EEE 321

Report: Signals and Systems Lab 5

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Introduction:

This report outlines the work completed for Lab Assignment 5. The primary focus of this assignment is to explore and apply the principles of Fourier Transform, a fundamental tool widely used in signal processing across various domains. This assignment is designed to enhance our understanding through practical application of Fourier Transform techniques in MATLAB.

Part 1

Images below are the related derivations and calculations for part 1 a and b.

Part 1)

a) $h_1(t) = \frac{d}{dt}\left[\frac{\sin(\omega_c t)}{2\pi t}\right]$, $H_3(j\omega) = e^{-j\frac{\pi}{\omega_1}\omega}$

$h_2(t) = -\frac{1}{2}e^{j\omega_1} + \cos(\omega_1 t)$, $h_4(t) = u(t)$

$[X(j\omega)H(j\omega) - X(j\omega)H_1(j\omega)H_2(j\omega)]H_3(j\omega)H_4(j\omega) = Y(j\omega)$

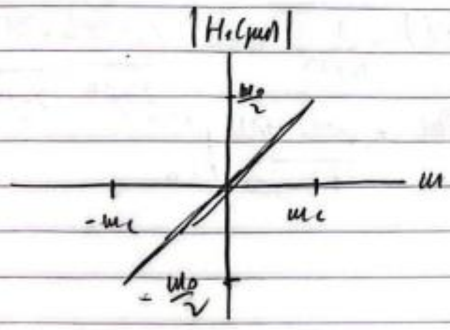
$$H_{eq}(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = H_3(j\omega)H_4(j\omega)[H_1(j\omega) - H_1(j\omega)H_2(j\omega)]$$

$$h_1^*(t) = \frac{\sin(\omega_c t)}{\pi t} \Rightarrow H_1^*(j\omega) = \begin{cases} \frac{1}{2} & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

$$h_1(t) = \frac{d}{dt}\left[\frac{\sin(\omega_c t)}{2\pi t}\right] \Rightarrow H_1^*(j\omega) = j\omega = H_1(j\omega) \begin{cases} \frac{j\omega}{2} & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

$$H_{eq} = H_1(j\omega)H_3(j\omega)H_4(j\omega) - H_1(j\omega)H_2(j\omega)H_3(j\omega)$$

$$|H_1(j\omega)| = \begin{cases} \frac{\omega}{2} & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$



$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$, $x(t)$ is real and even , $\int_{-\infty}^{\infty} x(t)\,dt = 1$

$X(j\omega) = c$ for $\omega \in [0, 2\pi]$, $X(j\omega) = c\, e^{2\omega t}$, $\omega < -2\pi$

$x(0) = \int_{-\infty} x(t)\,dt = 1 \Rightarrow c = 1.$

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega = \frac{1}{2\pi}\left[\int_{-2\pi}^{2\pi} e^{j\omega t}\, d\omega + \int_{-\infty}^{-2\pi} e^{2\omega} e^{j\omega t}\, d\omega + \int_{2\pi}^{\infty} e^{-2\omega} e^{j\omega t}\, d\omega\right.$

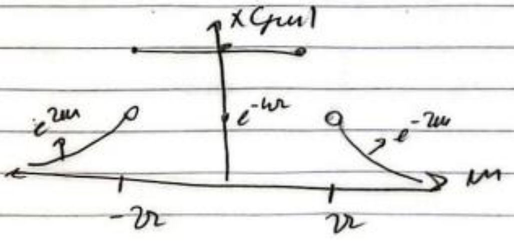
$= \frac{1}{2\pi}\frac{e^{j\omega t}}{jt}\Big|_{-2\pi}^{2\pi} + \frac{1}{2\pi}\frac{e^{\omega(2+jt)}}{2+jt}\Big|_{-\infty}^{-2\pi} - \frac{1}{2\pi}\frac{e^{-\omega(2-jt)}}{2-jt}\Big|_{2\pi}^{\infty}$

$= \frac{1}{jt2\pi}\left[e^{j2\pi t} - e^{-j2\pi t}\right] + \frac{1}{(2+jt)2\pi} e^{-2\pi(2+jt)} + \frac{1}{(2-jt)2\pi} e^{-2\pi(2-jt)}$

$\Rightarrow x(t) = \frac{\sin(2\pi t)}{\pi t} + \frac{e^{-4\pi}}{(4+t^2)\pi}\left[2\cos(2\pi t) - \sin(2\pi t)\, t\right]$

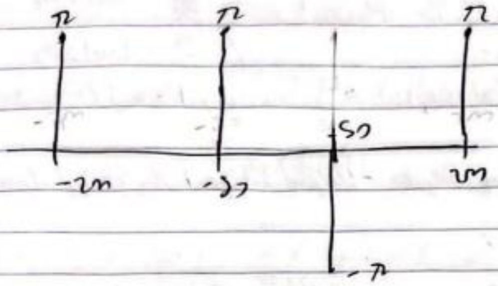
$= e^{-4\pi}\left(2e^{-2\pi t} + 2e^{2\pi jt} - jte^{-2\pi jt} + jte^{2\pi j}\right) \times \frac{1}{4+t^2} = e^{-4\pi}\left(\frac{e^{-2\pi jt}}{2+jt} + \frac{e^{2\pi jt}}{2-jt}\right)$

$= \frac{e^{-4\pi - 2\pi jt}}{2+jt} + \frac{e^{-4\pi + 2\pi jt}}{2-jt}$, $\int_{2\pi}^{\infty} e^{+2\omega} e^{j\omega t}\, d\omega = \frac{e^{\omega(2+jt)}}{(2+jt)}\Big|_{+\infty}^{+2\pi}$



$x(t) = \sin(5\pi t) + \cos(2\pi t)$

$X(j\omega) = \frac{\pi}{j}[\delta(\omega - 5\pi) - \delta(\omega + 5\pi)] + \pi[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)]$



$H(j\omega) = \frac{e^{-\frac{j\omega}{10}}}{2} - \frac{e^{-j\frac{\pi 5}{\pi}}}{2}$

$y(t) = -\sin(5)\pi\,\delta(t)$

$\frac{\pi}{j}\frac{e^{-j5}}{2} - \frac{e^{-j5j}}{2}\frac{\pi^2}{j} - \left(\frac{e^{5j}}{2} - \frac{e^{j\pi 5}}{2}\frac{\pi^2}{j}\right)$

$\Rightarrow X(j\omega) = \frac{\pi}{2j}(e^{-j5} - e^{j5}) = -\sin(5)\pi.$

$Y = (xH_1 - xH_1H_2)H_3H_4 \qquad A = xH_1,\ B = xH_1H_2,\ C = A - B$

$Y = CH_3H_4$

$H_4(j\omega) = \frac{1}{j\omega},\ H_3(j\omega) = e^{-j\frac{2\pi}{\omega_2}\omega},\quad H_1(j\omega) = \begin{cases} \frac{j\omega}{2} & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$

$H_2(j\omega) = \pi\delta(\omega-\omega_1) + \pi\delta(\omega+\omega_1) - \pi\delta(\omega_1-\omega_1) = \pi\delta(\omega+\omega_1)$

$H_{eq}(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = H_1(j\omega)H_3(j\omega)H_4(j\omega) - H_1(j\omega)H_2(j\omega)H_3(j\omega)H_4(j\omega)$

$= \frac{j\omega}{2} e^{-j\frac{2\pi}{\omega_2}\omega} \frac{1}{j\omega} - \frac{j\omega}{2}\pi\delta(\omega-\omega_1) e^{-\frac{2\pi j\omega}{\omega_2}} \frac{1}{j\omega}$, if $|\omega| < \omega_c$, 0 else

$= \frac{e^{-j\frac{2\pi}{\omega_2}\omega}}{2} - \frac{e^{j\frac{2\pi\omega_1}{\omega_2}}\pi}{2}$ if $|\omega| < \omega_c$ and $|\omega_1| < \omega_c$

$\frac{e^{-j\frac{2\pi}{\omega_2}\omega}}{2}$ if $|\omega| < \omega_c$ and else $|\omega_1| > \omega_c$

$h_{eq}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{eq}(j\omega) e^{j\omega t}\, d\omega$

$= \frac{1}{4\pi}\int_{-\omega_c}^{\omega_c}\left[e^{-\frac{2\pi\omega}{\omega_2}} - e^{j(2\pi\frac{\omega_1}{\omega_2}\pi)}\right] e^{j\omega t}\, d\omega$ if $|\omega_1| < |\omega_c|$

$= \frac{1}{4\pi}\left[\int_{-\omega_c}^{\omega_c} e^{(jt-\frac{2\pi}{\omega_2})\omega} - e^{j2\pi\frac{\omega_1}{\omega_2}}\pi \int_{-\omega_c}^{\omega_c} e^{j\omega t}\, d\omega\right.$

$= \frac{1}{4\pi}\left[\left.\frac{e^{(jt-\frac{2\pi}{\omega_2})\omega}}{jt-\frac{2\pi}{\omega_2}}\right|_{-\omega_c}^{\omega_c} - e^{j\frac{2\pi\omega_1}{\omega_2}\pi}\left.\frac{e^{j\omega t}}{jt}\right|_{-\omega_c}^{\omega_c}\right]$

$= \frac{1}{4\pi}\left[\frac{e^{(jt-\frac{2\pi}{\omega_2})\omega_c}}{jt-\frac{2\pi}{\omega_2}} - \frac{e^{(\frac{2\pi}{\omega_2}-jt)\omega_c}}{jt-\frac{2\pi}{\omega_2}} - e^{j\frac{2\pi\omega_1}{\omega_2}}\pi\left(\frac{e^{j\omega_c t}}{jt} - \frac{e^{-j\omega_c t}}{jt}\right)\right]$

Part 2

2.1 Implementing the Fourier Transform

```
function [frequency_array] = FourierTransform(x, t, Ts)
    N = length(x)/2;
    Fs = 1/Ts;
    ds = Fs/(2 * N);
    n = 1;
    for f = -Fs*0.5:ds:Fs*0.5
        e = exp(-1i*2*pi*f*t);
        frequency_array(n) = sum(x.*e* Ts);
        n = n + 1;
    end
end
```

“Related code for the function”

2.2 Testing the Function

FT of cos(2*30*pi)= $\pi\delta(w-60\pi)+\pi\delta(w+60\pi)$

So we should observe peaks at w = 60pi and w = -60pi. See the related plot for this part (figure 1).

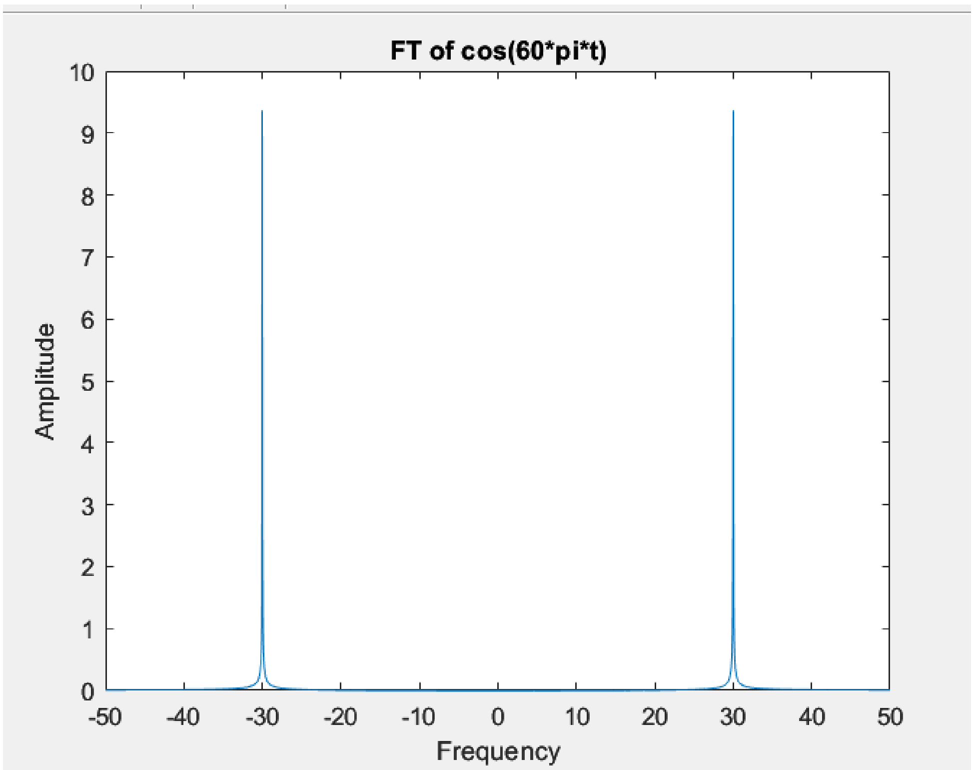


Figure 1: FT of cos(60*pi*t)

Comment: So, as expected we see peaks at frequency = 30 and frequency = -30 which means w = 60*pi and w = -60*pi. Note that amplitude can be as much as we want by changing simply the Ts value because theoretically the amplitude of impulse function is infinite.

Part 3

3.1 Derivation of Relations

Part 3)

$$r_1(t) = x_1\left(t - \frac{2d_1}{c}\right) + x_2\left(t - \left(\frac{d_1+d_2}{c}\right)\right)$$

$$r_2(t) = x_1\left(t - \left(\frac{d_1+d_2}{c}\right)\right) + x_2\left(t - 2\left(\frac{d_2}{c}\right)\right)$$

$$R_1(j\omega) = X_1(j\omega)\, e^{-j\omega\left(\frac{2d_1}{c}\right)} + X_2(j\omega)\, e^{-j\omega\left(\frac{d_1+d_2}{c}\right)}$$

$$R_2(j\omega) = X_1(j\omega)\, e^{-j\omega\left(\frac{d_1+d_2}{c}\right)} + X_2(j\omega)\, e^{-j\omega\left(\frac{2d_2}{c}\right)}$$

$$Y_1(t) = X_1\left(t - \frac{2d_1}{c}\right) \rightarrow Y_1(j\omega) = X_1(j\omega)\, e^{-j\omega\left(\frac{2d_1}{c}\right)}$$

$$Y_2(t) = X_2\left(t - \frac{2d_2}{c}\right) \rightarrow Y_2(j\omega) = X_2(j\omega)\, e^{-j\omega\left(\frac{2d_2}{c}\right)}$$

$$-j\omega\frac{2d_1}{c} = \ln\left(\frac{Y_1(j\omega)}{X_1(j\omega)}\right), \quad d_1 = \frac{jc}{2\omega}\ln\left(\frac{Y_1(j\omega)}{X_1(j\omega)}\right), \quad \omega = 2\pi f$$

$$d_2 = \frac{jc}{2\omega}\ln\left(\frac{Y_2(j\omega)}{X_2(j\omega)}\right), \quad \omega = 2\pi f$$

Derivation for part 3

3.2 Estimating Distances

See the related figure to see plots of this part.

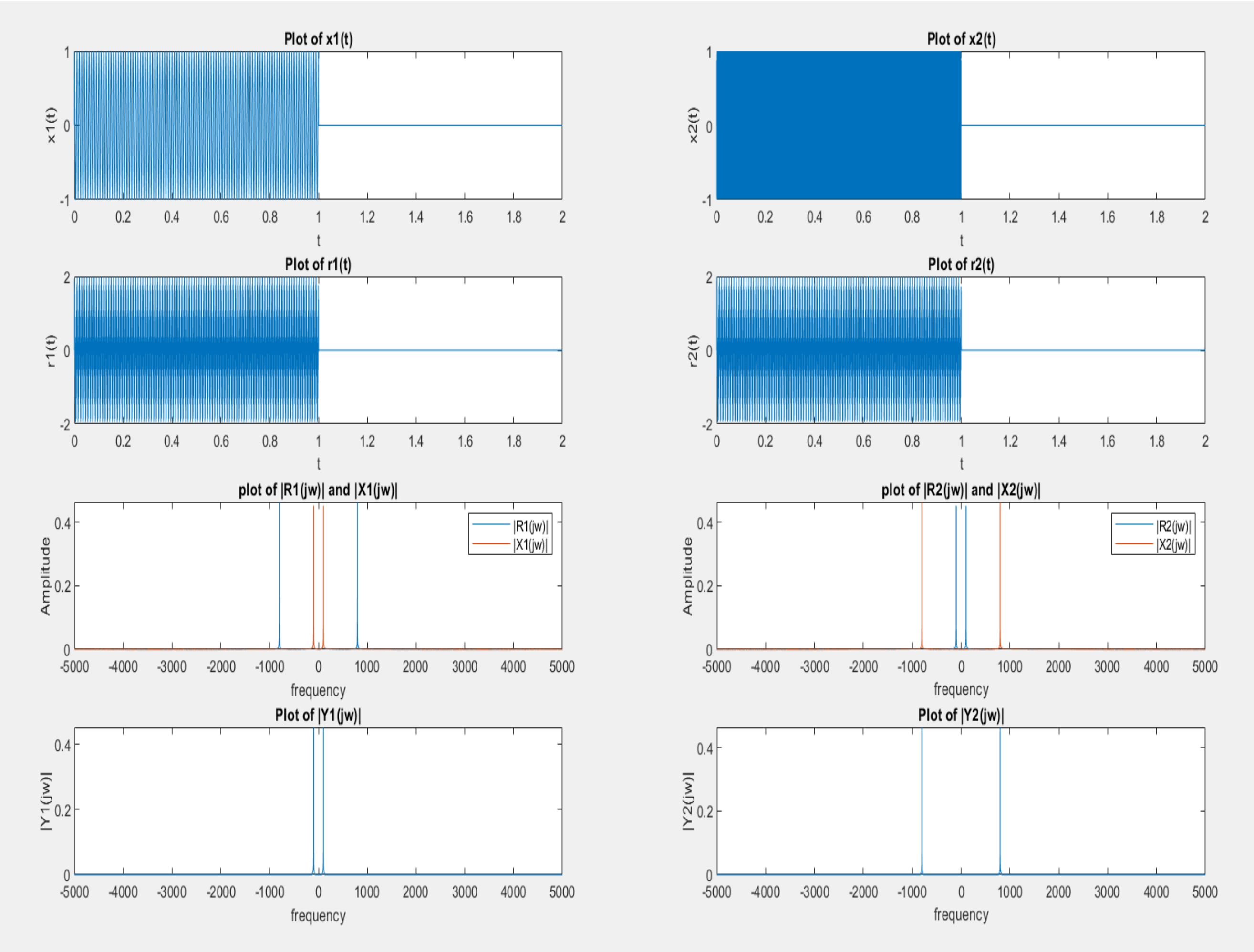


Figure 2: Plots for part 3

Estimated values for d1 and d2:

```
estimated d1
    0.0497

estimated d2
    0.1000
```

Comments:

So the estimated d1 is 0.0003 is smaller than the actual d1 and the estimated d2 is the same as the actual d2. Therefore, I can easily say that my work is parallel with the wanted results. To comment on the findings for part 3, matlab results are pretty parallel with the hands on calculated results.

Couple of ways to improve this system's location estimation performance can be utilizing of the Machine Learning models. Also we can use Multiple Frequency Bands to eliminate the signal distortions.

Part 4

Part4)

$$r_1(t) = x_1\left(t\left(\frac{c+u_1}{c-u_1}\right) - \frac{2d_1}{c}\right) + X_2\left(t\left(\frac{c+u_2}{c-u_2}\right)\right) - \left(\frac{d_1+d_2}{c}\right)\Big)$$

$$r_2(t) = X_1\left(t\left(\frac{c+u_1}{c-u_1}\right) - \frac{2(d_1+d_2)}{c}\right) + x_2\left(t\left(\frac{c+u_2}{c-u_2}\right) - \frac{2d_1}{c}\right)$$

$$\boxed{a = \frac{c+u_1}{c-u_1}, \quad b = \frac{c+u_2}{c-u_2}}$$

$$R_1(j\omega) = \frac{1}{|a|} X_1\left(\frac{j\omega}{a}\right) e^{-j\omega\left(\frac{2d_1}{c}\right)} + \frac{1}{|a|} X_2\left(\frac{j\omega}{a}\right) e^{-j\omega\left(\frac{d_1+d_2}{c}\right)}$$

$$R_2(j\omega) = \frac{1}{|b|} X_2\left(\frac{j\omega}{b}\right) e^{-j\omega\left(\frac{d_1+d_2}{c}\right)} + \frac{1}{|b|} X_2\left(\frac{j\omega}{b}\right) e^{-j\omega\left(\frac{2d_2}{c}\right)}$$

Derivation for Part 4

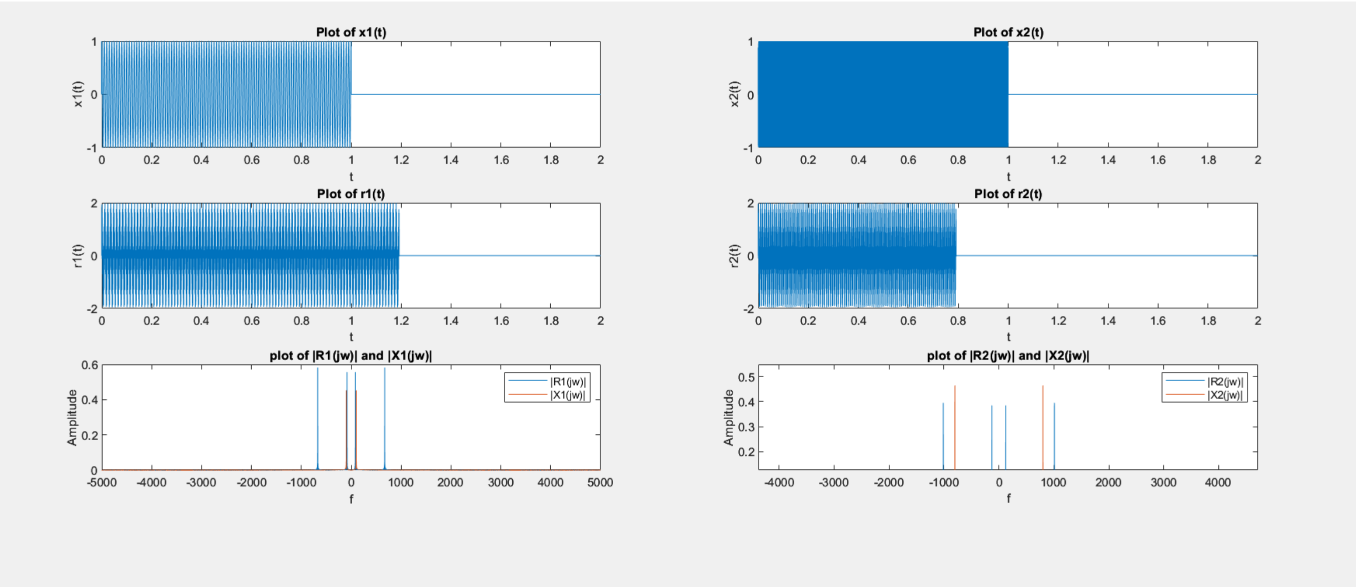


Figure 3: Plots for part 4 without labelled peaks

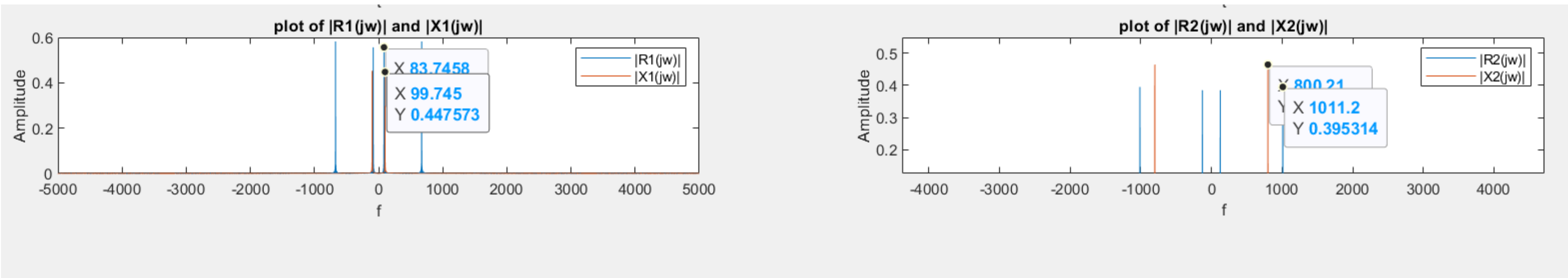


Figure 4: Plots to compare peaks

R1 frequency_peak = 83.7458

X1 frequency_peak = 99.745

R2 frequency_peak = 1011

X2 frequency_peak = 800

R1_f / X1_f = 0.84, also (c+v1)/(c-v1) = 0.84 where c = 343, v1 = -30.

Then estimated v1 is also -30 which shows almost zero error.

R2_f/X2_f = 1.26, also (c+v2)/(c-v2) = 1.26 where v2 = 40.

Then estimated v2 is also 40 which shows again almost zero error.

Comment: Another method to obtain the velocity *v*1 can be Cross corelation method becausetThis method is particularly effective for determining the time delay between a transmitted signal and its corresponding received signal, which has been altered due to motion, such as in Doppler radar systems.

Whole codes for the Assignment:

```
function [frequency_array] = FourierTransform(x, t, Ts)
    N = length(x)/2;
    Fs = 1/Ts;
    ds = Fs/(2 * N);
    n = 1;
    for f = -Fs*0.5:ds:Fs*0.5
        e = exp(-1i*2*pi*f*t);
        frequency_array(n) = sum(x.*e* Ts);
        n = n + 1;
    end
end
```

"FourierTransform.m"

```
Ts = 0.01;
time = 10.0;
t = -time:Ts:time;
frequency = 1/Ts;
f = -frequency * 0.5:(frequency/(length(t))):frequency * 0.5;

x1 = cos(60*pi*t);
FT_x1 = FourierTransform(x1,t,Ts);
plot(f,abs(FT_x1))
ylabel("Amplitude")
xlabel("Frequency")
title("FT of cos(60*pi*t)")
```

"Lab5part2.m

```
f1 = 100;
f2 = 800;
w_pass = 50;
T = 1;
c = 343;
Ts = 0.0001;
d1 = 0.05;
d2 = 0.1;

time = 2;
t_x = 0:Ts:time;

freq = 1/Ts;
f = -freq * 0.5:(freq/(length(t_x))):freq * 0.5;
% Initialization of functions
rec_pulse = @(t) abs(t) < T * 0.5;
x1 = @(t) cos(2*pi*f1*t) .* rec_pulse(t-T * 0.5);
x2 = @(t) cos(2*pi*f2*t) .* rec_pulse(t-T * 0.5);
r1 = @(t) x1(t-2*d1/c) + x2(t - (d1 +d2)/c);
r2 = @(t) x1(t-(d1+d2)/c) + x2(t - 2*d2/c);
BPF = @(f,fc,w_pass) (abs(f-fc)<=w_pass*0.5) + (abs(f+fc)<=w_pass*0.5);


func_x1 = x1(t_x);
func_x2 = x2(t_x);
func_r1 = r1(t_x);
func_r2 = r2(t_x);

bpf_f1 = BPF(f,f1,w_pass);
bpf_f2 = BPF(f,f2,w_pass);

FT_x1 = FourierTransform(func_x1,t_x,Ts);
FT_x2 = FourierTransform(func_x2,t_x,Ts);
FT_r1 = FourierTransform(func_r1,t_x,Ts);
FT_r2 = FourierTransform(func_r2,t_x,Ts);

FT_y1 = FT_r1.*bpf_f1;
FT_y2 = FT_r2.*bpf_f2;

figure;
subplot(4,2,1);
plot(t_x,func_x1)
title("Plot of x1(t)")
xlabel("t")
ylabel("x1(t)")
subplot(4,2,2);
plot(t_x,func_x2)
title("Plot of x2(t)")
xlabel("t")
ylabel("x2(t)")
```

```
subplot(4,2,3);
plot(t_x,func_r1)
title("Plot of r1(t)")
xlabel("t")
ylabel("r1(t)")
subplot(4,2,4);
plot(t_x,func_r2)
title("Plot of r2(t)")
xlabel("t")
ylabel("r2(t)")

subplot(4,2,5);
plot(f,abs(FT_r1));
hold on;
plot(f,abs(FT_x1));
title("plot of |R1(jw)| and |X1(jw)|");
legend("|R1(jw)|","|X1(jw)|");
xlabel("frequency")
ylabel("Amplitude")

subplot(4,2,6);
plot(f,abs(FT_r2));
hold on;
plot(f,abs(FT_x2));
title("plot of |R2(jw)| and |X2(jw)|");
legend("|R2(jw)|","|X2(jw)|");
xlabel("frequency")
ylabel("Amplitude")

subplot(4,2,7);
plot(f,abs(FT_y1));
title("Plot of |Y1(jw)|")
xlabel("frequency")
ylabel("|Y1(jw)|")

subplot(4,2,8);
plot(f,abs(FT_y2));
title("Plot of |Y2(jw)|")
xlabel("frequency")
ylabel("|Y2(jw)|")

w1 = 2 * pi * f1;
w2 = 2 * pi * f2;

loc_first = 10000 + f1 * 2.0;
estimated_d1 =abs((1i * c * 0.5 / w1 ) * log(FT_y1(loc_first)/FT_x1(loc_first)));
disp("estimated d1");
disp(estimated_d1);

loc_second = 10000 + f2 * 2.0;
estimated_d2 =abs((1i * c * 0.5 / w2 ) *
log(FT_y2(loc_second)/FT_x2(loc_second)));
disp("estimated d2");
disp(estimated_d2);
```

“Lab5part3.m”

```
f1 = 100;
f2 = 800;
w_pass = 50;
T = 1;
c = 343;
Ts = 0.0001;
d1 = 0.05;
d2 = 0.1;

v1 = -30;
v2 = 40;

difference_f1 = (c+v1)/(c-v1);
difference_f2 = (c+v2)/(c-v2);

time = 2;
t_x = 0:Ts:time;

frequency = 1/Ts;
f = -frequency * 0.5:(frequency/(length(t_x))):frequency * 0.5;

rec_pulse = @(t) abs(t) < T * 0.5;
x1 = @(t) cos(2*pi*f1*t) .* rec_pulse(t-T * 0.5);
x2 = @(t) cos(2*pi*f2*t) .* rec_pulse(t-T * 0.5);
r1 = @(t) x1(t*difference_f1-2*d1/c) + x2(t*difference_f1 - (d1+d2)/c);
r2 = @(t) x1(t*difference_f2-(d1+d2)/c) + x2(t*difference_f2 - 2*d2/c);

x1_seq = x1(t_x);
x2_seq = x2(t_x);
r1_seq = r1(t_x);
r2_seq = r2(t_x);

FT_x1 = FourierTransform(x1_seq,t_x,Ts);
FT_x2 = FourierTransform(x2_seq,t_x,Ts);
FT_r1 = FourierTransform(r1_seq,t_x,Ts);
FT_r2 = FourierTransform(r2_seq,t_x,Ts);


figure;
subplot(4,2,1);
plot(t_x,x1_seq)
title("Plot of x1(t)")
xlabel("t")
ylabel("x1(t)")
subplot(4,2,2);
plot(t_x,x2_seq)
title("Plot of x2(t)")
xlabel("t")
ylabel("x2(t)")
subplot(4,2,3);
```

```
plot(t_x,r1_seq)
title("Plot of r1(t)")
xlabel("t")
ylabel("r1(t)")
subplot(4,2,4);
plot(t_x,r2_seq)
title("Plot of r2(t)")
xlabel("t")
ylabel("r2(t)")

subplot(4,2,5);
plot(f,abs(FT_r1));
hold on;
plot(f,abs(FT_x1));
title("plot of |R1(jw)| and |X1(jw)|");
legend("|R1(jw)|","|X1(jw)|");
xlabel("f")
ylabel("Amplitude")

subplot(4,2,6);
plot(f,abs(FT_r2));
hold on;
plot(f,abs(FT_x2));
title("plot of |R2(jw)| and |X2(jw)|");
legend("|R2(jw)|","|X2(jw)|");
xlabel("f")
ylabel("Amplitude")
```

"Lab5part4.m"