Ray Fulkerson: Awe-inspiring Pioneer of Network Flows, Optimization, and Combinatorial Analysis

Thomas L Magnanti, MIT

September 20, 2024

Cornell University

D.R. Fulkerson Centennial

With gratitude

- Photos:

Bob Bland

- Biosketches:

Bob Bland and Jim Orlin

Lou Billera and Bill Lucas

Vasek Chvåtal

- Preparation of slides:

Yifu Ding

Agenda

- Personal and Career Lives
- Network Flows
  - Augmenting Flow Algorithm
  - Maximum Flows-Minimal Cuts
  - Machine/Project Scheduling
- Paths Between Two Nodes
  - Menger’s Theorem
  - Clutters and Blockers
- Primal-Dual Algorithms
  - Dijkstra’s Algorithm for Shortest Paths
- Column Generation
- Complementary Spanning Tree
- Traveling Salesman Problem and Integer Programming

Ray Fulkerson

- Born: August 14, 1924, in Tamms, Illinois.
- Died: January 10, 1976, in Ithaca, New York.
- Education: BA in Mathematics. Southern Illinois University, 1947
  M.S. in Mathematics. University of Wisconsin, 1948
  Ph.D. in Mathematics. University of Wisconsin, 1951.
- Key Positions: Rand Corporation, Mathematics Department. 1951-1971
  Cornell University, Dept. of Operations Research. 1971-1976
  Maxwell M. Upson Professor of Engineering and Professor of Operations Research and Applied Mathematics.
- Visiting Professor: U.C. Berkeley, Stanford University, University of Waterloo.

Personal and Career Lives

Personal and Career Lives

Personal and Career Lives

Ray Fulkerson, '47, and family.

Glen Fulkerson, '38, and family.

ounselor and advertising ac-
ount executive.

Merle Fulkerson Guthrie, old-
st of the girls, entered South-
rn in '36 and graduated in '40.
n '43 she received a master's
legree from the University of
llinois. At Southern she was
member of the Obelisk and

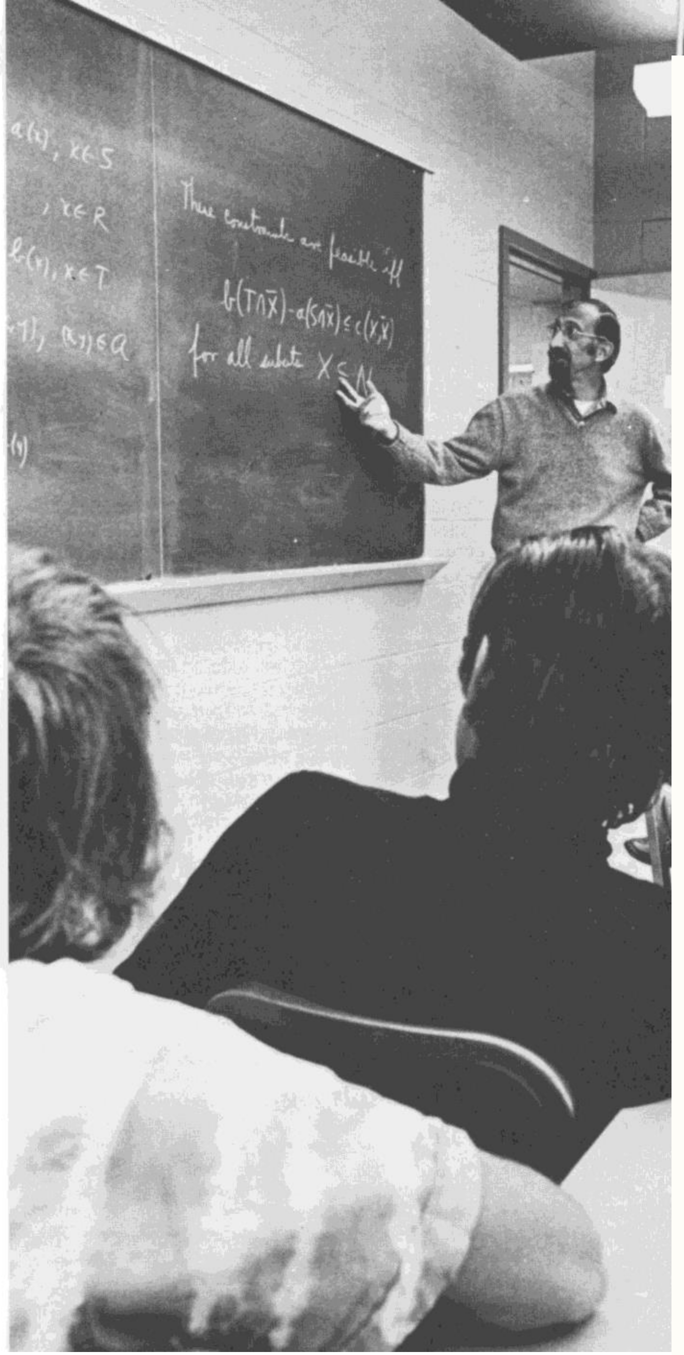
from SIU this August. In '49 she taught at Hurst-Bush and in '51 at Grayville. Her husband, Harold Todd, received a B. S. from Southern in '48, a B. A. in '49 and an M. A. in '50. He also is teaching at Athens.

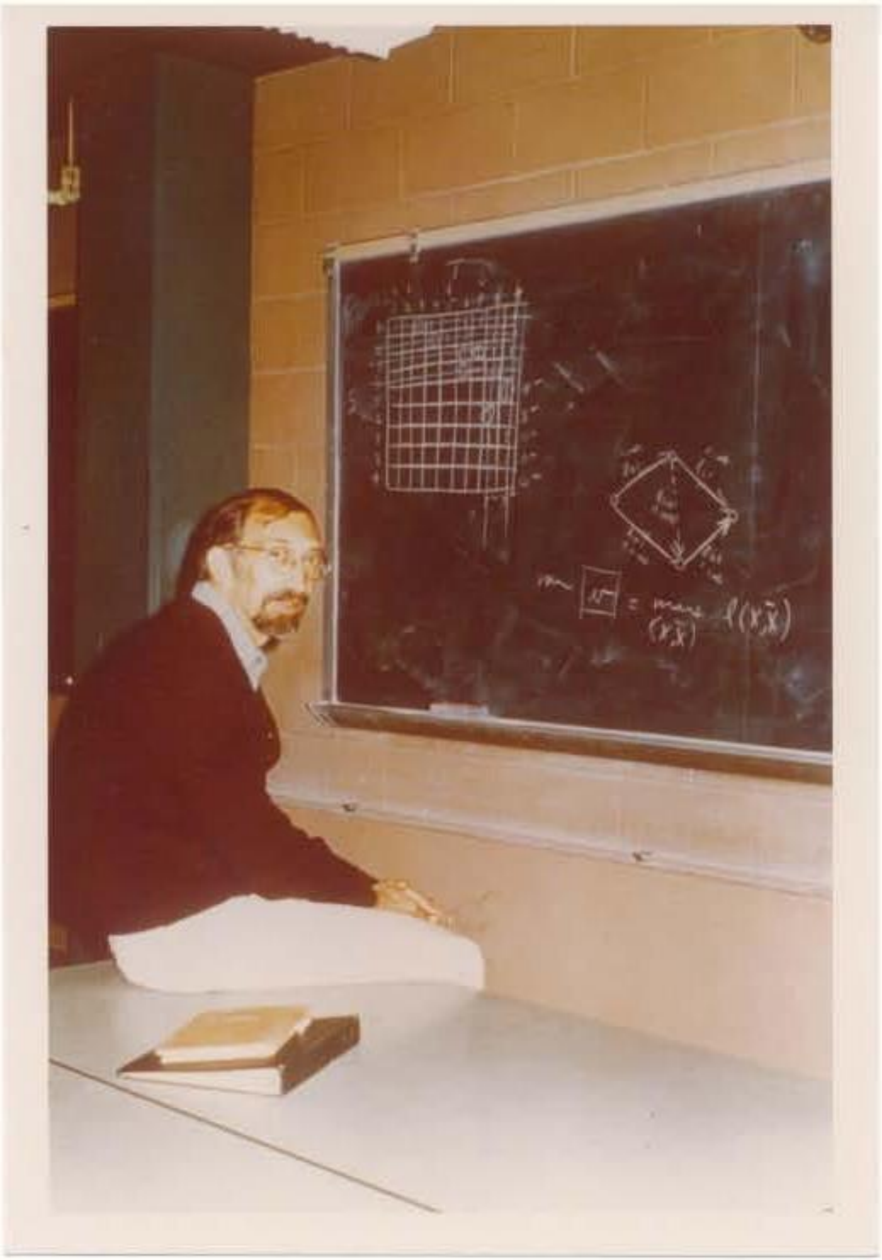
Grace Fulkerson Weshinskey, '52, was graduated from SIU

ternational Shoe company, Belle
ville.

Father of this alumni family
Elbert Fulkerson, has taught i
Southern Illinois since 191
From '27 to '44, he was princi
pal of the Carterville Communi
ty high school. Since '44 he ha
been a member of the SIU fac

Personal and Career Lives





PhD Students

David Weinberger

Robert Bland

Colleagues At Rand

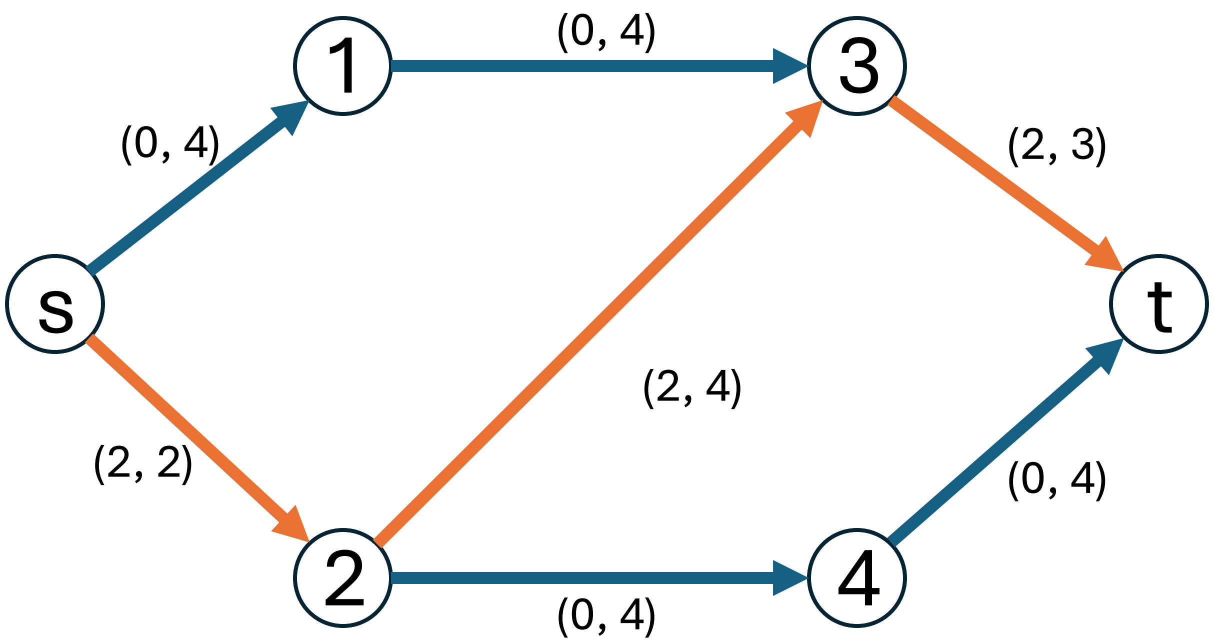
Staff
- Richard Bellman
- George Dantzig
- Merrill Flood
- Lester Ford
- Selmer Johnson
- Lloyd Shapley
- Philip Wolfe

Visitors
- Jack Edmonds
- Jon Folkman
- Herb Ryser
- John von Neumann
- ...

Maximum Flow

Ford & Fulkerson (1956)

From Source s to Terminal t



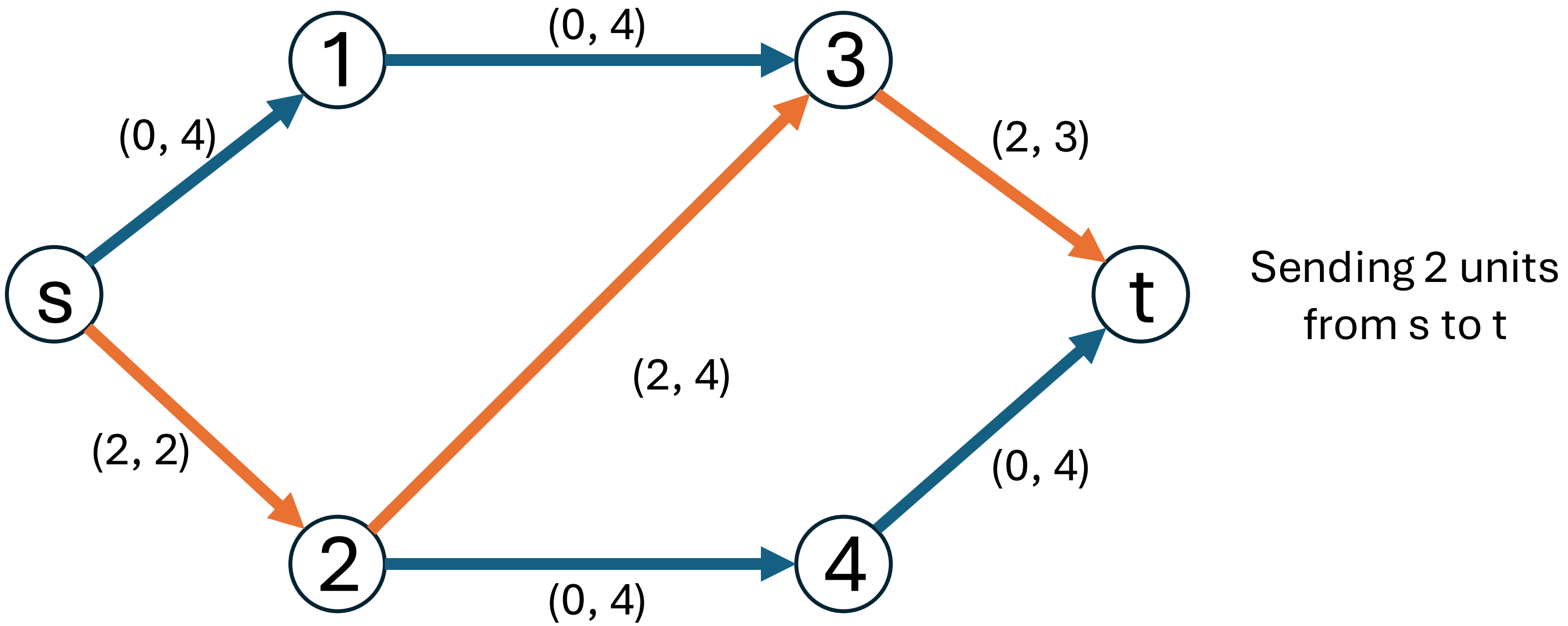
Sending 2 units (paths) from s to t

$$(x_{ij}, u_{ij})$$

Current flow $\leq$ Max flow capacity

Maximum Flow

From Source s to Terminal t

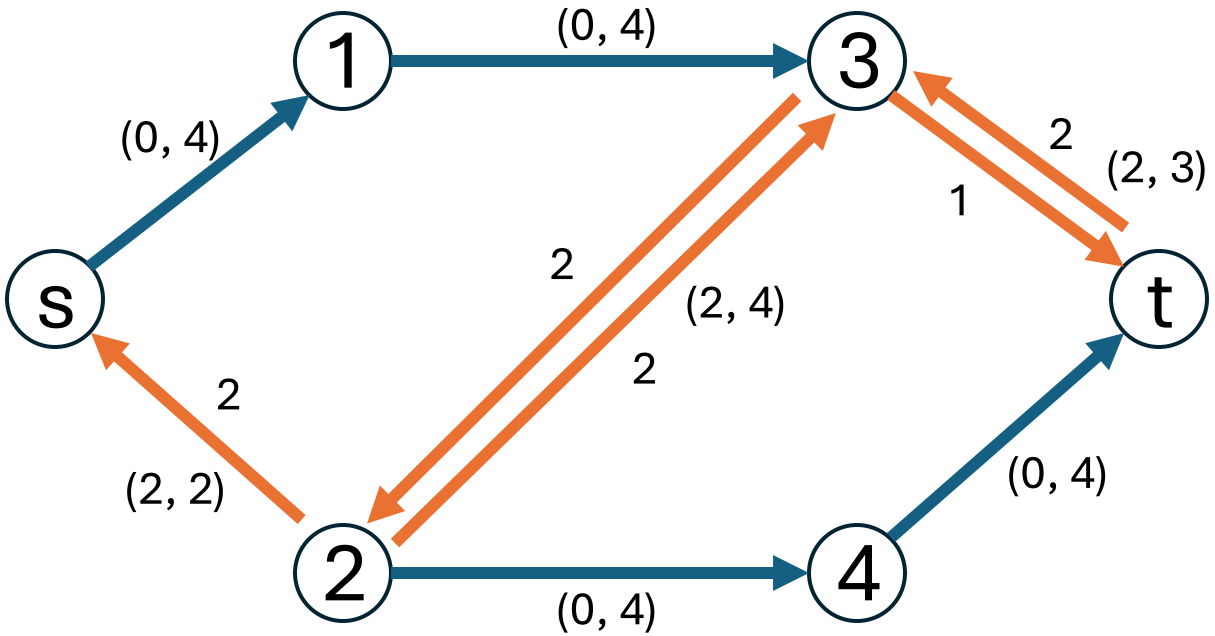


$$(x_{ij}, u_{ij})$$

Current flow $\leq$ Max flow capacity

Maximum Flow

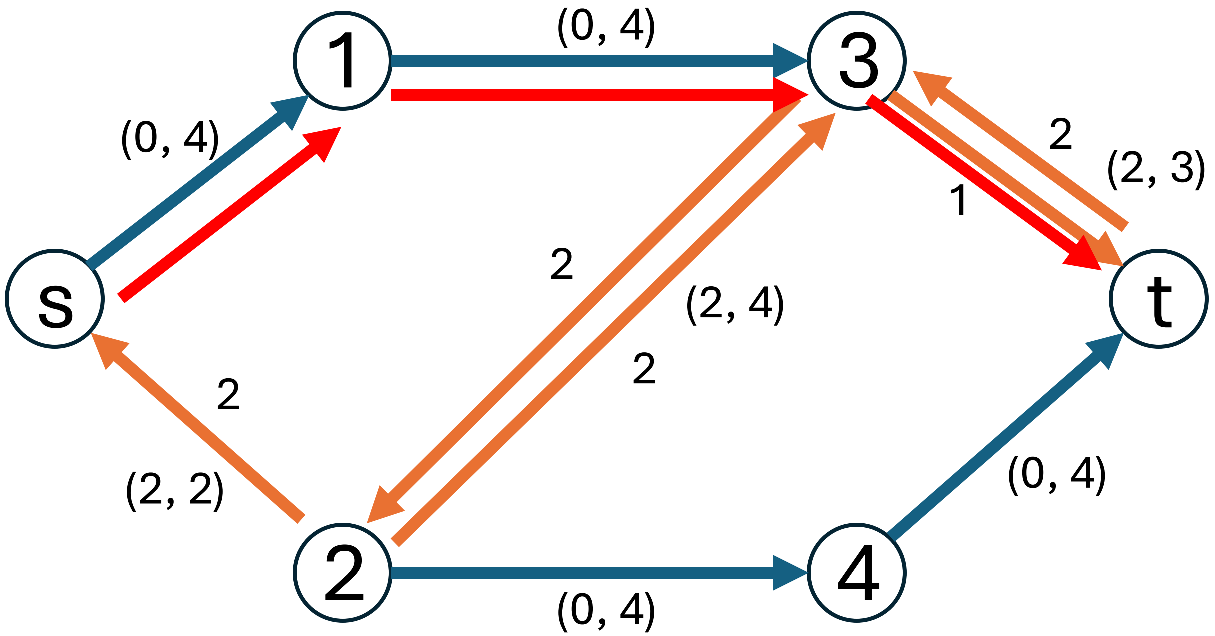
From Source s to Terminal t



Residual capacity network: can send back units that are flowing

Maximum Flow

From Source s to Terminal t

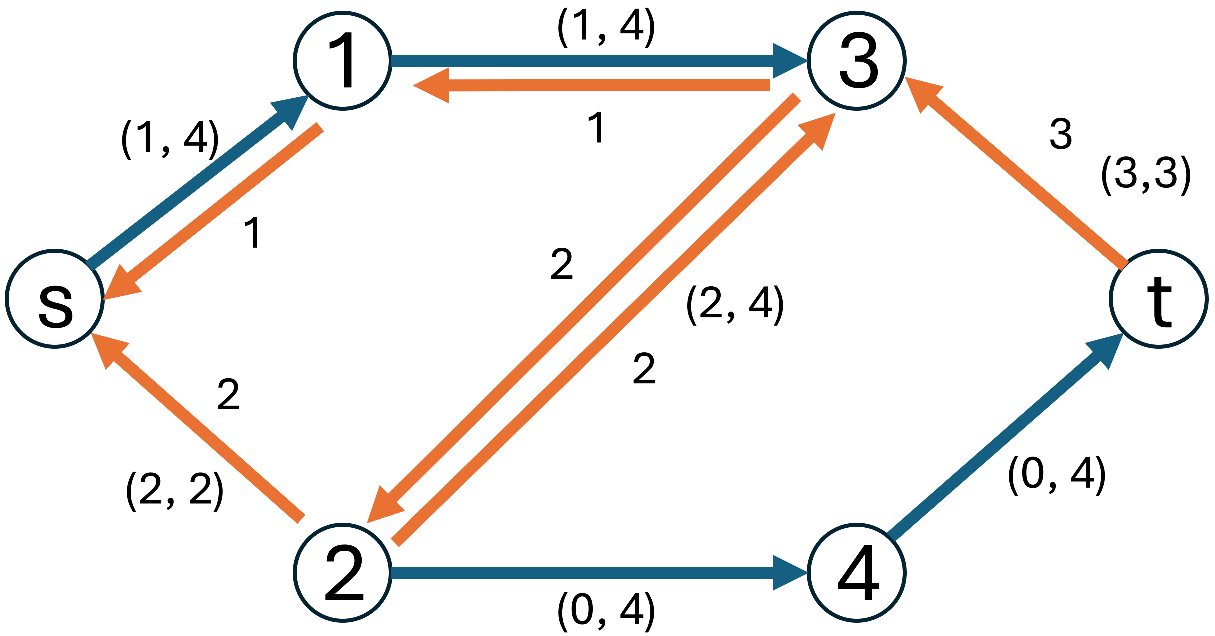


Sending 1 unit from s to t on red links

Probe and send one unit

Maximum Flow

From Source s to Terminal t

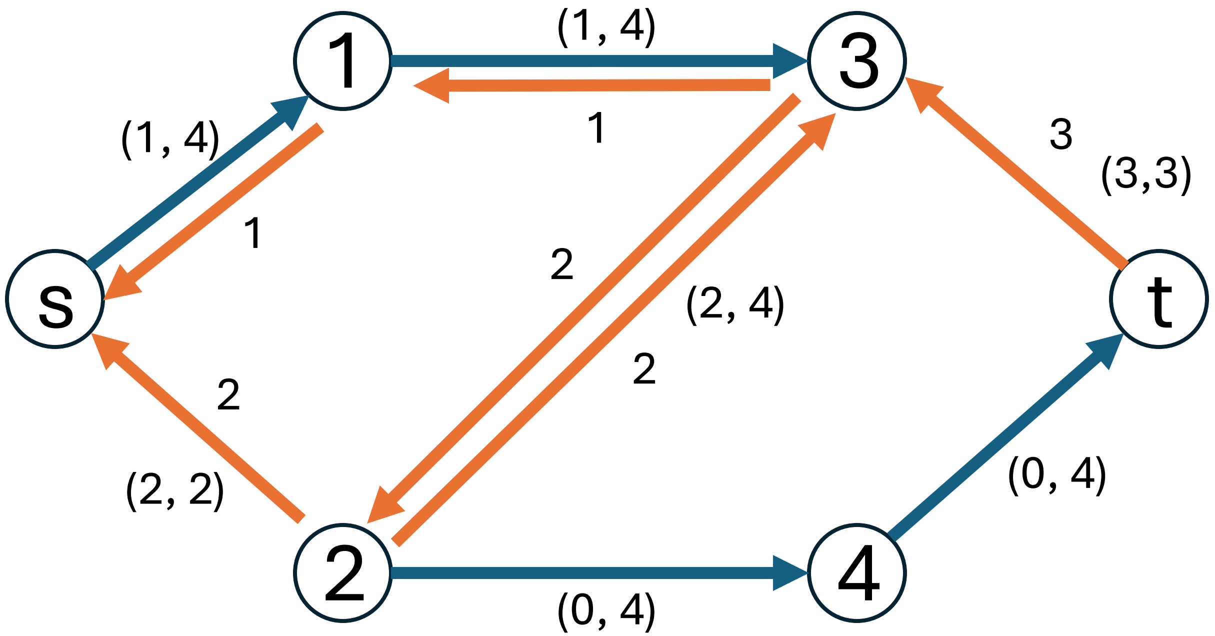


Sending 3 units from s to t

Residual capacity network: can send back units that are flowing

Maximum Flow

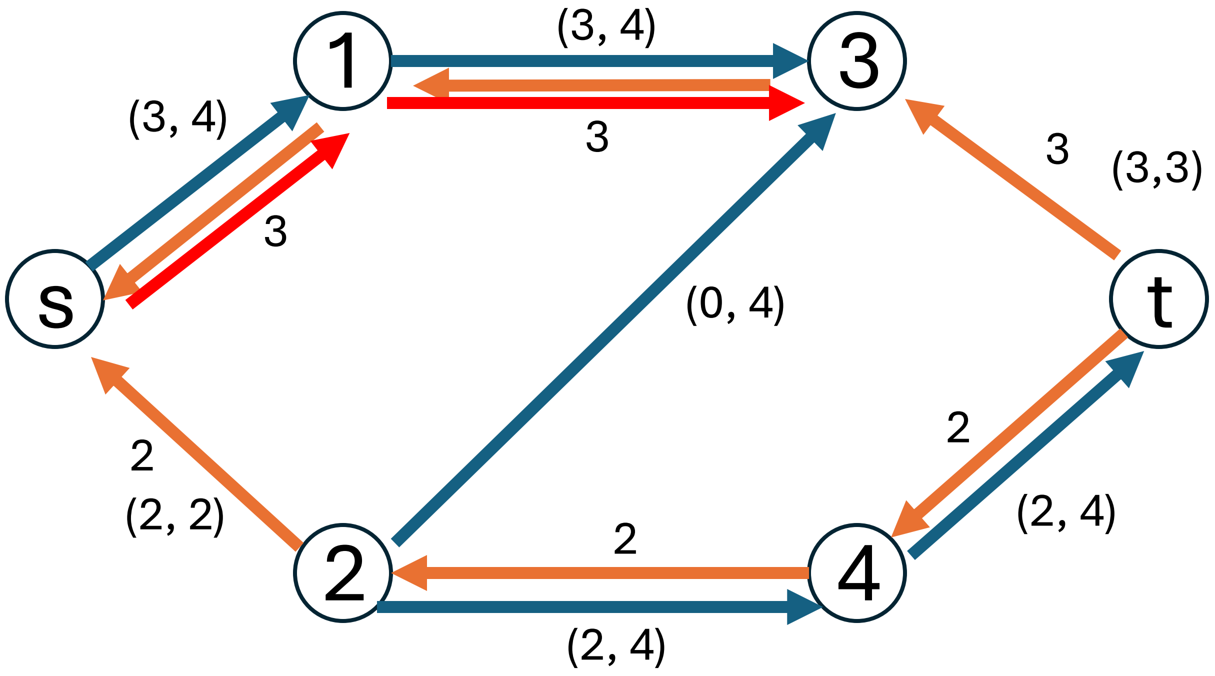
From Source s to Terminal t



Send 2 units on path s-1-3-2-4-t from s to t

Probe: s-1, 1-3, 3-2, 2-4, 4-t

Maximum Flow
From Source s to Terminal t

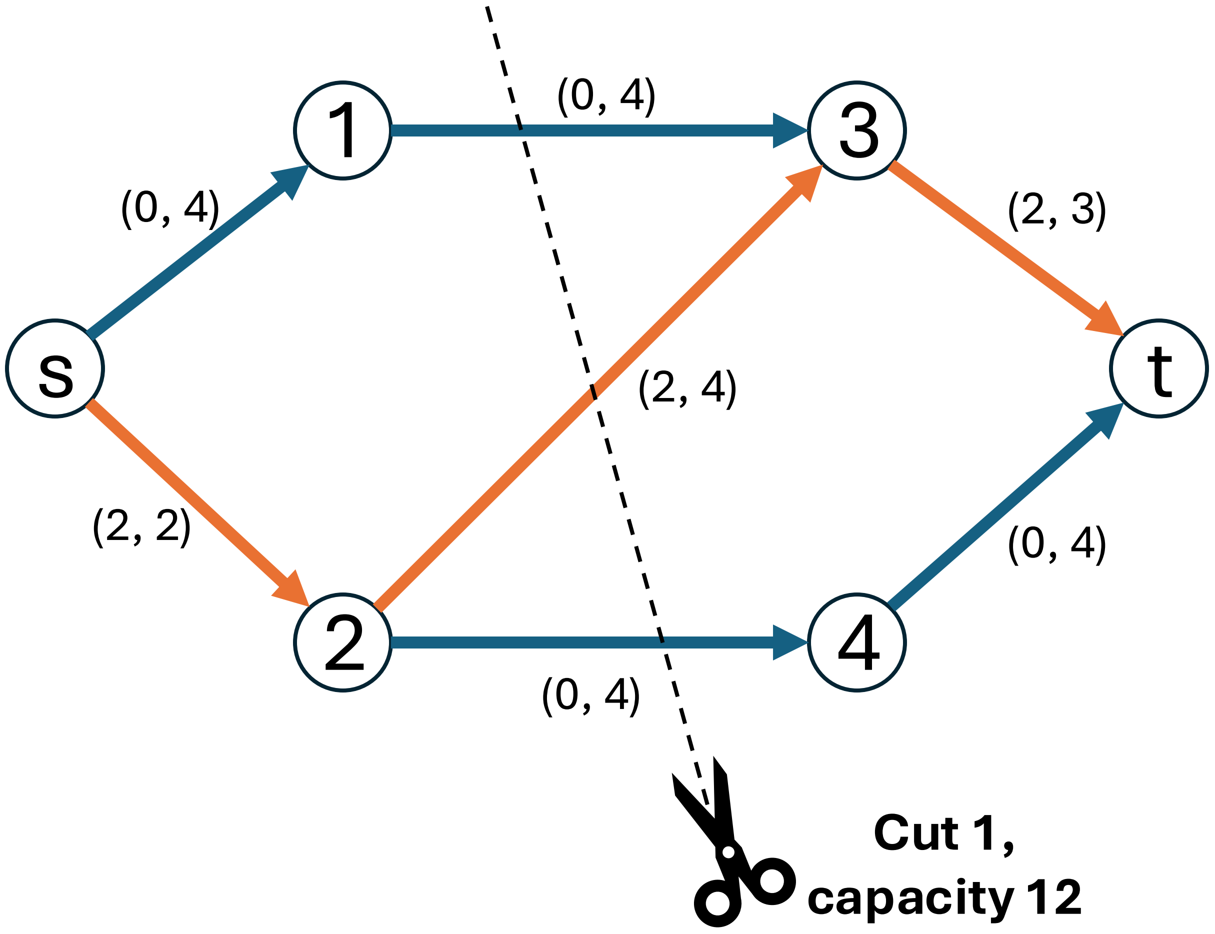


Now sending 5 units from s to t

Now probe s to 1, 1 to 3 and stop

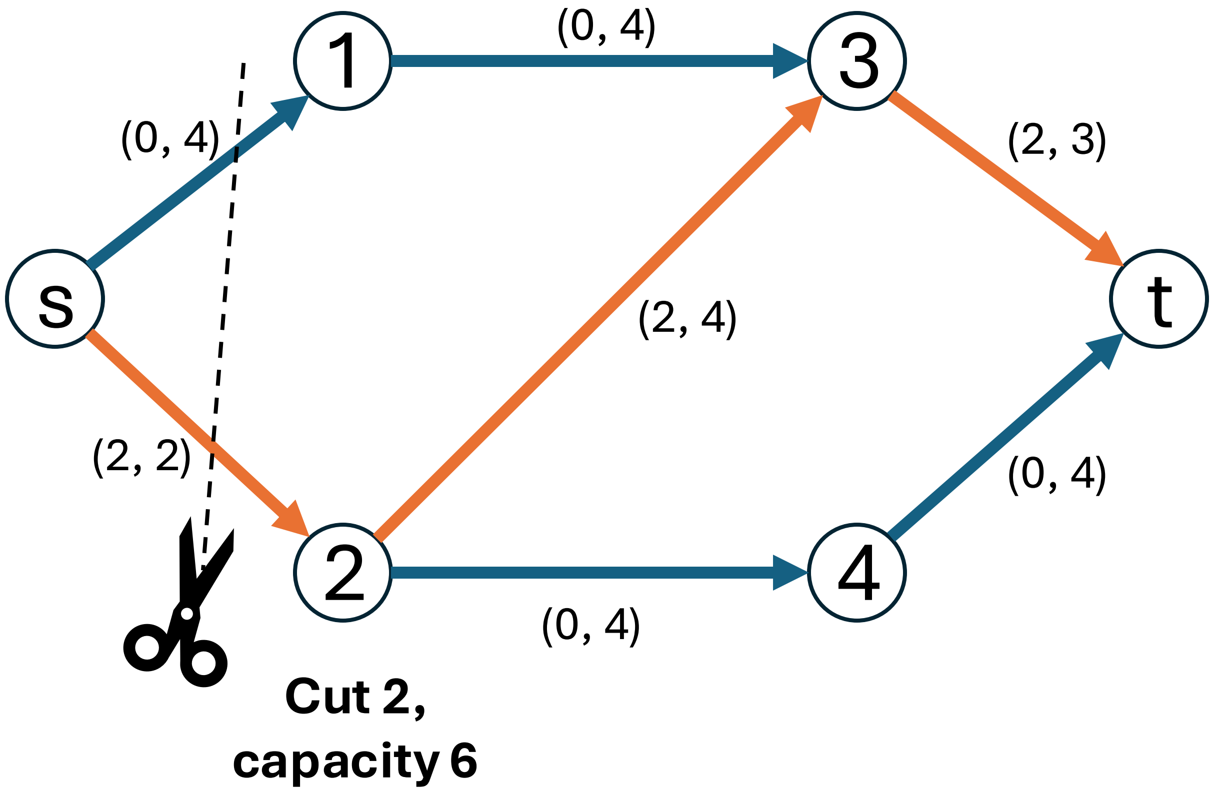
Maximum Flow

From Source s to Terminal t



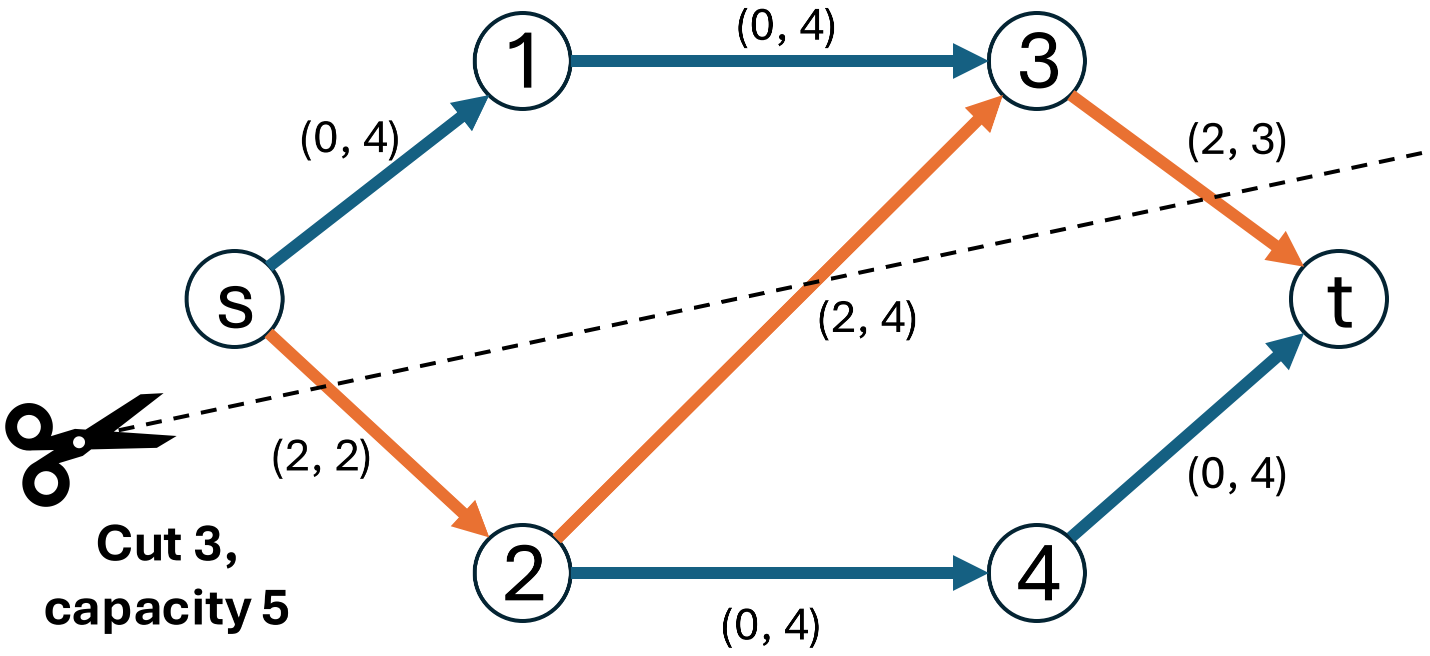
Maximum Flow

From Source s to Terminal t



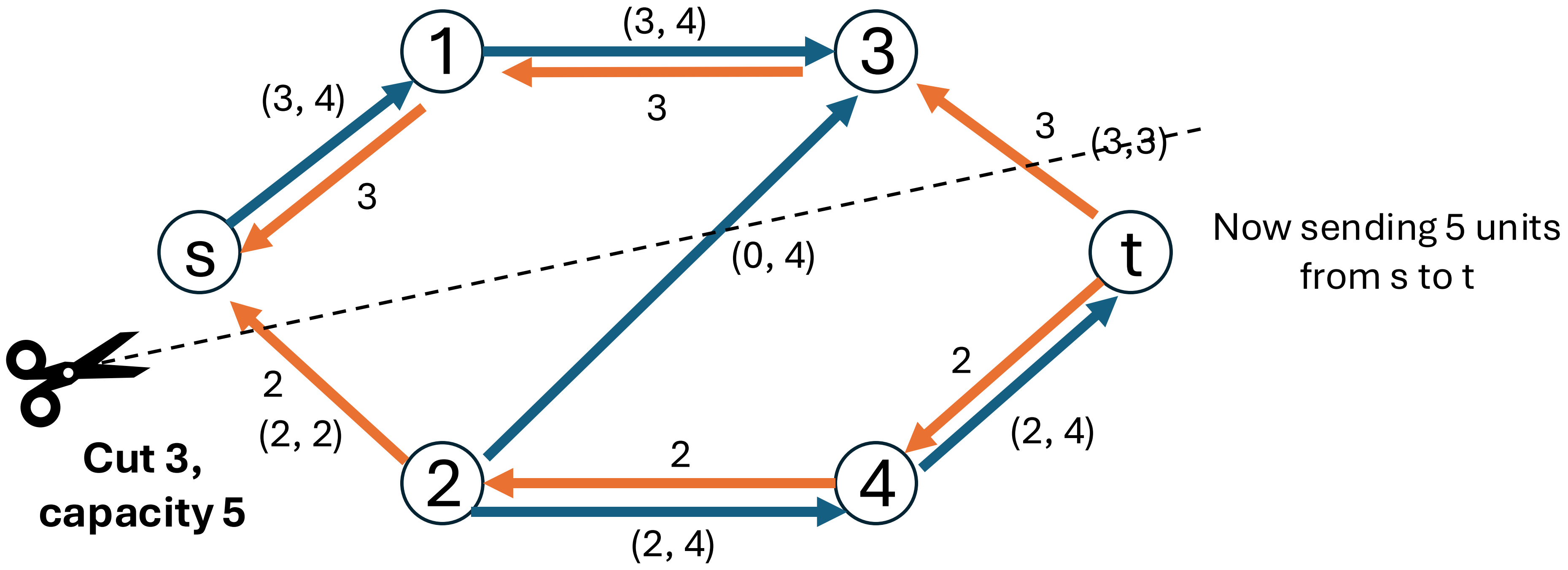
Maximum Flow

From Source s to Terminal t



Maximum flow = minimum cut

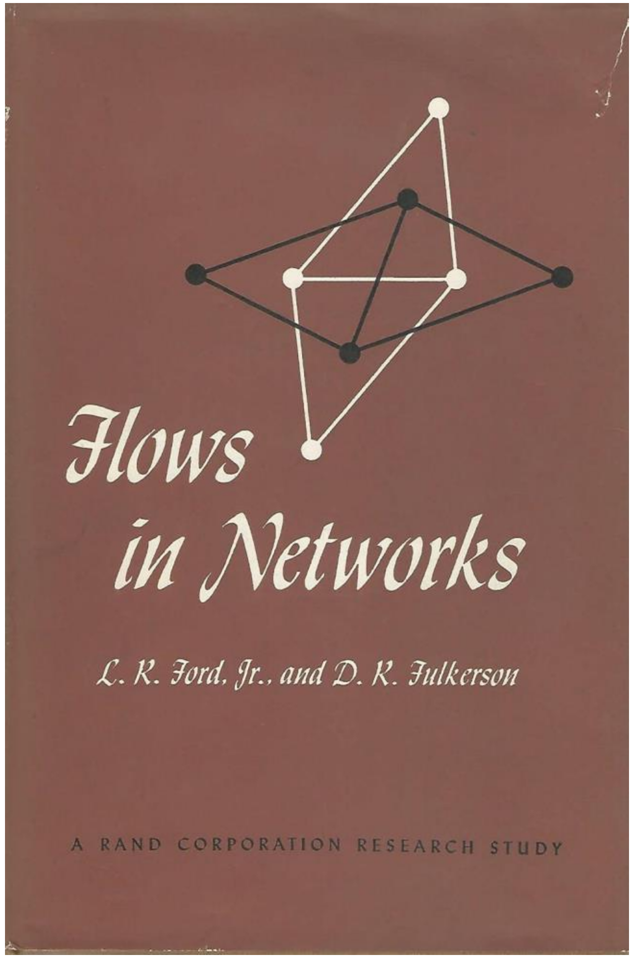
Maximum Flow
From Source s to Terminal t

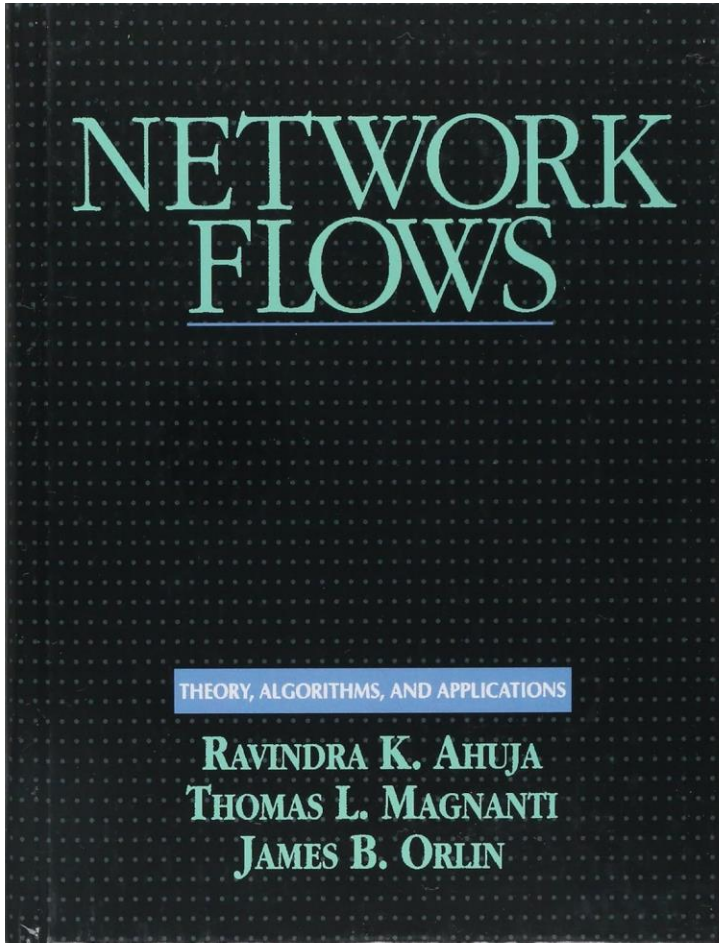


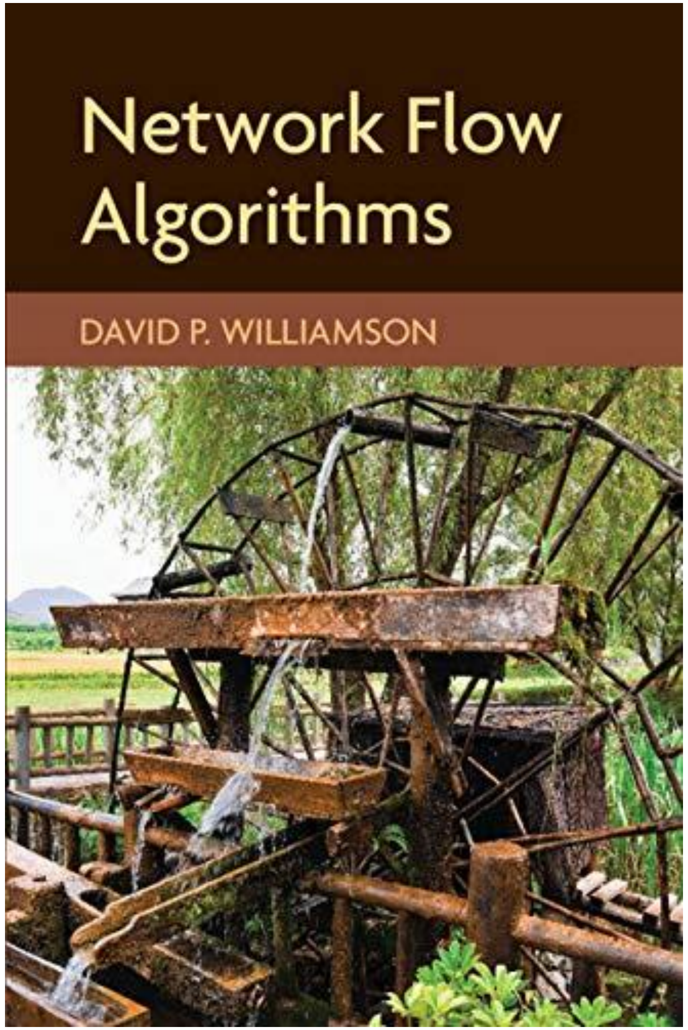
Maximum flow = minimum cut !!

Maximum Flow From Source s to Terminal t

- If capacities are integral (rational), algorithm finds an optimal solution
- The solution found is integral
- Max Flow = Min Cut
  - Has many combinatorial implications
- Breath search provides polynomial time algorithm
  - Edmonds and Karp, Dinic
- Much research on problem over decades

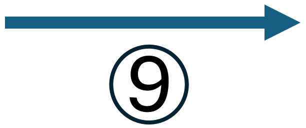




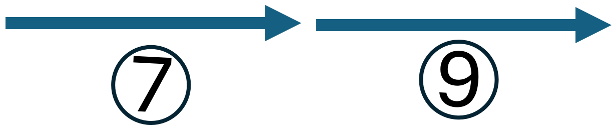


Pathbreaking Contribution

Machine Scheduling
Project Scheduling
Fulkerson, Rand 1964



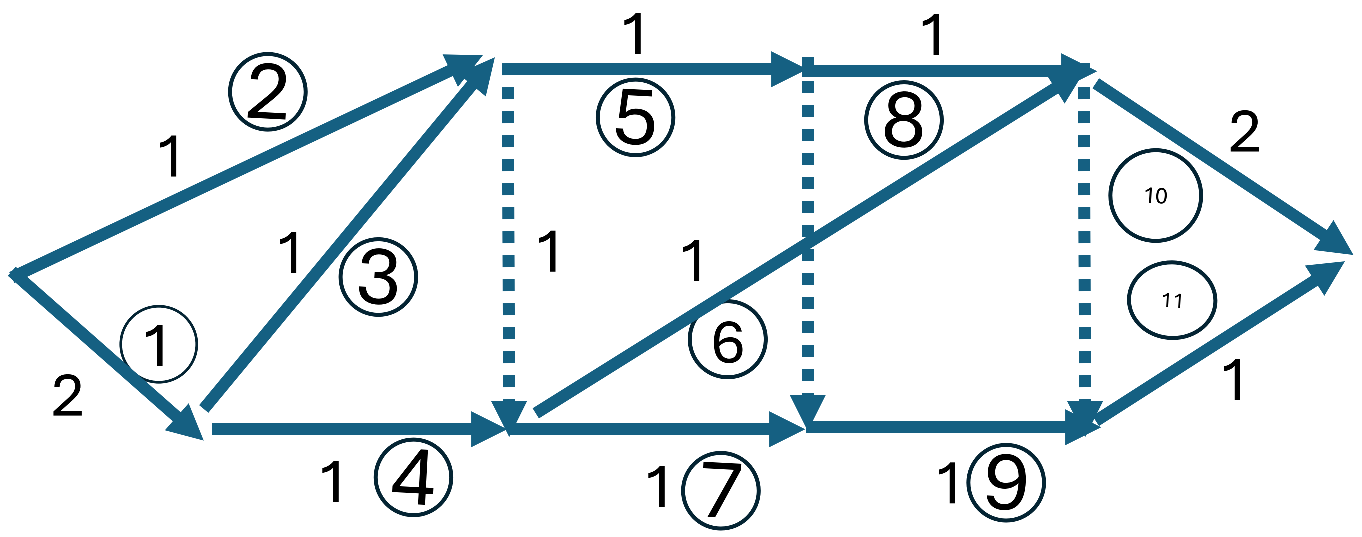
Job with job number



Precedence: if jobs 7 and 9 on same machine then 7 processed before 9

What order to process jobs and what is the smallest number of machines to process all the jobs?

Machine Scheduling
Project Scheduling



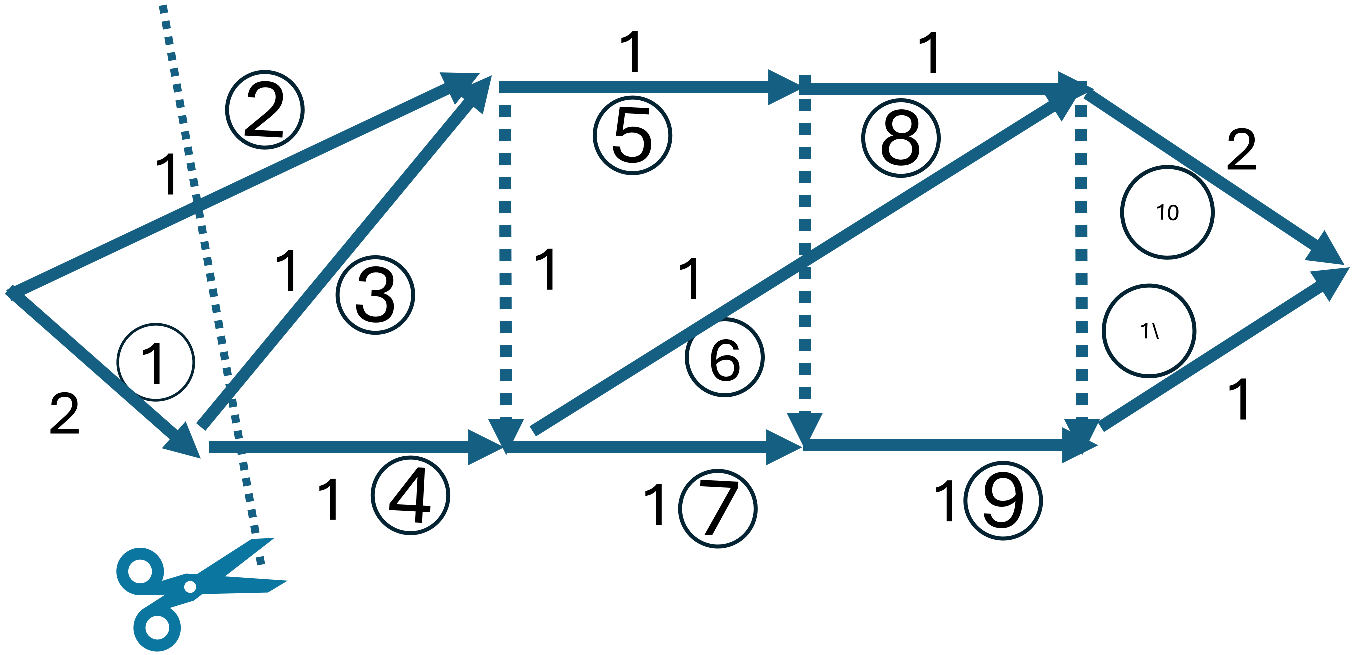
Sending 3 units
from left to right

Min flow 1 on each job arc with infinite capacity
No min or capacity on dashed timing arcs

Min 3 machines with processing jobs

2,5,8,10
1,4,7,9,11
3,6

Machine Scheduling
Project Scheduling



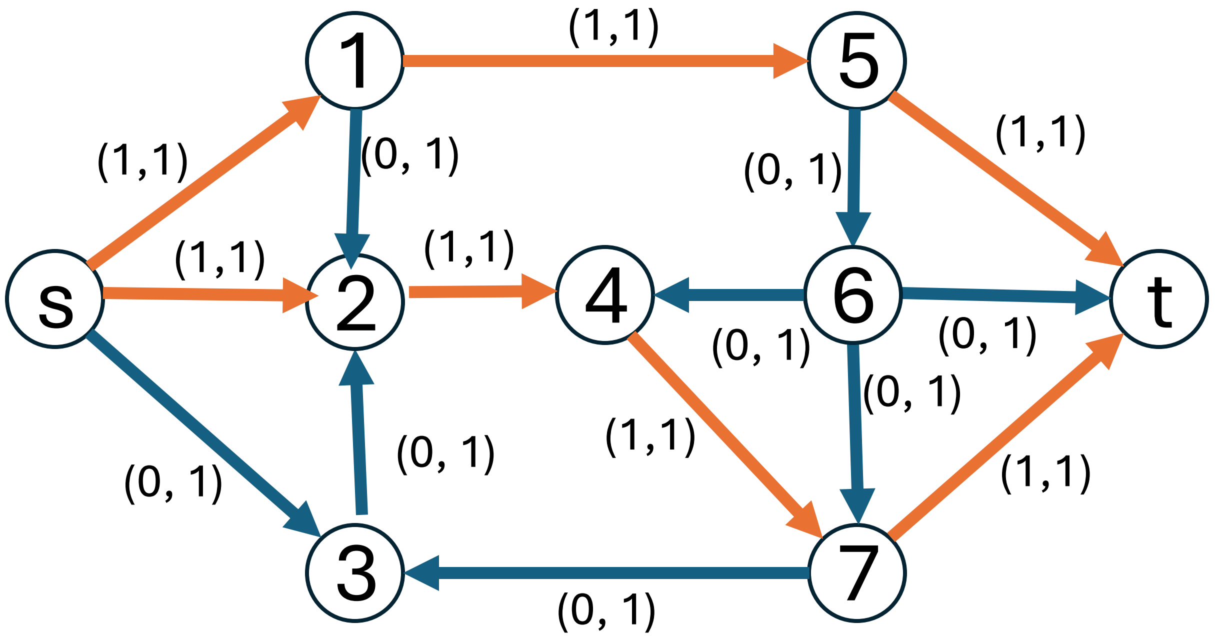
Sending 3 units
from left to right

Max number of jobs (2,3,4) that cannot be processed on same machine

Min number of machines (chains) = Max number of jobs (antichains) that cannot be processed on same machine

Form of Dilworth's Theorem

Menger's Theorem
From Source s to Terminal t



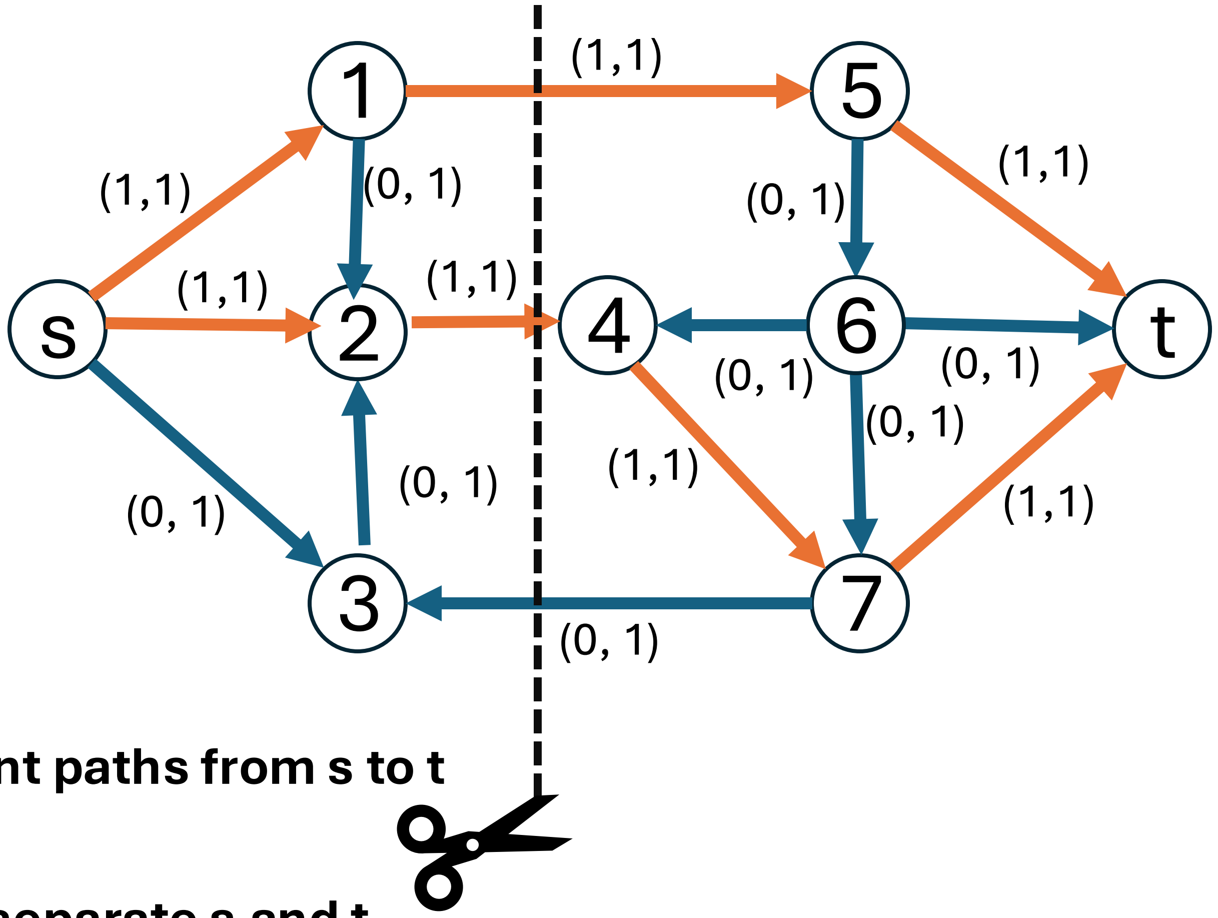
Sending 2 units (paths) from s to t

(x_{ij}, u_{ij})

Current flow $\leq$ Max flow capacity

Menger's Theorem

From Source s to Terminal t



Sending 2 units (paths) from s to t

Max number of arc disjoint paths from s to t

=

Min number of arcs that separate s and t

Menger’s Theorem

- Consider a max flow problem with each arc (edge) having a capacity of 1.
- Then each augmenting flow will send one unit between the source and sink.
- So the max flow will provide the largest number of paths between the source and sink that do not share an arc.
- Any source to sink cutset will be the number of arcs in the cutset.
- So, the max-flow-min-cut theorem states that the maximum number of arc-disjoint (edge-disjoint) paths equals the minimum number of arcs (edges) that have an arc (edge) in common with each path.

Clutters and Blockers

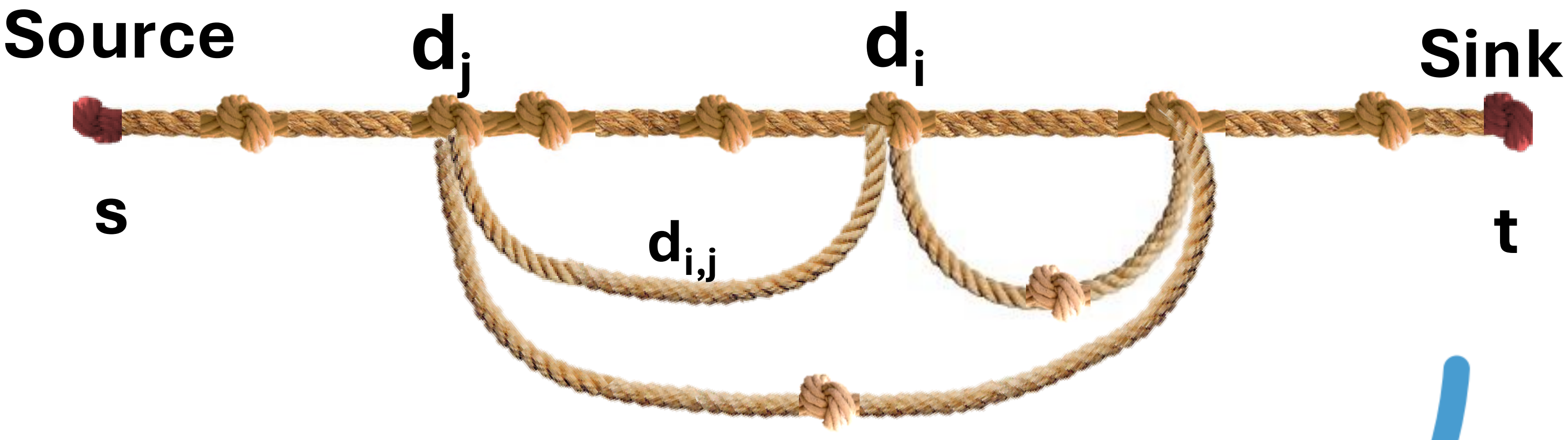
- A collection of sets this called a clutter (or *Sperner family*) if no member of the sets is contained in another.
- A blocker is a collection of minimal sets that intersect each element of a clutter.
- Menger's Theorem has arc-disjoint paths as a clutter and minimal cutsets of edges as blockers.
- Since the minimal cutsets is a clutter, it has a blocker, namely the arc-disjoint paths from the source to the sink. Thus, if C denotes the arc-disjoint paths and b(C) is its blocker, then b(b(C))=C.
- This is a special case of clutters and blockers more generally, an important type of duality in combinatorics proved by *Edmonds and Fulkerson (1970)*. For *Menger's Theorem*, the size of C and b(C) are the same. This is not true in general.

Fulkerson Also Examines

- Non fixed schedule of jobs
  - PERT (Program Evaluation and Review Technique)
  - CPM (Critical Path Method)
- Time compression of jobs

Primal-Dual Method

- Let d_j = shortest path distance from source to sink with arc lengths $d_{i,j}$



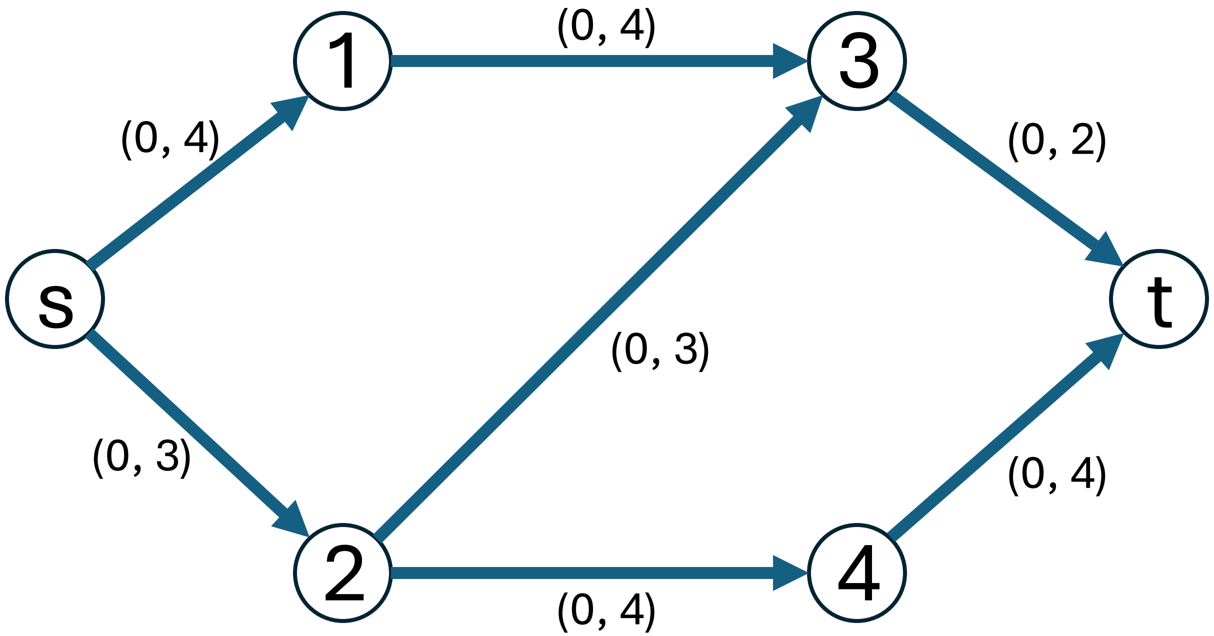
- Every node j, $d_j \leq d_i + d_{i,j}$

Primal-Dual Method

- Start at node t with all d_js = 0.
- Increase all d_js to d_j= 2.
- Add arc (3, t) in the (Primal) network.
- Keep d_3= 2, and increase other d_js to 4.

Dijkstra Shortest Path

From Source s to Terminal t



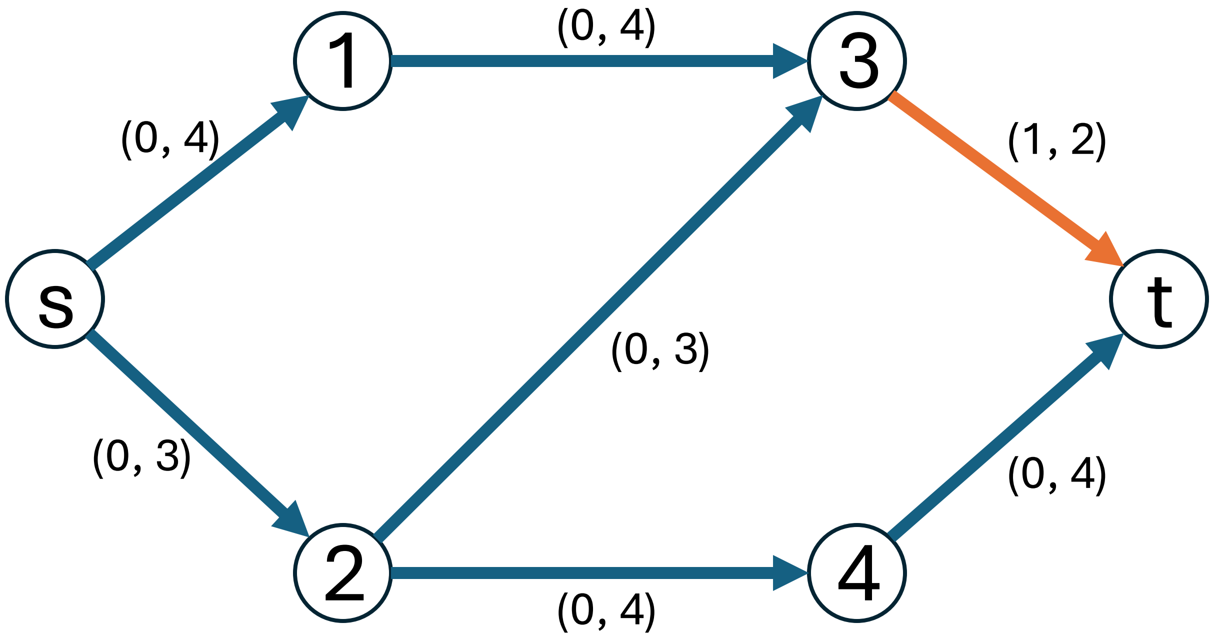
Sending 1 unit from s to t

$$(x_{ij}, dij)$$

Current flow $\leq$ Distance on arc (i,j)

Set all $d_j = 2$

Dijkstra Shortest Path
From Source s to Terminal t



Sending 1 unit from s to t

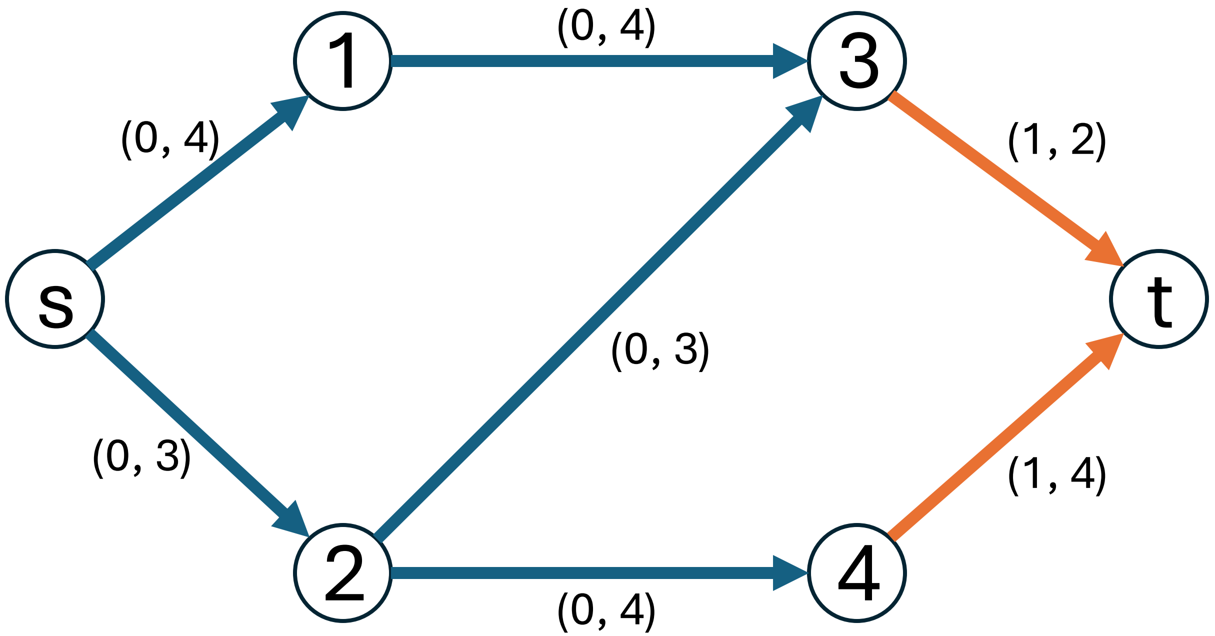
(x_{ij}, dij)

Current flow ≤ Distance on arc (i,j)

Keep d_3=2,
Set all other d_j = 4

Dijkstra Shortest Path

From Source s to Terminal t



Sending 1 unit from s to t

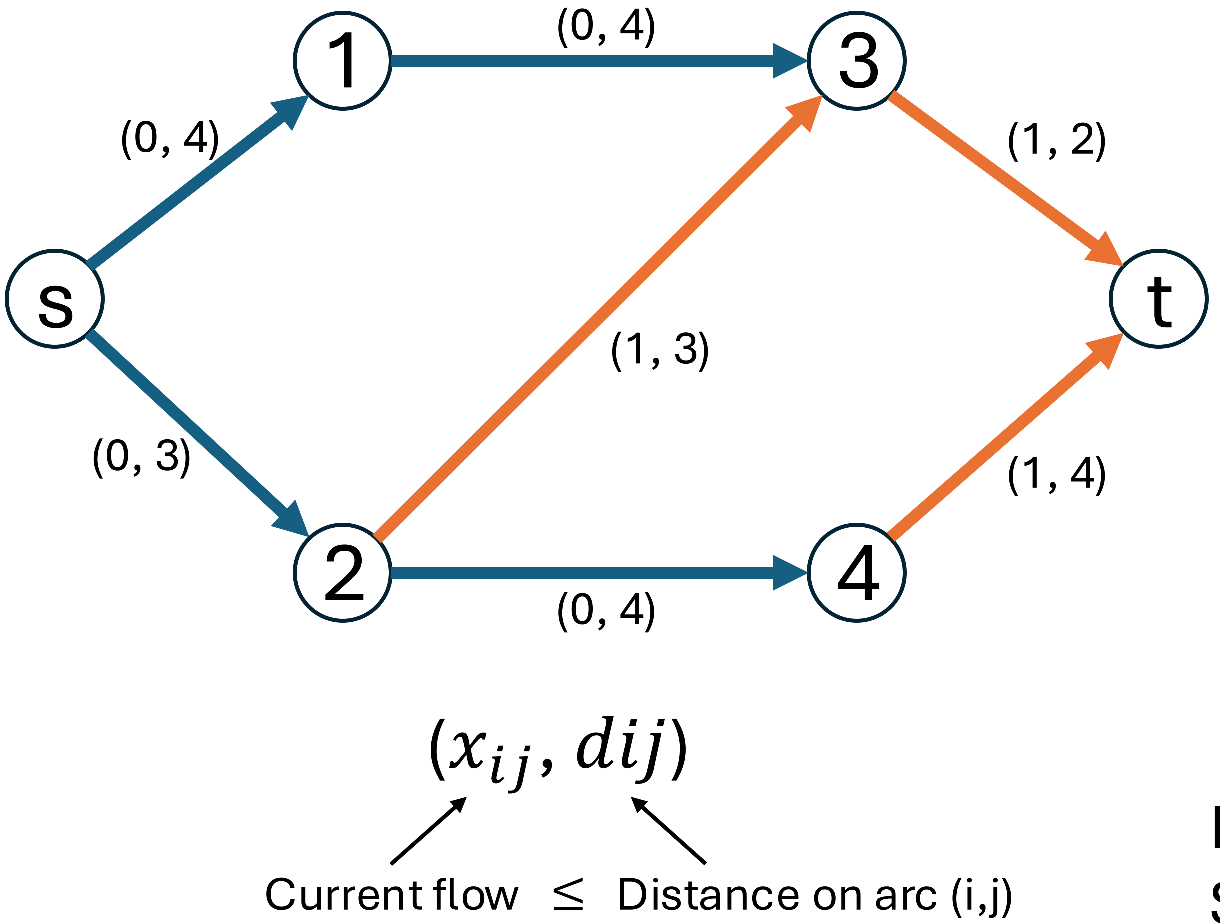
(x_{ij}, dij)

Current flow $\leq$ Distance on arc (i,j)

Keep d_3=2, d_4=4
Set all other d_j = 5

Dijkstra Shortest Path

From Source s to Terminal t



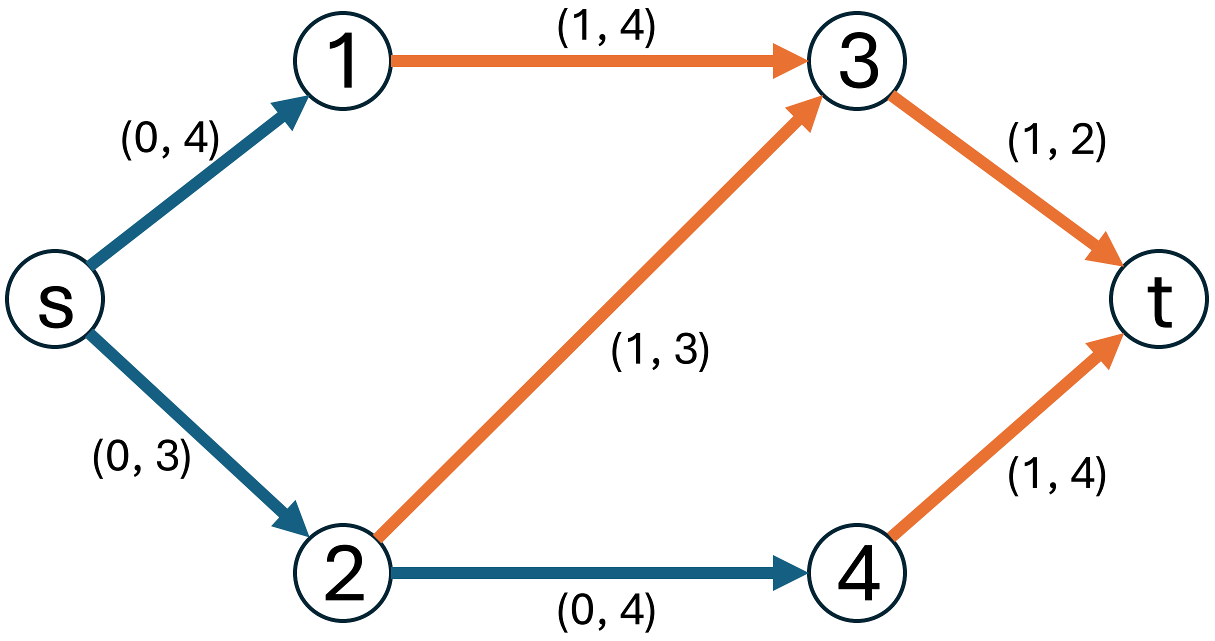
Sending 1 unit from s to t

Keep d_3=2, d_4=4, d_2=5

Set all other d_j = 6

Dijkstra Shortest Path

From Source s to Terminal t



Sending 1 unit from s to t

$$(x_{ij}, dij)$$

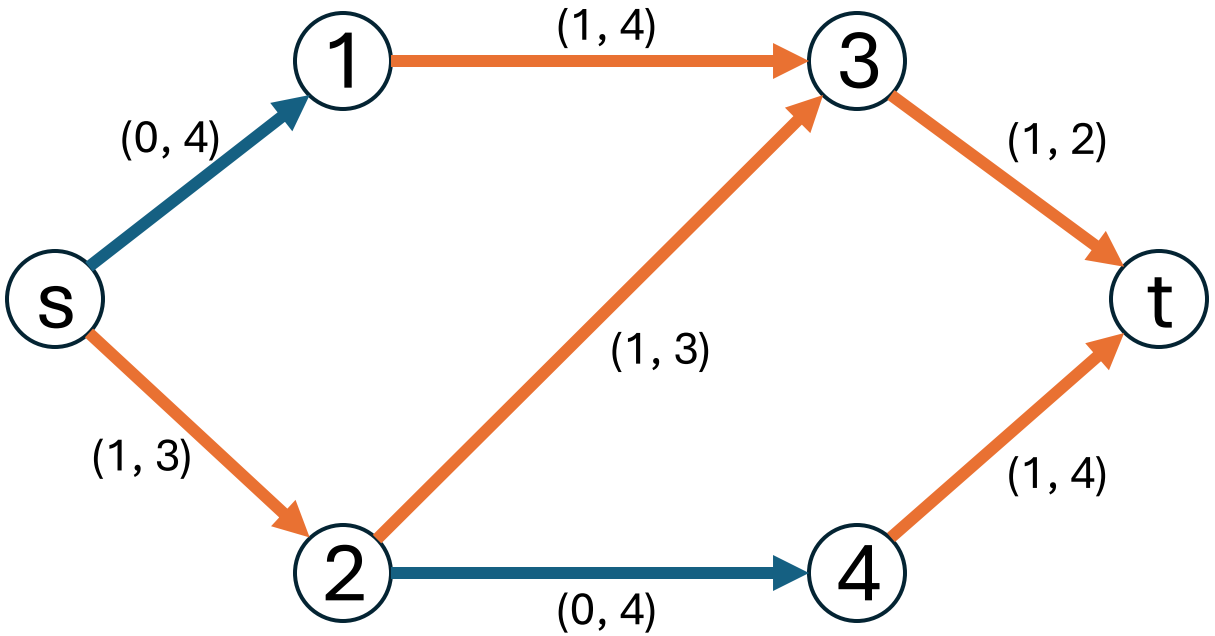
Current flow $\leq$ Distance on arc (i,j)

Keep d_3=2, d_4=4, d_2=5, d_1=6

Set d_s = 8

Dijkstra Shortest Path

From Source s to Terminal t



Sending 1 unit from s to t

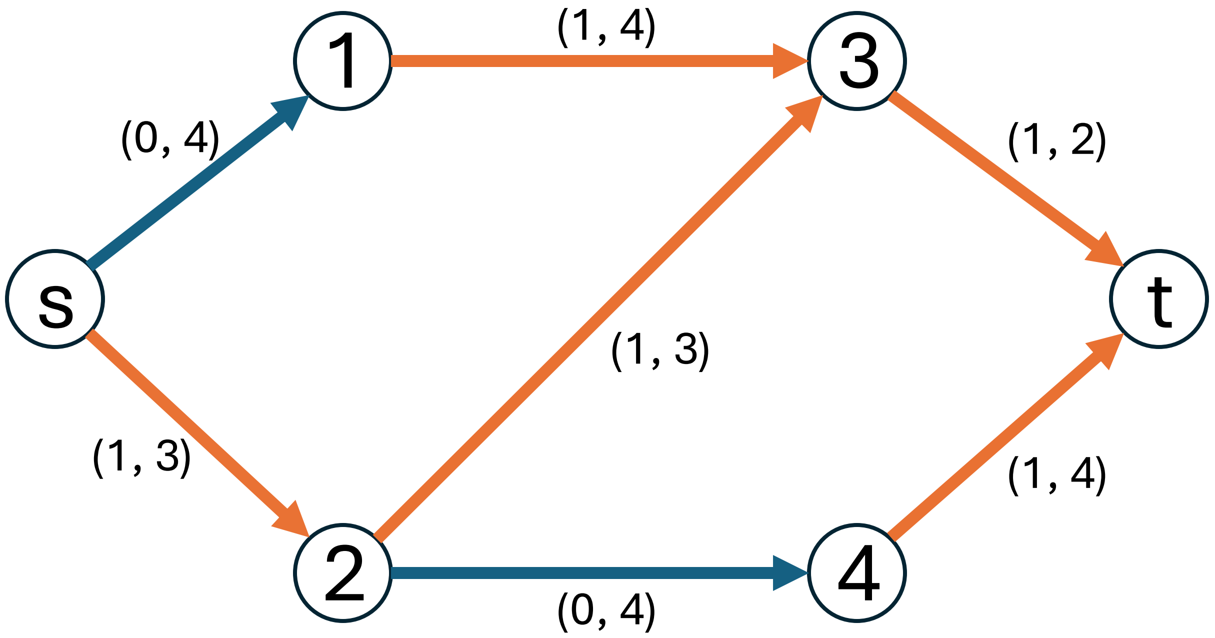
(x_{ij}, dij)

Current flow ≤ Distance on arc (i,j)

We now have the shortest path from s to t, s – 2 – 3 – t

Dijkstra Shortest Path

From Source s to Terminal t



Sending 1 unit from s to t

$$(x_{ij}, dij)$$

Current flow $\leq$ Distance on arc (i,j)

Primal-dual More Generally

Dantzig, Ford, Fulkerson (1956)

- For linear programming, start with a dual feasible solution and, from the solution, derive a partial feasible primal solution.
- Change the dual feasible solution and find another partial primal feasible solution.
- Continue until both dual and primal solutions are fully feasible.
- Approach uses extensively for combinatorial optimization approximation methods.

Column Generation (Ford and Fulkerson, 1958)

Multicommodity Maximum Flow Problem

- Variable for each path from each source to its destination.
- Constraints state that the sum of path flows on any arc cannot exceed the arc capacity.
- Number of paths exponential to the size of network.

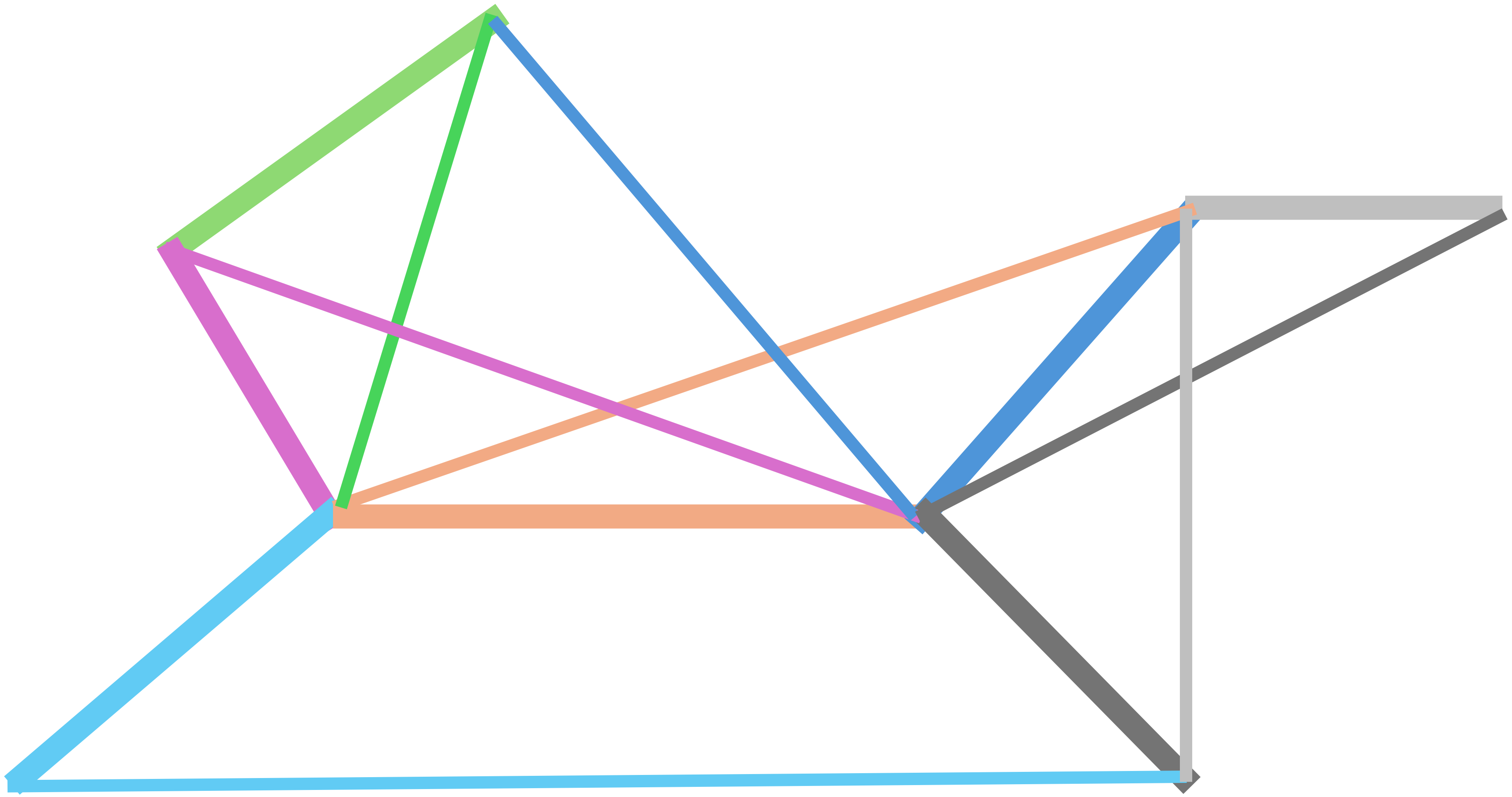
Approach

- Use some paths and solve the “master” problem.
- See if another path can provide a better solution and, if so, add the path (column) to the master problem. This can be done by solving a shortest path problem.

Some Applications of Column Generation

- Gilmore & Gomory Cutting Stock Problems (1965)
- Dantzig and Wolfe Decomposition (1960)
- Aircrew Scheduling
- Economic Planning

Complementary Spanning Trees of Network



Complementary Spanning Trees of Network

One morning, Fulkerson beats Dantzig one dollar for something Dantzig did not know about the network.

He describes necessary and sufficient conditions for a complementary tree exists.
Transversals and Matroid Partition, Edmonds and Fulkerson (1956)

The next day, Dantzig beats Fulkerson that he knows something about a network that Fulkerson did not know.

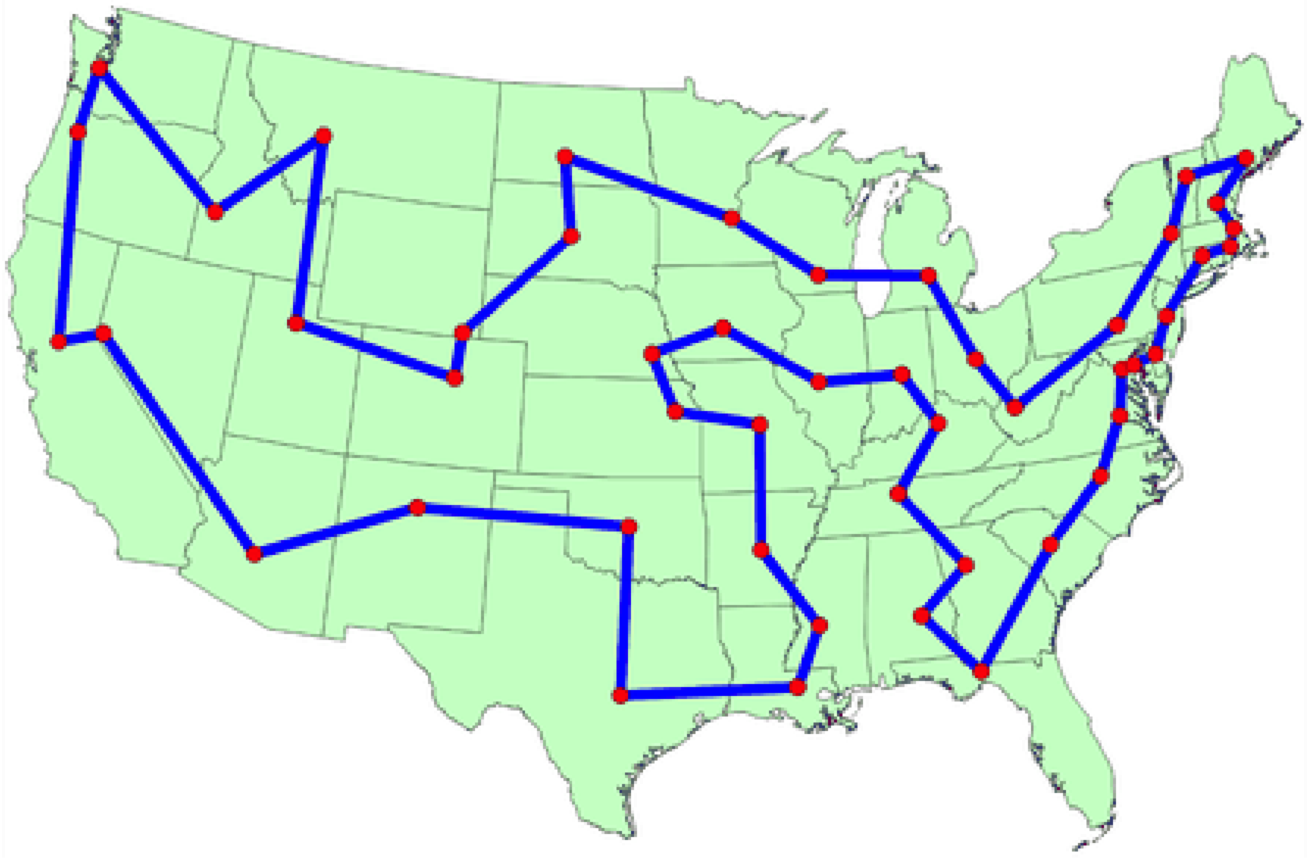
"***If there is one tree, there are at least two (complementary trees).*** "

Ian Adler later showed ***if there is a tree, there are at least four (complementary trees)***

Magnanti showed that ***if the smallest circuit has k edges, then at least*** 2^{k-1} ***(complementary trees) exist***

Dantzig, Fulkerson, Johnson (1954)

49 City Traveling Salesman Problem (TSP)

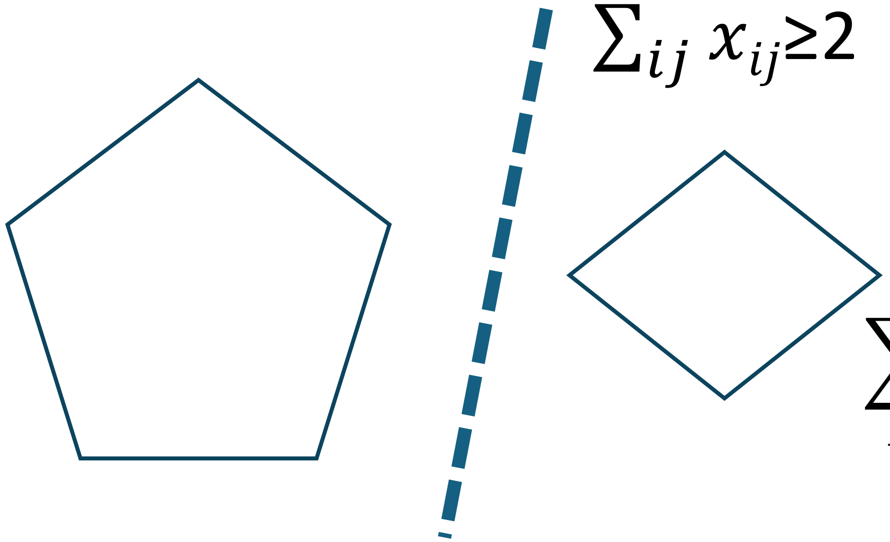


Optimal TSP Tour

Dantzig, Fulkerson, Johnson 49 City TSP

$\sum_{ij} x_{ij} \geq 2$

$$\sum_j x_{ij} = 2$$



$$\sum_j x_{ij} = 2$$

After adding 23 subtour breaking constraints and two others (comb. inequalities) find optimal solution

Used warm start (heuristic), preprocessing, LP, cuts, probing, variable fixing, reduced costs to eliminate arcs, elements of branch and bound

Chvảtal and Cook call this paper the 'Big Bang' of TSP and integer programming

Fulkerson Prize in Discrete Mathematics

Mathematical Optimization Society and American Mathematics Society

- Many famous mathematicians
- From Cornell
  - Eva Tardos
  - Louis Billera
  - David Williamson
- Also, Gerard Corneuljois

Awards and Recognition

Lester R. Ford Award from the Mathematical Association of America for his paper “Flow Networks and Combinatorial Operations Research.” 1967

Southern Illinois University Award for Outstanding Professional Achievement. 1972

Honorable mention for the Lanchester Prize in 1954 and 1961.

Fulkerson Prize

One thing not considered

Out-of-Kilter Method

- Fulkerson (1965)
- George Minty (1961)
- A general method for solving minimum cost network flow problems
- See Andrew Goldberg tomorrow.

To the magnificent
Ray Fulkerson

Happy Centennial!