Uygulama 3

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① $x(t) = 1 + \sin(w_0 t) + 2\cos(w_0 t) + \cos(2w_0 t + \pi/4)$ olarak verilen $x(t)$ işaretinin Fourier serisi katsayılarını bulunuz. Genlik ve faz spektrumlarını çiziniz.

$$* \; x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

$$* \; a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jkw_0 t}\, dt, \qquad w_0 = \frac{2\pi}{T}$$

$$* \; \sin(\alpha) = \frac{e^{j\alpha} - e^{-j\alpha}}{2j}$$

$$* \; \cos(\alpha) = \frac{e^{j\alpha} + e^{-j\alpha}}{2}$$

$$x(t) = 1 + \frac{1}{2j}\left[e^{jw_0 t} - e^{-jw_0 t}\right] + \left[e^{jw_0 t} + e^{-jw_0 t}\right] + \left[\frac{e^{j(2w_0 t + \pi/4)} + e^{-j(2w_0 t + \pi/4)}}{2}\right]$$

$$= 1 + \frac{1}{2j} e^{jw_0 t} - \frac{1}{2j} e^{-jw_0 t} + e^{jw_0 t} + e^{-jw_0 t} + \frac{e^{j\pi/4}}{2} \cdot e^{j2w_0 t} + \frac{e^{-j\pi/4}}{2} \cdot e^{-j2w_0 t}$$

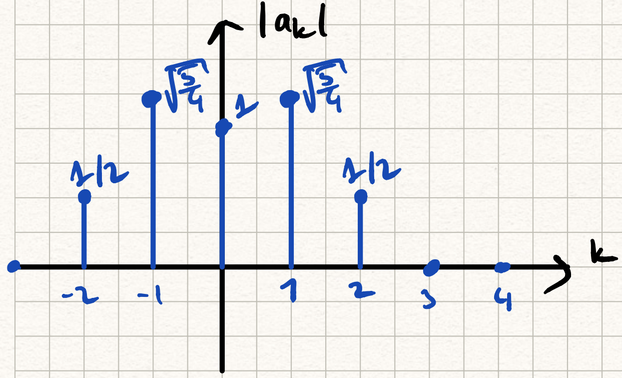
$$e^{j\pi/4} = \cos(\pi/4) + j\cdot\sin(\pi/4) = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

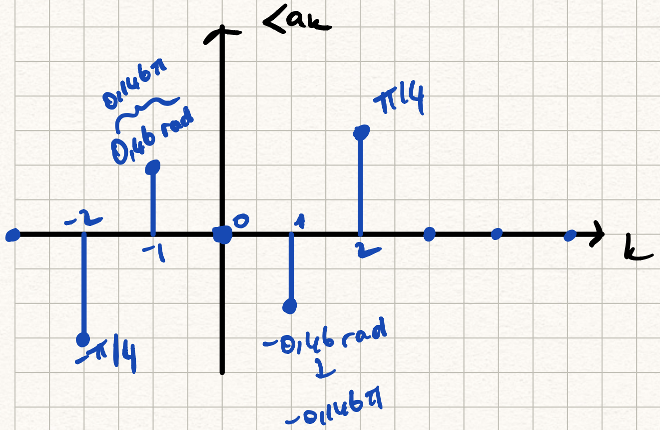
$$e^{-j\pi/4} = \cos(-\pi/4) + j\cdot\sin(-\pi/4) = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$x(t) = \underbrace{1}_{a_0} + \underbrace{\left(1+\frac{1}{2j}\right)}_{a_1} e^{j\omega_0 t} + \underbrace{\left(1-\frac{1}{2j}\right)}_{a_{-1}} e^{-j\omega_0 t} + \underbrace{\left(\frac{1}{2\sqrt{2}} + j\frac{1}{2\sqrt{2}}\right)}_{a_2} e^{j2\omega_0 t} + \underbrace{\left(\frac{1}{2\sqrt{2}} - j\frac{1}{2\sqrt{2}}\right)}_{a_{-2}} e^{-j2\omega_0 t}$$

- $a_0 = 1$
- $a_1 = 1 + \frac{1}{2j} = 1 - j\frac{1}{2}$
- $a_{-1} = 1 - \frac{1}{2j} = 1 + j\frac{1}{2}$
- $a_2 = \frac{\sqrt{2}}{4} + j\frac{\sqrt{2}}{4}$
- $a_2 = \frac{\sqrt{2}}{4} - j\frac{\sqrt{2}}{4}$
- $a_k = 0, \quad |k| > 2$

$$\boxed{\begin{array}{l} a_k = |a_k| e^{j\angle a_k} \rightarrow a_k = a + jb \\ |a_k| = \sqrt{a^2 + b^2} \\ \angle a_k = \tan^{-1}\left(\frac{b}{a}\right) \end{array}}$$





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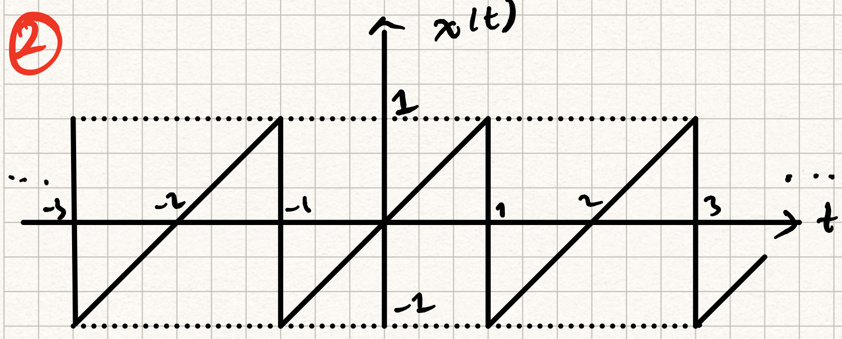

yukarıdaki periyodik x(t) işaretinin Fourier serisi katsayılarını bulunuz.

$x(t)=t,\ -1\le t\le 1$ ve $T=2$ ile periyodik.

$$* \ x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$* \ a_k=\frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}\,dt, \qquad \omega_0=\frac{2\pi}{T}$$

$$a_k=\frac{1}{2}\int_{-1}^{1} t\cdot e^{-j\frac{2\pi k}{T}t}\,dt$$

$t=u \qquad e^{-j\omega_0 kt}=dv$

$dt=du \qquad \frac{e^{-j\omega_0 kt}}{-j\omega_0 k}=v$

$$\left(uv-v\,du\right)\Big|_{-1}^{1}$$

$$a_k=\frac{1}{2}\left(\frac{t e^{-j\omega_0 k t}}{-j\omega_0 k}\Bigg|_{-1}^{1}-\int_{-1}^{1}\frac{e^{-j\omega_0 kt}}{-j\omega_0 k}\,dt\right)$$

$$a_k=\frac{1}{2}\left(\underbrace{\frac{e^{-j\omega_0 k}}{-j\omega_0 k}+\frac{e^{j\omega_0 k}}{-j\omega_0 k}}_{\cos}-\underbrace{\frac{e^{-j\omega_0 k}}{(-j\omega_0 k)^2}+\frac{e^{j\omega_0 k}}{(-j\omega_0 k)^2}}_{\sin}\right)$$

$$a_k = \frac{\cos(w_0 k)}{-jw_0 k} - j \cdot \frac{\sin(w_0 k)}{(w_0 k)^2} \qquad w_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

$$= \frac{\cos(\pi k)}{-j\pi k} - j\,\frac{\sin(\pi k)}{\pi^2 k^2}$$

$$a_k = \begin{cases} \dfrac{1}{j\pi k}, & k \text{ tek} \\ -\dfrac{1}{j\pi k}, & k \text{ çift} \end{cases}$$

③ Sürekli zamanlı $x(t)$ periyodik işareti gerçel değerlidir ve $T=8$ periyoduna sahiptir.
Aşağıda Fourier serisi katsayıları verilen bu $x(t)$ işaretini bulunuz.

- $a_1 = a_{-1}^* = j$
- $a_5 = a_{-5} = 2$ ise $x(t) = ?$
- $a_k = 0,\ k \neq |1|, |5|$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{-jkw_0 t}$$

$$x(t) = 2e^{j5w_0 t} + 2e^{-j5w_0 t} + je^{jw_0 t} - je^{-jw_0 t}$$

$$= 4\cos(5w_0 t) + j\,2j\sin w_0 t = 4\cos(5w_0 t) - 2\sin(w_0 t)$$

(4) Aşağıda verilen bilgileri kullanarak $x(t)$'yi bulunuz.

1) $x(t)$ gerçel bir işarettir.

2) $x(t)$ $T=4$ periyoduyla periyodiktir.

3) $a_k = 0, \quad |k| > 1$

4) $b_k = e^{-j\pi k/2} \cdot a_{-k}$ FS katsayılarına sahip olan işaret "tek" bir işarettir.

5) $\frac{1}{4} \int_{4} |x(t)|^2 dt = \frac{1}{2}$

Çözüm: 3)'ten anlıyoruz ki a_{-1}, a_0 ve a_1 dışındaki FS katsayıları "0".

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} = a_{-1} e^{-j2\pi/T t} + a_0 + a_1 e^{j2\pi/T t}$$

1)'den biliyoruz ki $\Rightarrow \quad a_1 = a_{-1}^*$

$$x(t) = \left(a_1 e^{j\frac{\pi}{2}t}\right)^* + a_0 + a_1 e^{j\frac{\pi}{2}t}$$

$$= a_0 + 2\,\mathrm{Re}\left\{a_1 e^{j\frac{\pi}{2}t}\right\}$$

4)'ten yola çıkarak b_k hakkında yorum yapalım:

→ Zamanda tersine çevirme (time-reversal) özelliğinden a_{-k}'nın $x(-t)$'nin FS katsayıları olduğu biliniyor.

$y(t) = x(-t)$ ise $FS\{y(t)\} = a_{-k}$

$\rightarrow$ Zamanda kayma özelliğinden

$y(t) = x(t-t_0)$ ise $FS\{y(t)\} = a_k e^{-jk\omega_0 t}$

Bu iki özellik kullanılarak

$$b_k = e^{-j\pi k/2} \cdot a_{-k} = FS\{x(-(t-1))\}$$

$$= FS\{x(-t+1)\}$$

$\rightarrow$ $x(t)$ gerçel ise $x(-t+1)$ de gerçel.

$x(-t+1)$ tek ise $b_0 = 0$ ve $b_1 = -b_{-1}$ olmalı
(purely imaginary)

Zamanda öteleme ve ters çevirme ortalama gücü değiştirmez.

$$\frac{1}{4}\int_4 |x(-t+1)|^2 dt = \frac{1}{2} \rightarrow \text{using Parseval's relation}$$

$$|b_1|^2 + |b_{-1}|^2 = \frac{1}{2} \rightarrow -b_1 = +b_{-1}$$

$$|b_1| = 1/2 \Rightarrow b_1 = j/2 \text{ ve}$$
$$b_{-1} = -j/2 \text{ olmalı.}$$

$b_0 = 0$ ise $a_0 = 0$ olmalı.

$$a_k = e^{-jk\frac{\pi}{2}} b_{-k} \Rightarrow a_1 = e^{-j\pi/2} b_{-1} = -j b_{-1} = j b_1 = -\frac{1}{2}$$

$$\Rightarrow$$

$$x(t) = a_0 + 2Re\{a_1 e^{j\frac{\pi}{2}t}\} = 2Re\{-\frac{1}{2} e^{j\frac{\pi}{2}t}\}$$

$$= 2 \cdot Re\{-\frac{1}{2}\cos(\frac{\pi}{2}t) - \frac{j}{2}\sin(\frac{\pi}{2}t)\}$$

$$= -\cos(\pi/2\, t)$$

Eğer $b_1 = -j/2$ olarak alınsaydı:

$x(t) = \cos(\frac{\pi}{2} t)$ olacaktı.

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t)$, $\alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathfrak{Re}\{a_k\} = \mathfrak{Re}\{a_{-k}\} \\ \mathfrak{Im}\{a_k\} = -\mathfrak{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathfrak{Re}\{a_k\}$ $j\mathfrak{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$