Supervised Machine Learning

Week 1: Introduction to Machine Learning

Learn from data `labeled` with `right answers`

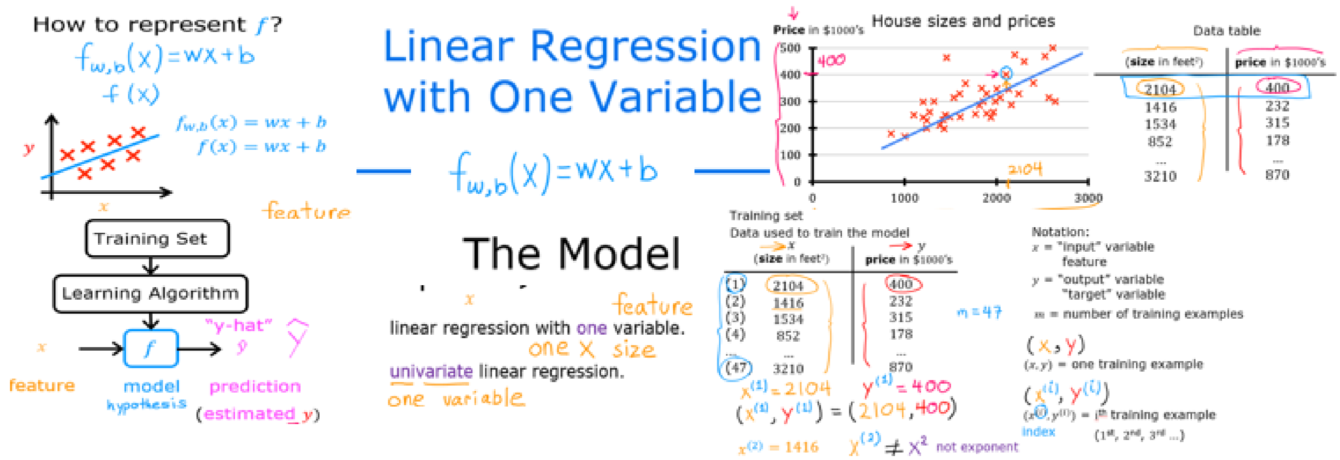
- **Regression**
  - Predict a `number`
  - `infinitely` many possible outputs
  - **Regression** model predicts **numbers**
- **Classification**
  - Predict `categories`
  - `small numbers` of possible outputs
  - **Classification** model predicts **categories**

  Training set - Data used to train the model
- Notation:
  - `x` = "input" variable/features
  - `y` = "output" variable/"target" variable
  - `m` = Number of training examples
  - `(x, y)` = one training example
  - $(\mathbf{x}^{(i)}, \mathbf{y}^{(i)})$ = $\mathbf{i}^{\mathbf{th}}$ training example
  - $\mathbf{x}_j^{(i)}$ = value of feature `j` in $\mathbf{i}^{\mathbf{th}}$ training example

Model:

$$y = f_{w,b}(x) = wx + b$$

- `w`, `b`: parameters of the model: weight/coefficient



Cost function: Squared error cost function

- **Cost function**: Measure the accuracy of our hypothesis function
  - $$J(w,b) = \frac{1}{2m}\sum_{i=0}^{m}(f_{w,b}(x^{(i)}) - y^{(i)})^2 \quad (1)$$
    - $$\hat{y}^{(i)} = f_{w,b}(x^{(i)}) = wx^{(i)} + b \quad (2)$$
    - m: number of training examples
    - reason for `1/2m` : make later calculations look `neater`
  - **Goal**: minimize `J(w,b)`
    - Find w, b that minimize `J(w,b)`
- **Cost function intuition:**
  - **Simplified hypothesis**:

$$f_{w,b}(x) = wx$$

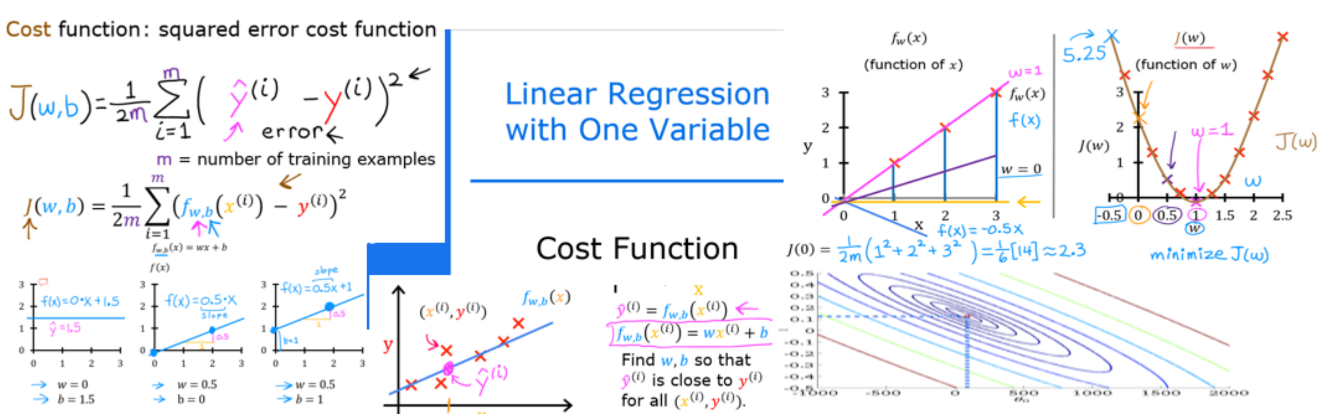
    - parameters: `w`
    - cost function:

$$J(w) = \frac{1}{2m}\sum_{i=0}^{m}(wx^{(i)} - y^{(i)})^2 \quad (3)$$

  - **Goal**: minimize `J(w)`
    - Find `w` that minimize `J(w)`

Visualizing cost function

- **Contour plot**
    - x-axis: `w`
    - y-axis: `J(w)`
    - z-axis: `J(w, b)`
    - **Goal**: find the lowest point in the plot



Gradient descent

1. Have some function `J(w,b)`
2. Want `w, b` that minimize `J(w,b)`
3. Outline:
    1. Start with some `w, b`
    2. Keep changing `w, b` to reduce `J(w,b)` (J is not always bowl-shaped)
    3. until we hopefully end up at a minimum(may have more than one minimum)

Gradient descent algorithm

Repeat until convergence:{

$$w = w - \alpha \frac{\partial}{\partial w} J(w, b) \tag{4}$$

$$b = b - \alpha \frac{\partial}{\partial b} J(w, b) \tag{5}$$

}

- `α`: learning rate
  - `α` is too small: slow convergence
  - `α` is too large: may not decrease on every iteration; may not converge
- $$\frac{\partial}{\partial w}J(w,b)$$
  - Derivative of `J(w,b)` with respect to `w`

$$\frac{\partial}{\partial b}J(w,b)$$

  - Derivative of `J(w,b)` with respect to `b`

`w` and `b` should be updated simultaneously

Simultaneous update:

$$tempw = w - \alpha\frac{\partial}{\partial w}J(w,b) \tag{6}$$

$$tempb = b - \alpha\frac{\partial}{\partial b}J(w,b) \tag{7}$$

$$w = tempw$$

$$b = tempb$$

Gradient descent intuition

$$w = w - \alpha\frac{d}{dw}J(w) \tag{8}$$

- **Derivative**: slope of the tangent line
  - **Positive slope**: `w` is too large

- **Negative slope**: `w` is too small
- **Zero slope**: `w` is just right

Learning rate

$$w = w - \alpha \frac{d}{dw} J(w) \tag{8}$$

- **Learning rate**: `α`
  - **Too small**: slow convergence
  - **Too large**:
    - may not decrease on every iteration
    - fail to converge, diverge
    - Overshoot and never reach the minimum

Can reach local minimum with fixed learning rate

- Near the local minimum
  - Derivative becomes smaller
  - Update steps become smaller
- Can reach global minimum without decreasing the learning rate

Gradient descent for linear regression

Linear regression model:

$$y = f_{w,b}(x) = wx + b \tag{9}$$

Cost function:

$$J(w, b) = \frac{1}{2m} \sum_{i=0}^{m} (f_{w,b}(x^{(i)}) - y^{(i)})^2 \tag{10}$$

Gradient descent algorithm:
repeat until convergence:
{

$$w = w - \alpha \frac{d}{dw} J(w, b) \tag{11}$$

$$b = b - \alpha \frac{d}{db} J(w, b) \tag{12}$$

}

$$\begin{aligned}
\frac{d}{dw} J(w, b) &= \frac{d}{dw} \frac{1}{2m} \sum_{i=0}^{m} (f_{w,b}(x^{(i)}) - y^{(i)})^2 && (1) \\
&= \frac{d}{dw} \frac{1}{2m} \sum_{i=0}^{m} (wx^{(i)} + b - y^{(i)})^2 && (2) \\
&= \frac{1}{2m} \sum_{i=0}^{m} 2(wx^{(i)} + b - y^{(i)})x^{(i)} && (3) \\
&= \frac{1}{m} \sum_{i=0}^{m} (f_{w,b}(x^{(i)}) - y^{(i)})x^{(i)} && (13)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{db} J(w, b) &= \frac{d}{db} \frac{1}{2m} \sum_{i=0}^{m} (f_{w,b}(x^{(i)}) - y^{(i)})^2 && (4) \\
&= \frac{d}{db} \frac{1}{2m} \sum_{i=0}^{m} (wx^{(i)} + b - y^{(i)})^2 && (5) \\
&= \frac{1}{2m} \sum_{i=0}^{m} 2(wx^{(i)} + b - y^{(i)}) && (6) \\
&= \frac{1}{m} \sum_{i=0}^{m} (f_{w,b}(x^{(i)}) - y^{(i)}) && (14)
\end{aligned}$$

Gradient descent algorithm:
repeat until convergence:{

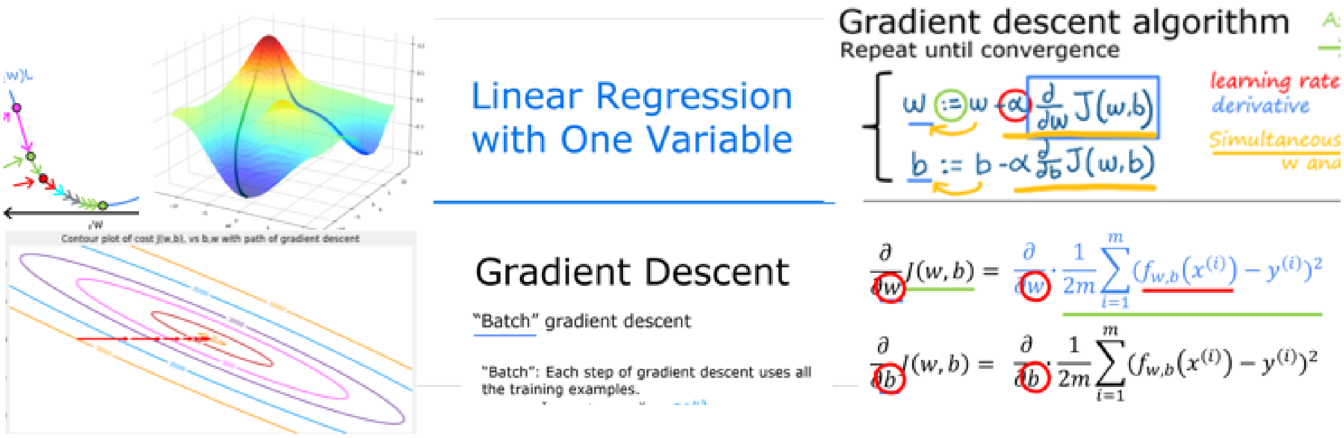
$$w = w - \alpha \frac{1}{m} \sum *i = 0^m (f * w, b(x^{(i)}) - y^{(i)})x^{(i)} \tag{15}$$

$$b = b - \alpha \frac{1}{m} \sum *i = 0^m (f * w, b(x^{(i)}) - y^{(i)}) \tag{16}$$

}

When implement gradient descent on a **convex function**, it is guaranteed to find the `global minimum`

- "Batch" gradient descent
  - Each step of gradient descent uses all the training examples



Week 2: Regression with multiple input variables

Multiple features

Model:

$$f_{w,b}(x) = w_1x_1 + w_2x_2 + ... + w_nx_n + b \tag{1}$$

Notation:

- `n` : number of features
- $$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

- `b` is a real number
- `w` and `b` are parameters of the model
- $$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Now we can rewrite the model as:

$$f_{\vec{w},b}(\vec{x}) = \vec{w} \cdot \vec{x} + b = w_1x_1 + w_2x_2 + \ldots + w_nx_n + b \tag{2}$$

Vectorization

- Without vectorization
  - $$f_{\vec{w},b}(\vec{x}) = \sum_{j=1}^{n} w_jx_j + b$$

  ```
  f = 0
  for j in range(n):
    f += w[j] * x[j]
  f += b
  ```

- With vectorization
  - $$f_{\vec{w},b}(\vec{x}) = \vec{w} \cdot \vec{x} + b$$

  ```
  f = np.dot(w, x) + b
  ```

1. Make code shorter
2. Run faster because of parallelization

Gradient descent for multiple variables

Cost function:

$$J(\vec{w}, b) = \frac{1}{2m} \sum_{i=1}^{m} (f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)})^2 \tag{3}$$

Gradient descent algorithm:
repeat until convergence:{

$$w_j = w_j - \alpha \frac{1}{m} \sum_{i=1}^{m} (f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)}) x_j^{(i)} \tag{4}$$

$$b = b - \alpha \frac{1}{m} \sum_{i=1}^{m} (f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)}) \tag{5}$$

}
Simultaneous update: `w(for j = 1, ..., n)` and `b`

An alternative to gradient descent

- Normal equation:
  - Only for linear regression
  - Solve for w, b without iteration
  - Disadvantages:
    - Does not generalize to other learning algorithms
    - Slow when number of features is large(more than 10,000)

Feature scaling

Example:

1. $300 <= x_1 <= 2000$ then

$$\frac{x_1}{2000}$$

2. $0 <= x_2 <= 5$ then

$$\frac{x_2}{5}$$

- $$x_i = \frac{x_i}{max}$$
- Get every feature into approximately a $-1 \leq x_i \leq 1$

Mean normalization

- $$x_i = \frac{x_i - \mu_i}{max - min}$$
  - μ_i is the average of all the values for feature
    - $$\mu_i = \frac{1}{m}\sum_{j=0}^{m-1} x_j^{(i)}$$

Z-score normalization

- $$x_j^{(i)} = \frac{x_j^{(i)} - \mu_j}{\sigma_j}$$
  - σ_i is the standard deviation of all the values for feature
    - $$\sigma_i = \sqrt{\frac{1}{m}\sum_{i=1}^{m}(x_j^{(i)} - \mu_j)^2}$$

Choosing the learning rate

- Plot $J(\vec{w}, b)$ as a function of the number of iterations of gradient descent
  - If the cost function is not just decreasing, but jumping up and down or even increasing, the reason maybe:
    1. code has bugs
    2. learning rate is too large
       With a small *learning rate*, *cost function* should **decrease** on every iteration
    - `Debug` :**Set the learning rate to a very small value, if the algorithm is not working correctly, then try to fix the code**
      - Values of α to try:
        - ... 0.001 0.003 0.01 0.03 0.1 0.3 1 ...
        - For each value of α **plot** the cost function, pick the learning rate

that causes the cost function to decrease the **fastest** and **consistently**

- **find a value that is too small and a value that is too large, slowly try to pick the largest possible learning rate**

- If learning rate is too small, gradient descent can be slow to converge

Feature engineering

Using `intuition` to design `new features` by transforming or combining the original features

Example:

$$f_{\vec{w},b}(\vec{x}) = w_1x_1 + w_2x_2 + b$$

If x_1 is the **frontage** of a house, x_2 is the **depth** of a house, then $x_3 = x_1x_2$ is the **area** of a house

$$x_3 = x_1x_2$$

is a new feature

$$f_{\vec{w},b}(\vec{x}) = w_1x_1 + w_2x_2 + w_3x_3 + b$$

Polynomial regression

- Linear regression with **higher order** polynomials

Example:

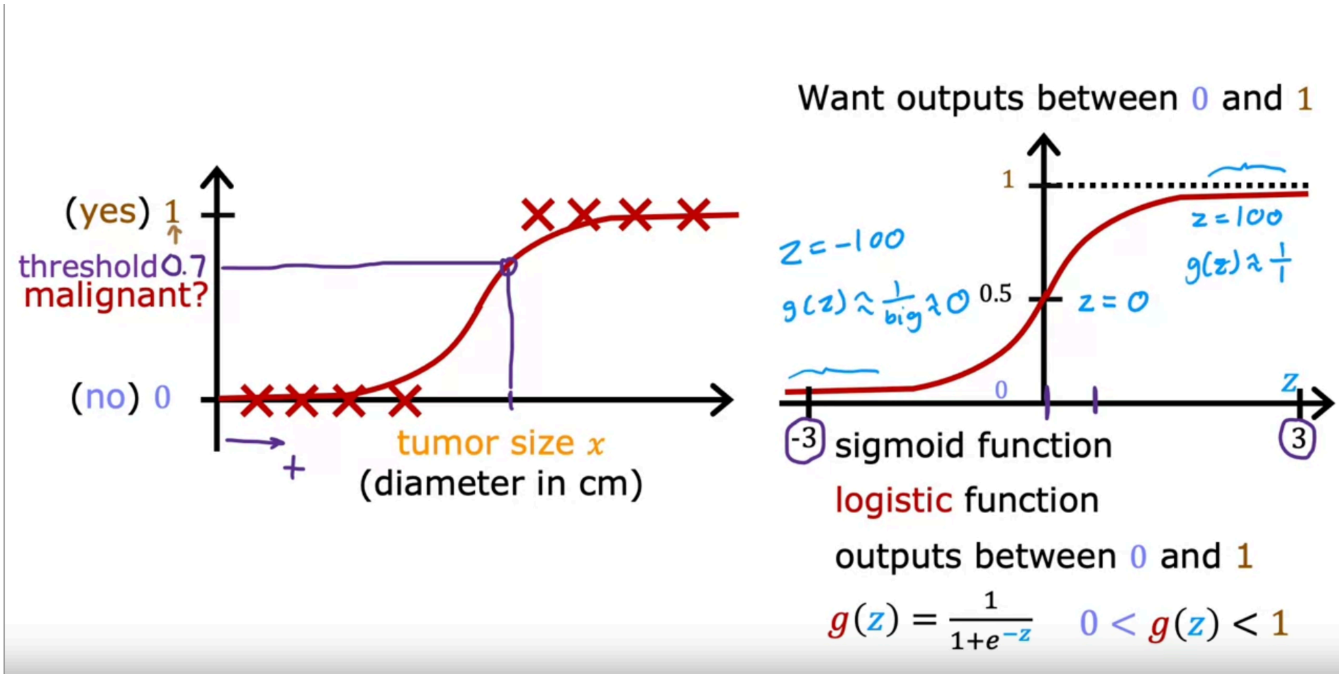
$$f_{\vec{w},b}(x) = w_0 + w_1x + w_2x^2 + w_3x^3 + b$$

Week 3: Classification

Classification with logistic regression

logistic regression

- Sigmoid funcion(`logistic function`)
    - output between 0 and 1
    - $$g(z) = \frac{1}{1+e^{-z}} (0 < g(z) < 1)$$



pass the value of linear regression into logistic regression

- $$z = \vec{w} \cdot \vec{x} + b$$
- $$f_{\vec{w},b}(\vec{x}) = g(z) = \frac{1}{1+e^{-(\vec{w}\cdot\vec{x}+b)}}$$

Due to **P(y=0) + P(y=1) = 1**

- $$f_{\vec{w},b}(\vec{x}) = P(y = 1|\vec{x}; \vec{w}, b)$$
    - Probability that y = 1, given x, parameterized by w, b

Decision boundary

$$z = \vec{w} \cdot \vec{x} + b = 0$$

- $$\vec{w} \cdot \vec{x} + b = 0$$

  is the **decision boundary**
  - We predict y = 1 if

    $$\vec{w} \cdot \vec{x} + b \geq 0$$

  - We predict y = 0 if

    $$\vec{w} \cdot \vec{x} + b < 0$$

Logistic regression cost function

- Cost function for logistic regression
  - $$J(\vec{w}, b) = \frac{1}{m} \sum_{i=1}^{m} \frac{1}{2} (f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)})^2$$
    - $$\frac{1}{2} (f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)})^2 = \mathcal{L}(f_{\vec{w},b}(\vec{x}^{(i)}), y^{(i)})$$
    - **L** is the **loss function**
  - $$f_{\vec{w},b}(\vec{x}) = \frac{1}{1 + e^{-(\vec{w} \cdot \vec{x} + b)}}$$

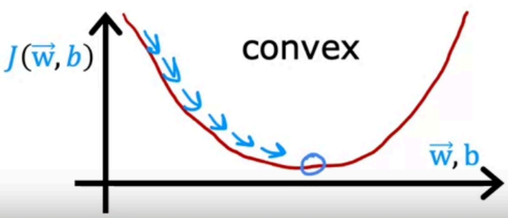
Squared error cost

$$\overset{\text{cost}}{J(\vec{w},b)} = \frac{1}{m}\sum_{i=1}^{m}\boxed{\frac{1}{2}\left(f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)}\right)^2}$$

loss $\boxed{L\left(f_{\vec{w},b}(\vec{x}^{(i)}), y^{(i)}\right)}$

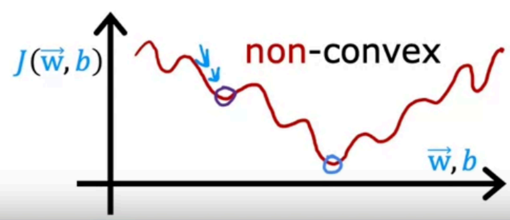
linear regression

$f_{\vec{w},b}(\vec{x}) = \vec{w}\cdot\vec{x} + b$



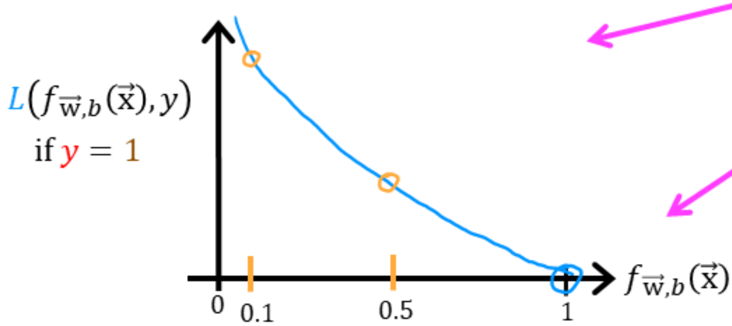
logistic regression

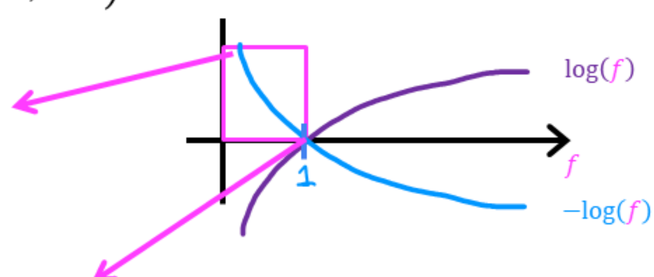
$$f_{\vec{w},b}(\vec{x}) = \frac{1}{1+e^{-(\vec{w}\cdot\vec{x}+b)}}$$



Logistic Loss Function

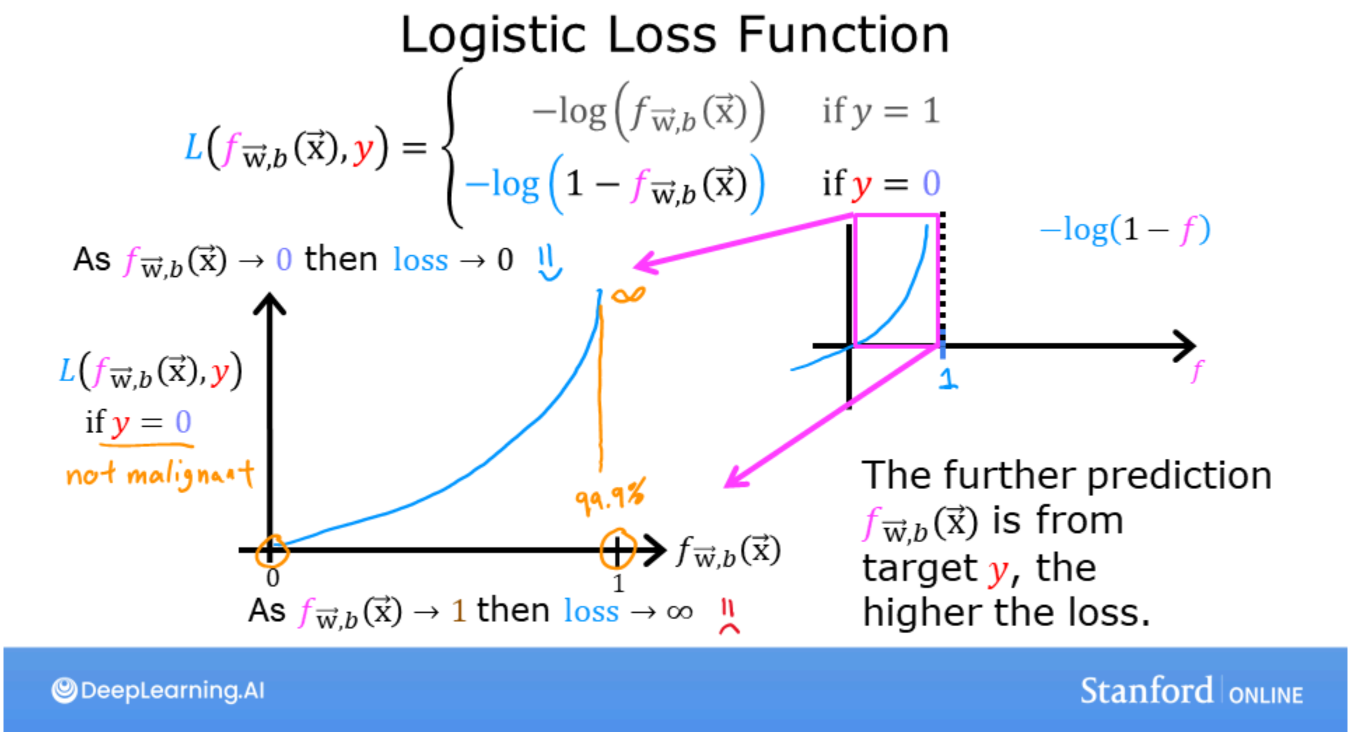
$$L\left(f_{\vec{w},b}(\vec{x}), y\right) = \begin{cases} -\log\left(f_{\vec{w},b}(\vec{x})\right) & \text{if } y = 1 \\ -\log\left(1 - f_{\vec{w},b}(\vec{x})\right) & \text{if } y = 0 \end{cases}$$





Loss is lowest when $f_{\vec{w},b}(\vec{x})$ predicts close to true label y.

As $f_{\vec{w},b}(\vec{x}) \to 1$ then loss $\to 0$



- Logistic loss function
  - $$\mathcal{L}(f_{\vec{w},b}(\vec{x}^{(i)}), y^{(i)}) = \begin{cases} -log(f_{\vec{w},b}(\vec{x}^{(i)}), & \text{if } y^{(i)} = 1 \\ -log(1 - f_{\vec{w},b}(\vec{x}^{(i)}), & \text{if } y^{(i)} = 0 \end{cases}$$

So the cost function is

- $$J(\vec{w}, b) = \frac{1}{m} \sum_{i=1}^{m} \mathcal{L}(f_{\vec{w},b}(\vec{x}^{(i)}), y^{(i)})$$
  - $$f_{\vec{w},b}(\vec{x}) = \frac{1}{1 + e^{-(\vec{w} \cdot \vec{x} + b)}}$$

Simplified cost function

- $$\mathcal{L}(f_{\mathbf{w},b}(\mathbf{x}^{(i)}), y^{(i)}) = (-y^{(i)} \log\left(f_{\mathbf{w},b}\left(\mathbf{x}^{(i)}\right)\right) - \left(1 - y^{(i)}\right) \log\left(1 - f_{\mathbf{w},b}\left(\mathbf{x}^{(i)}\right)\right)$$
  - when $y^{(i)} = 0$, the left-hand term is eliminated:

$$\begin{aligned} \mathcal{L}(f_{\mathbf{w},b}(\mathbf{x}^{(i)}), 0) &= (-(0) \log\left(f_{\mathbf{w},b}\left(\mathbf{x}^{(i)}\right)\right) - (1 - 0) \log\left(1 - f_{\mathbf{w},b}\left(\mathbf{x}^{(i)}\right)\right) \\ &= -\log\left(1 - f_{\mathbf{w},b}\left(\mathbf{x}^{(i)}\right)\right) \end{aligned} \tag{8}$$

- and when $y^{(i)} = 1$, the right-hand term is eliminated:

$$\mathcal{L}(f_{\mathbf{w},b}(\mathbf{x}^{(i)}), 1) = (-(1) \log\left(f_{\mathbf{w},b}\left(\mathbf{x}^{(i)}\right)\right) - (1-1) \log\left(1 - f_{\mathbf{w},b}\left(\mathbf{x}^{(i)}\right. \tag{9}$$

$$= -\log\left(f_{\mathbf{w},b}\left(\mathbf{x}^{(i)}\right)\right) \tag{10}$$

So the cost function can be simplified as

- $$J(\vec{w}, b) = -\frac{1}{m}\sum_{i=1}^{m}\left[y^{(i)} \log\left(f_{\vec{w},b}(\vec{x}^{(i)})\right) + (1-y^{(i)}) \log\left(1 - f_{\vec{w},b}(\vec{x}^{(i)})\right)\right]$$
  - $$f_{\vec{w},b}(\vec{x}) = \frac{1}{1+e^{-(\vec{w}\cdot\vec{x}+b)}}$$

The reason to choose this cost function is that it is **convex**.

Gradient descent for logistic regression

$$J(\vec{w}, b) = -\frac{1}{m}\sum_{i=1}^{m}\left[y^{(i)} \log\left(f_{\vec{w},b}(\vec{x}^{(i)})\right) + (1-y^{(i)}) \log\left(1 - f_{\vec{w},b}(\vec{x}^{(i)})\right)\right]$$

repeat {

$$w_j = w_j - \alpha\frac{\partial J(\vec{w}, b)}{\partial w_j}(\text{for j = 0..n-1})$$

$$b = b - \alpha\frac{\partial J(\vec{w}, b)}{\partial b}$$

} simultaneously update all w_j and b

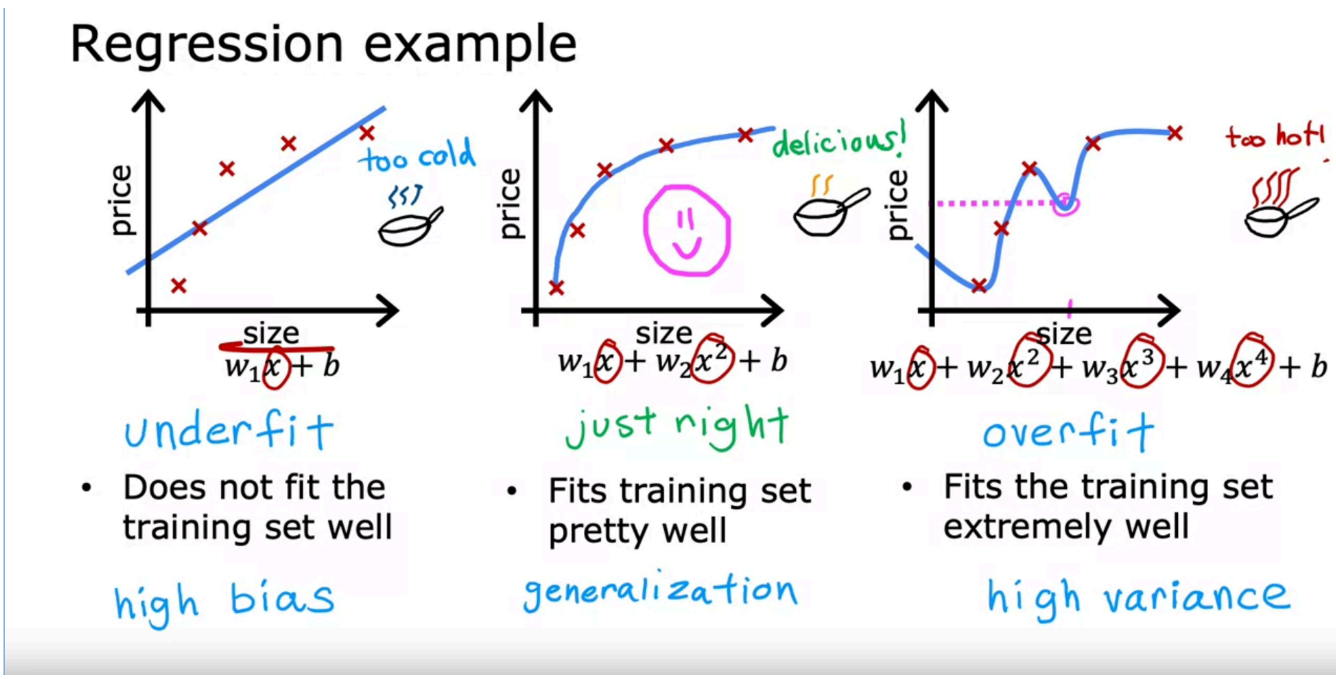
$$\frac{\partial J(\vec{w}, b)}{\partial w_j} = \frac{1}{m}\sum_{i=1}^{m}\left(f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)}\right) x_j^{(i)}$$

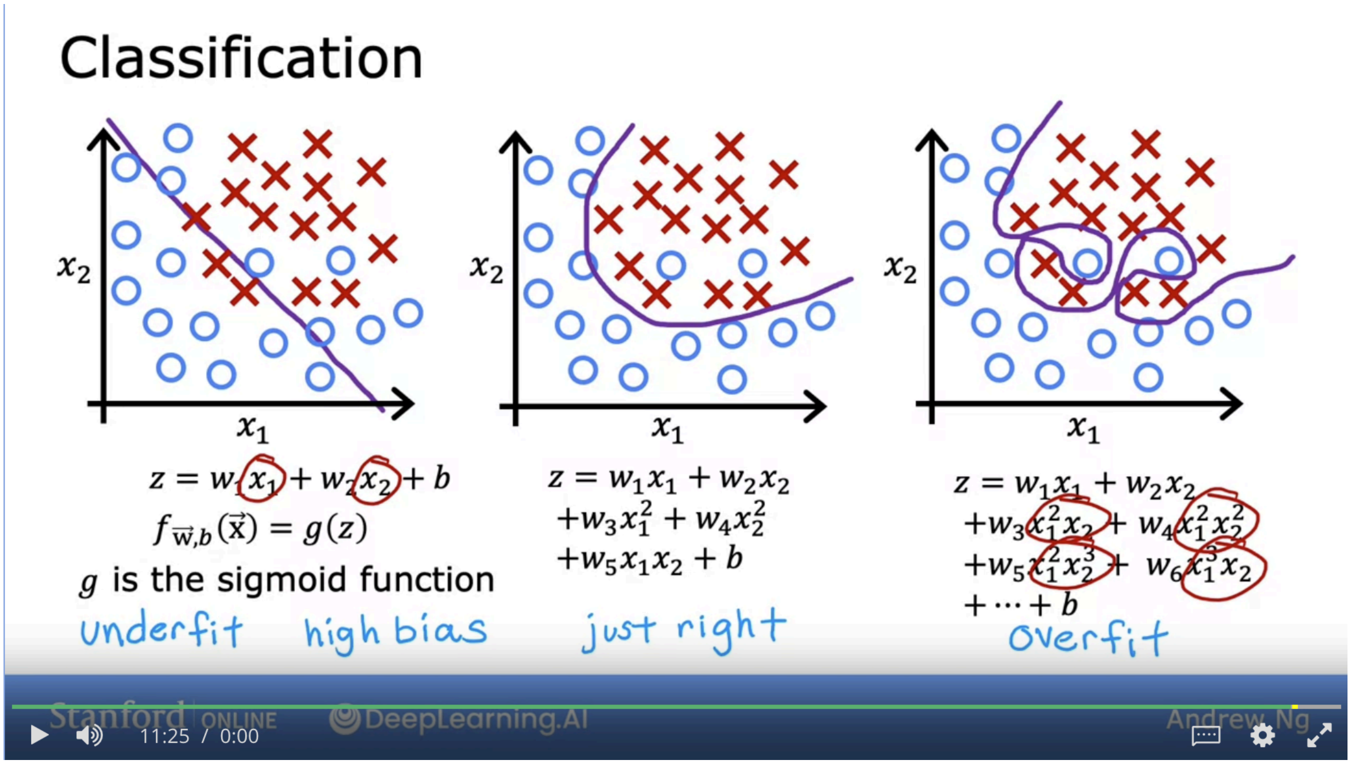
$$\frac{\partial J(\vec{w},b)}{\partial b} = \frac{1}{m}\sum_{i=1}^{m}\left(f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)}\right)$$

- Linear regression: $f_{\vec{w},b}(\vec{x}) = \vec{w}\cdot\vec{x} + b$
- logistic regression: $f_{\vec{w},b}(\vec{x}) = \frac{1}{1+e^{-(\vec{w}\cdot\vec{x}+b)}}$
- Same concepts:
  - Monitor gradient descent(learning curve) to make sure it is converging
  - Vectorized implementation
  - Feature scaling

Overfitting

- Overfitting
  - Underfitting: high bias
  - Overfitting: high variance
  - Just right: fitting well





- Addressing overfitting
    1. Collect more data
    2. Select fewer features
        1. Feature selection
    3. Reduce size of parameters
        1. Regularization

Cost function with regularization

- Regularization
    - Keep all the features, but reduce magnitude/values of parameters $\vec{w}$
    - Works well when we have a lot of features, each of which contributes a bit to predicting y
    - Reduces overfitting.
- $$J(\vec{w}, b) = \frac{1}{2m} \sum_{i=1}^{m} \left(f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)} \right)^2 + \frac{\lambda}{2m} \sum_{j=1}^{n} w_j^2$$
- We want to minimize $J(\vec{w}, b)$ with respect to $\vec{w}$ and b.
    - The first term is the same as the cost function of linear regression.

- The second term is the regularization term.
    - If λ is too large, it will smooth the function too much and cause underfitting.
    - If λ is too small, it will not smooth the function enough and cause overfitting.

Gradient descent for linear regression with regularization:

$$J(\vec{w}, b) = \frac{1}{2m} \sum_{i=1}^{m} \left(f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)} \right)^2 + \frac{\lambda}{2m} \sum_{j=1}^{n} w_j^2$$

Gradient descent for logistic regression with regularization:

$$J(\vec{w}, b) = -\frac{1}{m} \sum_{i=1}^{m} \left[y^{(i)} \log \left(f_{\vec{w},b}(\vec{x}^{(i)}) \right) + (1 - y^{(i)}) \log \left(1 - f_{\vec{w},b}(\vec{x}^{(i)}) \right) \right] + \frac{\lambda}{2m} \sum_{j=1}^{n} w_j^2$$

- **For logistic regression, $f(x)$ is sigmoid(logistic) functio, whereas for linear regression, $f(x)$ is linear function.**
    - Logistic regression: $f_{\vec{w},b}(\vec{x}) = \frac{1}{1+e^{-(\vec{w}\cdot\vec{x}+b)}}$
    - Linear regression: $f_{\vec{w},b}(\vec{x}) = \vec{w} \cdot \vec{x} + b$

repeat {

- $$w_j = w_j - \alpha \frac{\partial J(\vec{w}, b)}{\partial w_j}$$
    - $$\frac{\partial J(\vec{w}, b)}{\partial w_j} = \frac{1}{m} \sum_{i=1}^{m} \left(f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)} \right) x_j^{(i)} + \frac{\lambda}{m} w_j$$
- $$b = b - \alpha \frac{\partial J(\vec{w}, b)}{\partial b}$$
    - $$\frac{\partial J(\vec{w}, b)}{\partial b} = \frac{1}{m} \sum_{i=1}^{m} \left(f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)} \right)$$

} simultaneously update all w_j and b

Now, the **Gradient descent for logistic regression with regularization** is repeat {

$$w_j = w_j - \alpha \left[\frac{1}{m} \sum_{i=1}^{m} \left(f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)} \right) x_j^{(i)} + \frac{\lambda}{m} w_j \right]$$

$$b = b - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)} \right)$$

} simultaneously update all w_j and b

$$\begin{aligned} w_j &= w_j - \alpha \frac{\lambda}{m} w_j - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)} \right) x_j^{(i)} \\ &= \left(1 - \alpha \frac{\lambda}{m} \right) w_j - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)} \right) x_j^{(i)} \end{aligned}$$

So we can see that the regularization term is just a rescaling of the parameters.