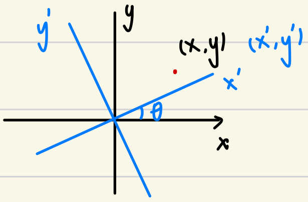
unitary image transform

In 2D, a rotation is like a change of basis of the coordinate system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



unitary transform:

$$v[k] = \sum_{n=0}^{N-1} a[k,n]\, u[n] \qquad 0 \le k \le N-1$$

output input

$\underset{n\times 1}{V} = \underset{n\times n}{A}\, \underset{n\times 1}{u}$

unitary: $A^{-1} = (A^T)^* = A^H$

Ex:

$$\underset{A}{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} \underset{A^T}{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

column of A are unit length and perpendicular to each other

In 1D, we also have the DFT basis:

$$F[k] = \sum_{n=0}^{N-1} f[n]\, e^{-j\frac{2\pi k}{N}n} \Rightarrow A(k,n) = e^{-j\frac{2\pi k}{N}n} \quad \text{fourier matrix}$$

$F = Af$ output signal, input sig

A: N*N fourier matrix complex exponentials

2D DFT: $\underset{M\times N \text{ output}}{F[u,v]} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f[x,y]\, e^{-j\left(\frac{2\pi ux}{M} + \frac{2\pi vy}{N}\right)}$ (M×N input image)

$= \underset{\substack{M\times M \\ \text{DFT Matrix}}}{A_M}\, f\, \underset{\substack{N\times N \\ \text{DFT matrix}}}{A_N}$

example of A 2D unitary transformation

spatial basis:

$$I(x,y) = I(1,1)B_{11} + I(1,2)B_{12} + \cdots$$

frequency basis:

$$I(x,y) = C_{11}\cdot DC + C_{12}\cdot F_{12} + \cdots$$

a unitary transform satisfies:

$$\sum_x \sum_y (f[x,y])^2 = \sum_u \sum_v |F[u,v]|^2$$ signal energy is preserved

2D DFT pro: 1. energy is usually packed into low frequency coefficients
2. convolution property
3. fast implementation

cons: 1. transform is complex even if image is real
2. basis functions span image height/ width

Discrete cosine transform

$$C(k,n) = \begin{cases} 1/\sqrt{N}, & k=0 \quad 0 \le n \le N-1 \\ \sqrt{2/N}\cos\frac{\pi(2n+1)k}{2N} & \text{otherwise} \quad k=1,\cdots N-1 \end{cases}$$

$\underset{\text{out}}{F} = \underset{M\times M}{C_M}\, \overset{\text{in}}{f}\, \underset{N\times N}{C_N}$ DCT matrix

C is real, $C^{-1} = C^T$ unitary trans

excellent energy compaction for natural image

fast transform

DCT is the critical part of the jpeg algorithm

input image -> - split in to 8*8 blocks
- take dct of each block
- quantize dct coefficients
- code the quantized coefficients

discrete sine transform:

$$S(k,n) = \sqrt{\tfrac{2}{N+1}}\sin\frac{\pi(k+1)(n+1)}{N+1} \qquad 0 \le k,n \le N-1$$

$S^{-1} = S^T$, real $\quad S = S^T$, symmetric

Hadamand Transform:
basis functions only contain ± 1
implement only requires addition/subtraction

wavelet transforms:
- unitary, - can represent both smooth and discontinuous images efficiency
- local basis functions
- computationally extremely efficient
O(N) operations vs O(NlogN)
- great compression
wavelet transform combines spatial and frequency domain