Ray Fulkerson: Awe-inspiring Pioneer of Network Flows, Optimization, and Combinatorial Analysis

Thomas L Magnanti, MIT

September 20, 2024

Cornell University

D.R. Fulkerson Centennial

With gratitude

Photos:

Bob Bland

Biosketches:

Bob Bland and Jim Orlin

Lou Billera and Bill Lucas

Vasek Chvåtal

Preparation of slides:

Yifu Ding

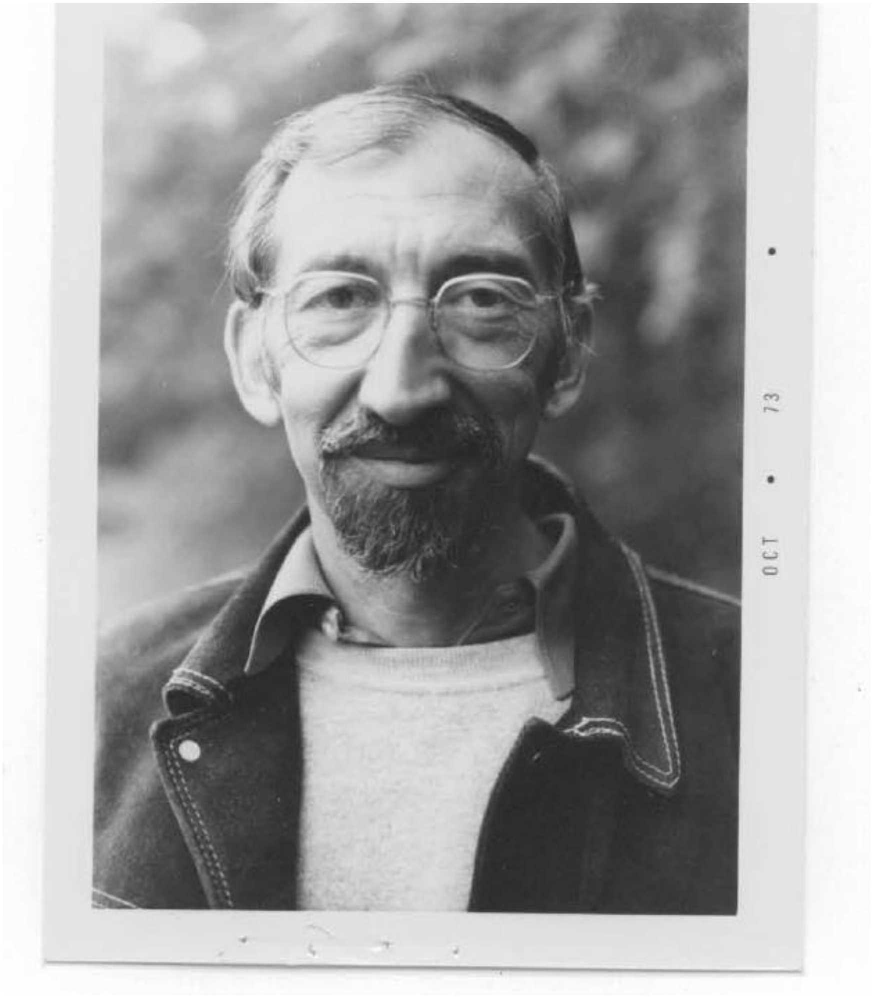
Agenda

- Personal and Career Lives
- Network Flows
  - Augmenting Flow Algorithm
  - Maximum Flows-Minimal Cuts
  - Machine/Project Scheduling
- Paths Between Two Nodes
  - Menger’s Theorem
  - Clutters and Blockers
- Primal Dual Algorithms
  - Dijkstra’s Algorithm for Shortest Paths
- Column Generation
- Traveling Salesman Problem and Integer Programming

Ray Fulkerson

- Born: August 14, 1924, in Tamms, Illinois.
- Died: January 10, 1976, in Ithaca, New York.
- Education: BA in Mathematics. Southern Illinois University, 1947
M.S. in Mathematics. University of Wisconsin, 1948
Ph.D. in Mathematics. University of Wisconsin, 1951.
- Key Positions: Rand Corporation, Mathematics Department. 1951-1971
Cornell University, Dept. of Operations Research. 1971-1976
Maxwell M. Upson Professor of Engineering and Professor of Operations Research and Applied Mathematics.

- Visiting Professor: U.C. Berkeley, Stanford University, University of Waterloo.



Ray Fulkerson, '47, and family.

Glen Fulkerson, '38, and family.

ounselor and advertising ac-
ount executive.

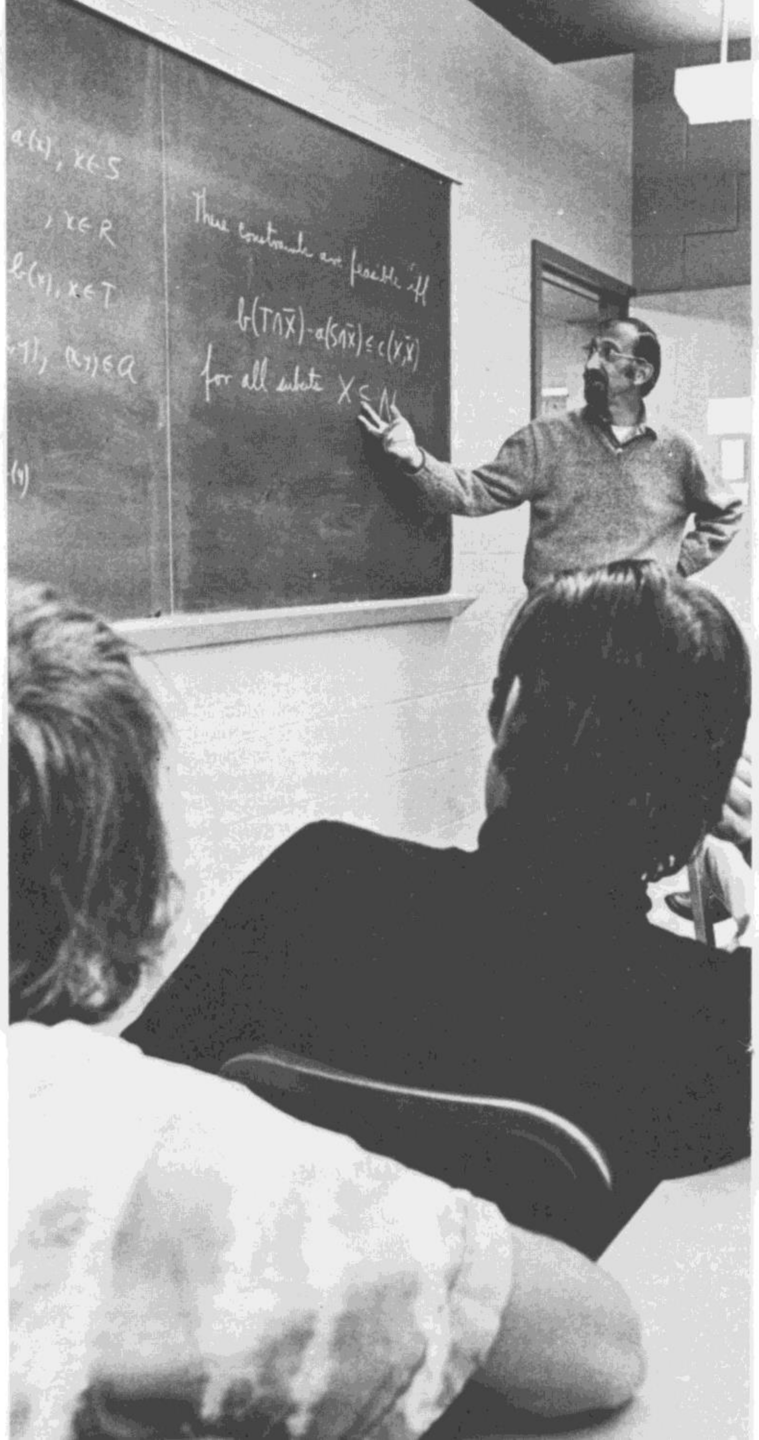
Merle Fulkerson Guthrie, old-
st of the girls, entered South-
rn in '36 and graduated in '40.
n '43 she received a master's
legree from the University of
llinois. At Southern she was
member of the Obelisk and

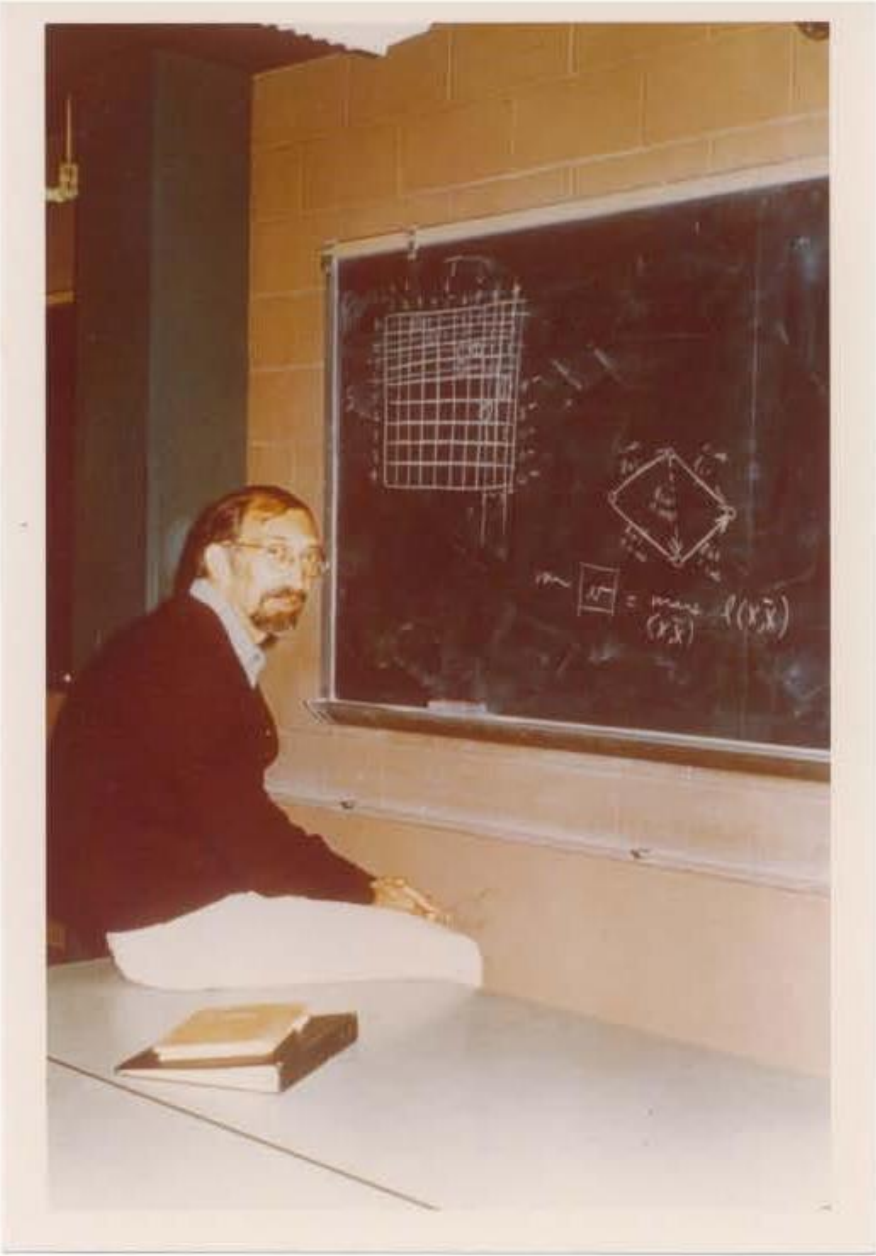
from SIU this August. In '49
she taught at Hurst-Bush and in
'51 at Grayville. Her husband,
Harold Todd, received a B. S.
from Southern in '48, a B. A.
in '49 and an M. A. in '50. He
also is teaching at Athens.

Grace Fulkerson Weshinskey,
'52, was graduated from SIU

ternational Shoe company, Belle
ville.

Father of this alumni family
Elbert Fulkerson, has taught i
Southern Illinois since 1913
From '27 to '44, he was princi
pal of the Carterville Communi
ty high school. Since '44 he ha
been a member of the SIU fac





PhD Students

David Weinberger

Robert Bland

Staff

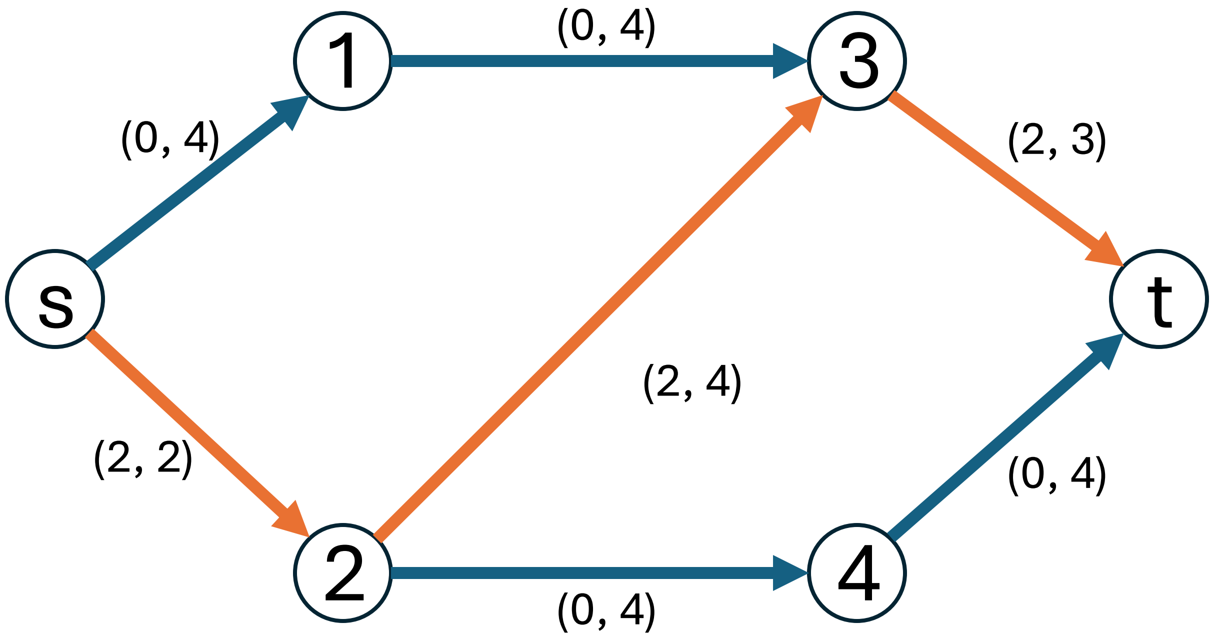
- Richard Bellman
- George Dantzig
- Merrill Flood
- Lester Ford
- Selmer Johnson
- Lloyd Shapley
- Philip Wolfe

Visitors

- Jack Edmonds
- Jon Folkman
- Herb Ryser
- John von Neumann
- ...

Maximum Flow

From Source s to Terminal t



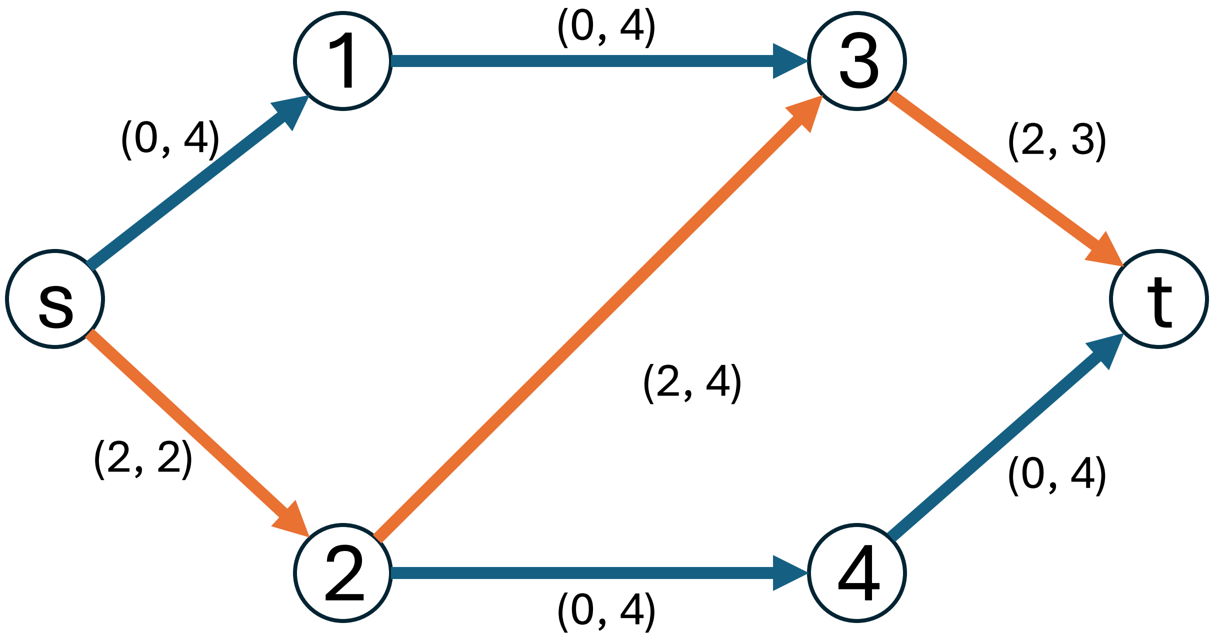
Sending 2 units (paths) from s to t

$$(x_{ij}, u_{ij})$$

Current flow $\leq$ Max flow capacity

Maximum Flow

From Source s to Terminal t



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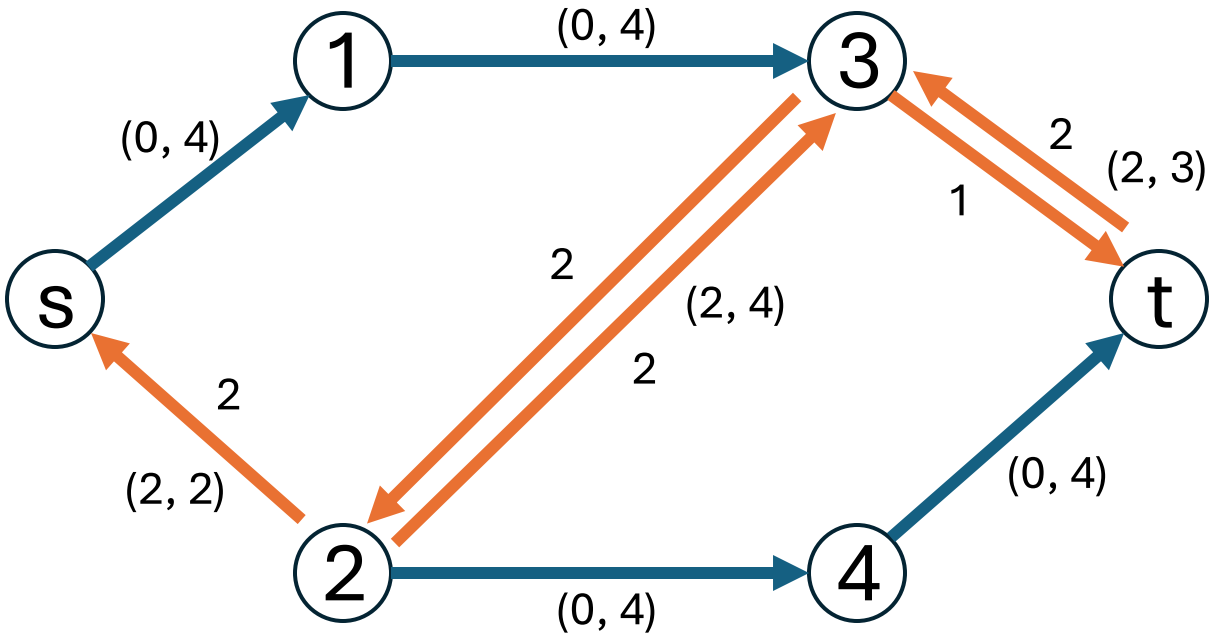
Maximum Flow

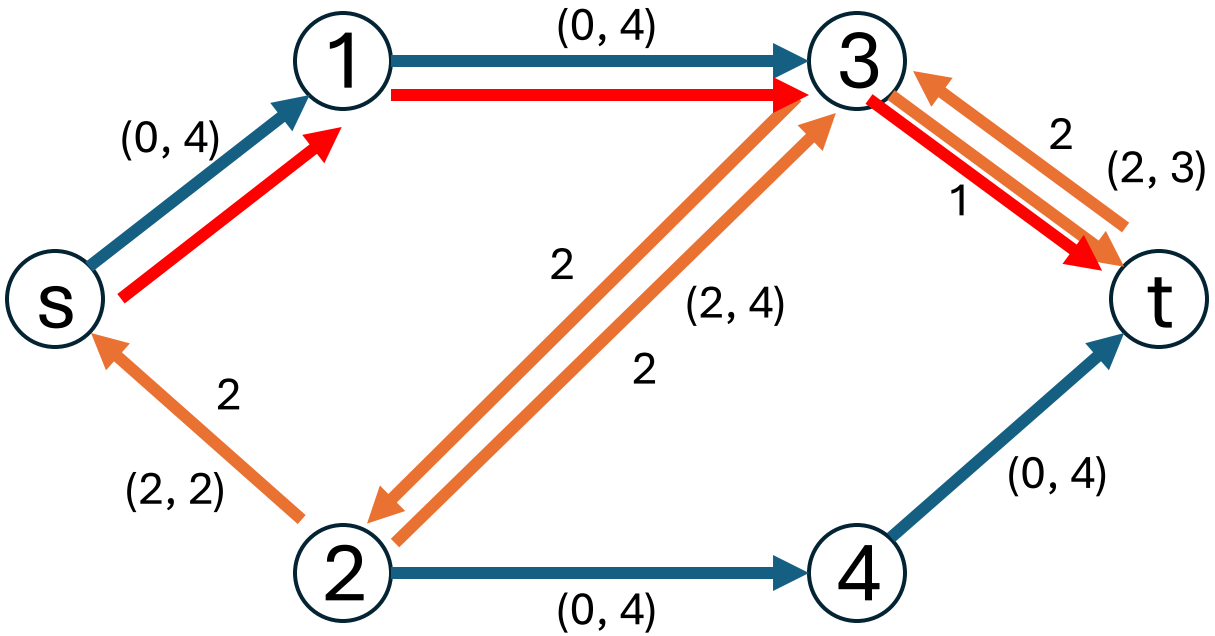
From Source s to Terminal t



Residual capacity network: can send back units that are flowing

Maximum Flow

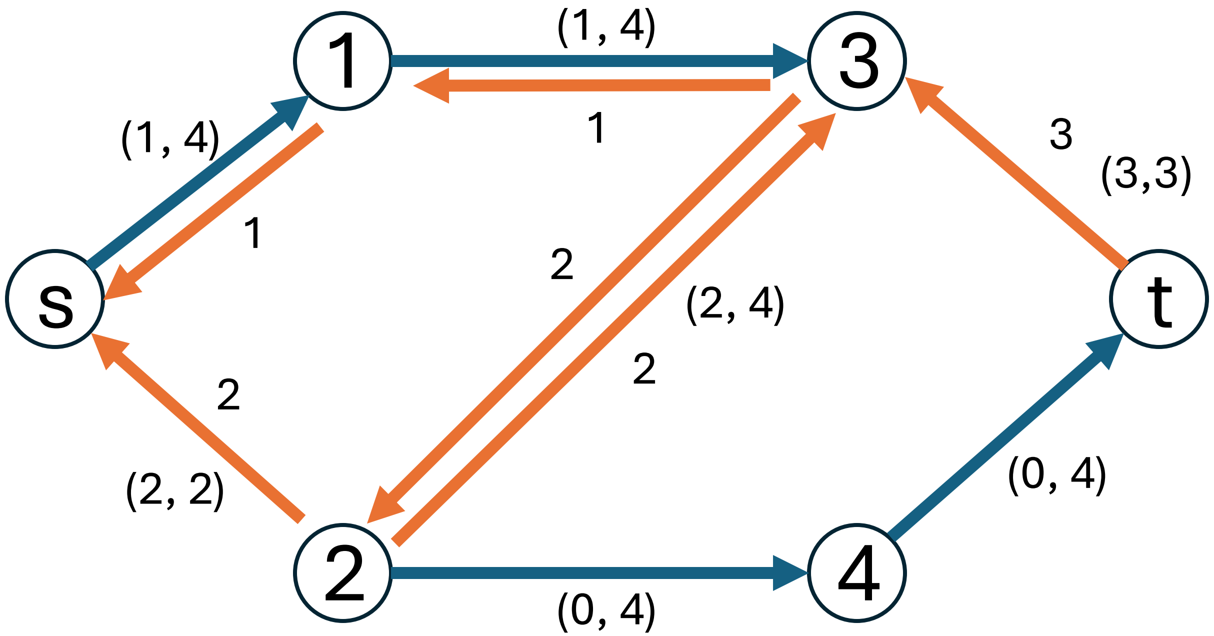
From Source s to Terminal t



Sending 1 unit from s to t on red links

Probe and send one unit

Maximum Flow
From Source s to Terminal t

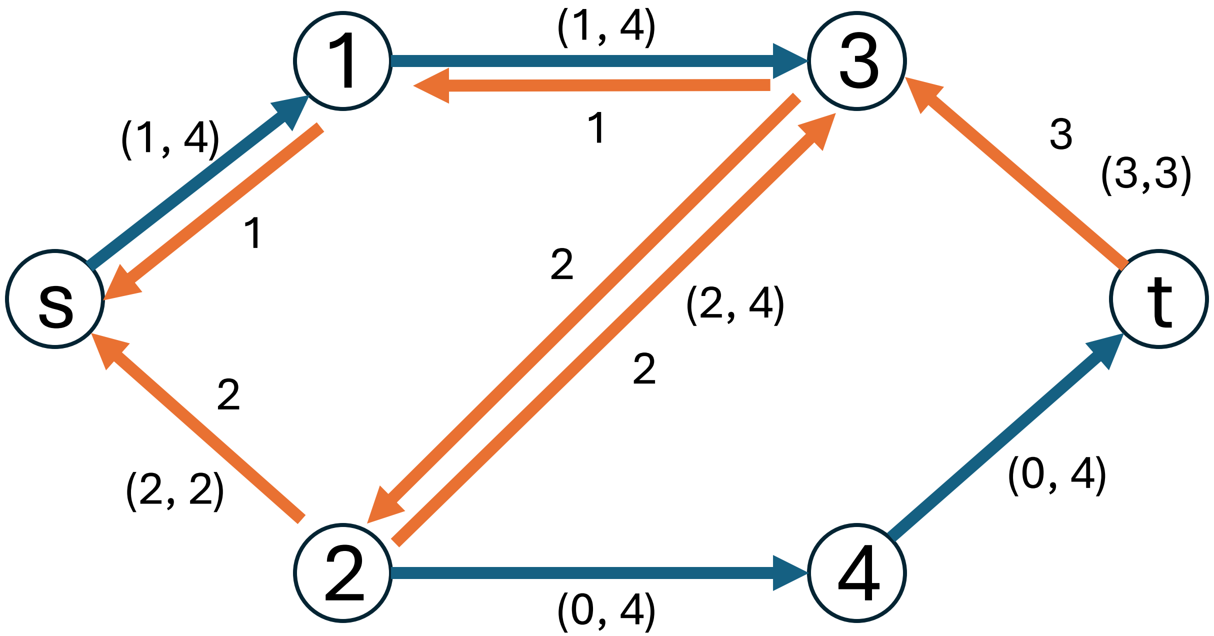


Sending 3 units from s to t

Residual capacity network: can send back units that are flowing

Maximum Flow

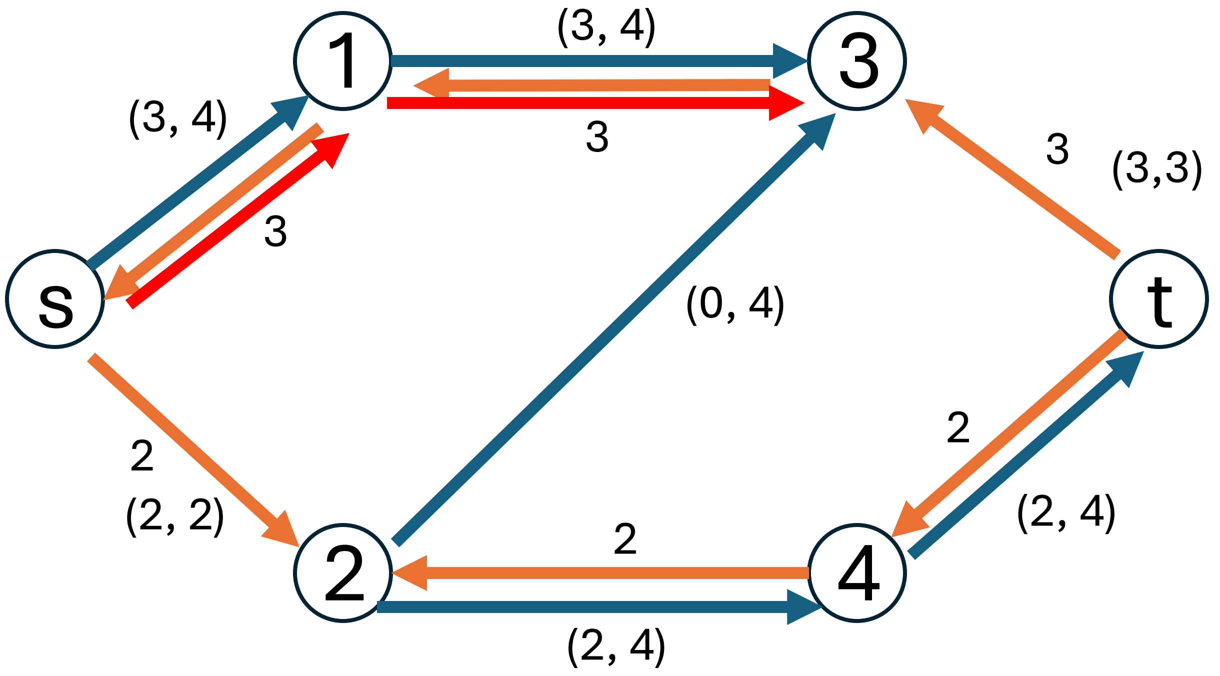
From Source s to Terminal t



Send 2 units on path s-1-3-2-4-t from s to t

Probe: s-1, 1-3, 3-2, 2-4, 4-5

Maximum Flow
From Source s to Terminal t

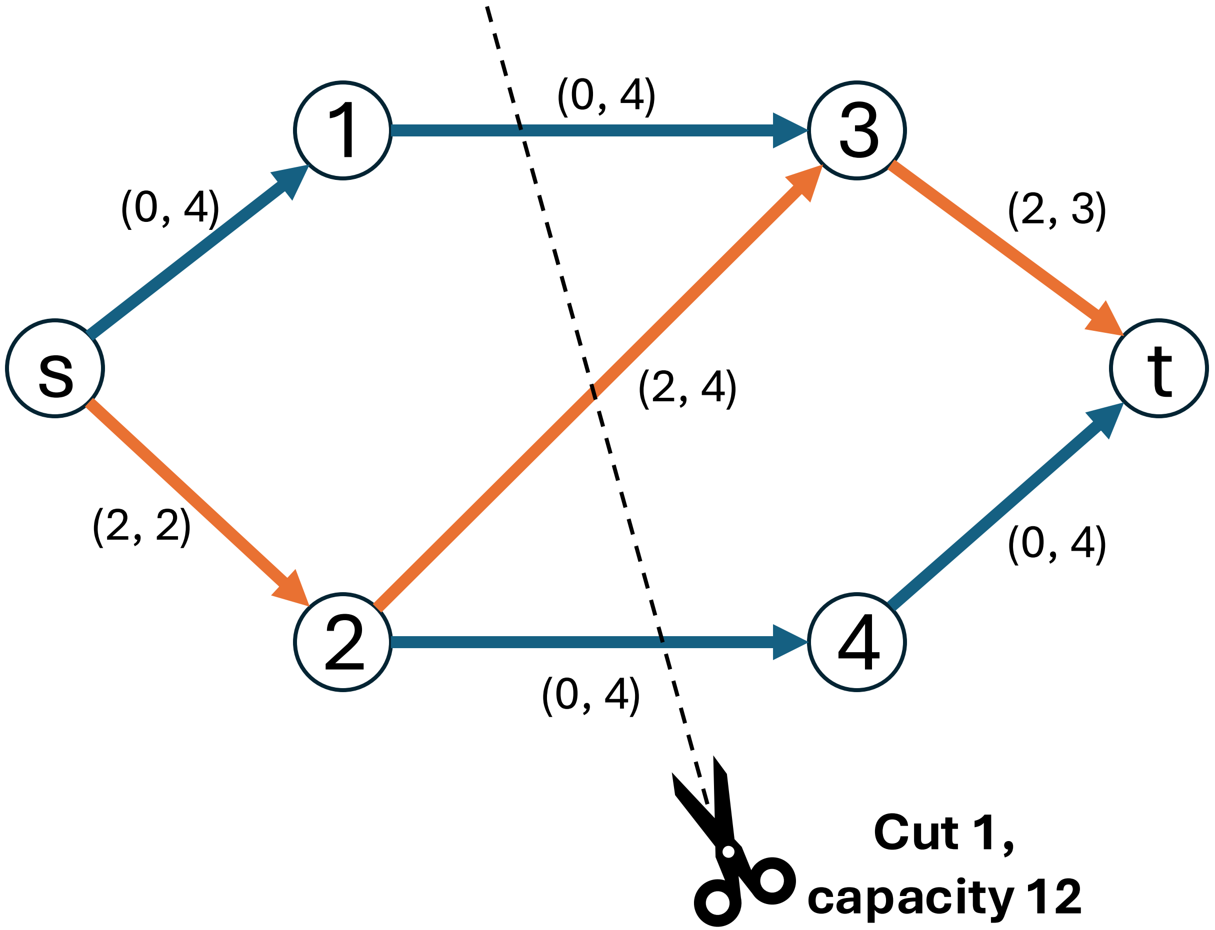


Now sending 5 units from s to t

Now probe s to 1, 1 to 3 and stop

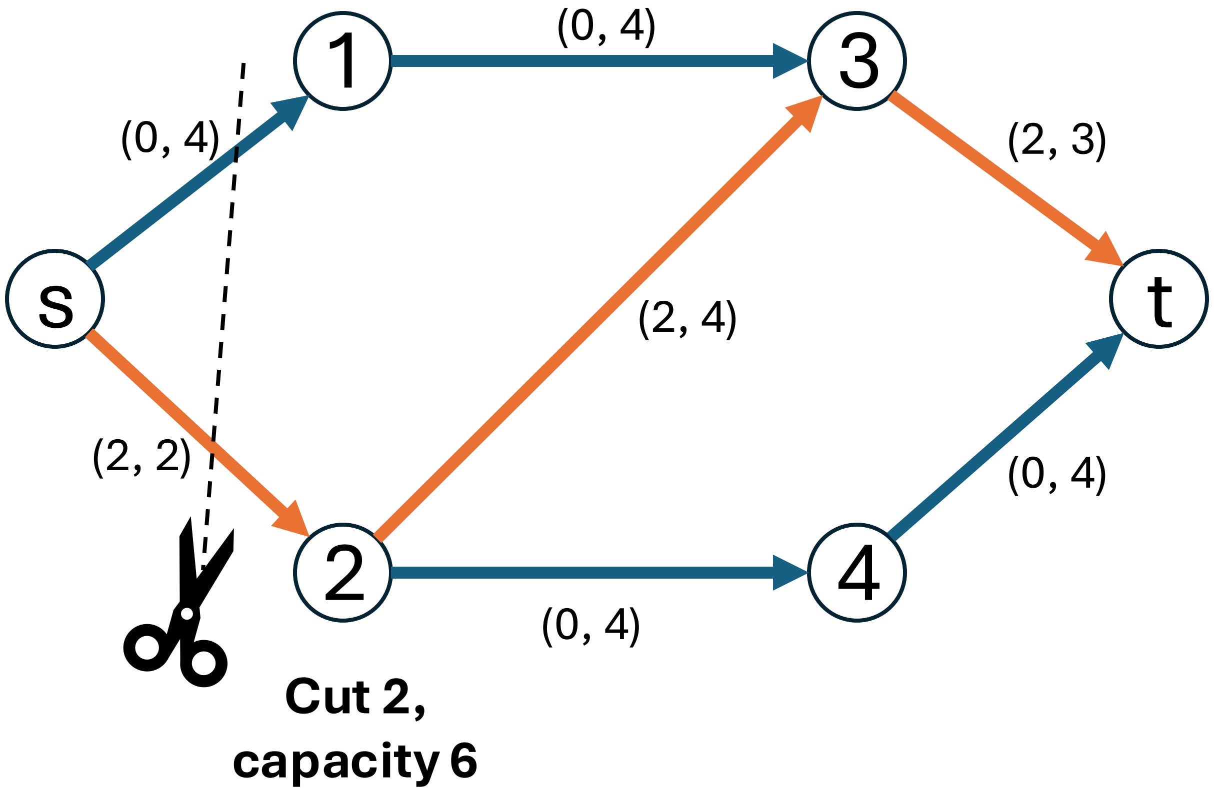
Maximum Flow

From Source s to Terminal t



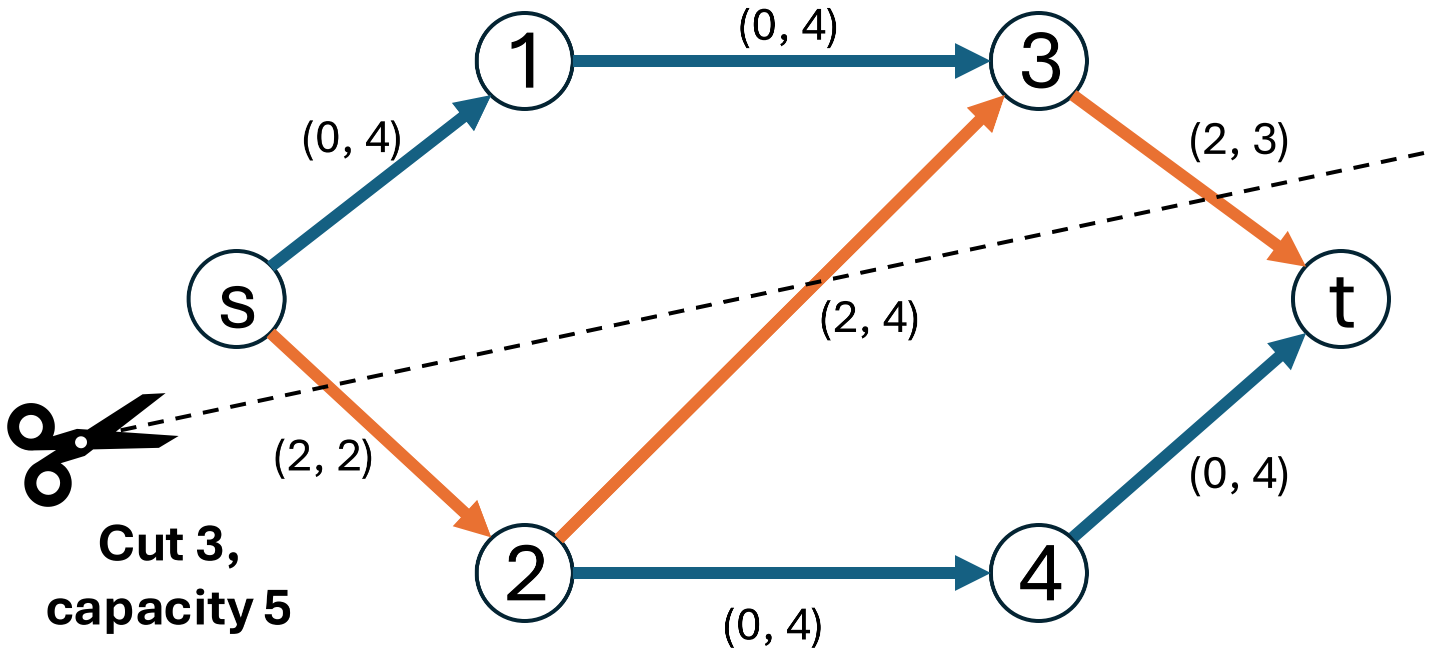
Maximum Flow

From Source s to Terminal t



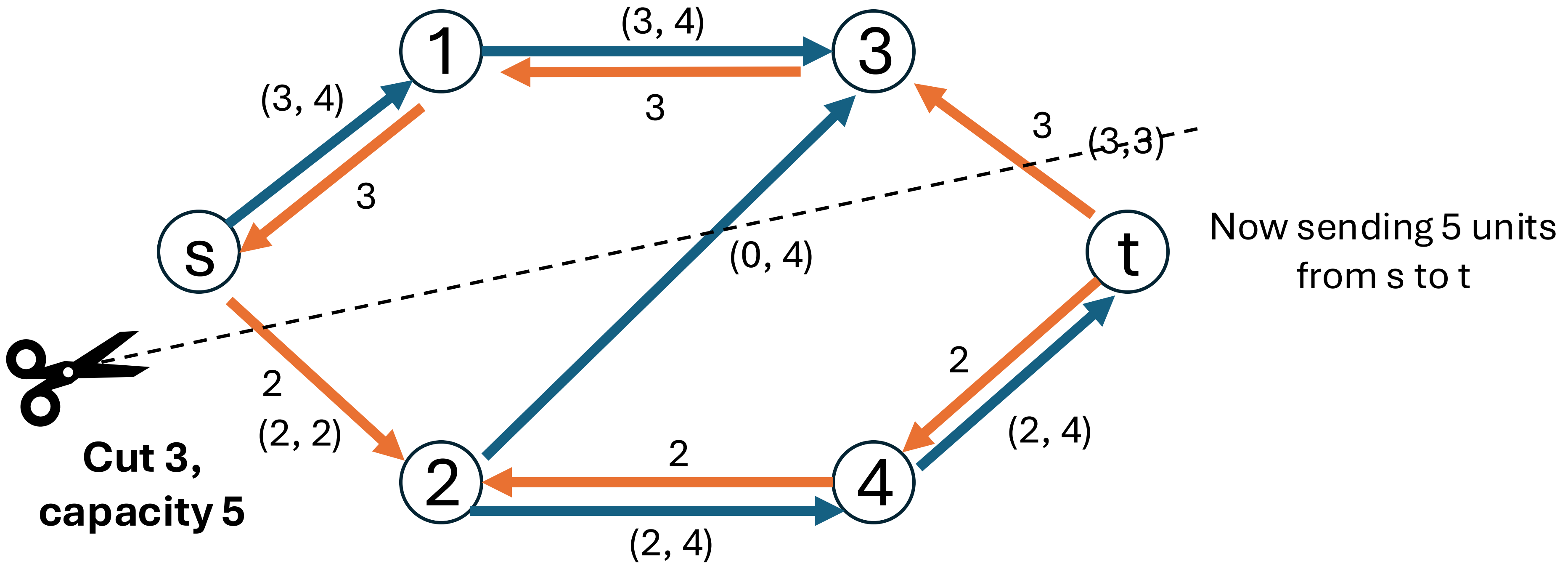
Maximum Flow

From Source s to Terminal t



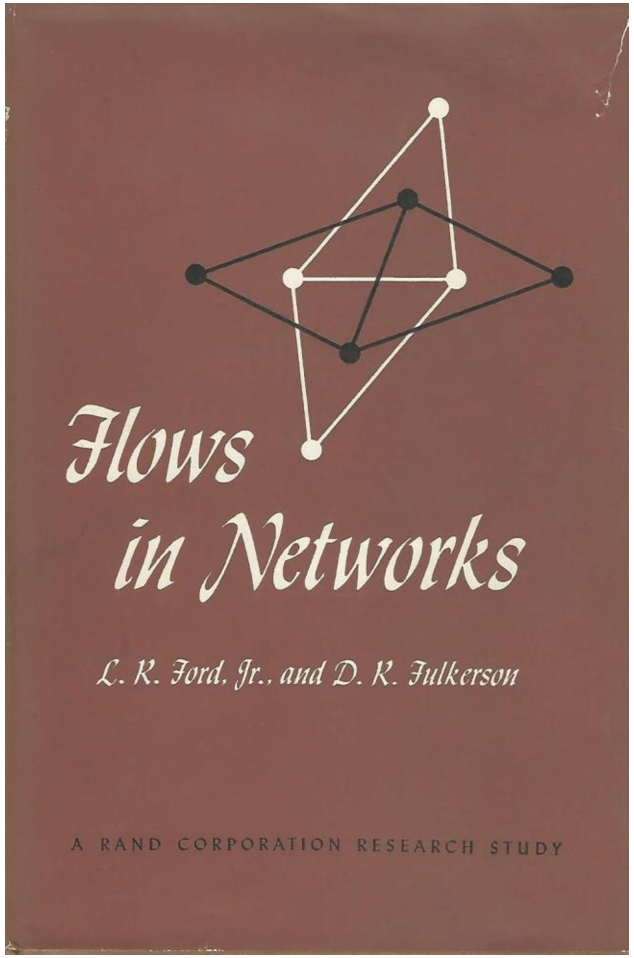
Maximum flow = minimum cut

Maximum Flow
From Source s to Terminal t

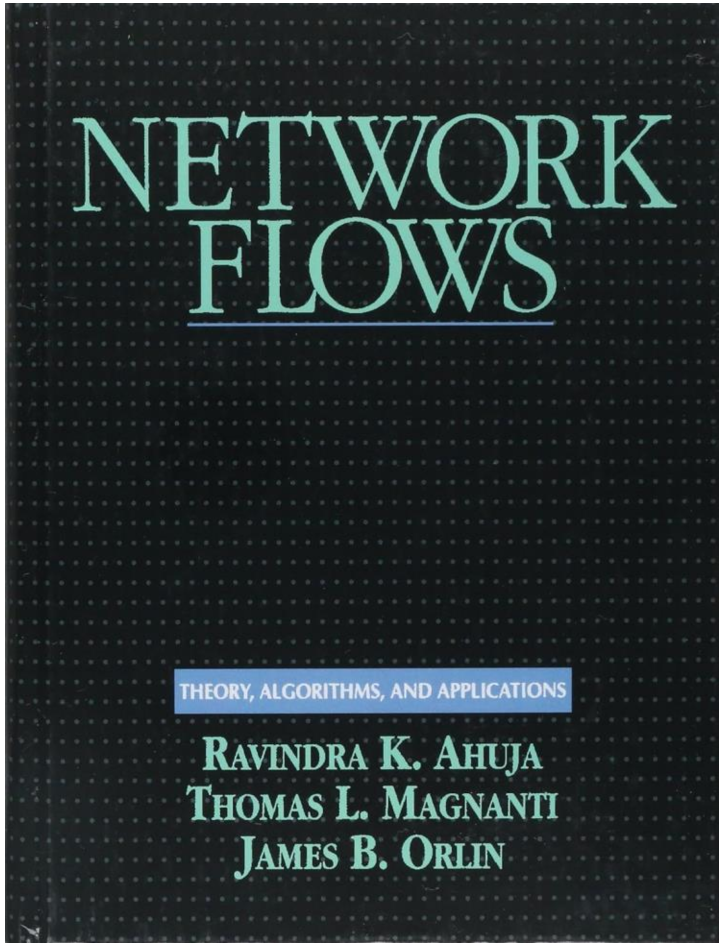


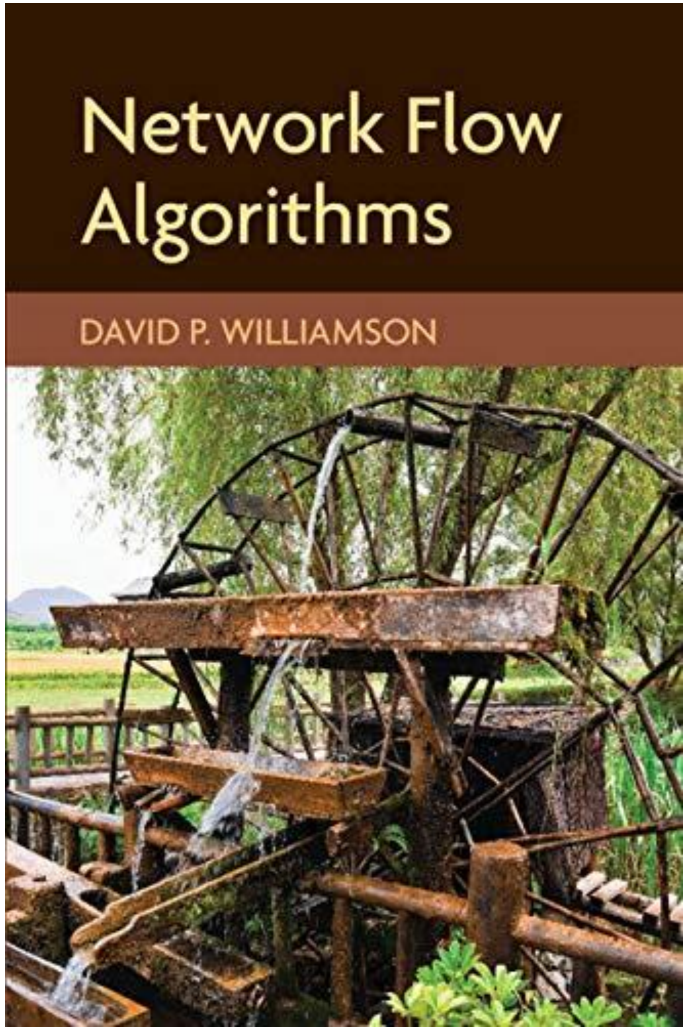
Maximum flow = minimum cut

- If capacities are integral (rational), algorithm finds an optimal solution
- The solution found is integral
- Max Flow = Min Cut
  - Has many combinatorial implications
- Breath search provides polynomial time algorithm
  - Edmonds and Karp, Dinic
- Much research on problem over decades



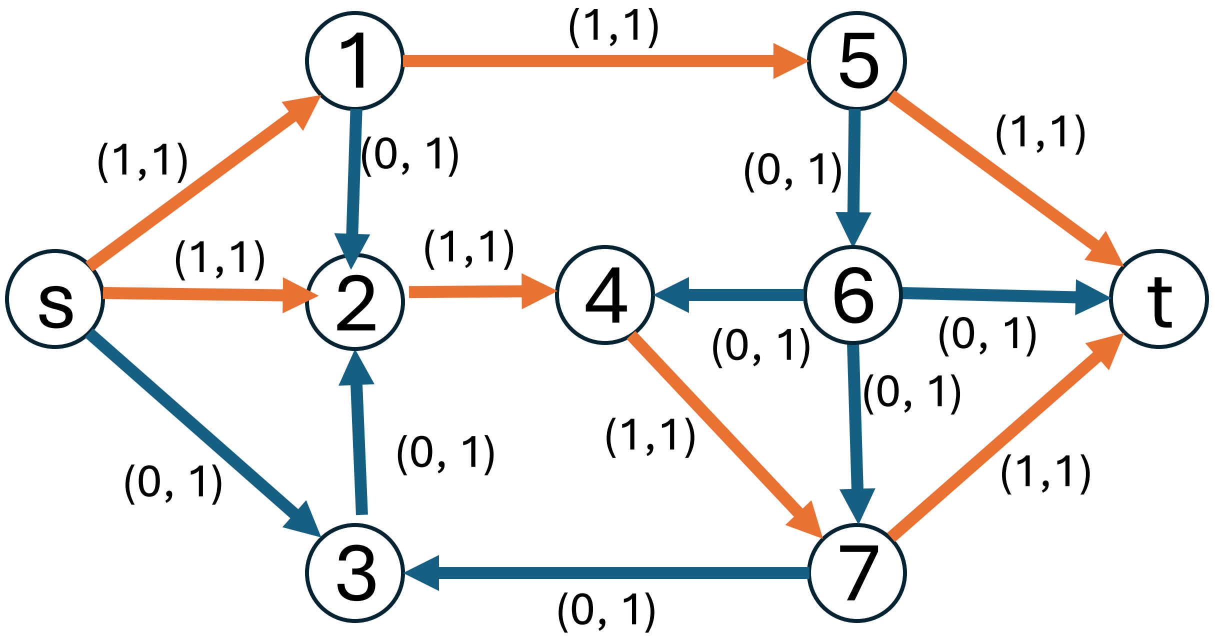
Pathbreaking Contribution





Menger's Theorem

From Source s to Terminal t



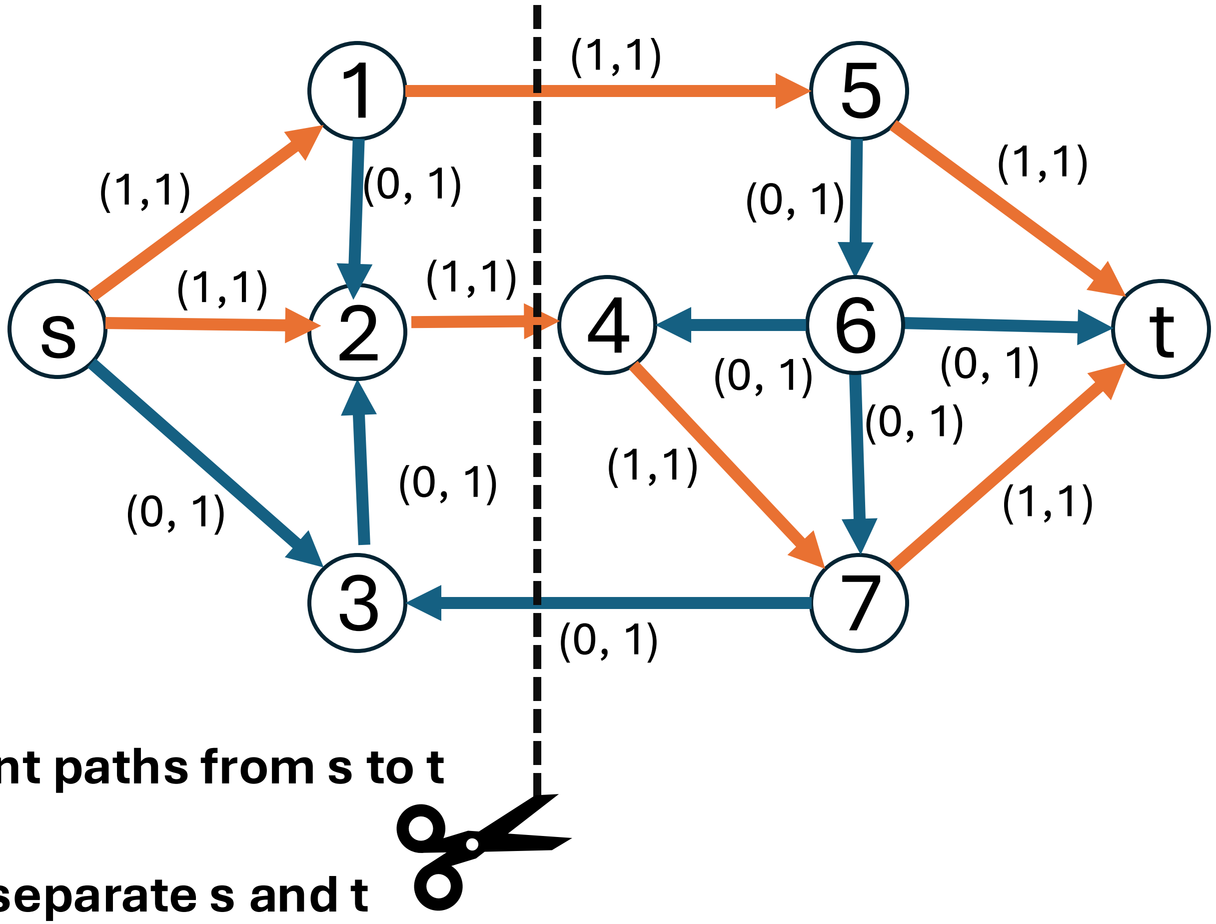
Sending 2 units (paths) from s to t

$$(x_{ij}, u_{ij})$$

Current flow $\leq$ Max flow capacity

Menger's Theorem

From Source s to Terminal t



Sending 2 units (paths) from s to t

Max number of arc disjoint paths from s to t

=

Min number of arcs that separate s and t

Menger's Theorem

Consider a max flow problem with each arc (edge) having a capacity of 1.

Then each augmenting flow will send one unit between the source and sink.

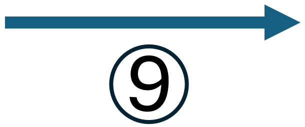
So the max flow will provide the largest number of paths between the source and sink that do not share an arc.

Any source to sink cutset will be the number of arcs in the cutset.

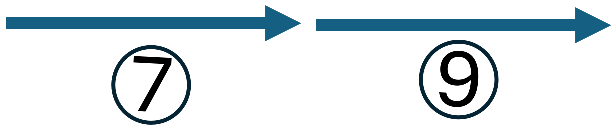
So, max flow min cut says that the maximum number arc-disjoint (edge-disjpint) paths equals the minimum number of arcs (edges) that have an arc (edge) in common with each path.

- A collection of sets this called a clutter (or Sperner family) if no member of the sets is contained in another.
- A blocker is a collection of minimal sets that intersect each element of a clutter.
- Menger's Theorem has arc-disjoint paths as a clutter and minimal cutsets of edges as blockers.
- Since the minimal cutsets is a clutter, it has a blocker, namely the arc-disjoint paths from the source to the sink. Thus if C denotes the arc-disjoint paths and b(C) is its blocker, then b(b(C))=C. This a special case of clutters and blockers more generally, an important type of duality in combinatorics proved by Edmonds and Fulkerson (1970}. For Menger's Theorem the size of C and b(C) are the same. This is not true in general.

Machine Scheduling
Project Scheduling
Fulkerson, Rand 1964



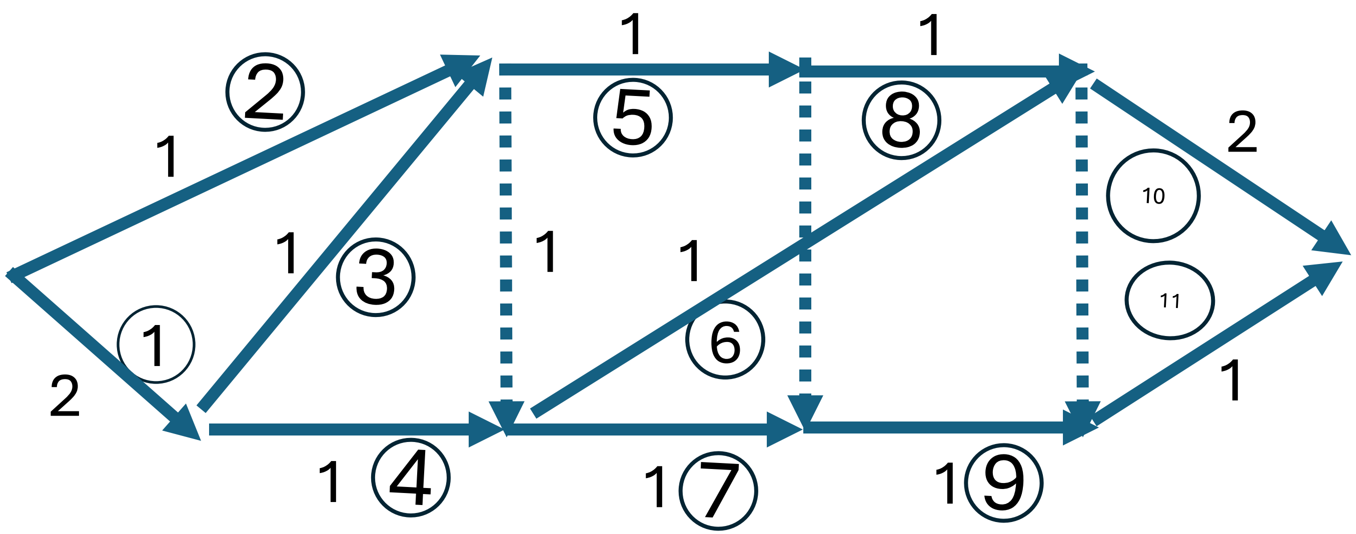
Job with job number



Precedence: if jobs 7 and 9 on same machine then 7 processed before 9

What order to process jobs and what is the smallest number of machines to process all the jobs?

Machine Scheduling
Project Scheduling



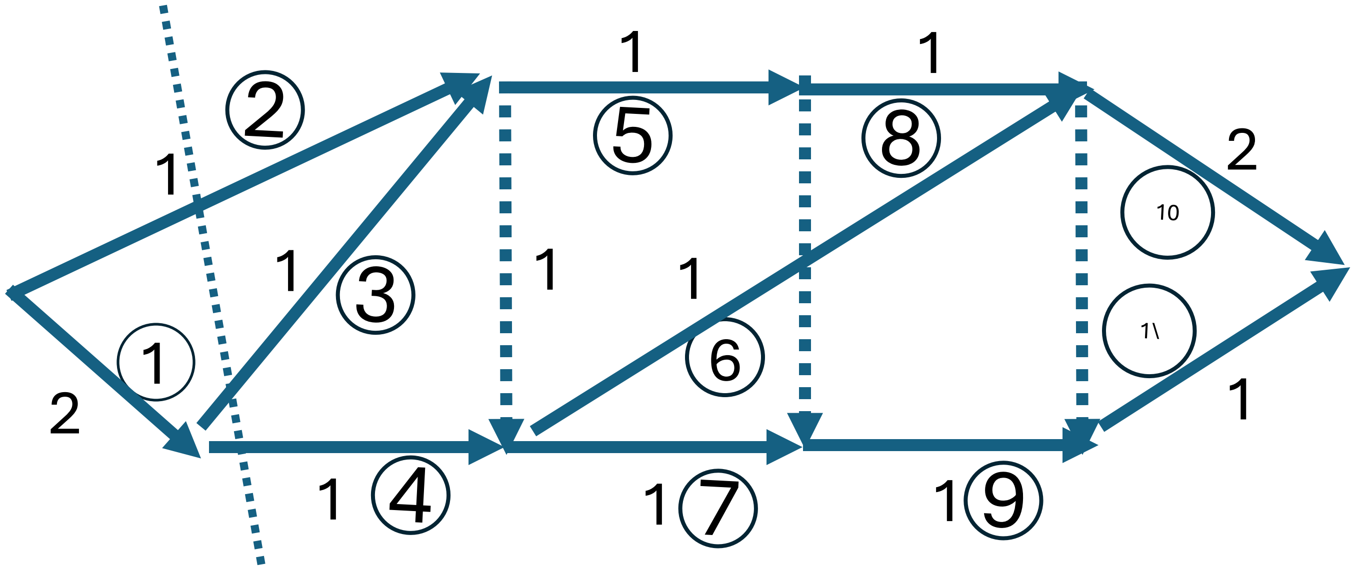
Sending 3 units
from left to right

Min flow 1 on each job arc with infinite capacity
No min or capacity on dashed timing arcs

Min 3 machines with processing jobs

2,5,8,10
1,4,7,9,11
3,6

Machine Scheduling
Project Scheduling



Sending 3 units from left to right

Max number of jobs (2,3,4) that cannot be processed on same machine

Min number of machines (chains) = Max number of jobs (antichains) that cannot be processed on same machine

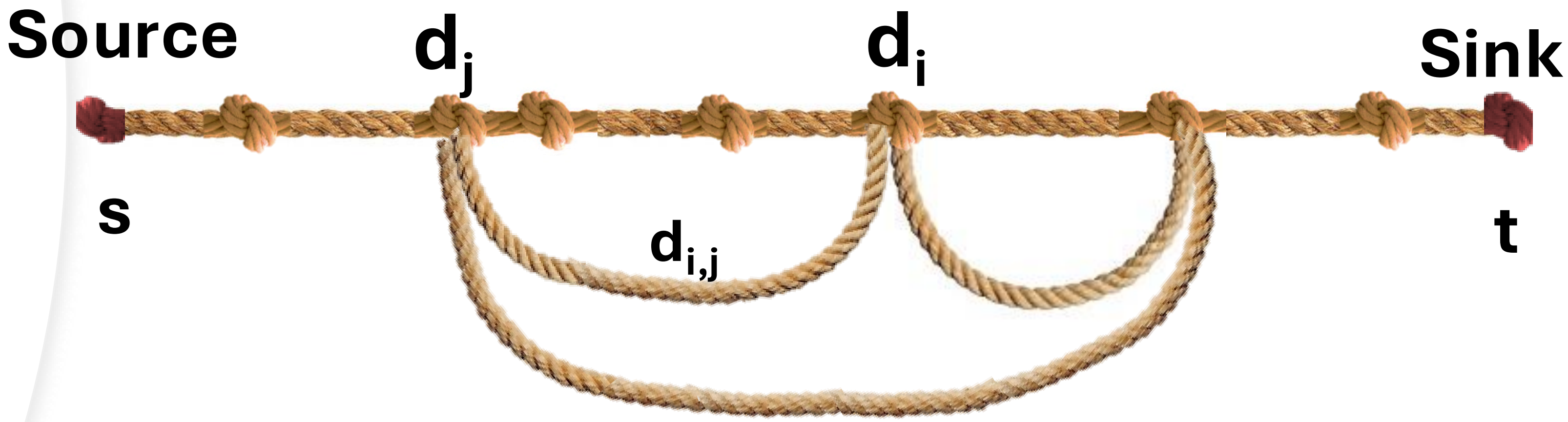
Form of Dilworth's Theorem

Fulkerson Also Examines

- Non fixed schedule of jobs
  - PERT (Program Evaluation and Review Technique)
  - CPM (Critical Path Method)
- Time compression of jobs

Primal-Dual Method

- Let d_j = shortest path distance from source to sink with arc length $d_{i,j}$



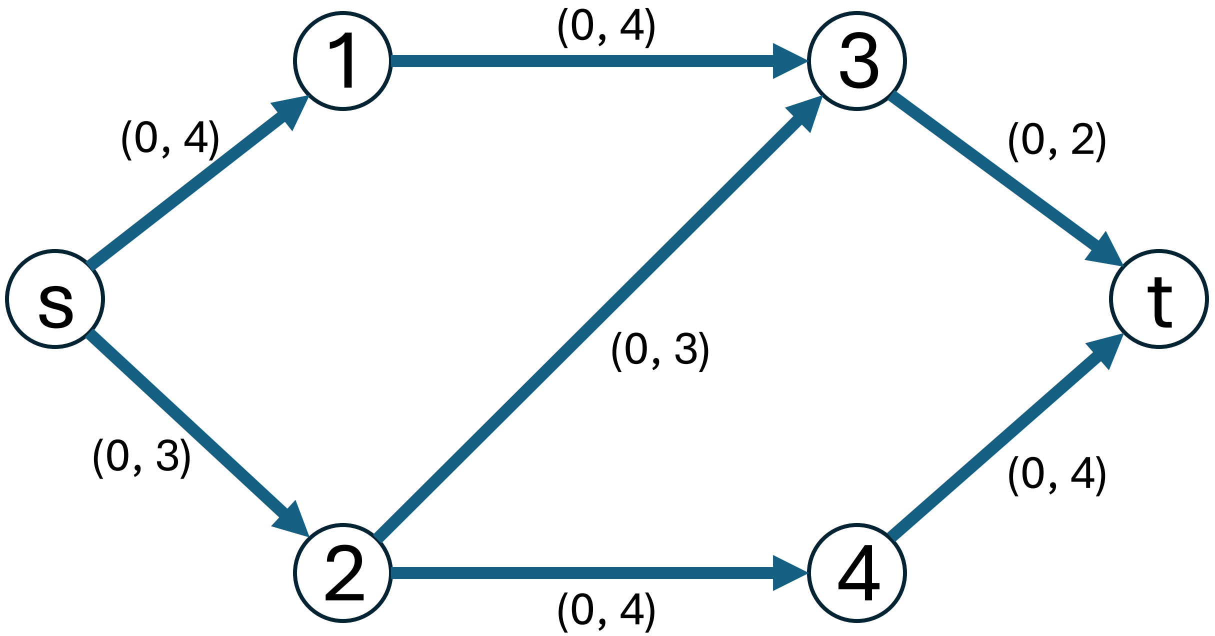
- Every node j, $d_j \leq d_i + d_{i,j}$

Primal-Dual Method

- Start at node t with all $d_{js} = 0$.
- Increase all d_{js} to $d_j = 2$.
- Add arc (3, t) in (Primal) network.
- Keep $d_3 = 2$, and increase other d_{js} to 4.

Dijkstra Shortest Path

From Source s to Terminal t



Sending 1 unit from s to t

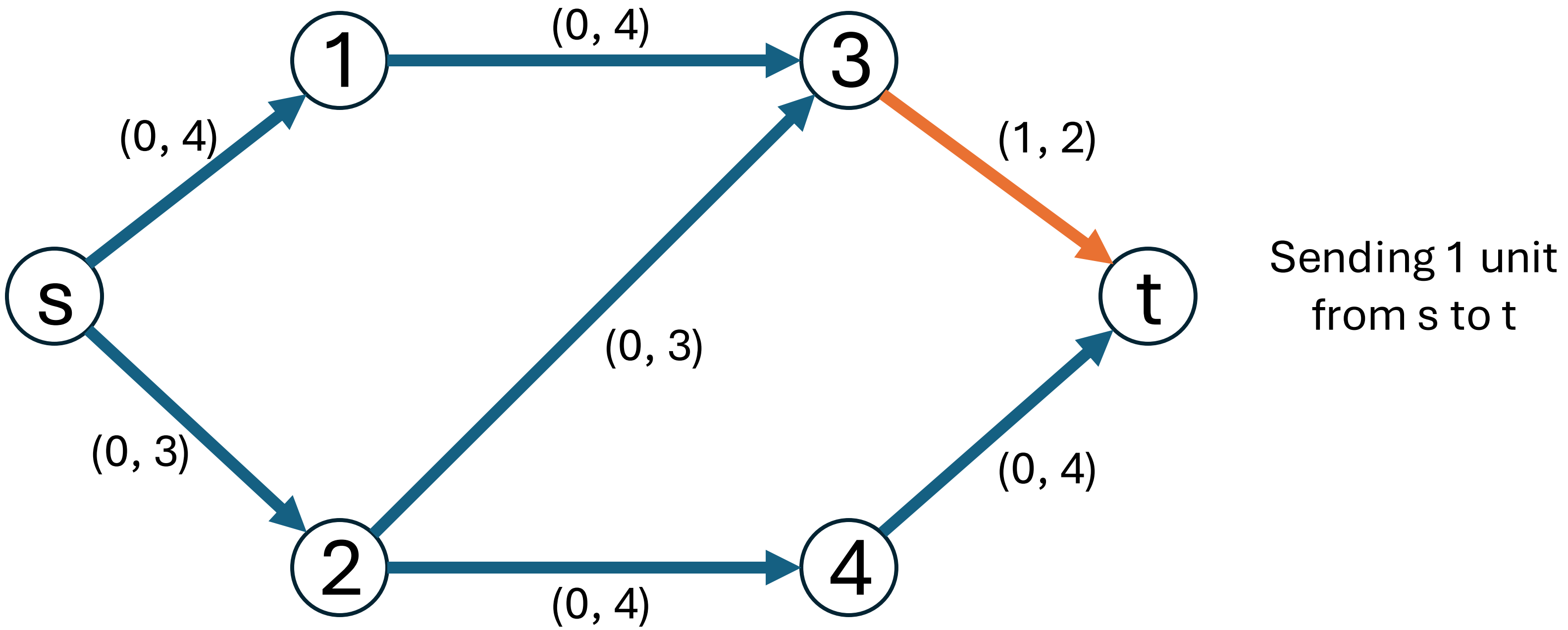
(x_{ij}, dij)

Current flow ≤ Distance on arc (i,j)

Set all $d_j = 2$

Dijkstra Shortest Path

From Source s to Terminal t



(x_{ij}, dij)

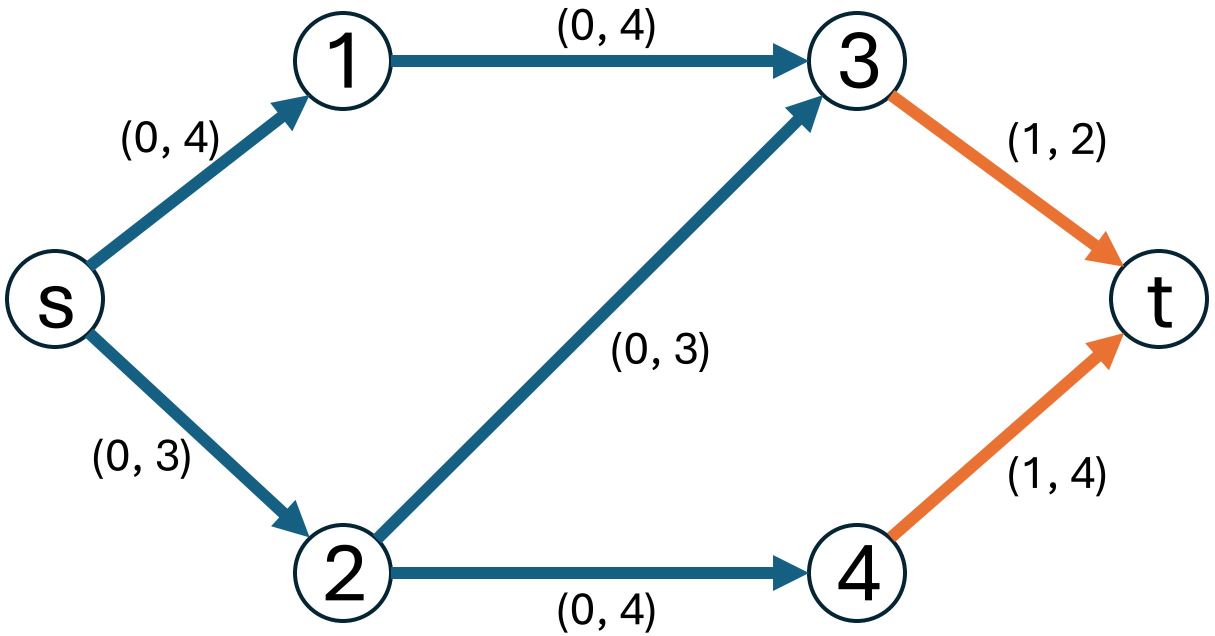
Current flow $\leq$ Distance on arc (i,j)

Keep d_3=2,

Set all other d_j = 4

Dijkstra Shortest Path

From Source s to Terminal t



Sending 1 unit from s to t

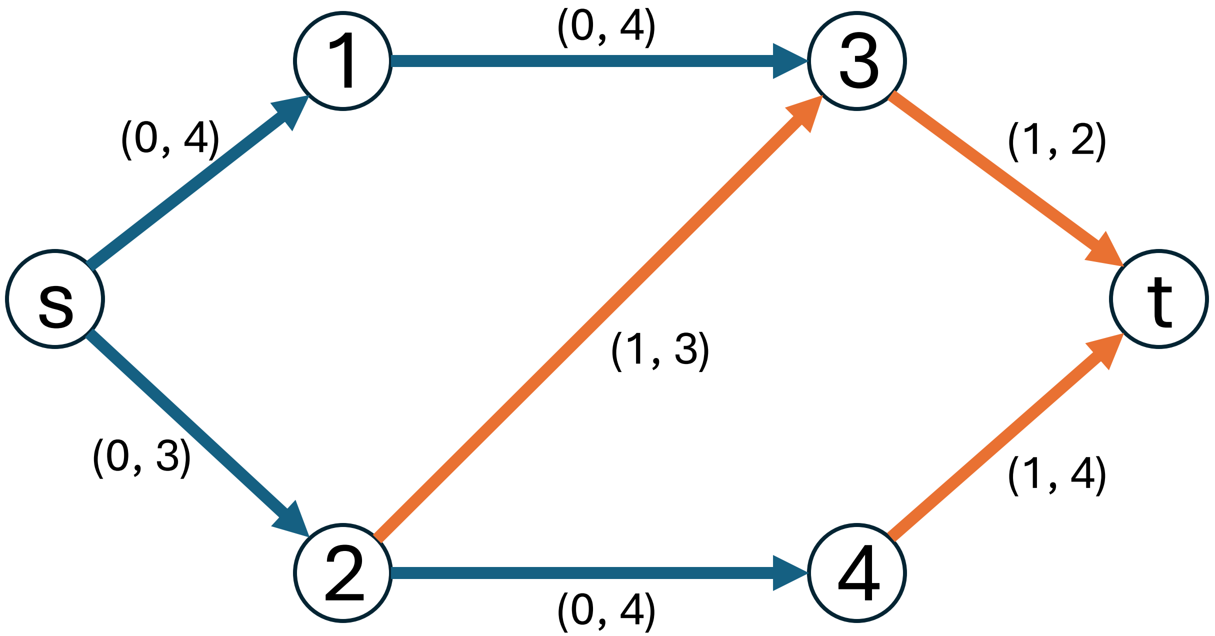
(x_{ij}, dij)

Current flow $\leq$ Distance on arc (i,j)

Keep d_3=2, d_4=4
Set all other d_j = 5

Dijkstra Shortest Path

From Source s to Terminal t



Sending 1 unit
from s to t

$$(x_{ij}, dij)$$

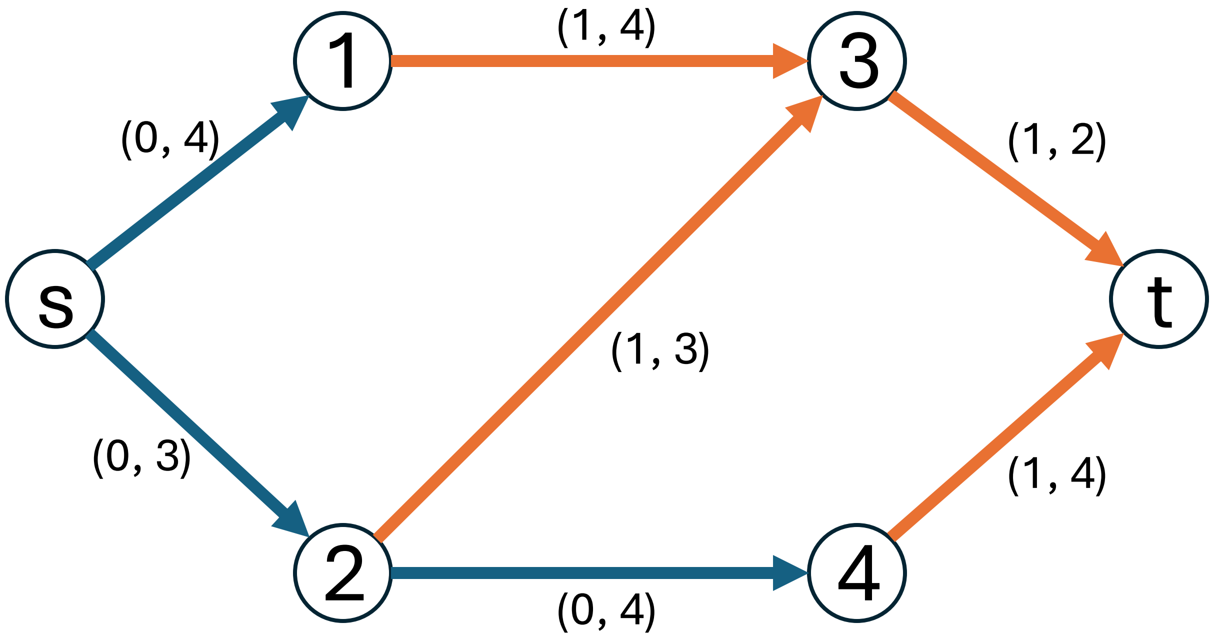
Current flow $\leq$ Distance on arc (i,j)

Keep d_3=2, d_4=4, d_2=5

Set all other d_j = 6

Dijkstra Shortest Path

From Source s to Terminal t



Sending 1 unit from s to t

$$(x_{ij}, dij)$$

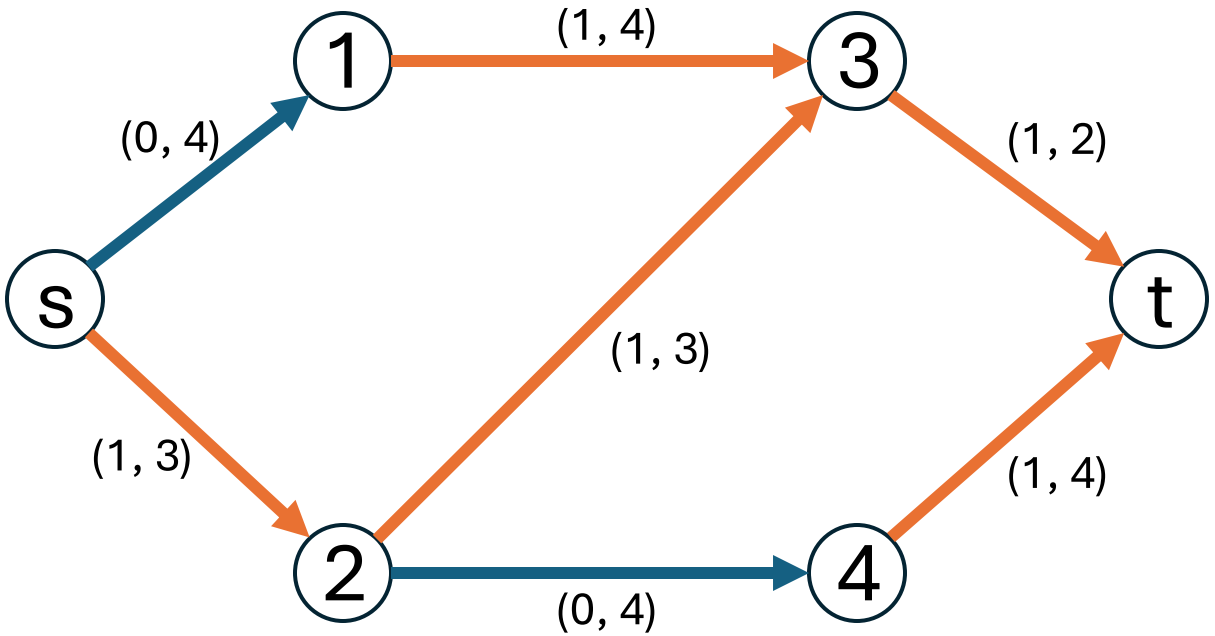
Current flow $\leq$ Distance on arc (i,j)

Keep d_3=2, d_4=4, d_2=5, d_1=6

Set d_s = 8

Dijkstra Shortest Path

From Source s to Terminal t



Sending 1 unit from s to t

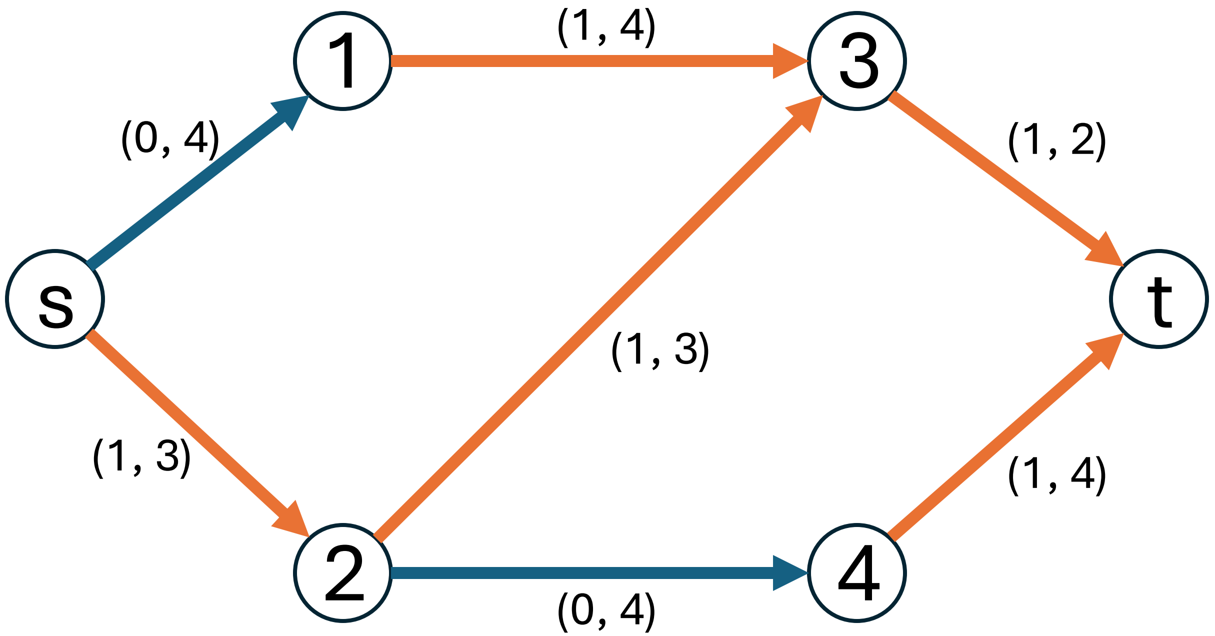
(x_{ij}, dij)

Current flow $\leq$ Distance on arc (i,j)

We now have the shortest path from s to t, s – 2 – 3 – t

Dijkstra Shortest Path

From Source s to Terminal t



Sending 1 unit from s to t

(x_{ij}, dij)

Current flow $\leq$ Distance on arc (i,j)

Primal-dual More Generally

Dantzig, Ford, Fulkerson (1956)

- For linear programming, start with a dual feasible solution and, from the solution, derive a partial feasible solution.
- Change the dual feasible solution and find another partial primal feasible solution.
- Continue until both dual and primal solutions are fully feasible.
- Approach uses extensively for combinatorial optimization approximation methods.

Column Generation (Ford and Fulkerson, 1958)

Multicommodity Maximum Flow Problem

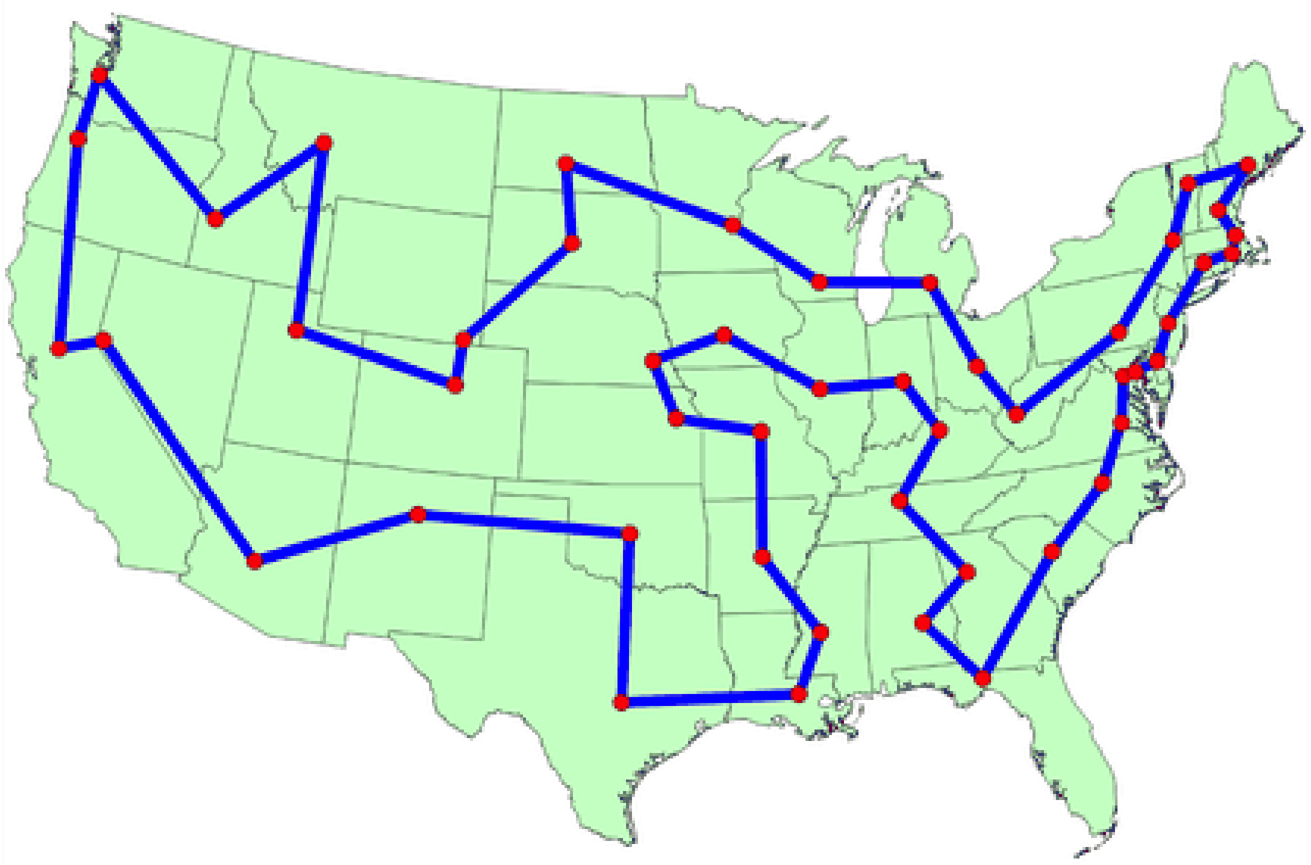
- Variable for each path from each source to its destination.
- Constraints state that the sum of path flows on any arc cannot exceed the arc capacity.
- Number of paths exponential is the size of network.

Approach

- Use some paths and solve the “master” problem.
- See if another path can provide a better solution and, if so, add the path (column) to the master problem. This can be done by solving a shortest path problem.

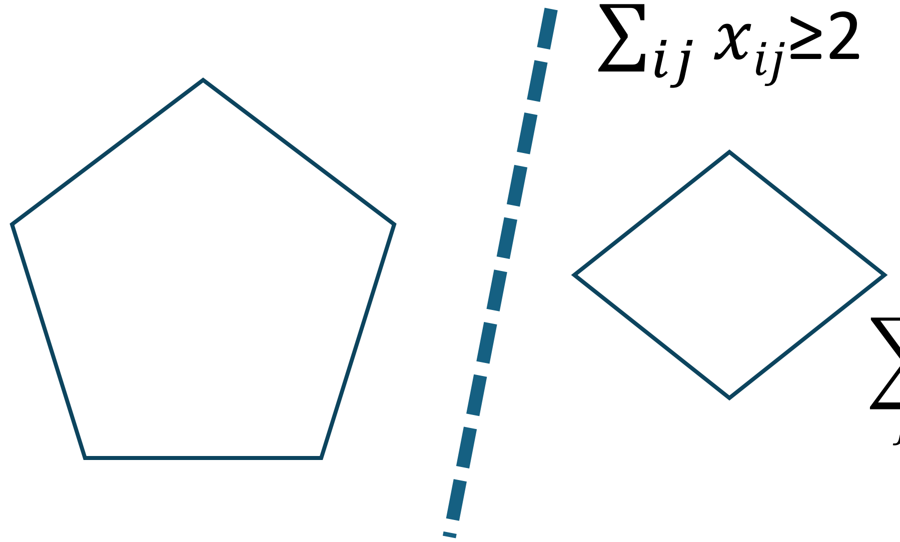
Dantzig, Fulkerson, Johnson (1954)

49 City TSP



Optimal TSP Tour

Dantzig, Fulkerson, Johnson 49 City TSP



After adding 23 subtour breaking constraints and two others (comb inequalities) find optimal solution

Used warm start (heuristic), preprocessing, LP, cuts, probing, variable fixing, reduced costs to eliminate arcs, elements of branch and bound

Chvatal and Cook call this paper the Big Bang of TSP and integer programming

Fulkerson Prize in Discrete Mathematics

Mathematical Optimization Society and American Mathematics Society

- Many famous mathematicians
- From Cornell
  - Eva Tardos
  - Louis Billera
  - David Williamson
- Also, Gerard Corneuljois

Awards and Recognition

Lester R. Ford Award from the Mathematical Association of America for his paper "Flow Networks and Combinatorial Operations Research." 1967

Southern Illinois University Award for Outstanding Professional Achievement. 1972

Honorable mention for the Lanchester Prize in 1954 and 1961.

Fulkerson Prize