CS440: Intro to Artificial Intelligence

Homework 1

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1. Let the domain of x be {-5, -3, -1, 1, 3}. Express using negations, disjunctions, and conjunctions, but without quantifiers, the proposition ∃xP(x).

   Answer:

$$P(-5) \lor P(-3) \lor P(-1) \lor P(1) \lor P(3)$$

2. Negate the statements below so that negation symbols immediately precedes predicates:

$$\exists x \forall y (P(x, y) \Rightarrow Q(x, y))) \Rightarrow \forall x \forall y \exists z (P(x, y) \Rightarrow P(y, z)).$$

   Answer:

$$\forall x \exists y (P(x, y) \land \neg Q(x, y)) \land \neg(\exists x \exists y \forall z P(x, y) \land \neg P(y, z))$$

3. Following the greatest common divisor algorithm, find $gcd(123, 46)$. Show your work.

   Answer:
   a) Neither 123 or 46 equal 0, so to solve $gcd(123, 46)$ we must compute the following steps:

$$123 mod 46 = 31$$

   b) $gcd(46, 31)$:

$$46 mod 31 = 15$$

   c) $gcd(31, 15)$:

$$31 mod 15 = 1$$

   Therefore, $gcd(123, 46) = 1$

4. Prove via induction that for an undirected simple graph with n vertices, there can be at most $|E| = \frac{n(n-1)}{2}$

   Answer:
   Base Case: This holds true for an undirected graph with $n = 0$ vertices, where we have $\frac{1(1-1)}{2} = 0$ edges.
   Inductive Step: Assume this holds true for a graph with up to n vertices, which would have a maximum of $\frac{(n)(n-1)}{2} = \frac{n^2-n}{n}$ edges. When we add the $(n+1)^{th}$ vertex, we need to connect it to our previous n vertices, requiring an additional n edges. This gives us:

$$\frac{(n)(n-1)}{2} + n$$

$$= \frac{(n+1)(n+1-1)}{2}$$

   This concludes our inductive step, proving our inductive hypothesis.

5. Recall that the 8-puzzle problem has eight movable pieces and one empty swap space on a 3×3 game board. What is the size of the state space if 3 of the pieces are labeled and 5 are unlabeled? That is, the game pieces can be thought of as being labeled $1, 2, 3, *, *, *, *, *$. You only need to provide the formula; there is no need to compute the final number.

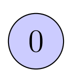
   Answer:
   Since we have four distinct tiles: the 3 labelled tiles and the one empty tile, we end up with a final formula of:

$$\frac{9!}{5!}$$

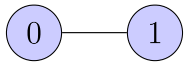
6. List all **complete graphs** and **complete bipartite graphs** that are planar. For each graph you list, provide a drawing showing that it is planar

   Answer:

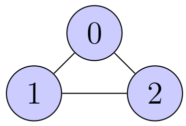
   - *k = 1*



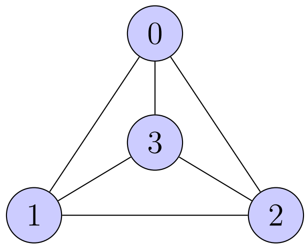
   - *k = 2*



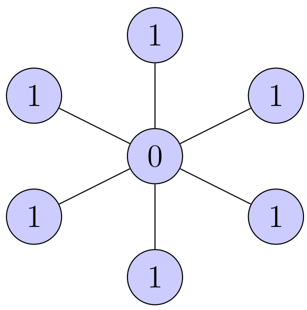
- *k = 3*



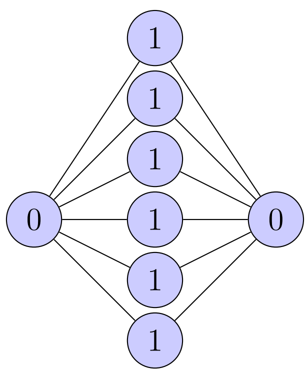
- *k = 3*



- $K_{1,n}$ $\rightarrow$ *As this is a star graph, it will always be planar if it is complete, eg;*



- $K_{2,n}$ $\rightarrow$ *Additionally, all complete bipartite* $K_{2,n}$ *graphs are always planar, since you can arrange the nodes of group* 0 *on either sides of group* 1 *nodes, eg;*



7. Give the smallest complete graph that is non-planar. Prove your claim. *[Hint: argue that edge crossings cannot be avoided somehow.]*

    Answer: The smallest complete graph that is non-planar is K_5. We can show this using Euler's formula:

    $$|V| - |E| + |F| = 2$$

    Where $|V|$ and $|E|$ are the number of vertices and edges in some graph $G = (V, E)$, and $|F|$ is the set of faces on that graph. From this theorem we can derive the lemma that If $G = (V, E)$ is a connected planar graph and $|V| > 2$, then $|E| \leq 3|V| - 6$.

    From this lemma we can see that K_5 is not planar, as $10 \nleq 3 \times 5 - 6$. In other words, with 5 vertices and 10 edges, there is no way of connecting all the vertices without a crossing.

8. Consider a bar consisting of n numbered squares. You are to break the bar into smaller ones, each of which must contain one or more complete numbered squares.

    (1) How many different bars can be obtained, including the original bar?

    Answer:
    Assuming you can't recombine separate pieces to form new bars (ie; you can't form a new bar out of square 1 and square 3), there are

    $$\sum_{k=1}^{n} k$$

    ways of dividing up the bar. This makes sense as there will be 1 bar with n pieces, 2 bars with $n-1$ pieces, 3 bars with $n-2$ pieces ... and n bars with 1 piece.

    (2) How many possible ways are there for doing the division? Extending the bar to be an $n \times m$ bar formed by nm uniquely numbered squares.
    Answer:
    There are:

    $$\sum_{k=1}^{n} 2(k-1)$$

    ways of doing the division as there are a total of $n-1$ slots to cut from. For example, a bar with $n = 4$ will have 6 possible ways it can be divided into single sized pieces (note: this answer assumes that getting a square from the middle of the larger bar will require two divisions).

(3) We are to obtain smaller rectangular bars consisting of adjacent squares. How many different bars can be obtained, including the original bar? *[Hint: for the first question, think about one different bar at a time and how a unique bar may be obtained.]*

Answer:
A 1×1 bar will have one possible rectangle that can be formed of it.
A 2×1 bar will have three possible rectangles that can be formed of it.
A 3×1 bar will have six possible rectangles that can be formed of it.

We can thus see that every individual column of the bar will have $\frac{n(n+1)}{2}$ possible bars that can be formed of it. We can generalize this to bar of size $m \times n$ we can determine the number of different possible bars using the following formula:

$$bars = \frac{m(m+1)n(n+1)}{4}$$