TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$ $x_1(t)$ $x_2(t)$	$X(s)$ $X_1(s)$ $X_2(s)$	R R_1 R_2
9.5.1	Linearity	$ax_1(t)+bx_2(t)$	$aX_1(s)+bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s-s_0)$	Shifted version of R (i.e., s is in the ROC if $s-s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\Re e\{s\}>0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t)=0$ for $t<0$ and $x(t)$ contains no impulses or higher-order singularities at $t=0$, then

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

If $x(t)=0$ for $t<0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re e\{s\}>0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\}<0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\}>0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\}<0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\}>-\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\}<-\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\}>-\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\}<-\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\}>0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\}>0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\}>-\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\}>-\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\}>0$

$\frac{1}{s+a} = e^{-at}.u(t)$

$-e^{-at}u(-t)$

$t.e^{-at}.u(t)$

$n=2$

$$\boxed{\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt, \quad s=\sigma+j\omega \\ x(t) &= \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\,e^{st}ds \\ X(\sigma+j\omega) &= \int x(t)\,e^{-(\sigma+j\omega)t}dt \end{aligned}}$$

Ör:

$$x(t) = e^{-at}.u(t)$$

$$X(j\omega) = \int_0^\infty e^{-at}e^{-j\omega t}dt = \frac{1}{j\omega+a}, \quad a>0$$

$$X(s) = \int_0^\infty e^{-at}.e^{-st}dt = \int_0^\infty e^{-(a+s)t}dt$$

$$X(\sigma+j\omega) = \int_0^\infty e^{-(\sigma+j\omega+a)t}dt$$

$$= \frac{1}{\sigma+a+j\omega}, \quad \sigma+a>0 \Rightarrow \Re e\{s\}>-a$$

$$\sigma = \Re e\{s\}$$

//

$X(s) = \frac{1}{s+a} \quad , \quad Re\{s\} > -a$

Lineerlik :

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)$$

$$x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$$

$$\boxed{a x_1(t) + b x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} a X_1(s) + b X_2(s), \text{ with ROC containing } R_1 \cap R_2.}$$

Zamanda Öteleme :

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{with ROC} = R,$$

$$\boxed{x(t - t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_0} X(s), \quad \text{with ROC} = R.}$$

S-domeninde Öteleme :

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{with ROC} = R,$$

$$\boxed{e^{s_0 t} x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s - s_0), \quad \text{with ROC} = R + \Re e\{s_0\}.}$$

Zamanda Ölçekleme :

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{with ROC} = R,$$

$$\boxed{x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right), \quad \text{with ROC } R_1 = aR.}$$

Eşlenik Alma :

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{with ROC} = R,$$

$$\boxed{x^*(t) \overset{\mathcal{L}}{\longleftrightarrow} X^*(s^*), \quad \text{with ROC} = R.}$$

$X(s) = X^*(s^*)$ reel x(t) için

Konvolüsyon :

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s), \quad \text{with } ROC = R_1,$$

$$x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s), \quad \text{with } ROC = R_2,$$

$$\boxed{x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) X_2(s), \quad \text{with ROC containing } R_1 \cap R_2.}$$

S Domeninde Türev :

$$X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt,$$

$$\frac{dX(s)}{ds} = \int_{-\infty}^{+\infty} (-t) x(t) e^{-st} dt.$$

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{with ROC} = R,$$

$$\boxed{-t x(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{dX(s)}{ds}, \quad \text{with ROC} = R.}$$

Zaman Domeninde Türev :

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{with ROC} = R,$$

$$\boxed{\frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX(s), \quad \text{with ROC containing } R.}$$

Zaman Domeninde İntegrasyon :

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{with ROC} = R,$$

$$\boxed{\int_{-\infty}^{t} x(\tau) d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} X(s), \text{ with ROC containing } R \cap \{\Re e\{s\} > 0\}.}$$

<u>LZD sistemlerin Laplace Domeni Analizi</u>

Nedensellik ve Kararlılık

-Nedensel bir sistemin yakınsaklık bölgesi en sağdaki kutbun sağındadır

-Kararlı bir sistemin yakınsaklık bölgesi jw eksenini içerir.

-Genel olarak kararlı ve nedensel bir sistemin kutupları sol yarı düzlemde bulunmalıdır.

1) Aşağıdaki işaretlerin Laplace dönüşümünü bulunuz. Yakınsaklık bölgelerini gösteriniz.

a) $x_1(t) = e^{-2(t-3)} u(t-3)$

$$\boxed{e^{-at}.u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+a}, \ Re\{s\}>-a}$$

$e^{-2t}.u(t)$

$$e^{-2t}.u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+2}, \quad Re\{s\}>-2$$

$$\boxed{x(t-t_0) \xrightarrow{\mathcal{L}} e^{-st_0} X(s)}$$

e^{-3s}

$$e^{-2(t-3)} u(t-3) \xrightarrow{\mathcal{L}} e^{-3s} \cdot \frac{1}{s+2}, \quad Re\{s\}>-2$$

$$X_1(s) = \int_{-\infty}^{\infty} x_1(t)e^{-st}dt = \int_{-\infty}^{\infty} e^{-2(t-3)}u(t-3)e^{-st}dt = e^6 \int_3^{\infty} e^{-(s+2)t}dt$$
$$= e^6 \left. \frac{e^{-(s+2)t}}{-(s+2)} \right|_3^{\infty} = \frac{e^{-3s}}{s+2}; \quad \text{Re}(s) > -2$$

b) $x(t) = \left(1 - (1-t)e^{-3t}\right) u(t)$

$$= \underbrace{u(t)}_{x_1(t)} - \underbrace{e^{-3t}.u(t)}_{x_2(t)} + \underbrace{t.e^{-3t}.u(t)}_{x_3(t)}$$

$$x_1(t) = u(t) \xleftrightarrow{\mathcal{L}} X_1(s) = \frac{1}{s}, \quad Re\{s\}>0 \ ✓$$

$$x_2(t) = e^{-3t}.u(t) \xleftrightarrow{\mathcal{L}} X_2(s) = \frac{1}{s+3}, \quad Re\{s\}>-3 \ ✓$$

$a=3$

$$\boxed{\frac{t^{n-1}}{(n-1)!} e^{-at} u(t) \xrightarrow{\mathcal{L}} \frac{1}{(s+a)^n}, \quad Re\{s\} > -a}$$

$$x_3(t) = t \cdot e^{-3t} u(t) \xrightarrow{\mathcal{L}} \frac{1}{(s+3)^2}, \quad Re\{s\} > -3$$

$n=2$
$a=3$

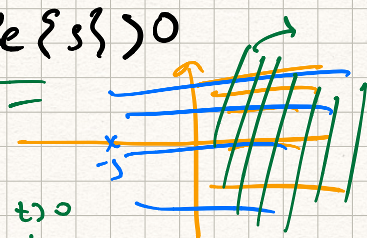
$$X_3(s) = \int_0^\infty t e^{-3t} e^{-st} dt = t \frac{e^{-(s+3)t}}{-(s+3)}\Bigg|_0^\infty - \int_0^\infty \frac{e^{-(s+3)t}}{-(s+3)} dt = \frac{1}{(s+3)^2} \; ; \; \mathrm{Re}(s) > -3$$

$$X(s) = X_1(s) - X_2(s) + X_3(s)$$

$$= \frac{1}{s} - \frac{1}{s+3} + \frac{1}{(s+3)^2}, \quad \underbrace{R_1 \cap R_2 \cap R_3}_{R}$$

R_1: $Re\{s\} > 0$
R_2: $Re\{s\} > -3$
R_3: $Re\{s\} > -3$

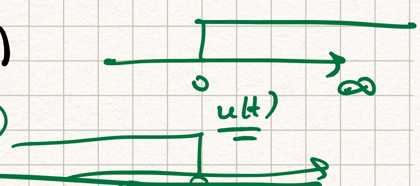
$\Big\}$ R : $Re\{s\} > 0$



ⓒ $x(t) = |t| e^{-|t|}$

$t<0$, $|t| = -t$; $t>0$, t

$$x(t) = \underbrace{t e^{-t} u(t)}_{x_1(t)} - \underbrace{t e^{t} u(-t)}_{x_2(t)}$$



$$\boxed{-\frac{t^{n-1}}{(n-1)!} \cdot e^{-at} \cdot u(-t) \xrightarrow{\mathcal{L}} \frac{1}{(s+a)^n}; \quad Re\{s\} < -a}$$

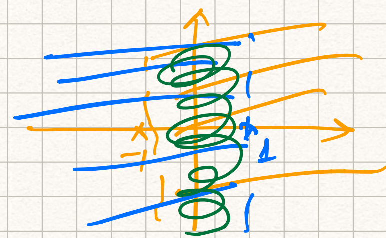
$$-t \cdot e^{t} u(-t) \xrightarrow{\mathcal{L}} \frac{1}{(s-1)^2}, \quad Re\{s\} < 1$$

$a=-1$
$n=2$

$$X_1(s) = \frac{1}{(s+1)^2} \rightarrow R_1: Re\{s\} > -1 \quad , \quad X_2(s) = \frac{1}{(s-1)^2} \rightarrow R_2: Re\{s\} < 1$$

$$X(s) = X_1(s) + X_2(s) = \frac{1}{(s+1)^2} + \frac{1}{(s-1)^2} , \quad R_1 \cap R_2$$

$$= \frac{2(s^2+1)}{(s+1)^2(s-1)^2}$$



$$R_1: Re\{s\} > -1 \quad , \quad R_2: Re\{s\} < 1$$

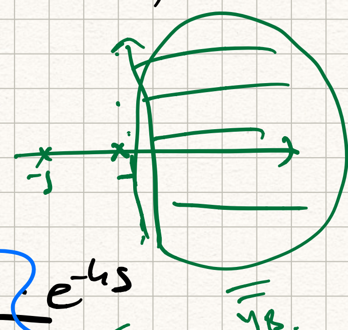
$$R: \; -1 < Re\{s\} < 1$$

2) $$X(s) = \frac{2 + 2se^{-2s} + 4e^{-4s}}{s^2 + 4s + 3} , \; Re\{s\} > -1$$

$s^2+4s+3 \rightarrow (s+1)(s+3)$

Yukarıda Laplace dönüşümü verilen $x(t)$ işaretini bulunuz

$$X(s) = \frac{2 + 2se^{-2s} + 4e^{-4s}}{s^2 + 4s + 3}$$



$$= \underbrace{\frac{2}{s^2+4s+3}}_{X_1(s)} + \underbrace{\frac{2s}{s^2+4s+3}}_{X_2(s)} \cdot e^{-2s} + \underbrace{\frac{4}{s^2+4s+3}}_{X_3(s)} \cdot e^{-4s}$$

$$X(s) = X_1(s) + X_2(s)e^{-2s} + X_3(s)e^{-4s}$$

$$x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s), \quad x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s), \quad x_3(t) \xleftrightarrow{\mathcal{L}} X_3(s)$$

→ Lineerlik özelliğinden :

$$x(t-t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0} X(s)$$

$$x(t) = x_1(t) + x_2(t-2) + x_3(t-4)$$

* $$X_1(s) = \frac{2}{(s+1)(s+3)} = \frac{1}{s+1} - \frac{1}{s+3} \xleftrightarrow{\mathcal{L}} x_1(t) = \left(e^{-t} - e^{-3t}\right) u(t)$$

$$\frac{1}{s+a}$$

* $$X_2(s) = \frac{2s}{(s+1)(s+3)} = \frac{-1}{s+1} + \frac{3}{s+3} \xleftrightarrow{\mathcal{L}} x_2(t) = \left(-e^{-t} + 3e^{-3t}\right) u(t)$$

* $$X_3(s) = \frac{4}{(s+1)(s+3)} = \frac{2}{s+1} - \frac{2}{s+3} \xleftrightarrow{\mathcal{L}} x_3(t) = 2\left(e^{-t} - e^{-3t}\right) u(t)$$

$$\boxed{\begin{aligned} x(t) = {} & \left(e^{-t} - e^{-3t}\right) u(t) + \left[-e^{-(t-2)} + 3e^{-3(t-2)}\right] u(t-2) \\ & + 2\left[e^{-(t-4)} - e^{-3(t-4)}\right] u(t-4) \end{aligned}}$$

3) Nedensel LTI sistemin giriş çıkış ilişkisi aşağıda verilmiştir.

$$y''(t) + y'(t) - 2y(t) = x(t)$$

i) Sistemin transfer fonksiyonu $H(s)$'i bulunuz.

ii) Sistemin birim dürtü yanıtı $h(t)$'yi bulunuz.

(DZD) LTI

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

$$X(s) \qquad H(s) \qquad Y(s)$$

$$y(t) = x(t) * h(t)$$

$$Y(s) = X(s)\,H(s)$$

$$H(s) = \frac{Y(s)}{X(s)}$$

$$y(t) \xrightarrow{\mathcal{L}} Y(s)$$

$$y'(t) = \frac{dy(t)}{dt} \xrightarrow{\mathcal{L}} sY(s)$$

$$y''(t) = \frac{d^2 y(t)}{dt^2} \xrightarrow{\mathcal{L}} s^2 Y(s)$$

$$s^2Y(s) + sY(s) - 2Y(s) = X(s)$$

$$Y(s)\left[s^2 + s - 2\right] = X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{1}{s^2+s-2} = H(s)$$

$$H(s) = \frac{1}{(s+2)} \cdot \frac{1}{(s-1)} = \frac{A}{(s+2)} + \frac{B}{(s-1)}$$

$\frac{1}{s+a}$

$A = -1/3$

$B = 1/3$

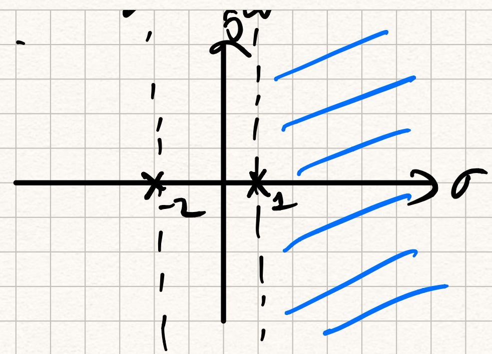
A, B ✓

$$H(s) = \frac{-1/3}{(s+2)} + \frac{1/3}{(s-1)}$$

YB için ihtimaller: $Re\{s\} > -2$, $Re\{s\} > 1$

$Re\{s\} < -2$, $Re\{s\} < 1$

Nedensel olduğundan YB tüm kutupların sağında yer almak.



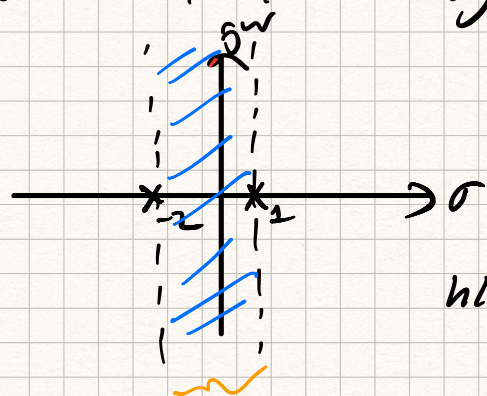
$R: R_1 \cap R_2$

$R_1: Re\{s\} > -2$

$R_2: Re\{s\} > 1$ olmalı.

$$H(s) = \frac{-1/3}{(s+2)} + \frac{1/3}{s-1} \xrightarrow{\mathcal{L}^{-1}} h(t) = -\frac{1}{3}e^{-2t}u(t) + \frac{1}{3}e^{t}u(t)$$

→ Eğer kararlı olması isteseydi: $j\omega$ eksenini kapsamalıydı.



$R: R_1 \cap R_2$

$R_1: Re\{s\} > -2$

$R_2: Re\{s\} < 1$

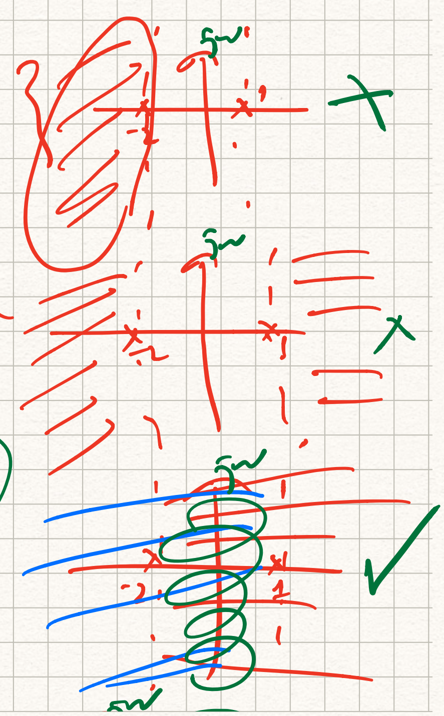
$$h(t) = -\frac{1}{3}e^{-2t}u(t) - \frac{1}{3}e^{t}u(-t)$$

→ $Re\{s\} < -2$
$Re\{s\} < 1$

→ $Re\{s\} < -2$
$Re\{s\} > 1$

→ $Re\{s\} > -2$
$Re\{s\} < 1$

→ $Re\{s\} > -2$
$Re\{s\} > 1$



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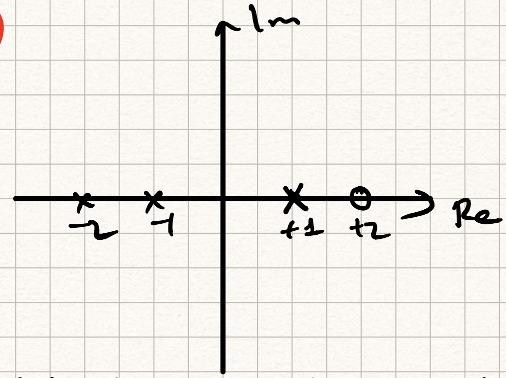
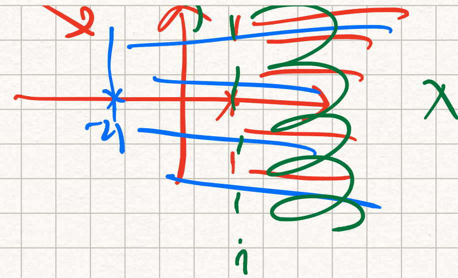
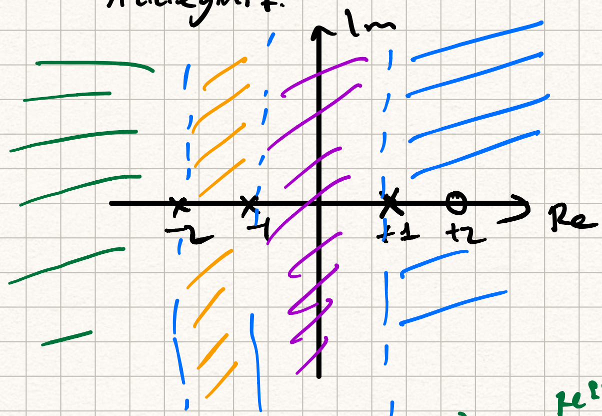




Yukarıda sıfır/kutup gösterimi verilen H(s) transfer fonksiyonuna sahip bir LTI sistem için

a) Tüm olası yakınsaklık bölgelerini belirtiniz.

b) Bu yakınsaklık bölgeleri için nedensellik ve kararlılık özelliklerini irdeleyiniz.



Re{s} > -1 ×

Re{s} > -2, Re{s} > -1

Re{s} > 1, Re{s} < 1

Re{s} < -2, Re{s} < -1, Re{s} < 1

i) Re{s} < -2 → sol taraflı, jω eksenini içermiyor

nedensel değil, kararlı değil

→ Re{s} > -2, Re{s} < -1, Re{s} < 1

ii) -2 < Re{s} < -1 → hepsi sağ taraflı değil, jω eksenini içermiyor

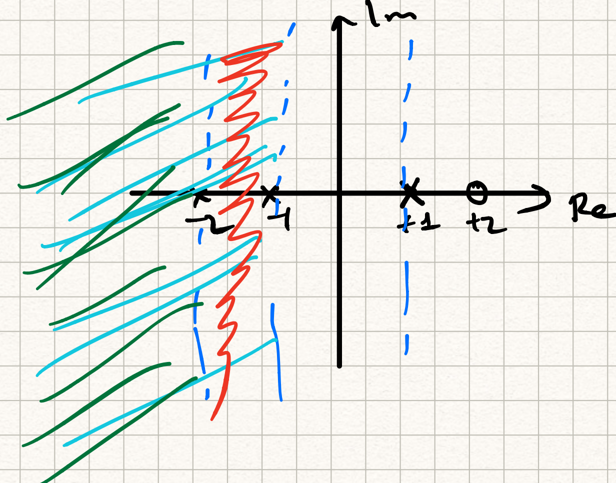
nedensel değil, kararlı değil

iii) -1 < Re{s} < 1 → hepsi sağ taraflı değil, jω eksenini içeriyor

nedensel değil, kararlı

iv) Re{s} > 1 → sağ taraflı, jω eksenini içermiyor

nedensel, kararlı değil



$R : Re\{s\} < -1$

$R_1 : Re\{s\} < 1$

$R_2 : Re\{s\} < -1$

$\Big\}\ R_1 \cap R_2 : Re\{s\} < 1$

$R_3 : Re\{s\} < -2 \rightarrow Re\{s\} < -2$

$Re\{s\} > -2 \rightarrow -2 < Re\{s\} < -1$