HW 02 Yifu Huang yh9692

1. $f(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-t^2}{\sigma^2}\right)$

$$\int_{-\infty}^{+\infty} f(t)\,dt = \frac{1}{\sigma\sqrt{2\pi}} \int e^{-\frac{t^2}{\sigma^2}} dt =$$

$$\boxed{\int_{-\infty}^{+\infty} e^{-A^2x^2} dx = \frac{\sqrt{\pi}}{A}}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \cdot \sigma \cdot \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2}}$$

define: $g(t) = \exp\left(-\frac{t^2}{\sigma^2}\right)$

$$\Rightarrow G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{\sigma^2}} e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{\sigma^2}} (\cos\omega t - i\sin\omega t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{\sigma^2}} \cos\omega t\, dt$$

$$G'(\omega) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} t e^{-\frac{t^2}{\sigma^2}} \sin\omega t\, dt$$

Define: $u = \sin\omega t \quad v' = -t\exp(-t^2/\sigma^2)$

$\Rightarrow v = \frac{\sigma^2}{2} e^{-\frac{t^2}{\sigma^2}} \quad u' = \omega\cos\omega t$

$$\Rightarrow G'(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u v' \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left(uv - \int_{-\infty}^{+\infty} u' v \, dt \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{\sigma^2}{2} e^{-\frac{t^2}{\sigma^2}} \sin \omega t - \frac{\sigma^2}{2} \omega \int_{-\infty}^{+\infty} e^{-\frac{t^2}{\sigma^2}} \cos \omega t \, dt \right)$$

$$= \frac{-\sigma^2 \omega}{2} G(\omega)$$

$$\Rightarrow G(\omega) = D \cdot \exp\left(-\frac{1}{2}\sigma^2 \omega^2\right)$$

$$G(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{\sigma^2}} dt = \frac{1}{\sqrt{2\pi}} \sqrt{\pi} \cdot \sigma$$

$$= \frac{\sigma}{\sqrt{2}} = D$$

$$\Rightarrow G(\omega) = \frac{\sigma}{\sqrt{2}} e^{-\frac{1}{2}\sigma^2 \omega^2}$$

for $f(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{\sigma^2}}$

Fourier transform: $F(\omega) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{2}\sigma^2 \omega^2}$ *

2. $f(t) = \begin{cases} e^{\alpha t}, & t<0 \\ e^{-\alpha t}, & t>0 \end{cases}$

$$\Rightarrow f(t) = e^{-\alpha|t|}$$

$$F(\omega) = \int_{-\infty}^{+\infty} e^{-\alpha|t|} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{\alpha t} e^{-j\omega t} dt + \int_{0}^{+\infty} e^{-\alpha t} e^{-j\omega t} dt$$

$$= \frac{1}{\alpha - j\omega} + \frac{1}{\alpha + j\omega}$$

$$= \frac{2\alpha}{\alpha^2 + \omega^2} \quad *$$

3. Compton equation

① $\begin{cases} h(\nu_1-\nu_2)+m_e c^2 = E_e \\ \vec{p}_e = \vec{p}_1 - \vec{p}_2 \end{cases}$

$p = \frac{m_0}{\sqrt{1-\beta^2}} V = m_e V$, here $\beta = \frac{V}{c}$ is the relative constant for mass of electron.

The kinetic energy of electron.

$$E_k = m_e c^2 - m_0 c^2 = m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)$$

$\because \beta \ll 1$

$$\therefore E_k = m_0 c^2 \left(1 + \frac{1}{2}\cdot\frac{V^2}{c^2} + \cdots\right) - m_0 c^2 \approx \frac{m_0^2 V^2}{2m_0}$$

$$= \frac{p^2}{2m_0}$$

$\therefore$ ② $\begin{cases} E_e = m_e c^2 \\ E_e^2 = p_e^2 c^2 + m_e^2 c^4 \end{cases}$

② $\longrightarrow$ ①

$$\Rightarrow (h(\nu_1-\nu_2)+m_e c^2)^2 = |\vec{p}_1 - \vec{p}_2|^2 + m_e^2 c^4$$

$$\Rightarrow -2h^2\nu_1\nu_2 + 2m_e^2 hc^2(\nu_1-\nu_2) = -2h^2\nu_1\nu_2\cos\theta$$

$\Rightarrow \begin{cases} m_e c^2(\nu_1-\nu_2) = h\nu_1\nu_2(1-\cos\theta) \\ m_e c^2\left(\frac{1}{\nu_2}-\frac{1}{\nu_1}\right) = h(1-\cos\theta) \\ \frac{1}{\nu_2}-\frac{1}{\nu_1} = \frac{h}{m_e c^2}(1-\cos\theta) \quad (*) \end{cases}$

$\because \lambda = \frac{c}{\nu}$

$\therefore$ from $(*)$ we have

$$\Delta\lambda = \frac{c}{\nu_2} - \frac{c}{\nu_1} = \lambda_2 - \lambda_1 = \frac{h}{m_e c}(1-\cos\theta)$$

(m_e is the mass of electron)

4.

$$\boxed{C_p} \underset{k_2}{\overset{k_1}{\rightleftharpoons}} \boxed{C_{EES}} \xrightarrow{k_3} \boxed{C_{EIS}}$$

(a)

$$\begin{cases} \frac{dC_p}{dt} = -k_1 C_p + k_2 C_{EES} \\ \frac{dC_{EES}}{dt} = k_1 C_p - k_2 C_{EES} - k_3 C_{EES} \\ \frac{dC_{EIS}}{dt} = k_3 C_{EES} \end{cases}$$

(b)

$$\begin{cases} \frac{dC_p}{dt} = -k_1 C_p + k_2 C_{EES} \\ \frac{dC_{EES}}{dt} = k_1 C_p - k_2 C_{EES} \end{cases}$$

Laplace transform:

$$\begin{cases} s C_p' = -k_1 C_p' + k_2 C_{EES}' \\ s \cdot C_{EES}' = k_1 C_p' - k_2 C_{EES}' \end{cases}$$

$$C_p' = \frac{k_2}{s+k_1} C_{EES}'$$

$$\Rightarrow C_p(t) = k_2 e^{-k_1 t} * C_{EES}(t)$$

(c) In FDG-PET study, by solving the equations and get the value of C_p, C_{EES} and C_{EIS}, we can get the peramaters for reconstruction of the PET image.

5. $\phi(x,t) = f(x-ct)$

(a)

$$\begin{cases} \dfrac{\partial \phi}{\partial t} = -cf'(x-ct) \\ \dfrac{\partial \phi}{\partial x} = f'(x-ct) \end{cases}$$

$$\Rightarrow \frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = -cf'(x-ct) + cf'(x-ct) = 0$$

This represents wave propagation to the direction of speed c because $\phi(x,t) = f(x-ct) =$ Constant when the input $x-ct$ is a constant.

Assume $x-ct = u$

$$\begin{cases} \dfrac{\partial \phi}{\partial t} = \dfrac{df}{du} \cdot \dfrac{\partial u}{\partial t} = \dfrac{df}{du} \cdot (-c) \\ \dfrac{\partial \phi}{\partial x} = \dfrac{df}{du} \cdot \dfrac{\partial u}{\partial x} = \dfrac{df}{du} \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial t^2} = c^2 \dfrac{d^2 f}{du^2} \\ \dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{d^2 f}{du^2} \end{cases}$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \longrightarrow \text{Wave equation!}$$

$\phi(x,t) = f(x-ct)$ is the wave function. It describes the status of space and time of a given point by using x and t.

(b) We have the wave equation

$$\frac{\partial^2\phi}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=0$$

Define $\phi(x,t)=\Phi(x)f(t)$

$$\Rightarrow f\frac{d^2\Phi}{dx^2}-\frac{1}{c^2}\cdot\Phi\frac{d^2f}{dt^2}=0$$

$$c^2f\frac{d^2\Phi}{dx^2}=\Phi\frac{d^2f}{dt^2}=-\lambda$$

$$\begin{cases} f''(t)+\lambda f(t)=0 \\ \Phi''(x)+\frac{\lambda}{c^2}\Phi(x)=0 \end{cases} \quad (*)$$

Solve the equation set $(*)$, and the form of solution should be $y=Ae^{at+bx}$

$$\Rightarrow \begin{aligned} &f(t)=a_1\sin(\omega t+\theta) \\ &\Phi(x)=a_2\sin kx+a_3\cos kx \\ &\omega^2=\lambda,\ k=\frac{\omega}{c} \end{aligned}$$

$$\Rightarrow \phi(x,t)=A\exp[i(\omega t\pm kx)]$$

6. $\alpha = \frac{(Z_2 - Z_1)^2}{(Z_2 + Z_1)^2}$

$= \left(\frac{1.66 - 1.35}{1.66 + 1.35}\right)^2$

$= 1.06\%$

It doesn't matter from which direction the incident energy arrives.

The intensity reflection coefficient α is decided only by the difference between 2 materials the wave is going through. The greater the difference in acoustic impedence between the two media, the greater the reflection and the smaller the transmission.

7. a). $t = \dfrac{2 \times 2\,cm}{1450\,m/s}$

$= \dfrac{0.04\,m}{1450\,m/s}$

$= 2.76 \times 10^{-5}\,s$

$= 27.6\,ns$

b) The amplitude attenuation factor of fat is 0.63 dB/cm/MHz

At 5 MHz in Np

$-\mu_a = \dfrac{0.63 \times 5}{8.686}\ Np/cm$

$= 0.363\ Np/cm$

$\dfrac{A_z}{A_0} = \exp(-\mu_a \cdot d)$

$= \exp(0.363 \times 4)$

$= 0.234$

$R = 0.0106^{0.5} = 0.103$

$LOSS = 20 \log_{10} \dfrac{A_z}{A_0} R$

$= 20 \log_{10}(0.234 \times 0.103)$

$= -32.4\ dB$