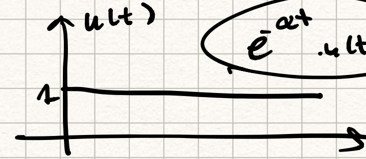
İbrahim Yıldırım

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$



$e^{-at} . u(t)$

1) $x(t) = e^{-3t}[u(t+2) - u(t-3)]$ işaretinin Fourier dönüşümünü hesaplayınız.

$a.x(t) + b.y(t) \xrightarrow{F} A.X(j\omega) + B.Y(j\omega)$

$$e^{-at}.u(t) \xrightarrow{F} \frac{1}{a+j\omega},\ a>0$$

$$x(t-t_0) \xrightarrow{F} e^{-j\omega t_0} X(j\omega)$$

$$a x(t) + b y(t) \to a X(j\omega) + b Y(j\omega)$$

$a(t) = a(t+2)$

* $e^{-3t} u(t) \longrightarrow A(j\omega) = \dfrac{1}{3+j\omega}$

$a(t+2) = e^{-3(t+2)} u(t+2) \longrightarrow \dfrac{e^{2j\omega}\, 1}{3+j\omega}$

$\dfrac{1}{e^{-6}} e^{-6} e^{-3t} u(t+2) \longrightarrow \dfrac{1}{e^{-6}} \dfrac{e^{2j\omega}}{3+j\omega}$

$e^{-3t} u(t+2) \longrightarrow e^{6} \dfrac{e^{2j\omega}}{3+j\omega}$ $A(j\omega)$

* $e^{-3t} u(t) \longrightarrow \dfrac{1}{3+j\omega}$

$e^{-3(t-3)} . u(t-3) \longrightarrow \dfrac{e^{-3j\omega}}{3+j\omega}$

$\dfrac{1}{e^{9}} e^{9} . e^{-3t} u(t-3) \longrightarrow \dfrac{1}{e^{9}} \dfrac{e^{-3j\omega}}{3+j\omega}$

$e^{-3t} u(t-3) \longrightarrow \dfrac{e^{-9} e^{-3j\omega}}{3+j\omega}$ $B(j\omega)$

$x(t) \xrightarrow{F} A(j\omega) - B(j\omega)$

$$X(j\omega) = \frac{e^{6} e^{2j\omega}}{3+j\omega} - \frac{e^{-9} e^{-3j\omega}}{3+j\omega} = \frac{e^{2(j\omega+3)} - e^{-3(j\omega+3)}}{3+j\omega}$$

② $x(t) = e^{-2|t-1|}$ işaretinin Fourier dönüşümünü hesaplayınız.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{-2|t-1|}\, e^{-j\omega t}\, dt$$

$x(t) = e^{-2|t|} \rightarrow A(j\omega)$

$x(t-1) \rightarrow e^{-j\omega} A(j\omega)$

$t-1 > 0$ için $|t-1| = t-1 \rightarrow t > 1$

$t-1 < 0$ için $|t-1| = -t+1 \quad t < 1$

$$X(j\omega) = \int_{-\infty}^{1} e^{2(t-1)}\, e^{-j\omega t}\, dt + \int_{1}^{\infty} e^{-2(t-1)}\, e^{-j\omega t}\, dt$$

$$= e^{-2} \int_{-\infty}^{1} e^{t(2-j\omega)} \cdot dt + e^{2} \int_{1}^{\infty} e^{t(-2-j\omega)}\, dt$$

$$= e^{-2} \left(\frac{1}{2-j\omega}\, e^{t(2-j\omega)} \Big|_{-\infty}^{1} \right) + e^{2} \left(\frac{1}{(2-j\omega)}\, e^{t(2-j\omega)} \Big|_{1}^{\infty} \right)$$

$$= e^{-2} \left(\frac{e^{2-j\omega}}{2-j\omega} \right) + e^{2} \left(\frac{e^{(-2-j\omega)}}{2+j\omega} \right)$$

$$X(j\omega) = \underset{(2+j\omega)}{\frac{e^{-j\omega}}{2-j\omega}} + \underset{(2-j\omega)}{\frac{e^{-j\omega}}{2+j\omega}} = \frac{2e^{-j\omega} + \cancel{j\omega e^{-j\omega}} + 2e^{-j\omega} - \cancel{j\omega e^{-j\omega}}}{4+\omega^2}$$

$$X(j\omega) = \frac{4e^{-j\omega}}{4+\omega^2} \quad //$$

Find the Fourier transform of a periodic signal $x(t)$ with period T_0.

We express $x(t)$ as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

$x(t) \xrightarrow{FS} c_k$

$X(j\omega)$

Taking the Fourier transform of both sides and using Eq. (5.142) and the linearity property (5.49), we get

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} c_k \delta(\omega - k\omega_0) \tag{5.146}$$

which indicates that the Fourier transform of a periodic signal consists of a sequence of equidistant impulses located at the harmonic frequencies of the signal.

3) Aşağıda verilen periyodik $x(t)$ işaretinin Fourier dönüşümünü bulunuz.

$6\pi = \frac{2\pi}{T}$

$$x(t) = 1 + \cos\left(6\pi t + \frac{\pi}{8}\right).$$

$\cos(\alpha) = \frac{e^{j\alpha} + e^{-j\alpha}}{2}$

$\frac{2\pi}{T} = 6\pi \rightarrow T = 1/3$ temel periyoduyla periyodik.

$$x(t) = 1 + \frac{1}{2} e^{j(6\pi t + \pi/8)} + \frac{1}{2} e^{-j(6\pi t + \pi/8)}$$

$$= \underset{k=0}{1} + \frac{1}{2} e^{j\pi/8} \underset{k=1}{e^{j6\pi t}} + \frac{1}{2} e^{-j\pi/8} \underset{k=-1}{e^{-j6\pi t}}$$

$e^{j\omega_0 k t} = e^{j6\pi t}$, $\omega_0 = \frac{2\pi}{T} = 6\pi$

$x(t)$'nin sıfır olmayan Fourier serisi katsayıları:

$c_0 = 1$, $c_1 = \frac{1}{2} e^{j\pi/8}$, $c_{-1} = \frac{1}{2} e^{-j\pi/8}$ $\quad k \neq 0, \mp 1 \Rightarrow c_k = 0$

$k\omega_0 = \omega_0$ $\quad -k\omega_0 = k\omega_0$

$$X(j\omega) = 2\pi c_0 \delta(\omega) + 2\pi c_1 \delta(\omega - \omega_0) + 2\pi c_{-1} \delta(\omega + \omega_0)$$

$$= 2\pi \delta(\omega) + \pi e^{j\pi/8} \delta(\omega - 6\pi) + \pi e^{-j\pi/8} \delta(\omega + 6\pi)$$

④ Aşağıda Fourier dönüşümü verilen $x(t)$ işaretini bulunuz.

$$X(j\omega) = \begin{cases} 2, & 0 \le \omega \le 2 \\ -2, & -2 \le \omega < 0 \\ 0, & |\omega| > 2 \end{cases}$$

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \int_{0}^{2} 2\, e^{j\omega t}\, d\omega + \frac{1}{2\pi} \int_{-2}^{0} (-2)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{\pi} \left. \frac{e^{j\omega t}}{jt} \right|_{0}^{2} - \frac{1}{\pi} \cdot \left. \frac{e^{j\omega t}}{jt} \right|_{-2}^{0}$$

$$= \left(\frac{e^{j2t} - 1}{\pi j t} \right) - \left(\frac{1 - e^{-j2t}}{\pi j t} \right)$$

$$\cos(2x) = 1 - 2\sin^2 x$$
$$2\sin^2 x = 1 - \cos(2x)$$

$$= \frac{e^{j2t} + e^{-j2t} - 2}{\pi j t} = \frac{2(\cos(2t) - 1)}{\pi j t}$$

$$= -\frac{(4j \sin^2 t)}{\pi t}$$

(5) $x(t) = e^{-2t}u(t)$, $h(t) = e^{-4t}u(t)$ için konvolusyon fourier dönüşümü özelliğini kullanarak $y(t)=x(t)*h(t)$'yi bulunuz.

$$y(t)=x(t)*h(t) \xrightarrow{\mathcal{F}} Y(j\omega)=X(j\omega)H(j\omega)$$

$$e^{-at}\cdot u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a+j\omega}$$

$$X(j\omega) = \frac{1}{2+j\omega}, \quad H(j\omega) = \frac{1}{4+j\omega}$$

$$Y(j\omega) = \frac{1}{(2+j\omega)}\,\frac{1}{(4+j\omega)}$$

$$Y(j\omega) = \frac{A}{2+j\omega} + \frac{B}{4+j\omega} = \frac{1}{(2+j\omega)(4+j\omega)}$$

$(4+j\omega)$ $\quad$ $(2+j\omega)$

$\rightarrow 4A + Aj\omega + 2B + Bj\omega = 1$

$4A+2B=1$

$Aj\omega + Bj\omega = 0 \rightarrow A=-B$

$A = 1/2$

$B = -1/2$

$$Y(j\omega) = \frac{1/2}{2+j\omega} - \frac{1/2}{4+j\omega}$$

$$e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$$

$\updownarrow \mathcal{F}^{-1}$

$$\frac{1}{2}\cdot e^{-2t}\cdot u(t) \longleftrightarrow \frac{1}{2}\frac{1}{2+j\omega}$$

$$y(t) = \frac{1}{2}\cdot e^{-2t}\cdot u(t) - \frac{1}{2}e^{-4t}u(t)$$

$-\frac{1}{2}\cdot e^{-4t}\cdot u(t)$

Duality:

$$x(t) \xrightarrow{\mathcal{F}} X(j\omega)$$

$$X(t) \xrightarrow{\mathcal{F}} 2\pi x(-j\omega)$$

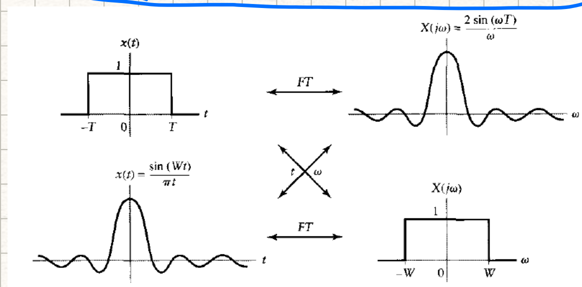


FIGURE 3.42 Duality of rectangular pulses and sinc functions.

6) Dualite özelliğini kullanarak aşağıda verilen $x(t)$ işaretinin Fourier dönüşümünü bulunuz.

$$x(t) = \frac{1}{a^2+t^2} \rightarrow$$

$f(t) = e^{-a|t|}$

$F(j\omega) = \frac{2a}{a^2+\omega^2}$

$F(t) = \frac{2a}{a^2+t^2}$

Fourier dönüşüm tablosundan bildiğimiz üzere

$$e^{-a|t|} \xleftrightarrow{\mathcal{F}} \frac{2a}{a^2+\omega^2}$$

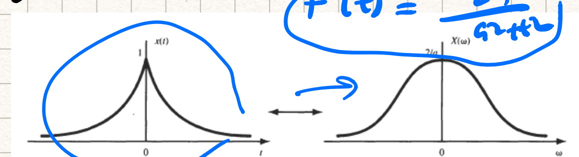


Fig. 5-18 $e^{-a|t|}$ and its Fourier transform.

Dualite özelliğini kullanırsak

$$\frac{2a}{a^2+t^2} \longleftrightarrow 2\pi e^{-a|-\omega|}$$

İki tarafı 2a'ya bölelim.

$$\underbrace{\frac{1}{a^2+t^2}}_{x(t)} \longrightarrow \underbrace{\frac{\pi}{a} e^{-a|\omega|}}_{X(j\omega)}$$

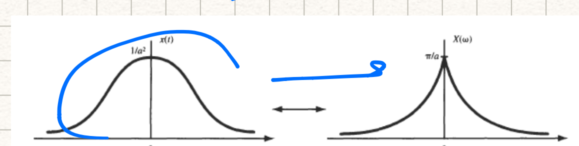


Fig. 5-19 $1/(a^2+t^2)$ and its Fourier transform.

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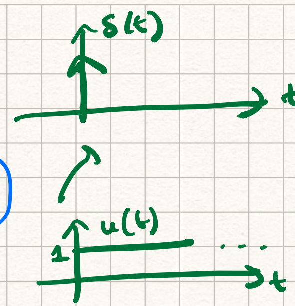
$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

$$\int_{-\infty}^{t} x(t)\,dt \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} X(j\omega) + \pi X(0)\,\delta(\omega)$$



Örnek :

$x(t) = u(t)$ işaretinin Fourier dönüşümünü bulalım

$u(t) = \int_{-\infty}^{t} \delta(t)\,dt$

$\frac{du(t)}{dt} = \delta(t)$

$g(t) = \delta(t) \xleftrightarrow{\mathcal{F}} G(j\omega) = 1.$

$$x(t) = \int_{-\infty}^{t} g(\tau)\,d\tau$$

$g(t)$

$g(t) \rightarrow G(j\omega)$

$\int_{-\infty}^{t} \delta(t)\,dt \rightarrow \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega)$

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega),$$

$G(j\omega) = 1$

$\delta(t) \xrightarrow{\mathcal{F}} 1$

$G(j\omega) = 1$

$$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

$G(j\omega)$

$\rightarrow \frac{1}{j\omega} + \pi\delta(\omega)$

$$\delta(t) = \frac{du(t)}{dt} \xleftrightarrow{\mathcal{F}} \cancel{j\omega}\left[\frac{1}{\cancel{j\omega}} + \pi\cancel{\delta(\omega)}\right] = 1$$

$j\omega\pi\delta(\omega)$

Table 4.1

A Short Table of Fourier Transforms

	$f(t)$	$F(\omega)$	
1	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\frac{1}{a-j\omega}$	$a>0$
3	$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\frac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\frac{2}{j\omega}$	
13	$\cos\omega_0 t\, u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0 t\, u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\frac{t}{\tau}\right)$	$\tau\operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$	
18	$\frac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\frac{\omega}{2W}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\omega\tau}{4}\right)$	
20	$\frac{W}{2\pi}\operatorname{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\frac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	

Properties of the Continuous-Time Fourier Transform

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

Property	Aperiodic Signal	Fourier Transform
	$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
Time-shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency-shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time-Reversal	$x(-t)$	$X(-j\omega)$
Time- and Frequency-Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}X(j\omega)*Y(j\omega)$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases}X(j\omega)=X^*(-j\omega)\\ \Re e\{X(j\omega)\}=\Re e\{X(-j\omega)\}\\ \Im m\{X(j\omega)\}=-\Im m\{X(-j\omega)\}\\ \|X(j\omega)\|=\|X(-j\omega)\|\\ \sphericalangle X(j\omega)=-\sphericalangle X(-j\omega)\end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition for Real Signals	$x_e(t)=\mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t)=\mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega$$

$x(t)\rightarrow X(j\omega)$

$\int_{-\infty}^{t}x(t)\,dt$

$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$