)$.
2. Let f' be Bounded and Integrable on $[a, b]$. $\int_a^b f' = f(b) - f(a)$.

Infinite Series

Let $\{a_n\}$ be a sequence.

Define the **Sequence of Partial Sums** $\sum a_n = s_n = \sum_{i=1}^n a_i$.

If $\{s_n\}$ is convergent, then $\{a_n\}$ is convergent to 0.

Cauchy's Convergence Test

$\{s_n\}$ is convergent $\iff \forall \epsilon > 0, \exists N \in \mathbb{N}$ so that $\forall n, m \in \mathbb{N}, N < m < n \rightarrow \sum_{i=m}^n a_i < \epsilon$.

Comparison Test

Suppose $\forall n, 0 \leq a_n \leq b_n$.

If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.

Limit Comparison Test

Suppose $\forall n, a_n$ and b_n are non-negative. Suppose $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \in [0, \infty]$.

1. If $L \in [0, \infty)$ and $\sum b_n$ is convergent, then $\sum a_n$ is convergent.
2. If $L \in (0, \infty]$ and $\sum b_n$ is divergent, then $\sum a_n$ is divergent.
3. If $L \in (0, \infty)$, then $(\sum a_n \text{ is convergent} \iff \sum b_n \text{ is convergent})$.

Calculus

Differentiation and Integration TODO

Integration By Substitution: $\int f(g(x))g'(x)dx = \int f(u)du$, where $u = g(x)$. (Note: Remember to change domain for definite integrals.)

Integration By Parts: $\int f g' dx = f g - \int f' g dx$.

Multiple Integration

Double Integral

Polar Coordinates: Let $x = r \cos(\theta)$, $y = r \sin(\theta)$, such that $r^2 = x^2 + y^2$.

$$\iint_D f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos(\theta), r \sin(\theta)) r dr d\theta$$

Triple Integral

Spherical Coordinates: Let $x = p \sin(\phi) \cos(\theta)$, $y = p \sin(\phi) \sin(\theta)$, $z = p \cos(\phi)$, such that $x^2 + y^2 + z^2 = p^2$.

$$\iiint_E f(x, y, z) dV = \int_{\delta}^{\gamma} \int_{\alpha}^{\beta} \int_a^b p^2 \sin(\phi) f(p \sin(\phi) \cos(\theta), p \sin(\phi) \sin(\theta), p \cos(\phi)) dp d\theta d\phi$$

Cylindrical Coordinates: Let $x = r \cos(\theta)$, $y = r \sin(\theta)$, $z = z$.

$$\iiint_E f(x, y, z) dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos(\theta), r \sin(\theta))}^{u_2(r \cos(\theta), r \sin(\theta))} r f(r \cos(\theta), r \sin(\theta), z) dz dr d\theta$$

Multivariable TODO

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$. The **Jacobian Matrix** is the $m \times n$ matrix J , where $J_{ij} = \frac{\partial f_i}{\partial x_j}$. The **Jacobian (Determinant)** is the determinant of J , denoted as $|J|$.

To integrate $f(x, y)$ over the region R . Let $x = g(u, v)$, $y = h(u, v)$ be the transformation where the region becomes S . Then the integral becomes: $\iint_R f(x, y) dA = \iint_S f(g(u, v), h(u, v)) |J| dA$.

Lagrange Multipliers

For finding the Absolute Extrema of a $f(x, y, z)$ given $g(x, y, z) = c$, $h(x, y, z) = k$ constraints, solve

$$\begin{aligned}\nabla f(x, y, z) &= \lambda_1 \nabla g(x, y, z) + \lambda_2 \nabla h(x, y, z) \\ g(x, y, z) &= c \\ h(x, y, z) &= k\end{aligned}$$

where the constant λ_1, λ_2 are the **Lagrange Multipliers**. This should yield 5 equations with 5 variables.

Dimensions for f other than 3 works similarly. Number of constraints other than 2 works similarly.

(Note: using $\nabla f(x, y, z) + \lambda_1 \nabla g(x, y, z) + \lambda_2 \nabla h(x, y, z) = 0$ works as well.)(?)

Vector

Let $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ be a vector function.

$\vec{r}'(t)$ is the **Tangent Vector**.

$\vec{N}(t)$ is the **Unit Normal Vector** where $\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$.

$\vec{T}(t)$ is the **Unit Tangent Vector** where $\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$. $\vec{B}(t)$ is the **Binormal Vector** where $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$.

Define the operator $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ where $\nabla f(x, y, z) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$.

∇f is a vector with the gradient vectors of f .

Let $\vec{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$ be a vector field.

\vec{F} is **Conservative** $\iff \exists f$ such that $\vec{F} = \nabla f$, where f is the **Potential Function** of \vec{F} .

Line Integral

Let \vec{F} be a continuous vector field with domain D . Let C be a curve. Let f be a function of x_1, x_2, \dots, x_n .

The **Line Integral** along C of $\begin{cases} f \text{ is } \int_C f(x_1, \dots, x_n) ds = \int_a^b f(x_1(t), \dots, x_n(t)) \|\vec{r}'(t)\| dt \\ f \text{ respect to } x_k \text{ is } \int_C f(x_1, \dots, x_n) dx_k = \int_a^b f(x_1(t), \dots, x_n(t)) x'_k(t) dt \\ \vec{F} \text{ is } \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt \end{cases}$

Fundamental Theorem of Calculus (for Line Integrals):

If C is a Smooth curve $\vec{r}(t)$ and ∇f is continuous on C , then $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

Green's Theorem

Let C positively oriented smooth simple closed curve, and D be the region enclosed by the curve (TODO).

$$\int_C P dx + Q dy = \iint_D (Q_x - P_y) dA$$

$$A = \oint_C x dy = - \oint_C y dx = \frac{1}{2} \oint_C x dy - y dx$$

Surface Integral

Define $\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k}$.

If \vec{F} is Conservative, then $\text{curl}(\vec{F}) = \vec{0}$.

If $\text{curl}(\vec{F}) = \vec{0}$ and TODO, then \vec{F} is Conservative.

Define $\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

$\text{div}(\text{curl}(\vec{F})) = 0$.

For a surface S : Let $z = g(x, y)$. Let $\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$. Depending on which exists (TODO)
 $dS = \sqrt{g_x^2 + g_y^2 + 1} dA$:

To Integrate $f(x, y, z)$ on a surface S ,

$$\iint_S f(x, y, z) dS = \iint_D f(x, y, g(x, y)) \sqrt{g_x^2 + g_y^2 + 1} dA = \iint_D f(\vec{r}(u, v)) \|\vec{r}_u \times \vec{r}_v\| dA$$

To Integrate \vec{F} on the surface S (TODO (\vec{n} is the unit normal of S)),

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS = \iint_D \vec{F} \cdot \nabla h dA = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA = \iint_D -Pg_x - Qg_y + RdA$$

where $h(x, y, z) = z - g(x, y)$.

Divergence Theorem

Let E be a bounded solid with S being the surface. Let \vec{F} be a vector field with continuous first partial derivatives on E . (TODO)

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div} \vec{F} dV$$

Stokes' Theorem

Let S be an oriented smooth surface that is bounded by a Smooth Simple Closed curve C with positive orientation. Let \vec{F} be a vector field with continuous first partial derivatives on S .

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$$

Linear Algebra TODO

TODO Define determinant, vector magnitude, etc.

Let u, v be vectors of size n . Let θ be the angle between the u and v .

Dot Product of u and v , $u \cdot v = \sum_{i=1}^n u_i v_i = \|u\| \|v\| \cos \theta$.

If u, v are of size 3, then the **Cross Product** $u \times v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ where $u \times v$ is orthogonal to both u and v ,

and $\|u \times v\| = \|u\| \|v\| \sin \theta$.

$proj_v(u) = \frac{u \cdot v}{\|v\|^2} v$.

Unorganized

$$(AC)^T = C^T A^T \quad (AC)^{-1} = C^{-1} A^{-1}$$

$$(pA)^{-1} = 1/p * A^{-1}$$

A is $m \times n$ b is $n \times p$ $ab_i, j = \text{sum of } a_i, k * b_k, j \text{ from } k = 1 \text{ to } n$

$$Qm \text{ on } y = mx \quad Qm = 1/(1 + m^2) \quad 1 - m^2, 2m, 2m, m^2 - 1$$

$QmQn = R\theta$ where θ is 2x the angle between lines $y=mx$ and $y=nx$?

Effects of the 3 elementary row operations on the determinant of triangular matrices

Co-factors is the determinant after deleting the col and row. That's it

Cramer's rule on invertible matrices. Unique solution

The Vandermonde Determinant $\prod_{1 \leq j < i \leq n} (a_i - a_j)$.

Cofactor expansion theorem

Orthogonal matrix $Q^T = Q^{-1}$

Adjugate $A(\text{adj}A) = (\det A)I$

Elementary matrices

dynamic systems

Proof that if A is invertible it's RREF is I

Interpolating polynomials

cofactor expansion theorem

quadrilateral parallelograms

fundamental theorem

$$\text{law of cosine } c^2 = a^2 + b^2 - 2ab * \cos(C)$$

Statistics DRAFT

Introduction TODO

Define Independent, E, $P(A|B)$, etc.

Let X, Y be random variables.

$\sigma_X^2 = \text{Var}(X) = E[(X - \mu_X)^2] = E(X)^2 - E(X^2)$,
where σ_X^2 is the **Variance** of X , and σ_X is the **Standard Deviation**.

The **Covariance** of X is $\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$.

Pearson Correlation Coefficient $\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$.

$\forall X, \forall Y, -1 \leq \rho(X, Y) \leq 1$.

Let $\{X_1, \dots, X_n\}$ be any set of random variables. Let $\{a_1, \dots, a_n\}$ be constants.

$\text{Var}(\sum_{i=1}^n a_i X_i) = \sum_{i=1}^n a_i^2 \text{Var}(x_i) + 2 \sum_{i < j} a_i a_j \text{Cov}(X_i, X_j)$.

If X_i and X_j are Independent, then $\text{Cov}(X_i, X_j) = 0$.

Law of Total Variance

$\text{Var}(Y) = E(\text{Var}(Y|X)) + \text{Var}(E(Y|X))$.

Markov Chain (MC)

State j is **Accessible** from state $i \iff P_{ij}^n > 0$ for some $n \geq 0$.

If two states i and j are accessible to each other, then they are **Communicate**, denoted as $i \longleftrightarrow j$.

A Markov Chain is **irreducible** \iff there is only one class.

Let f_i denote the probability that, starting in state i , the process will ever reenter state i .

State i is **Recurrent** if $f_i = 1$, and **Transient** if $f_i < 1$.

Recurrent and Transient are class properties.

State i has a **Period** of $d \iff n \not\equiv 0 \pmod{d} \rightarrow P_{ii}^n = 0$, where d is the largest integer with this property.

A state is **Aperiodic** \iff it has a period of 1.

Periodicity is a class property.

If state i is recurrent, and starting in i , the expected time until the process returns to state i is finite, then it is **Positive Recurrent**.

Positive Recurrency is a class property.

A state is **Ergodic** \iff it is positive recurrent and aperiodic.

If a Markov Chain is irreducible and ergodic, then $\lim_{n \rightarrow +\infty} P_{ij}^n$ exists and is independent of i . Let π_j be the **Ergodic/Limiting Probabilities**, where $\pi_j = \lim_{n \rightarrow +\infty} P_{ij}^n$.

TODO:

$$S = (I - P_T)^{-1} \text{ (Lec 10).}$$

Continuous-time Markov Chain (CTMC)

A stochastic process is a **CTMC** \iff When the process enters state i ,

1. The amount of time it spends in i before making a transition into a different state is Exponentially Distributed (where mean is $1/v_i$).
2. The probability of entering state j from i is P_{ij} .
3. $\forall i, P_{ii} = 0$ and $\sum_j P_{ij} = 1$.

Birth and Death Processes (BDP) are a CTMC with states $(0, 1, 2, \dots)$ and state n can go to either state $n - 1$ or $n + 1$. The transition probabilities are such that:

$$v_0 = \lambda_0, P_{0,1} = 1, \text{ and for } i > 0, v_i = \lambda_i + \mu_i. P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$$

The **Transition Probability Function (TPF)** $P_{ij}(t)$ for a CTMC is defined as $P_{ij}(t) = P(X(t+s) = j | X(s) = i)$.

The **Instantaneous Transition Rates** v_{ij} is defined as $v_{ij} = v_i P_{ij} = \lim_{h \rightarrow 0} \frac{P_{ij}(h)}{h}$.

v_i is the **Rate of Transition**, where $v_i = \lim_{h \rightarrow 0} \frac{1 - P_{ii}(h)}{h}$.

$$v_i = \sum_j v_{ij} P_{ij} = \sum_j v_{ij} \text{ and } P_{ij} = \frac{v_{ij}}{v_i} = \frac{v_{ij}}{\sum_j v_{ij}}$$

Defined the matrix R such that $r_{ij} = \begin{cases} v_{ij} & \text{for } i \neq j. \\ -v_i & \text{for } i = j. \end{cases}$ So $\forall i, \sum_j r_{ij} = 0$.

Kolmogorov's Backward Equation: $P'_{ij}(t) = \left(\sum_{k \neq i} v_{ik} P_{kj}(t) \right) - v_i P_{ij}(t) = \sum_k r_{ik} P_{kj}(t)$

Kolmogorov's Forward Equation: $P'_{ij}(t) = \left(\sum_{k \neq j} P_{ik}(t) v_{kj} \right) - P_{ij}(t) v_i = \sum_k P_{ik}(t) r_{kj}$

Define the matrices $P(t), P'(t)$ using $P_{ij}(t)$ and $P'_{ij}(t)$.

The Kolmogorov equations become $P'(t) = RP(t) = P(t)R$, and

$$P(t) = e^{Rt} = \sum_{n=0}^{\infty} R^n \frac{t^n}{n!} = \lim_{n \rightarrow \infty} \left(I + R \frac{t}{n} \right)^n = \lim_{n \rightarrow \infty} \left(I - R \frac{t}{n} \right)^{-n}$$

If R is a $n \times n$ matrix with n Eigenvectors ($R = VDV^{-1}$), then e^{Rt} can be solved using a diagonal matrix ($e^{Rt} = Ve^{Dt}V^{-1}$), where e^{Dt} has the elements $e^{\lambda_i t}$, and λ_i is the i th Eigenvalue.

The **Limiting/Stationary Probability** $P_j = \pi_j = \lim_{t \rightarrow \infty} P_{ij}(t)$ exists \iff the CTMC is Ergodic.

$$v_j P_j = \sum_{k \neq j} v_{kj} P_k, \sum_j P_j = 1 \text{ and } \vec{\pi} \times R = \vec{0}$$

Poisson Process (PP)

$N(t)$ for $t \geq 0$ is a **Counting Process (CP)** if

1. $N(t) \geq 0$.
2. $N(t) \in \mathbb{Z}$.
3. If $s < t$, then $N(s) \leq N(t)$.
4. For $s < t$, $N(t) - N(s)$ equals the number of events that occur in the interval $(s, t]$.

A CP has **Independent Increments** if the number of events that occur in disjoint intervals are independent.

A CP has **Stationary Increments** if the distribution of the number of events that occur in any interval depends only on the length of the interval.

Let $o(h)$ be any function of h that satisfies the condition: $\lim_{h \rightarrow 0} (o(h)/h) = 0$.

$N(t)$ is a **Poisson Process (PP)** with rate $\lambda > 0$, If

1. $N(0) = 0$.
2. $N(t)$ has Independent and Stationary increments.
3. $P(N(t+s) - N(s) = n) = e^{-\lambda t} (\lambda t)^n / n!^{-1}, n \geq 0, n \in \mathbb{Z}$.
4. $P(N(h) = 1) = \lambda h + o(h)$.
5. $P(N(h) \geq 2) = o(h)$.

Where $4 \wedge 5 \leftrightarrow 3$.

Let $\{T_n\}$ be the sequence of **Interarrival Times** of $N(t)$. $P(T_n > t) = e^{-\lambda t}$.

T_n has an Exponential Distribution with a mean of $1/\lambda$ for all n .

Let S_n be the **Arrival/Waiting Time** of the n th event. Where $S_n = \sum_{i=1}^n T_i, n \geq 1$.

S_n has a Gamma Distribution with parameters n and λ . So $f_{S_n}(t) = \lambda e^{-\lambda t} (\lambda t)^{n-1} / (n-1)!^{-1}, t \geq 0$.

$S_n \leq t = N(t) \geq n$. The formula for probability density can also be obtained using this fact.

$$P(T_1 < s | N(t) = 1) = s/t.$$

Two Poisson Processes

Suppose a PP with rate λ is split into two types, $N_1(t)$ with probability p or $N_2(t)$ with probability $1-p$. Where $N(t) = N_1(t) + N_2(t)$.

$N_1(t)$ has a rate of λp . $N_2(t)$ has a rate of $\lambda(1-p)$.

Let S_n^1, S_m^2 denote the time of the n th/ m th event of $N_1(t)$ and $N_2(t)$ respectively.

$$P(S_1^1 < S_1^2) = \lambda_1 / (\lambda_1 + \lambda_2)^{-1}.$$

Each event that occurs is going to be $N_1(t)$ with probability $\lambda_1 / (\lambda_1 + \lambda_2)^{-1}$, and $N_2(t)$ with probability $\lambda_2 / (\lambda_1 + \lambda_2)^{-1}$. So $P(S_n^1 < S_m^2)$ is based on the Binomial Distribution with $p = \lambda_1 / (\lambda_1 + \lambda_2)^{-1}$.

n Poisson Processes

Defined n processes similarly to two processes.

$$\sum_{i=1}^n P_i(y) = 1.$$

$$E(N_i(t)) = \lambda \int_0^t P_i(s) ds.$$

Non-homogeneous Poisson Process (NHPP)

Every event that occurs has a $p(t)$ chance of being recorded. Let $N_c(t)$ be the NHPP, then it has a rate of $\lambda(t) = \lambda p(t)$.

A NHPP with $\lambda(t) = \lambda$ is a regular PP.

Let $m(t)$ be the mean value function $= E(N_c(t)) = \lambda \int_0^t p(s) ds$.

The increments of a NHPP are Independent, but not necessarily stationary.

$\forall s, t, 0 \leq s < t, N(t) - N(s)$ has a Poisson Distribution with mean $m(t) - m(s) = \int_s^t \lambda(x) dx$.

Compound Poisson Process (CPP)

$X(t)$ is a CPP if $X(t) = \sum_{i=1}^{N(t)} Y_i$, where $N(t)$ is a PP, and $Y_i, i \geq 1$ is some i.i.d.r.v.'s that are also independent of $N(t)$.

$$E(X(t)) = \lambda t E(Y_1).$$

$$\text{Var}(X(t)) = \lambda t E(Y_1^2) \text{ (Derived from the Law of Total Variance) (?)}$$

$$M_X(s) = \exp(\lambda t (M_Y(s) - 1)) \text{ (Derived from the Law of Total Probability) (?)}$$

If $Y_i = 1$ then $X(t)$ is a normal PP.

TODO:

$$f(y_1, y_2 \dots y_n) = f(s_1, s_2 \dots s_n | n) = n! / t^n \text{ (lec 15)}.$$

Renewal Process (RP) DRAFT

Let $N(t)$ be a CP. Let X_n be the time between any two consecutive events.

If X_n are i.i.d.r.v.'s, then $N(t)$ is a **Renewal Process**.

$m(t) = E[N(t)]$ is the **Renewal/Mean-value Function**.

Let $F(x) = P(X_n < x)$.

The PP is an example of a RP with $F(x) = 1 - e^{-\lambda x}, x \geq 0$ and $m(t) = \lambda t$.

$m(t) = F(t) + \int_0^t m(t-x)f(x)dx$ is the **Renewal Equation**.

Let $\{X_n\}$ be some sequence of r.v.'s. This stochastic process is **Martingale** $\iff \forall n$,

1. $E(|X_n|) < \infty$.
2. $E(X_{n+1} | X_u, 0 \leq u \leq n) = X_n$.

Finance

Market Model DRAFT

Let time $t \geq 0, t \in \mathbb{Q}$. Let $t = n\tau$ where $n \in \mathbb{N} \cup \{0\}$ (Discrete Time).

Let $m \in \mathbb{N}, \forall i \in \mathbb{N}, 1 \leq i \leq m$: $S_i(n)$ is the price of the i th **Risky Asset** at time n , with a total of m risky assets.

$A(n)$ is the price of the **Risk-Free Asset** at time n .

Unless stated otherwise, $A(0)$ is used as a reference, with $A(0) = 1$ or 100.

A **Portfolio** is a vector $\vec{v}(n) = \langle x_1(n), \dots, x_m(n), y(n) \rangle$ representing the amount of assets held between time $n - 1$ and time n , where $\vec{v}(0)$ is undefined. Let the **Value** of the Portfolio at time n being

$$V(n) = \begin{cases} y(1)A(0) + \sum_{j=1}^m x_j(1)S_j(0) & \text{for } n = 0 \text{ (Initial Value/Wealth)} \\ y(n)A(n) + \sum_{j=1}^m x_j(n)S_j(n) & \text{for } n > 0 \end{cases}$$

Rate of Return TODO**** K_i = return of S_i is $K_F = \frac{F(1)-F(0)}{F(0)}$.

Six Assumptions:

1. $\forall n > 0, S(0), A(0), A(n)$ are known, while $S(n)$ is a random variable (Randomness).
2. $\forall n, S(n) > 0 \wedge A(n) > 0$ (Positivity of Prices).
3. $\forall i, \forall n, x_i(n) \in \mathbb{R}, y(n) \in \mathbb{R}$ (Divisibility, Liquidity and Short Selling).
4. $\forall n, V(n) \geq 0$ (Solvency).
5. $\forall n, S(n)$ can only have a finite number of values (Discrete Unit Prices).
6. $\forall n > 0, V(0) = 0 \rightarrow V(n) = 0$ (No-Arbitrage Principle).

An Investment Strategy is **Self-Financing** $\iff \forall n \geq 1, V(n) = y(n+1)A(n) + \sum_{j=1}^m x_j(n+1)S_j(n)$

An Investment Strategy is **Predictable** $\iff \forall n$, the Portfolio $\vec{v}(n+1)$ constructed at time n only depends on previous Portfolios.

An Investment Strategy is **Admissible** \iff it is Predictable, Self-Financing and Assumption 6 (?).

TODO (Fundamental Theorem of Asset Pricing, Page 83) (Extended Model with Derivative securities, very similar)

A **Forward Contract** is an agreement to exchange a Risky Asset at a future **Delivery Date** for a **Forward Price** F . To sell is to take a **Short Forward Position**. To buy is to take a **Long Forward Position**.

A **Call Option** is a contract that gives the holder a right/option to purchase a Risky Asset at a future **Exercise Time** for a **Strike/Exercise Price**. Let $C(t)$ denote the price of the option at time t .

The prices for Options and Forward Contracts are determined by the No-Arbitrage Principle.

Options are a derivative asset.

Risk-Free Asset DRAFT

Let $V(s) = P$ be the **Principle** investment at time s .

Let $K(s, t)$ denote the return on an investment from time s to time t , where $K(s, t) = \frac{V(t) - V(s)}{V(s)}$.

The **Growth Factor** is $V(t)/V(s)$, while **Discount Factor** is the multiplicative inverse.

Simple Interest

Let r be the annual interest for exactly 365 days.

For **Simple Interest**: $V(t) = (1 + (t - s)r)V(s)$, $K(s, t) = (t - s)r$.

Periodic Compound Interest

Let r be the annual interest for exactly 365 days.

For **Compound Interest**: $V(t) = \left(1 + \frac{r}{m}\right)^{(t-s)m} V(s)$, $K(s, t) = \left(1 + \frac{r}{m}\right)^{(t-s)m} - 1$, where m is the number of interest payments per annum/year.

A **Perpetuity** is a constant infinite sequence of payments made at equal time intervals.

A **Annuity** is a constant finite sequence of payments made at equal time intervals.

A **Amortised Loan** is a Annuity from the point of view of the borrower.

Let C be the constant.

Let $PA(r, n) = \sum_{i=1}^n (1 + r)^{-i} = \left(\frac{1}{r}\right) - \left(\frac{1}{r} \frac{1}{(1+r)^n}\right) = \frac{1 - (1+r)^{-n}}{r}$, where $PA(r, n)$ returns the **Present Value Factor** for an Annuity.

$C \times PA(r, n)$ is the present value of an Annuity that produces n annual payments of C .

$\lim_{n \rightarrow \infty} C \times PA(r, n) = \frac{C}{r}$ is the present value of a Perpetuity that produces annual payments of C .

The above formulas are adjusted accordingly for $m \neq 1$.

Continuous Compound Interest

Let r be the interest rate for the continuous compounding method.

The **Effective Rate** $r_e = e^r$. And r_e is defined similarly for periodic compounding as well.

For **Continuous Compounding**: $V(t) = e^{(t-s)r} V(s)$.

Define $k(s, t)$ to be the **Logarithmic Return**, where $k(s, t) = \ln\left(\frac{V(t)}{V(s)}\right)$.

For $a \leq b \leq c$, $k(a, c) = k(a, b) + k(b, c)$ (Not true for $K(a, c)$).

Coupon Bond

A **Zero-Coupon Bond** promises one payment for a **Face Value** F on a **Maturity Date** T .

$B(t, T)F$ denotes the price of the Zero-Coupon Bond at time t , where $B(t, T) = e^{-r(T-t)}$ depending on the compounding method.

A **Coupon Bond** promises a sequence of payments. The value of a Coupon Bond at time t can be calculated by the values and times of the payments discounted by a constant interest rate.

The values for the payments are of the form $\{C, C, \dots, C, C + F\}$.

Risky Asset TODO

TODO, Binomial Tree, u, d .

Let r be the one-step return for a risk-free investment, where $d < r < u$.

Define the **Risk-Neutral Probability** $p_* = \frac{r-d}{u-d}$, where $0 < p_* < 1$ and $p_*(u-r) + (1-p_*)(d-r) = 0$.

$$E_*(S(t+1)|S(t)) = S(t)(1+r)$$

TODO r_*, q_* .

TODO Continuous-Time Limit

$$\ln(1+u) = m\tau + \sigma\sqrt{\tau}, \ln(1+d) = m\tau - \sigma\sqrt{\tau},$$

Portfolio Management TODO

Let K be the return on a risky investment, where K is a random variable.

Let $\sigma_K^2 = \text{Var}(K)$.

σ_K and σ_K^2 are both used to measure **Risk**.

Risk-Free assets have no Risk, therefore is ignored for the following ($\sigma = 0$ messes up Pearson's Correlation Coefficient):

Weight $w_i = \frac{x_i S_i(0)}{V(0)}$, where $\sum w_i = 1$ and $K_V = \sum w_i K_i$.

$E(K_V)$ and $\text{Var}(K_V)$ can be calculated using the properties of Var and E .

TODO Logarithmic return for Chapter 3

TODO Page 101

Forward and Futures Contracts TODO

Options TODO

Financial Engineering TODO

Notations and Tables TODO

$x \in [a, b]$ means $a \leq x \leq b$.

$x \in (a, b)$ means $a < x < b$.

$\{a \text{ such that } a = b\} = \{a | a = b\} = \{a : a = b\}$.

n choose $k = \binom{n}{k} = \frac{n!}{k!(n-k)!} = C_n^k = {}_nC_k = {}^nC_k = C(n, k)$.

For Trigonometric Functions, $\sin^2(x) = (\sin(x))^2$, not $\sin(\sin(x))$.

Unless the exponent is -1 in which case, $y = \sin^{-1}(x) = \arcsin(x)$, is defined $x = \sin(y)$.

i.i.d.r.v.'s means "Independent Identically Distributed Random Variables".