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1. a.

$$\frac{d\vec{\mu}}{dt} = \gamma \vec{\mu} \times \vec{B_0}$$
$$= \vec{\mu} \gamma \times |B_0| \vec{z}$$
$$= \omega_0 \vec{\mu} \times \vec{z}$$

$$\Longrightarrow \begin{cases} \frac{d\mu_x}{dt} = \omega_0 \mu_y \\ \frac{d\mu_y}{dt} = -\omega_0 \mu_x \\ \frac{d\mu_z}{dt} = 0 \end{cases}$$

b. $$\begin{cases} \frac{d^2\mu_x}{dt^2} = \omega_0 \cdot \frac{d\mu_y}{dt} = -\omega_0^2 \mu_x \\ \frac{d^2\mu_y}{dt} = -\omega_0 \cdot \frac{d\mu_x}{dt} = -\omega_0^2 \mu_y \end{cases}$$

c. $$\frac{d^2\mu_x}{dt^2} = -\omega_0^2 \mu_x$$

$$\Rightarrow \frac{d^2\mu_x}{dt^2} + \omega_0^2 \mu_x = 0$$

The solution form is $\mu = e^{kt}$

$$k^2 e^{kt} + \omega_0^2 e^{kt} = 0$$
$$k^2 = -\omega_0^2$$
$$k = -\omega_0 i$$

$$\begin{cases} \mu_x = e^{-\omega_0 i t} \\ \mu_y = e^{\omega_0 i t} \end{cases}$$

2. $\theta = \cos^{-1}(\exp(-TR/T_1))$

Here, θ is so-called Ernst Angle. In MRI applications, we may assume that T_1 is a constant in a specific organ. When we know the value of TR, the optimal flip angle, which is the Ernst angle, is given by Ernst Equation. At these angles, the MR signal will be maximized

3. $$P_m \propto \exp(-E_m/kT)$$

$$\Rightarrow P_m = A \cdot \exp(-E_m/kT)$$

for protons in anti-parallel & parallel state

$$\begin{cases} P_{anti} = A\exp(-E_{anti}/kT) \\ P_{para} = A\exp(-E_{para}/kT) \end{cases}$$

$$\Rightarrow \frac{P_{anti}}{P_{para}} = \exp\left(-\frac{E_{anti}}{kT} + \frac{E_{para}}{kT}\right)$$
$$= \exp\left(-\frac{E_{anti} - E_{para}}{kT}\right)$$

4. According to Faraday's Law, EMF is given by the rate of change of the magnetic flux

$$\mathcal{E} = -\frac{d\phi_B}{dt} \qquad ①$$

$\mathcal{E}$ is electromotive force (EMF), ϕ_B is magnetic flux

Given $\vec{B}(t) = B(\sin\theta\,\vec{y} + \cos\theta\,\vec{z})\sin\omega t \qquad ②$

$$d\vec{S} = dx\,dy \cdot \vec{z} \qquad ③$$

②③ → ①

$$\Rightarrow \mathcal{E} = -\frac{d}{dt}\int_{\mathbb{R}} \vec{B}(t)\cdot d\vec{S}$$

$$= -\frac{d}{dt}\left(\vec{B}(t)\cdot L^2 \cdot \vec{z}\right)$$

$$= -L^2 B\,\vec{z}\cdot\omega(\sin\theta\,\vec{y} + \cos\theta\,\vec{z})\cos\omega t$$

$$\because \vec{y}\cdot\vec{z} = 0,\ (\vec{z})^2 = 1$$

$$\therefore \mathcal{E} = -L^2 B\omega\cdot\cos\theta\cdot\cos\omega t$$

QED