Proof of the Atmospheric Greenhouse Effect

Arthur P. Smith[*]

American Physical Society, 1 Research Road, Ridge NY, 11961

A recently advanced argument against the atmospheric greenhouse effect is refuted. A planet without an infrared absorbing atmosphere is mathematically constrained to have an average temperature less than or equal to the effective radiating temperature. Observed parameters for Earth prove that without infrared absorption by the atmosphere, the average temperature of Earth's surface would be at least 33 K lower than what is observed.

I. INTRODUCTION

The results presented here are not new. However the form of presentation is designed to clearly and accurately respond to recent claims[1] that a physics-based analysis can "falsify" the atmospheric greenhouse effect. In fact, the standard presentation in climatology textbooks[2] is accurate in all material respects. The following explores in more detail certain points that seem to have been cause for confusion.

First presented are the definitions of basic terms and the relevant equations for the flow of energy. The situation for a planet with no infrared-absorbing atmosphere is then examined, and a constraint on average temperature is proved.

Several specific models of planets with no infrared-absorbing atmosphere are then solved, including one presented by Gerlich and Tscheuschner[1], and it is verified that all satisfy this constraint.

A simple infrared-absorbing atmospheric layer is added to these models, and it is proved that the temperature constraint is easily violated, as is shown by the observational data for Earth.

II. DEFINITIONS AND BASIC EQUATIONS

Define the incoming irradiance S as the energy per unit area and per unit time arriving at a planet from a stellar source. The actual radiation field is characterized by a spectrum of wavelengths and (depending on the size of the star(s) and distance to the planet) a small spread in directions. S is an integral over all wavelengths and propagation directions of the radiant specific energy at the distance of the planet from the star. As the planet moves through its yearly orbit, the value of S will vary, so it is strictly a function $S(t)$ of time.

The energy per unit time arriving at the planet is the product of $S(t)$ with the area the planet subtends in the plane perpendicular to the radiant propagation direction. For a spherical planet of radius r, this area is simply πr^2. So total arriving energy (energy per unit time, or power) from space is

$$E_{in}(t) = \pi r^2 S(t) \tag{1}$$

For a given location $\mathbf{x}$ on the planetary surface, the normal to the plane of the local surface makes an angle $\theta(\mathbf{x}, t)$ with the radiation propagation direction of a given stellar source. Only one side of the planet will be lit at any given time from that source; this can be generally indicated by those angles θ from 0 to $\pi/2$ radians. Angles from $\pi/2$ to π would be unlit. So local incoming irradiance would be:

$$s(\mathbf{x}, t) = \cos(\theta(\mathbf{x}, t))S(t) \tag{2}$$

Integrating this on a spherical planet (including only the lit side) gives the πr^2 factor in equation 1.

Define the albedo a of the planet as the fraction of incoming irradiance that is reflected. a is also a local property (for Earth much is reflected by clouds and ice), so the locally reflected energy is $a(\mathbf{x}, t)s(\mathbf{x}, t)$. Integration across the lit side of the planet gives a well defined reflected energy:

$$E_{reflected}(t) = S(t) \int a(\mathbf{x}, t) \cos(\theta(\mathbf{x}, t)) d\mathbf{x} \tag{3}$$

An effective albedo a_{eff} can then be defined by the ratio of reflected to incoming energy across the planet as a whole:

$$E_{reflected}(t) = a_{eff}(t)E_{in}(t) = \pi r^2 a_{eff} S(t) \qquad (4)$$

The difference between incoming and reflected energy is what the planet absorbs (again, per unit time):

$$E_{absorbed}(t) = \pi r^2 (1 - a_{eff}(t))S(t) \qquad (5)$$

The fundamental characteristic of a planet in space is that of no material interchanges with its surroundings. The only substantive way energy can come in is through electromagnetic radiation, and the only way energy can leave is similarly through the planet's own electromagnetic emissions. There is a very small correction from gravitational tidal forces, and a planet also receives a small net energy input from internal radioactive decay, but for planets like the Earth these are thousands of times smaller than the stellar input.

A planet with no incoming absorbed energy would reach thermodynamic equilibrium with the cosmic microwave background, with a uniform temperature of about 2 K. Absorption of incoming stellar irradiance results in heating until a steady state with equal incoming and outgoing energy (measured outside the atmosphere, and averaged over one planetary revolution, or whatever the most important variation in time) is reached. Define $T(\mathbf{x}, t)$ as the local surface temperature of the planet, and $\epsilon(\mathbf{x}, t)$ as the local emissivity. Thermal radiation from the surface is then given by the Stefan-Boltzmann law[3]:

$$E_{emitted}(t) = \sigma \int \epsilon(\mathbf{x}, t)T(\mathbf{x}, t)^4 d\mathbf{x} \qquad (6)$$

Similar to the effective albedo, an effective emissivity and effective radiative temperature can be defined as averages over the planetary surface:

$$T_{eff}(t)^4 = \frac{1}{4\pi r^2} \int T(\mathbf{x}, t)^4 d\mathbf{x} \qquad (7)$$

and

$$\epsilon_{eff}(t) = \frac{1}{4\pi r^2 T_{eff}(t)^4} \int \epsilon(\mathbf{x}, t)T(\mathbf{x}, t)^4 d\mathbf{x} \qquad (8)$$

Total radiated thermal energy from the surface can then be written in terms of the effective temperature and albedo:

$$E_{emitted}(t) = 4\pi r^2 \sigma \epsilon_{eff}(t)T_{eff}(t)^4 \qquad (9)$$

For a planet with no atmosphere, or with an atmosphere that doesn't absorb electromagnetic radiation to any significant degree, all this surface-emitted thermal radiation escapes directly into space, just as all the absorbed stellar radiation reaches the ground. So the net rate of change in energy of the planet at time t is:

$$\dot{E}_{planet}(t) = E_{absorbed}(t) - E_{emitted}(t) = \pi r^2 (1 - a_{eff}(t))S(t) - 4\pi r^2 \sigma \epsilon_{eff}(t)T_{eff}(t)^4 \qquad (10)$$

We will look at relevant constraints associated with an absorbing atmosphere via a simple model later in the discussion.

The orbital processes for a planet (and any internal variability in the star) determine the variation in $S(t)$; that combined with rotation and internal dynamics gives variability in $a_{eff}(t)$, $\epsilon_{eff}(t)$, and $T_{eff}(t)$. As a result the planet may experience natural periods of warming or cooling as $\dot{E}_{planet}(t)$ goes positive or negative, respectively. On average, however, over time, this rate of energy change should come very close to zero as long as all the input parameters are reasonable stable over the long term. If it didn't average to zero for a long period of time, the energy of the planet would cumulatively build or decline.

In addition to the effective temperature obtained from averaging temperature to the fourth power, relevant for the thermal radiation problem, we should also look at the more natural average temperature for the planetary surface:

$$T_{ave}(t) = \frac{1}{4\pi r^2} \int T(\mathbf{x}, t) d\mathbf{x} \tag{11}$$

As Gerlich and Tscheuschner note[1] in their section 3.7, thanks to Hölder's inequality, this average temperature $T_{ave}(t)$ is always less than or equal to the effective thermal radiation temperature $T_{eff}(t)$, so T_{ave}^4 is less than or equal to T_{eff}^4, and rearranging Eq. 10 gives the following constraint on average temperature.

$$T_{ave}(t)^4 \le \frac{1}{\sigma \epsilon_{eff}(t)}((1 - a_{eff}(t))S(t)/4 - \dot{E}_{planet}(t)/4\pi r^2) \tag{12}$$

III. SOME EXAMPLES

A. Model 1: Nonrotating planet

First let's look at the simple model planet solved by Gerlich and Tscheuschner (section 3.7.4) This is a non-rotating planet (or a planet with a rotation axis parallel to the incoming radiation) with no internal heat transport in constant local radiative equilibrium so that $\dot{E}_{planet}$ is always zero. The non-rotation removes all time-dependences. Emissivity is assumed to be 1 everywhere; likewise S and a are uniform. Also the microwave background is ignored so the un-lit side of the planet is always at absolute zero temperature. From Eq. 2 for the local irradiance $s(\mathbf{x})$ we quickly obtain the local temperature for the lit side of the planet:

$$T_{model1}(\mathbf{x}) = \{(1 - a)\cos(\theta(\mathbf{x}))S/\sigma\}^{1/4} \tag{13}$$

The average temperature is obtained by integrating over the sphere:

$$T_{ave} = \frac{1}{4\pi}((1 - a)S/\sigma)^{1/4} \int_0^{\pi/2} \cos(\theta)^{1/4} 2\pi \sin(\theta)d\theta = \frac{2}{5}((1 - a)S/\sigma)^{1/4} \tag{14}$$

The effective temperature similarly is given by

$$T_{eff}^4 = \frac{1}{4\pi}((1 - a)S/\sigma) \int_0^{\pi/2} \cos(\theta) 2\pi \sin(\theta)d\theta = \frac{1}{4}((1 - a)S/\sigma) \tag{15}$$

which is what it has to be to ensure (Eq. 10) that $\dot{E}_{planet}$ is zero.

So in this case the ratio T_{ave}/T_{eff} is $2\sqrt{2}/5$ or about 0.566, and the planet's average temperature is indeed well below the effective temperature in this simple model. Plugging in numbers appropriate for Earth, T_{eff} comes to 255 K and T_{ave} would be 144 K, for this non-rotating atmosphere-free version of the planet.

B. Model 2: Simple rotating planet

To this simple model let us now add rotation, including a local heat capacity effect that accounts for some heat transport vertically, while still leaving out any transport of heat horizontally from one location to another. Assume the radiation direction is in the plane of rotation. Define the rotation period D (a day for the planet) and a thermal inertia coefficient[4] c with units of J/Km^2. On a real planet, c depends on temperature and on D (a time- or frequency-dependence); physically it represents the product of the volumetric heat capacity and the depth or height to which the incident heat energy is circulated or conducted during a daily thermal cycle. c then determines the local rate of change of temperature based on the local version of the net energy equation:

$$c\dot{T}(\mathbf{x}, t) = E_{absorbed}(\mathbf{x}, t) - E_{emitted}(\mathbf{x}, t) = (1 - a)S\cos(\theta(\mathbf{x}, t))\Theta(\cos(\theta(\mathbf{x}, t))) - \sigma T(\mathbf{x}, t)^4 \tag{16}$$

Represent $\mathbf{x}$ by angular coordinates ϕ for longitude and ξ for latitude. Thanks to the rotation, the position of $\mathbf{x}$ relative to the incoming sunlight changes as if ϕ were steadily incremented at a rate $2\pi t/D$. The sun angle θ then is found from:

$$\cos(\theta) = \cos(\phi + 2\pi t/D)\cos\xi \tag{17}$$

The driving forces in the equation repeat with period D, so under steady state conditions the solution(s) of Eq. 16 should also repeat with that period. At any point ϕ, ξ on the surface this solution would follow exactly the same curve of temperature as a function of time for every longitude ϕ, at the given latitude ξ. In the following we replace $2\pi/D$ with the symbol ω and $(1-a)S\cos\xi$ with A, and for simplicity set $\phi = -\pi/2$ (for all other points on the surface the solution is just shifted forward or back a bit in time). Then the step function $\Theta(\cos(\phi + \omega t))$ becomes equivalent to a square wave $W(\omega t)$ which is 1 for ωt between 0 and π, 0 for ωt between π and 2π, and repeating periodically after that.

So Eq. 16 is reduced to:

$$c\dot{T} = A\sin(\omega t)W(\omega t) - \sigma T^4 \tag{18}$$

Any non-transient solution for $T(t)$ will be periodic in time so that $T(D) = T(0)$. Integrating Eq. 18 over a planetary day (t = 0 to t = D) gives:

$$0 = \frac{2A}{\omega} - \sigma \int_0^D T^4 dt \tag{19}$$

Define an effective radiative temperature $T_{eff}(\xi)$ for latitude ξ based on the average fourth power (whether averaged over time or over longitudes is the same):

$$T_{eff}(\xi)^4 = \frac{1}{D}\int_0^D T^4 dt \tag{20}$$

then rearranging Eq. 19 and substituting in the definitions of ω and A gives:

$$T_{eff}(\xi)^4 = \frac{(1-a)S\cos\xi}{\pi\sigma} \tag{21}$$

Note that the peak T_{eff} for $\xi = 0$ (the equator) is a factor of $1/\pi^{1/4} \approx 0.75$ times the peak temperature on the non-rotating planet (from the point directly under the sun).

Once again we can check that the rate of change in net energy for the planet as a whole (Eq. 10) comes to zero by finding the effective radiative temperature for the entire surface:

$$T_{eff}^4 = \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} \frac{(1-a)S\cos\xi}{\pi\sigma} \cdot 2\pi\cos\xi d\xi = \frac{(1-a)S}{2\pi\sigma}\int_{-\pi/2}^{\pi/2}\cos^2\xi d\xi = \frac{(1-a)S}{4\sigma} \tag{22}$$

as it has to be, the same as for the non-rotating case in Eq. 15.

While we won't find a full analytic form for the temperature as a function of latitude and time in this model, we can learn a bit by examining the time dependence of temperature in Eq. 18 more closely. First, define $x = \omega t$ and $y(x) = T(\xi, t)/T_{eff}(\xi)$ for a given latitude ξ, with $T_{eff}(\xi)$ determined by Eq. 21. Eq. 18 then reduces to:

$$\frac{dy}{dx} = \frac{(1-a)S\cos\xi}{c\,\omega T_{eff}(\xi)}\sin(x)W(x) - \frac{\sigma T_{eff}(\xi)^3}{c\,\omega}y^4 = \lambda(\sin(x)W(x) - \frac{1}{\pi}y^4) \tag{23}$$

where we define the dimensionless parameter $\lambda = (1-a)S\cos\xi/c\,\omega T_{eff}(\xi)$ Physically this roughly represents the ratio of the quantity of incoming energy absorbed in a day to the total heat content of the surface (to a relevant depth) at the effective radiative temperature. If λ is small (heat capacity or rotation frequency high or latitude close to the poles), heating or cooling will occur only slowly, and the temperature will stay close to T_{eff} throughout the day (y will be close to 1). If λ is large (heat capacity or rotation frequency low, latitude closer to the equator) then heating and cooling are rapid, and the temperature variation is more significant.

We can find an analytic solution for the night side of the planet, where the square wave $W(x) = 0$ (x between $(2n-1)\pi$ and $2n\pi$ for integer n). Eq. 23 loses all dependence on x and is easily integrated:
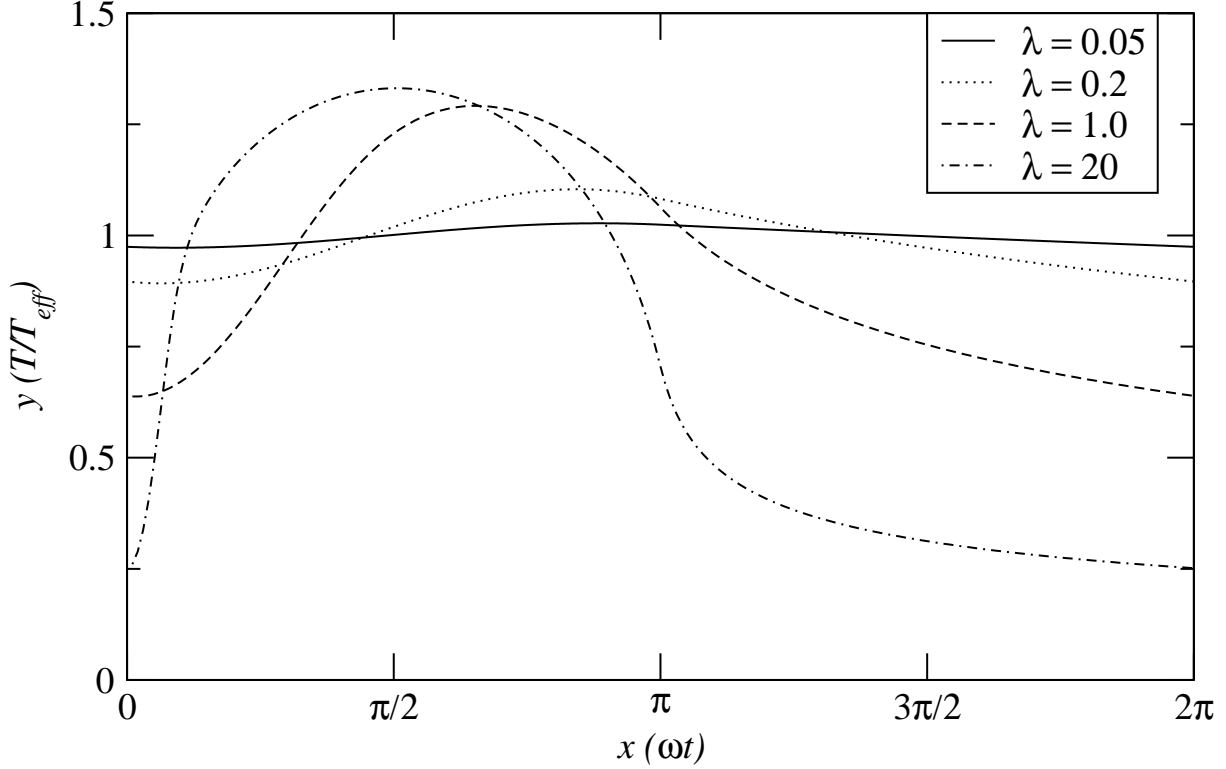
Temperature on a rotating planet

FIG. 1: Temperature relative to the effective radiating temperature (T_{eff}) for the simple rotating planet model for various values of the thermal response parameter λ. The plots are of temperature against time where 0 is sunrise, π is sunset, and 2π is sunrise again.

$$\frac{dy}{dx} = -\frac{\lambda}{\pi} y^4 \Rightarrow y(x) = (\frac{\pi}{3\lambda(x-a)})^{1/3} \tag{24}$$

where a is a constant determined by the initial condition $y(\pi)$ (the value of the temperature when night begins for $x = \pi$):

$$a = \pi(1 - \frac{1}{3\lambda y(\pi)^3}) \Rightarrow y(2\pi)/y(\pi) = 1/(1 + 3\lambda y(\pi)^3)^{1/3} \tag{25}$$

which gives us the night-time temperature drop. When λ is small and y is not too large to start with, the change is small - a fractional decline of roughly $\lambda y(\pi)^3$. For large values of λ, the night-time temperature drop is limited by this slow inverse $1/3$ power. This makes sense as the rate of temperature decrease must decline sharply as temperature gets lower and the T^4 radiative term drops.

For the daytime, if the temperature starts out low with $y << 1$, then the y^4 term is negligible, at least at first, and Eq. 23 can be integrated easily enough:

$$\frac{dy}{dx} \approx \lambda \sin x \Rightarrow y(x) \approx b - \lambda \cos x \tag{26}$$

where the inegration constant b again is set by the initial condition $b = y(0) + \lambda$, which $y(\pi) = y(0) + 2\lambda$, i.e. the temperature increments by 2λ on a sinusoidal curve during the day. Of course when λ is large or $y(0)$ starts close to one, this approximation breaks down.
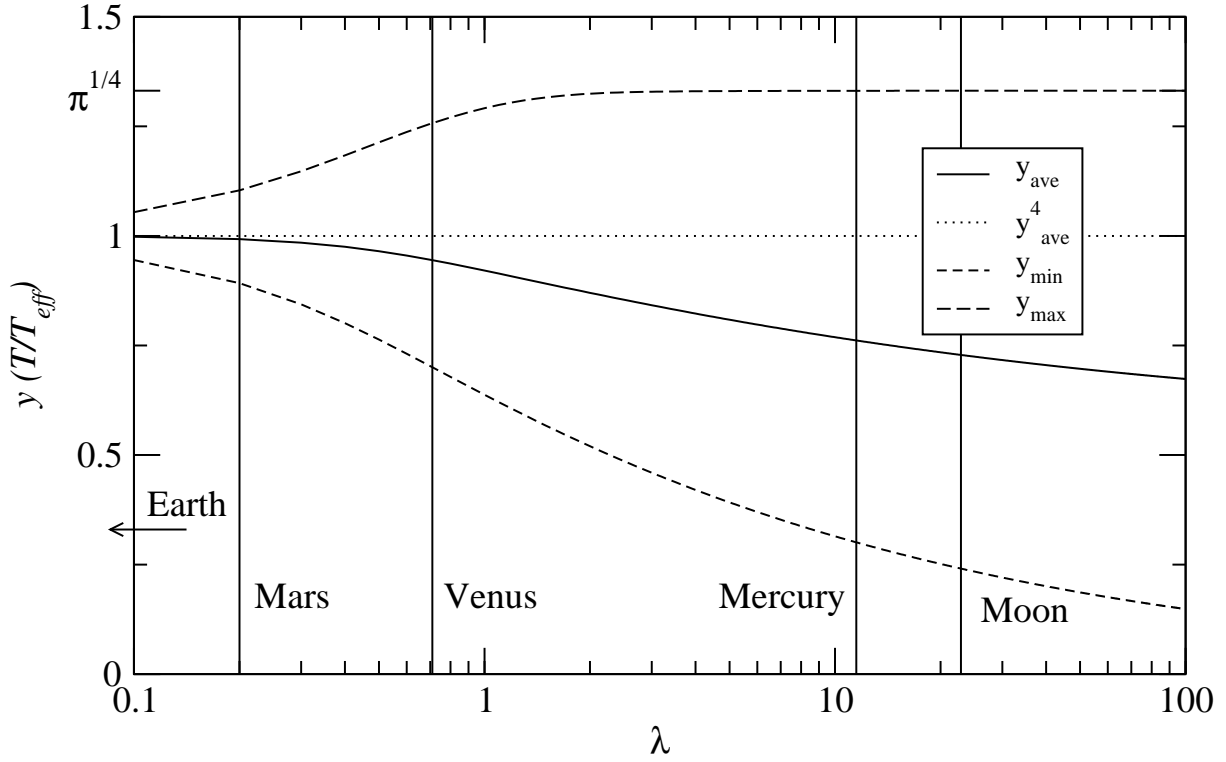
FIG. 2: The average, minimum, and maximum values of the relative temperature y from numerically integrating the equations for different values of λ. The numerically computed average value of y^4 is also shown; this should always be exactly 1. Lines are shown indicating the approximate values of λ corresponding to the equators of the terrestrial planets, which gives a picture of their most extreme temperature profiles if they had infrared-transparent atmospheres.

Another approximation is to assume variations in y are small and that we can linearize about some chosen value y_0. This gives:

$$\frac{dy}{dx} \approx \lambda \sin x - \frac{\lambda y_0^4}{\pi} - 4\frac{\lambda y_0^3}{\pi}(y - y_0) \tag{27}$$

which as a linear ordinary differential equation yields a solution:

$$y(x) = \frac{3}{4}y_0 + \beta e^{-4\lambda y_0^3 x/\pi} + \frac{\lambda}{1 + (4\lambda y_0^3/\pi)^2}(\frac{4\lambda y_0^3}{\pi} \sin x - \cos x) \tag{28}$$

β here is another constant of integration to be determined by an appropriate initial condition. The linearization fails once y deviates significantly from y_0, but the result should be generally valid if λ is small, and can be used to generate step-wise solutions for the daytime temperatures under any value of λ; numerical intergration of the basic equation of course can do the same.

Numerical computed solutions for various values of λ are shown in Fig. 1; Fig. 2 shows the trends for average y (T/T_{eff}), average fourth power of y, and the minimum and maximum y values as λ increases. As expected, the average fourth power is fixed at 1, while the average y decreases as λ increases, eventually approaching the non-rotating value of $2\sqrt{2}/5$ as $\lambda \to \infty$. Also note the maximum temperature quickly approaches the nonrotating value of $y = \sqrt[4]{\pi}$ for large λ. The minimum temperature drops slowly as $y \sim 1/(3\lambda)^{1/3}$.

Approximate formulas for the average temperature are, for large λ:

$$y_{ave} \sim 2\sqrt{2}/5 + 0.392\lambda^{-0.279}; \lambda \to \infty \tag{29}$$

and for small λ:

$$y_{ave} \sim 1 - 0.196\lambda^2; \lambda \to 0 \tag{30}$$

Note that from Eq. 21 the temperature scale varies with latitude as $\cos(\xi)^{1/4}$, while the value of λ varies as $\cos(\xi)^{3/4}$. So finally, integrating over the whole planet we have an average temperature value of:

$$T_{ave} = (\frac{(1-a)S}{\pi\sigma})^{1/4} \int_0^{\pi/2} \cos(\xi)^{1/4} y_{ave}(\lambda(\xi)) \cos(\xi) d\xi \tag{31}$$

For small λ we could substitute the expression from Eq. 30; in any case, we know $y_{ave} < 1$ for all latitudes, so we have an upper bound on the average temperature of the entire rotating planet T_{ave} by substituting in the numerical value for the $\cos(\xi)^{5/4}$ integral:

$$T_{ave} < 0.69921(\frac{(1-a)S}{\sigma})^{1/4} \tag{32}$$

and note that this bound is a little over 1% less than T_{eff} for the entire planet (Eq. 22) which has a constant $(1/4)^{1/4} = 0.7071...$ instead of 0.69921 in the same expression.

So no matter the rotation rate, no matter the surface heat capacity, the average temperature of the planet in this rotating example, with only radiative energy flows and no absorbing layer in the atmosphere, is always less than the effective radiating temperature. For very slow rotation or low heat capacity it can be significantly less; for parameters in the other direction it can come as close as 1% (i.e. up to 252 K on a planet like Earth).

C. Model 3: Rotating planet with varying albedo

While the variability in infrared emissivity is relatively small across the surface of a realistic planet, the albedo can be significantly different from place to place. One of these involves taking into account the effect of ice, by which the high latitudes reflect more incoming radiation back into space than equatorial latitudes do. What effect does this have on effective radiating temperature and total temperature?

We can model this by a slight change in model 2, by making the value of a dependent on latitude ξ. For example let $a = \sin^2(\xi)$ so it is zero at the equator, and approaches 1 at the poles. This changes nothing in most of the analysis of the preceding section, until we integrate over latitudes. For Eq. 22 we now have:

$$T_{eff}^4 = \frac{1}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{(1-a)S\cos\xi}{\pi\sigma} \cdot 2\pi\cos\xi d\xi = \frac{S}{\pi\sigma} \int_0^{\pi/2} \cos^4\xi d\xi = \frac{3S}{16\sigma} \tag{33}$$

This just means that our $\sin^2$ albedo has the same effect on the total radiation absorbed by the planet as would a uniform albedo value of 1/4. However, it redistributes that energy, putting more near the equator and less near the poles. The effect on average temperature across the planet, for this modified version of Eq. 31 is:

$$T_{ave} = (\frac{S}{\pi\sigma})^{1/4} \int_0^{\pi/2} \cos(\xi)^{3/4} y_{ave}(\lambda(\xi)) \cos(\xi) d\xi \tag{34}$$

which then, putting in the numerical value for the $\cos^{7/4}$ integral, gives the inequality

TABLE I: Relevant parameters for the planets. See http://nssdc.gsfc.nasa.gov/planetary/factsheet/. λ for Eq. 23 (at the equator, $\xi = 0$) estimated from thermal inertia, solar day, and the other parameters. It is particularly small for Earth thanks to rapid rotation and the high heat capacity of water covering most of the surface.

Planet	solar constant (W/m^2)	albedo	solar day (Earth days)	T_{eff} (K)	T_{ave} (K)	Difference (K)	λ
Mercury	9127	0.12	176	434	?	?	11
Venus	2615	0.75	117	232	737	505	0.7
Earth	1367	0.306	1	255	288	33	0.04
Moon	1367	0.11	29.53	270	253	-17	20
Mars	589	0.25	1.03	210	210	0	0.2

$$T_{ave} < 0.6206(\frac{S}{\sigma})^{1/4} \tag{35}$$

which is about 5% below the effective temperature (the numerical coefficient is $(3/16)^{1/4}$ or 0.6580).

IV.   INFRARED ABSORPTION IN THE ATMOSPHERE

The examples of these simple models show that vertical energy transport for a planet with a transparent atmosphere only smooths out the daily temperature curve, without being able to bring the surface temperature higher than the effective radiative temperature. The same is true if we were to add in more realistic horizontal energy transport from larger-scale atmospheric and oceanic circulation - of course getting much more realistic means entering the realm of more full-scale general circulation models[5], which we have no intention of doing here.

On a planet with significant internal energy sources the effective temperature for radiative balance could be exceeded even with a transparent atmosphere. For example a planet still losing its initial heat of formation, or a planet remote from its sun with a high enough radioactive content, or on a planet or moon with very large tidal forces, you will have a net outward flow of energy to space, and may well have an average temperature above the limit. But for the terrestrial worlds of our solar system, these internal sources of heat are thousands of times too small to have any noticeable effect on surface temperature.

And yet the observed average surface temperature on Earth and Venus significantly exceeds the effective radiative temperature set by the incoming solar radiation. This is not observed for the Moon or Mars. What makes Venus and Earth so different?

Net energy flux is determined by the radiation that gets into space, not what leaves the surface. The only way for a planet to be radiatively warmer than the incoming sunlight allows is for some of that thermal radiation to be blocked from leaving. That means some layer above the surface must be absorbing or reflecting a significant fraction of the outgoing infrared radiation. I.e. the atmosphere must not be transparent to infrared.

So, let's add to our rotating planet model a simple model of this blocking effect: a fraction f (between 0 and 1) of the outgoing radiation $E_{emitted}$ from the surface is absorbed by a thin layer of the atmosphere. This layer will have its own temperature but for simplicity we make the assumption that the heat capacity of the atmospheric layer is low so that it remains essentially radiatively balanced through the day, and the specific temperature becomes irrelevant. That means that this atmospheric layer continuously emits an amount $f \cdot E_{emitted}$ equal to what it absorbs from the ground.

Since thermal re-emission is randomly directed, half the radiation from this atmospheric layer will go up, and half down. Assuming the surface is fully absorbing and the rest of the atmosphere is transparent, total outgoing radiation from the planet (above the atmospheric layer) is then:

$$E_{out} = (1 - f)E_{emitted} + \frac{1}{2}fE_{emitted} = (1 - f/2)E_{emitted} \tag{36}$$

while absorbed radiation on the surface is now:

$$E_{absorbed}(t) = \pi r^2(1 - a_{eff}(t))S(t) + \frac{1}{2}fE_{emitted} \tag{37}$$

Incoming solar radiation still drives everything - if the solar constant S drops, then so does everything else. But the effect of the absorbing layer is to reduce the final outgoing energy for a given temperature, so the planet heats up until things are back in balance again.

Generalizing Eq. 10 we have net energy change (which can be calculated either at the surface or above the absorbing layer of the atmosphere):

$$\dot{E}_{surface}(t) = E_{absorbed}(t) - E_{emitted}(t) = \pi r^2(1 - a_{eff}(t))S(t) + \frac{1}{2}fE_{emitted}(t) - 4\pi r^2\sigma\epsilon_{eff}(t)T_{eff}(t)^4 \tag{38}$$

$$= \pi r^2(1 - a_{eff}(t))S(t) - 4\pi r^2\sigma\epsilon_{eff}(t)(1 - f/2)T_{eff}(t)^4 \tag{39}$$

We then end up with essentially the same equations as in the previous section, for example Eq. 16 is the same, except that effectively the solar input S and thermal inertia c in those equations are increased by the factor $1/(1-f/2)$.

That means the surface effective radiative temperature T_{eff} in those equations is increased by a factor $(1/(1 - f/2))^{1/4}$, or as much as $2^{1/4}$ for a fully absorbing atmospheric layer. The parameter λ is then reduced by that same ratio (the increases in S and c cancel out, leaving a $1/T_{eff}$ term). So the temperature curve of this radiatively insulated planet is even more uniform than without the insulating layer. The average temperature can come within a few percent of this higher T_{eff}, or well above the limits for a planet with a transparent atmosphere.

A more realistic atmosphere would be characterized by more than one absorbing layer (or a thick layer with a temperature differential and limited conductivity from bottom to top), which will further decrease outgoing thermal radiation and increase surface temperatures. Details of absorption in the real atmosphere also depend on pressure; nevertheless, the presence of any absorption at all is what qualitatively distinguishes a greenhouse-effect planet from one with a transparent atmosphere, and is what allows surface temperatures to climb above the effective radiative limit.

V. CONCLUSION

Gerlich and Tscheuschner[1] state, among more extravagant claims, that "Unfortunately, there is no source in the literature, where the greenhouse effect is introduced in harmony with the scientific standards of theoretical physics." The above analysis I believe completely establishes, within perfectly simple and appropriate theoretical physics constructs, the main points. Namely that assuming "the atmosphere is transparent for visible light but opaque for infrared radiation" leads to "a warming of the Earth's surface" relative to firm limits established by basic physical principles of energy conservation, for the case of an atmosphere transparent to both visible and infrared.

In particular, it has been shown that:

1. An average surface temperature for a planet is perfectly well defined with or without rotation, and with or without infrared absorbing gases

2. This average temperature is mathematically constrained to be less than the fourth root of the average fourth power of the temperature, and can in some circumstances (a planet with no or very slow rotation, and low surface thermal inertia) be much less

3. For a planet with no infrared absorbing or reflecting layer above the surface (and no significant flux of internal energy), the fourth power of the surface temperature always eventually averages to a value determined by the incoming stellar energy flux and relevant reflectivity and emissivity parameters.

4. The only way the fourth power of the surface temperature can exceed this limit is to be covered by an atmosphere that is at least partially opaque to infrared radiation. This is the atmospheric greenhouse effect.

5. The measured average temperature of Earth's surface is 33 degrees C higher than the limit determined by items (2) and (3). Therefore, Earth is proved to have a greenhouse effect of at least 33K.

The specific contributions of individual gases such as CO_2 to Earth's greenhouse effect are covered well by the standard treatments of the subject[2,4,5].

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[*] E-mail:apsmith@aps.org

[1] "Falsification Of The Atmospheric CO2 Greenhouse Effects Within The Frame Of Physics" by Gerhard Gerlich and Ralf D. Tscheuschner, arXiv:0707.1161 (2007).

[2] See for example, *An Introduction to Atmospheric Radiation, second edition* by K.N. Liou (2002), section 4.1..

[3] For example, *Thermal Physics, second edition* by P.C. Riedi (1988), section 10.2 on Black Body radiation.

[4] See *Principles of Planetary Climate*, by R. T. Pierrehumbert (retrieved February 2008 from http://geosci.uchicago.edu/~rtp1/ClimateBook/ClimateBook.html) - in particular section 8.3 on thermal inertia.

[5] See *The Discovery of Global Warming*, by Spencer Weart - http://www.aip.org/history/climate/ - for a good discussion of the development of more and more detailed climate models.