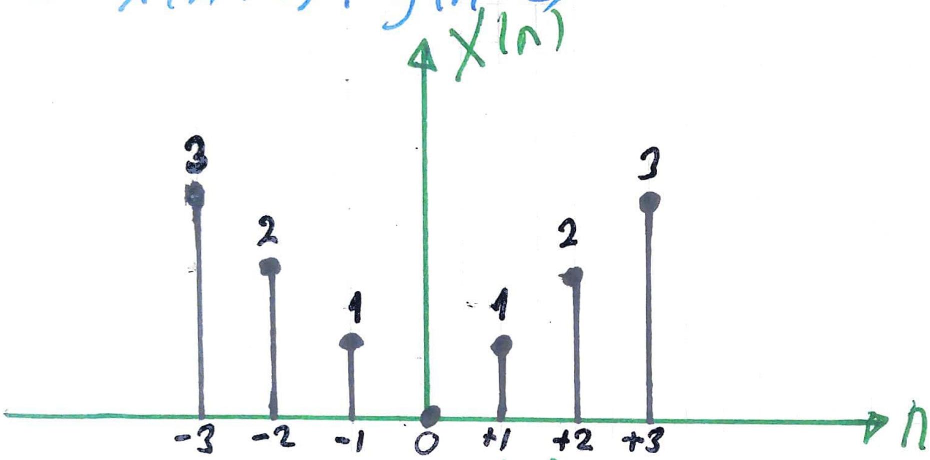
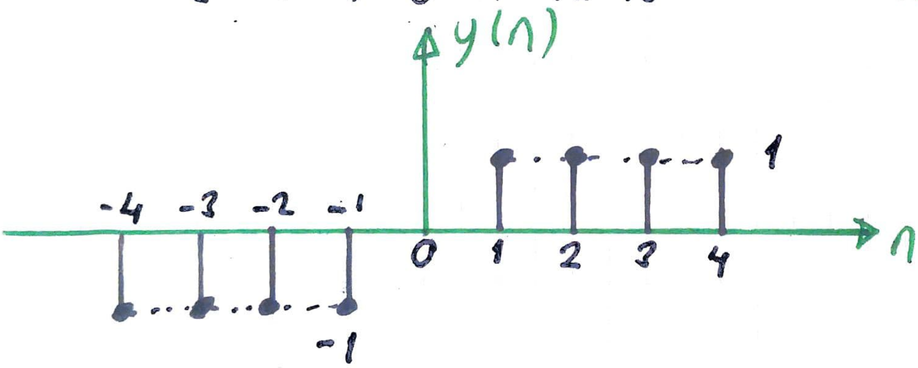
Question 1: Let $x(n)$ and $y(n)$ be given in the figures below. Carefully sketch and label the following signals.

a. $x(3n-1)$

b. $y(1-n)$

c. $x(n+2)+y(n-2)$



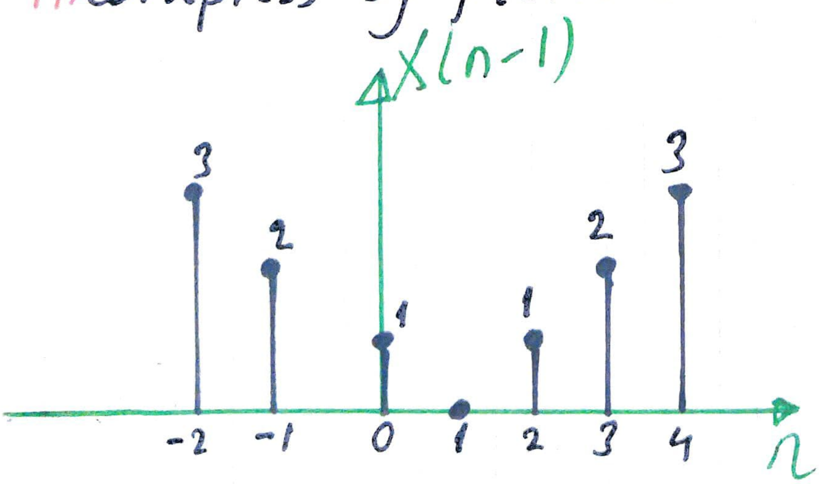


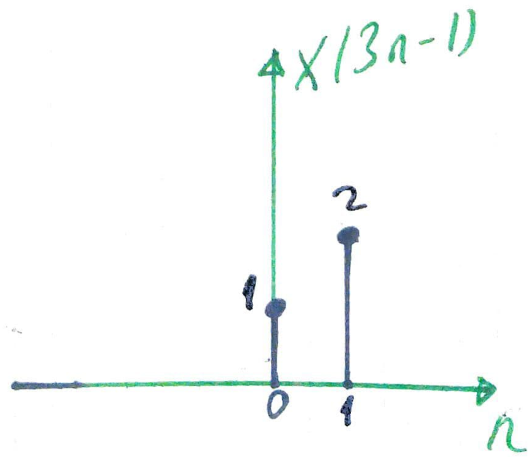
SOLUTIONS

a. $x(3n-1)$ can be drawn in two steps:

i. shift right by 1 unit on the x-axis

ii. compress by factor 3

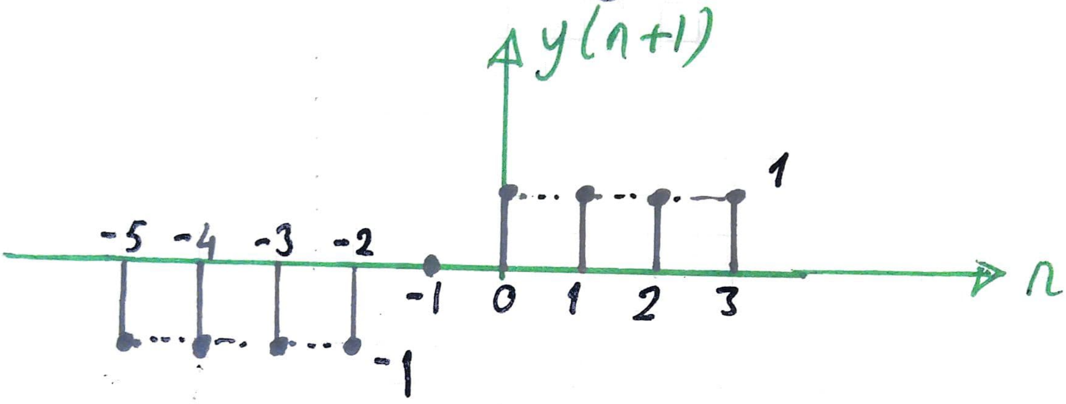


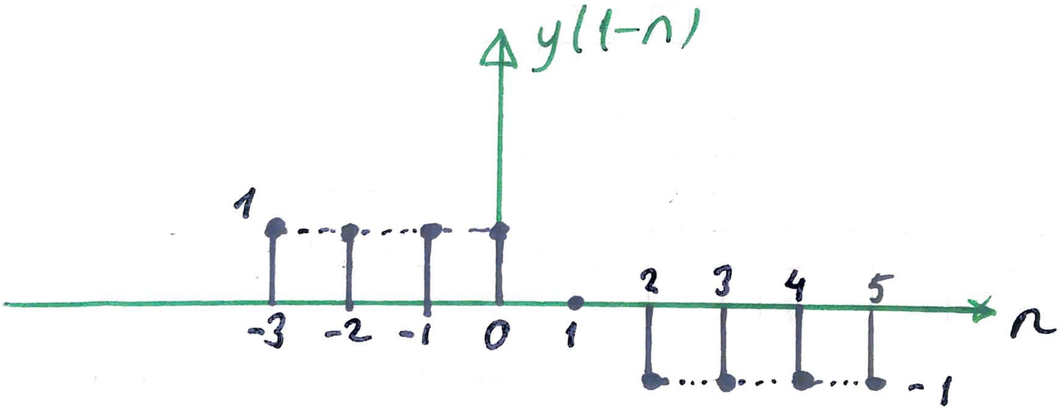


b. $y(1-n)$ is drawn in two steps:

i. shift left by 1 unit on the x-axis

ii. flip over the y-axis



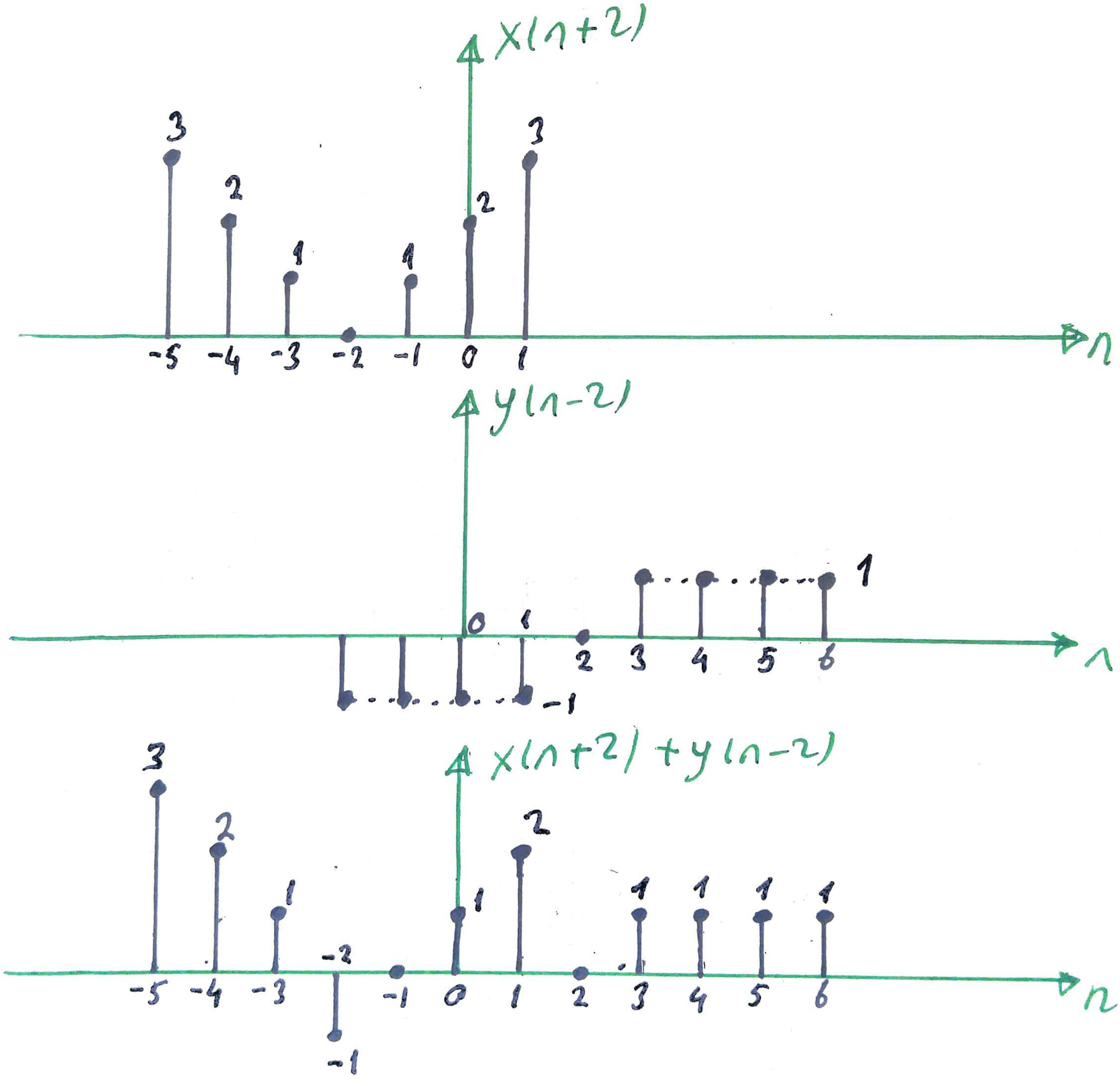


c. $x(n+2) + y(n+2)$ can be drawn by following the steps below.

i. shift $x(n)$ left by 2 unit on the x-axis

ii. shift $y(n)$ right by 2 unit on the y-axis

iii. Sum the signals obtained by i and ii



Question 2: Determine whether or not the signals below are periodic, and for each signal that is periodic, determine the fundamental period

a. $x(n) = \cos(\pi/8\, n)$

b. $x(n) = \cos\left(\frac{3\pi}{7}n - \frac{\pi}{8}\right)$

c. $x(n) = e^{j(n/8 - \pi)}$

d. $x(n) = \mathrm{Re}(e^{j\pi n/12}) + \mathrm{Im}(e^{j\pi n/18})$

SOLUTIONS

a. $x(n) = \cos\left(\frac{\pi}{8}n\right)$

$$\boxed{w_0 N = 2\pi k}$$

$$N\frac{\pi}{8} = 2\pi k$$

$$\boxed{N = 16} \longrightarrow x(n) \text{ is periodic with } N = 16.$$

b. $x(n)$ is periodic if $x(n) = x(n+N)$

$$\cos\left(\frac{3\pi}{7}n - \frac{\pi}{8}\right) = \cos\left(\frac{3\pi}{7}n + \frac{3\pi}{7}N - \frac{\pi}{8}\right)$$

$x(n+N) = x(n)$ if $\frac{3\pi}{7}N$ is an integer multiple of 2π.

The smallest value of N for this signal is

$$\boxed{N = 14} \longrightarrow x(n) \text{ is periodic with } N = 14.$$

c. $x(n)$ is periodic if $x(n) = x(n+N)$

$$e^{j(n/8 - \pi)} = e^{j(n/8 + N/8 - \pi)}$$
$$= e^{j(n/8 - \pi)} e^{jN/8}$$
$$= x(n) \underbrace{e^{jN/8}}$$

$$\boxed{\frac{N}{8} = 2\pi k}$$

If this factor is unity, $x(n)$ is periodic

where N and k are both integers. This is not possible since π is an irrational. Therefore, this sequence is not periodic.

d. $x(n) = Re(e^{j\pi n/12}) + Im(e^{j\pi n/18})$

$$x(n) = \underbrace{\cos\left(\frac{\pi}{12} n\right)}_{x_1(n)} + \underbrace{\sin\left(\frac{\pi}{18} n\right)}_{x_2(n)}$$

N_1, N_2 and N are the fundamental period of $x_1(n)$, $x_2(n)$ and $x(n)$, respectively

$$\frac{\pi}{12} \cdot N_1 = 2\pi k \longrightarrow N_1 = 12 \cdot 2$$
$$N_1 = 24$$

$$\frac{\pi}{18} \cdot N_2 = 2\pi k \longrightarrow N_2 = 18 \cdot 2$$
$$N_2 = 36$$

$$N = \frac{N_1 \cdot N_2}{gcd(N_1, N_2)} = \frac{24 \cdot 36}{gcd(24, 36)} = \boxed{72}$$

"greatest common divisor"

Question 3: For the following system below, $y(n)$ denotes the output and $x(n)$ the input. Determine whether the specified input-output relationship is linear and/or shift invariant.

$$y(n) = 2x(n) + 3$$

Solution 3:

Assume that $T[\cdot]$ denotes the system. For linearity,

$$T[x_1(n)] = 2x_1(n) + 3$$

$$T[x_2(n)] = 2x_2(n) + 3$$

i. $T[ax_1(n) + bx_2(n)] = 2[ax_1(n) + bx_2(n)] + 3$

ii. $aT[x_1(n)] + bT[x_2(n)] = 2ax_1(n) + 2bx_2(n) + 3(a+b)$

The equations i and ii are not equal to themselves. Because of this the system is not linear.

For time invariant property,

i. $T[x_1(n)] = T[x(n-n_0)] = 2(x(n-n_0)) + 3$

ii. $y(n-n_0) = 2(x(n-n_0)) + 3$

The equations i and ii are equal to themselves. Therefore, the system is time invariant

Question 4: The systems that follow have input $x(n)$ and output $y(n)$. For each system, determine whether it is (i) memoryless, (ii) stable, (iii) causal, (iv) linear and (v) time-invariant

a. $y(n) = n x(n)$

b. $y(n) = e^{x(n)}$

c. $y(n) = x(n) + 3u(n+1)$

SOLUTIONS:

a) $y(n) = n x(n)$

i. $y(n)$ only depends on the input at time n. Therefore, the system is $\boxed{\text{MEMORYLESS}}$

ii. If $|x(n)| \leq B_x < \infty$ and $|y(n)| \leq B_y < \infty$, the system is called BIBO stable.

$$|y(n)| = |n x(n)| \leq |n| \, |x(n)|$$

If n goes to infinity, the system will produce unbounded output. Hence, $y(n) = n x(n)$ is $\boxed{\text{UNSTABLE}}$

iii. The output depends on only the present value of the input. Therefore, it is $\boxed{\text{CAUSAL}}$

iv. $T[x_1(n)] = n x_1(n)$

$T[x_2(n)] = n x_2(n)$

If $T[a x_1(n) + b x_2(n)] = a T[x_1(n)] + b T[x_2(n)]$, the system is called LINEAR

iv. (cont.) $T[x_1(n)] = n x_1(n)$

$T[x_2(n)] = n x_2(n)$

$$T[a x_1(n) + b x_2(n)] = n\,(a x_1(n) + b x_2(n))$$
$$= \underbrace{a n x_1(n)}_{T[x_1(n)]} + \underbrace{b n x_2(n)}_{T[x_2(n)]}$$
$$= a T[x_1(n)] + b T[x_2(n)]$$

Therefore, the system is $\boxed{\text{LINEAR}}$

v. $T[x_1(n)] = T[x(n-n_0)] = n\,x(n-n_0)$

$y(n-n_0) = (n-n_0)\,x(n-n_0)$

$(n-n_0)(x(n-n_0)) \neq n\,x(n-n_0)$

Hence the system is $\boxed{\text{time varying}}$

(b) $y(n) = e^{x(n)}$

i. The output depends on only the input at time n. $\longrightarrow$ $\boxed{\text{MEMORYLESS}}$

ii. $|x(n)| \leq B_x < \infty$

$$|y(n)| = |e^{x(n)}| = e^{B_x} \leq B_y < \infty$$

Even if n goes to infinity, it will produce bounded output. $\longrightarrow$ $\boxed{\text{BIBO STABLE}}$

iii. The output depends only on the present value of the input $\longrightarrow$ $\boxed{\text{CAUSAL}}$

iv. $T[x_1(n)] = e^{x_1(n)}$

$T[x_2(n)] = e^{x_2(n)}$

$T[ax_1(n) + bx_2(n)] = e^{ax_1(n) + bx_2(n)}$

$= e^{ax_1(n)} \cdot e^{bx_2(n)}$

$aT[x_1(n)] + bT[x_2(n)] = ae^{x_1(n)} + be^{x_2(n)}$

$e^{ax_1(n)} e^{bx_2(n)} \neq ae^{x_1(n)} + be^{x_2(n)}$

the system is $\boxed{\text{not linear.}}$

v. $T[x_1(n)] = T[x(n-n_0)] = e^{x(n-n_0)}$

$y(n-n_0) = e^{x(n-n_0)}$

$y(n-n_0) = T[x(n-n_0)]$ $\longrightarrow$ $\boxed{\text{TIME INVARIANT}}$

---

c) $y(n) = x(n) + 3u(n+1)$

i. $y(n)$ depends on the n^{th} value of x only, so this is $\boxed{\text{MEMORYLESS}}$

ii. $|y(n)| = \begin{cases} |x(n)| & , n < -1 \\ |x(n) + 3| & , n \geq -1 \end{cases}$

The system produces bounded outputs for bounded inputs if $n \geq -1$ or not. $\longrightarrow$ $\boxed{\text{STABLE}}$

iii. Since it doesn't use future values of $x(n)$, it is $\boxed{\text{CAUSAL}}$

iv. $$T[ax_1(n)+bx_2(n)] = (ax_1(n)+bx_2(n))+3u(n+1)$$

$$aT[x_1(n)] + bT[x_2(n)] = ax_1(n)+3u(n+1) + bx_2(n) + 3u(n+1)$$

$$T[ax_1(n)+bx_2(n)] \neq aT[x_1(n)] + bT[x_2(n)]$$

This is $\boxed{\text{NOT LINEAR}}$

v. $$T[x_1(n)] = T[x(n-n_0)] = x(n-n_0) + 3u(n+1)$$

$$y(n-n_0) = x(n-n_0) + 3u(n-n_0+1)$$

$$y(n-n_0) \neq T[x(n-n_0)] \longrightarrow \boxed{\text{NOT TIME INVARIANT}}$$

Question 5: Consider a discrete time LTI system described by the rule

$$y[n] = x(n-5) + \frac{1}{2} x(n-7)$$

What is the impulse response of the system?

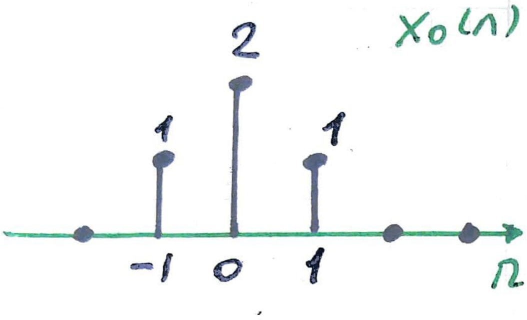
Solution: output signal of any LTI system can be defined by convolution sum as follows:

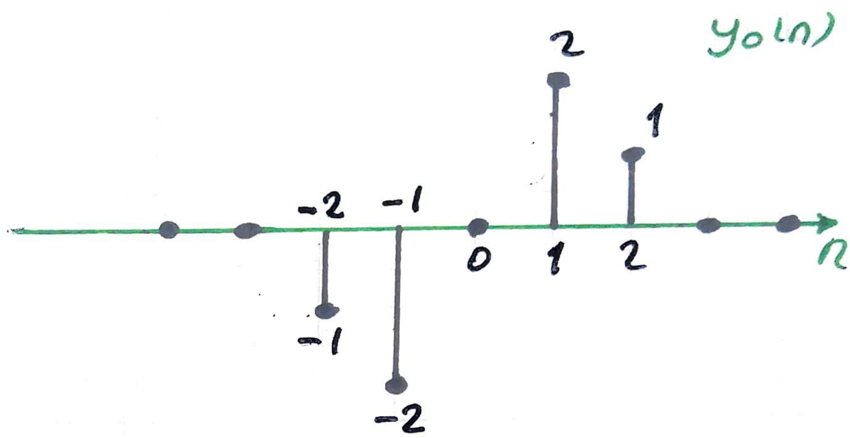
$$y[n] = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

If we look at the system above, we find the impulse response of the system:

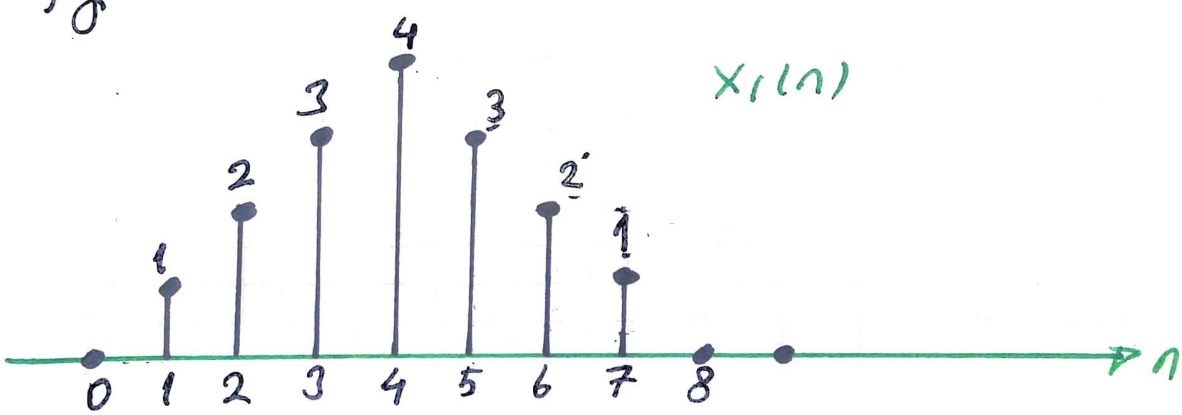
$$h[n] = \delta(n-5) + \frac{1}{2}\delta(n-7)$$

Question 6: The input-output pair shown in the figure below is given for stable LTI system





a. Determine the response to the input $x_1(n)$ in the figure below.



b. Determine the impulse response of the system

SOLUTIONS

a. Notice that

$$x_1(n) = x_0(n-2) + 2x_0(n-4) + x_0(n-6)$$

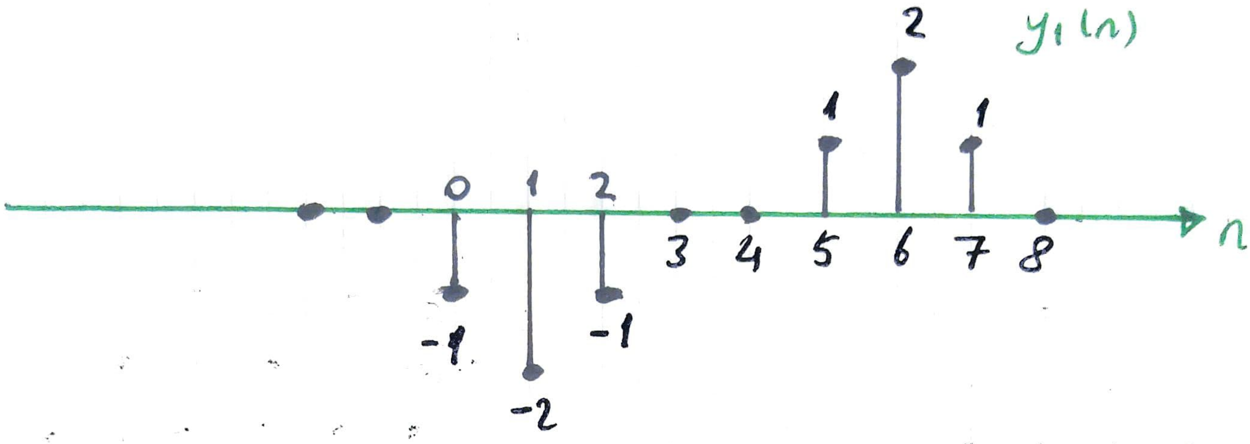
Since the system is LTI,

$$y_1(n) = y_0(n-2) + 2y_0(n-4) + y_0(n-6)$$

We get sequence shown here

$$y_1(n) = -\delta(n) + (-2)\delta(n-1) - \delta(n-2) + \delta(n-5) + 2\delta(n-6) + 7\delta(n-7)$$

a. (cont.)



b. $y_0(n)$ can be written as follows

$$y_0(n) = -x_0(n+1) + x_0(n-1)$$

$$y_0(n) = x_0(n) * \underbrace{\left(-\delta(n+1) + \delta(n-1)\right)}_{h(n)}$$

Q1 : Let us perform the convolution of the two signals:

$$x(n) = \begin{cases} a^n & , n \geq 0 \\ 0 & , n < 0 \end{cases} \longrightarrow x(n) = a^n u(n)$$

and

$$h(n) = u(n)$$

S1:

We can find the output of the system with the direct evaluation

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} a^k u(k)\, u(n-k)$$

Because $u(k)$ is equal to zero for $k<0$ and $u(n-k)$ is equal to zero for $k>n$, when $n<0$, there are no nonzero terms in the sum, and $y(n)=0$. On the other hand, if $n \geq 0$

$$y(n) = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \quad , n \geq 0$$

$$\boxed{y(n) = \frac{1-a^{n+1}}{1-a}\, u(n)}$$

Q2: An input signal $x(n)$ and the impulse response of an LTI system $h(n)$ is given as

i. $x(n) = \left(\frac{1}{2}\right)^n u(n-2) \quad , h(n) = u(n)$

ii. $x(n) = \cos\left(\frac{n\pi}{2}\right) u(n) \; , \quad h(n) = u(n-1)$

S2:

i.

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k-2)\, u(n-k)$$

* for $n < 2$, $\quad y(n) = 0$

* for $n \geq 2$, $\quad y(n) = \sum_{k=2}^{n} \left(\frac{1}{2}\right)^k$

$\boxed{k = m+2} \longrightarrow y(n) = \sum_{m=0}^{n-2} \left(\frac{1}{2}\right)^{m+2}$

$$= \sum_{m=0}^{n-2} \frac{1}{4}\left(\frac{1}{2}\right)^m$$

$$= \frac{1}{4}\, \frac{1-\left(\frac{1}{2}\right)^{n-2+1}}{1-\frac{1}{2}}$$

$$= \frac{1}{2}\, \frac{1-\left(\frac{1}{2}\right)^{n-1}}{2\cdot\left(\frac{1}{2}\right)}$$

$$= \left(\frac{1}{2}\right)\left(1-\left(\frac{1}{2}\right)^{n-1}\right)$$

$$= \frac{1}{2} - \left(\frac{1}{2}\right)^n$$

$$\boxed{y(n) = \left[\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^n\right] u(n-2)}$$

ii.

$$x(n) = \cos\left(\frac{n\pi}{2}\right) u(n)$$

$$h(n) = u(n-1)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \cos\left(\frac{k\pi}{2}\right) u(k)\, u(n-k-1)$$

* for $n<1$, $y(n) = 0$

* for $n \geq 1$, $y(n) = \sum_{k=0}^{n-1} \cos\left(\frac{\pi}{2} k\right)$

$n=1 \rightarrow y(n) = \cos(0) = 1$

$n=2 \rightarrow y(n) = \cos(0) + \cos(\pi/2) = 1$

$n=3 \rightarrow y(n) = \cos(0) + \cos(\pi/2) + \cos(\pi) = 0$

$n=4 \rightarrow y(n) = \cos(0) + \cos(\pi/2) + \cos(\pi) + \cos\left(\frac{3\pi}{2}\right)$ =0

$n=5 \rightarrow y(n) = 1$

$n=6 \rightarrow y(n) = 1$

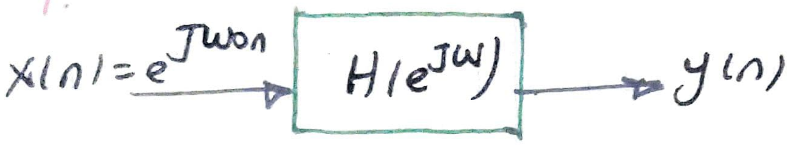
$\vdots$

$$y(n) = \begin{cases} 1 \ , & \mathrm{mod}_4(n) \ , \ n \in \{1,2\} \ , \ n \geq 1 \\ 0 \ , & \mathrm{mod}_4(n) \ , \ n \in \{0,3\} \ , \ n \geq 1 \end{cases}$$

$$y(n) = \begin{cases} u(n-1) \ , & \mathrm{mod}_4(n) \ , \ n \in \{1,2\} \\ 0 \ , & \mathrm{mod}_4(n) \ , \ n \in \{0,3\} \end{cases}$$

Q3 : Consider systems given in the figures. For each system, determine the output $y(n)$ for all n

i.



ii.

S3:

i. $y(n) = x(n) * h(n) = \boxed{h(n) * x(n)}$

$$= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, e^{Jw_0(n-k)}$$

$$= e^{Jw_0 n} \underbrace{\sum_{k=-\infty}^{\infty} h(k)\, e^{-Jw_0 k}}_{H(e^{Jw_0})}$$

$$\boxed{y(n) = e^{Jw_0 n} H(e^{Jw_0})}$$

ii.

$$y(n) = x(n) * h(n) = \boxed{h(n) * x(n)}$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, e^{Jw_0(n-k-n_0)}$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, e^{Jw_0(n-n_0)}\, e^{-Jw_0 k}$$

$$= e^{Jw_0(n-n_0)} \underbrace{\sum_{k=-\infty}^{\infty} h(k)\, e^{-Jw_0 k}}_{H(e^{Jw_0})}$$

$$\boxed{y(n) = e^{Jw_0(n-n_0)}\, H(e^{Jw_0})}$$

Q4: Which of the following discrete-time signals could be eigenfunctions of any stable LTI systems?

i. $5^n u(n)$

ii. e^{J2wn}

iii. $e^{Jwn} + e^{J2wn}$

iv. 5^n

v. $5^n e^{J2wn}$

S4: Recall that an eigenfunction of a system is an input signal which appears at the output of the system scaled by a complex constant.

i. $5^n u(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, 5^{(n-k)}\, u(n-k)$$

$$= 5^n \sum_{k=-\infty}^{n} h(k)\, 5^{-k}$$

Because the summation depends on n, $x(n)$ is

$\boxed{\text{Not an EIGENFUNCTION.}}$

ii

$$x(n) = e^{jwn}$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, e^{j2w(n-k)}$$

$$= e^{j2wn} \underbrace{\sum_{k=-\infty}^{\infty} h(k)\, e^{-j2wk}}_{H(e^{j2w})}$$

$$= e^{j2wn} H(e^{j2w})$$

$\boxed{\text{YES, EIGENFUNCTION}}$

iii. $x(n) = e^{jwn} + e^{j2wn}$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, e^{jw(n-k)} + \sum_{k=-\infty}^{\infty} h(k)\, e^{j2w(n-k)}$$

$$= e^{jwn} \underbrace{\sum_{k=-\infty}^{\infty} h(k)\, e^{-jwk}}_{H(e^{jw})} + e^{j2wn} \underbrace{\sum_{k=-\infty}^{\infty} h(k)\, e^{-j2wk}}_{H(e^{j2w})}$$

$$\boxed{y(n) = e^{jwn} H(e^{jw}) + e^{j2wn} H(e^{j2w})}$$

Since the input cannot be extracted from the above expression, the sum of complex exponentials is $\boxed{\text{NOT AN EIGENFUNCTION}}$. Although, seperately the inputs are eigenfunctions. In general, complex exponentrial signals are always eigenfunctions of LTI systems.

iv. $x(n) = 5^n$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, 5^{(n-k)}$$

$$= 5^n \sum_{k=-\infty}^{\infty} h(k)\, 5^{-k}$$

$\boxed{\text{YES, EIGENFUNCTION}}$

v. $x(n) = 5^n e^{j2wn}$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, 5^{(n-k)}\, e^{j2w(n-k)}$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, 5^n 5^{-k}\, e^{j2wn}\, e^{-j2wk}$$

$$= 5^n e^{j2wn} \sum_{k=-\infty}^{\infty} h(k)\, 5^{-k} e^{-j2wk}$$

$\boxed{\text{YES, EIGENFUNCTION}}$

Q6: Let

$$h(n) = 3\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{3}\right)^{n-1} u(n)$$

be unit sample response of an LTI system. If the input to the system is a unit step,

$$x(n) = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

find $\lim_{n \to \infty} y(n)$ where $y(n) = h(n) * x(n)$

S6:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

If $x(n)$ is a unit step,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, u(n-k)$$

$$= \sum_{k=-\infty}^{n} h(k)$$

Therefore,

$$\lim_{n \to \infty} y(n) = \sum_{k=-\infty}^{\infty} h(k)$$

$$\Rightarrow \quad = \sum_{k=-\infty}^{\infty} 3\left(\frac{1}{2}\right)^k u(k) - 2\left(\frac{1}{3}\right)^{n-1} u(k)$$

$$= 3\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k - 2\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^{k-1}$$

$$= \frac{3 \cdot 1}{1 - \frac{1}{2}} - \frac{6}{1 - 1/3} = \boxed{-3}$$

Q7: If the response of an LTI system to a unit step (i.e. step response) is

$$s(n) = n\left(\frac{1}{2}\right)^n u(n)$$

Find the unit sample response, $h(n)$

S7: In this problem, we begin by noting that

$$\boxed{\delta(n) = u(n) - u(n-1)}$$

Therefore, the unit sample response, $h(n)$, is related to the step response, $s(n)$, as follows:

$$\boxed{h(n) = s(n) - s(n-1)}$$

Thus, given $s(n)$, we have

$$h(n) = s(n) - s(n-1)$$

$$\boxed{h(n) = n\left(\frac{1}{2}\right)^n u(n) - (n-1)\left(\frac{1}{2}\right)^{n-1} u(n-1)}$$

Q5 in Detail

(ii) $e^{jn/8}u(n) \rightarrow$ Sys. B $\rightarrow 2e^{jn/8}u(n)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} e^{jn/8}u(n)e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} e^{jn/8}e^{-j\omega n}$$

$$\boxed{X(e^{j\omega}) = \frac{1}{1-e^{-j(\omega-1/8)}}}$$

Analogously

$$\boxed{Y(e^{j\omega}) = \frac{2}{1-e^{-j(\omega-1/8)}}}$$

$$\boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 2}$$

Whatever ω takes any value, $H(e^{j\omega}) = 2$. Because of this, LTI system is unique.

(iii) $e^{jn/8} \rightarrow$ Sys. C $\rightarrow 2e^{jn/8}$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} e^{jn/8}e^{-j\omega n}$$

$$\boxed{X(e^{j\omega}) = 2\pi\delta(\omega - 1/8)}$$

Similarly, we can find the Discrete Fourier Tran[sform] of the output signal:

$$\boxed{\begin{aligned} Y(e^{j\omega}) &= 2\cdot 2\pi\delta(\omega-1/8) \\ &= 4\pi\delta(\omega-1/8) \end{aligned}}$$

The frequency response of the system C is

$$H_1(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{4\pi\delta(\omega-1/8)}{2\pi\delta(\omega-1/8)}$$

$$\boxed{H(e^{j\omega}) = \begin{cases} 2, & \omega = \frac{1}{8} \\ \text{indefinite}, & \omega \neq \frac{1}{8} \end{cases}}$$

Except for the frequency $\omega = \frac{1}{8}$, at the other frequency point, $H(e^{j\omega})$ is indefin[ite]. Therefore it is not unique, but the syste[m] [is] LTI.

Q1: Show that each of the unit sample responses listed below corresponds to a stable system.

i. $h(n) = \delta(n+2)$

ii. $h(n) = \left(\frac{1}{2}\right)^n u(n)$

iii. $h(n) = 3^n u(-n)$

S1:

i. To correspond to a stable system, the unit sample response must be absolutely summable. For each system

$$\sum_{n=-\infty}^{\infty} |h(n)| \text{ is given by}$$

$$\sum_{n=-\infty}^{\infty} |\delta(n+2)| = 1$$

Therefore, the system is STABLE

ii. $$\sum_{n=-\infty}^{\infty} \left|\left(\frac{1}{2}\right)^n u(n)\right| = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2$$

Therefore, the system is STABLE

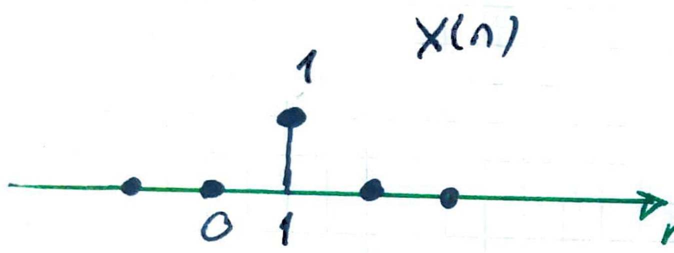
iii. $$\sum_{n=-\infty}^{\infty} |3^n u(-n)| = \sum_{n=-\infty}^{0} 3^n$$

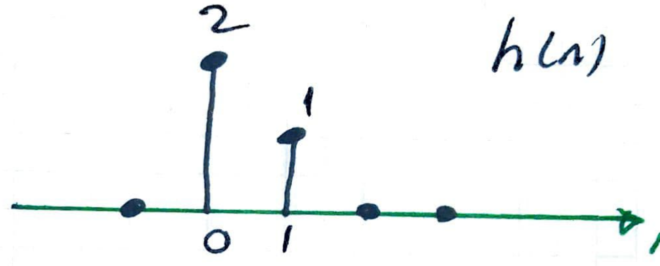
$$= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

Therefore, the system is STABLE

Q2: For each of the pairs of sequences in figures below, use discrete convolution to find the response to the input $x(n)$ of the LTI system with impulse response $h(n)$:

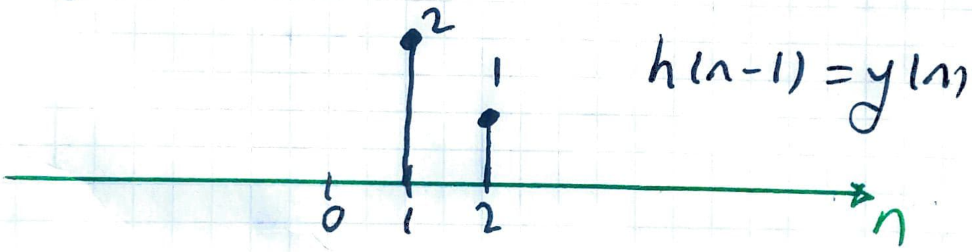
a.



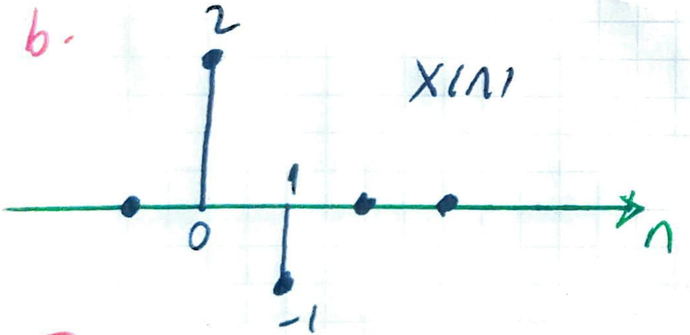


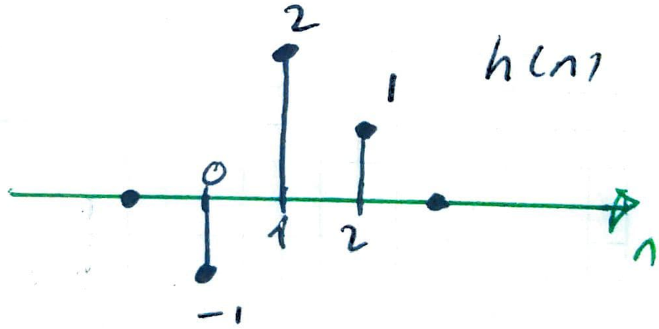
5 a.

$$y(n) = x(n) * h(n)$$
$$x(n) = \delta(n-1)$$
$$y(n) = \delta(n-1) * h(n) = h(n-1)$$



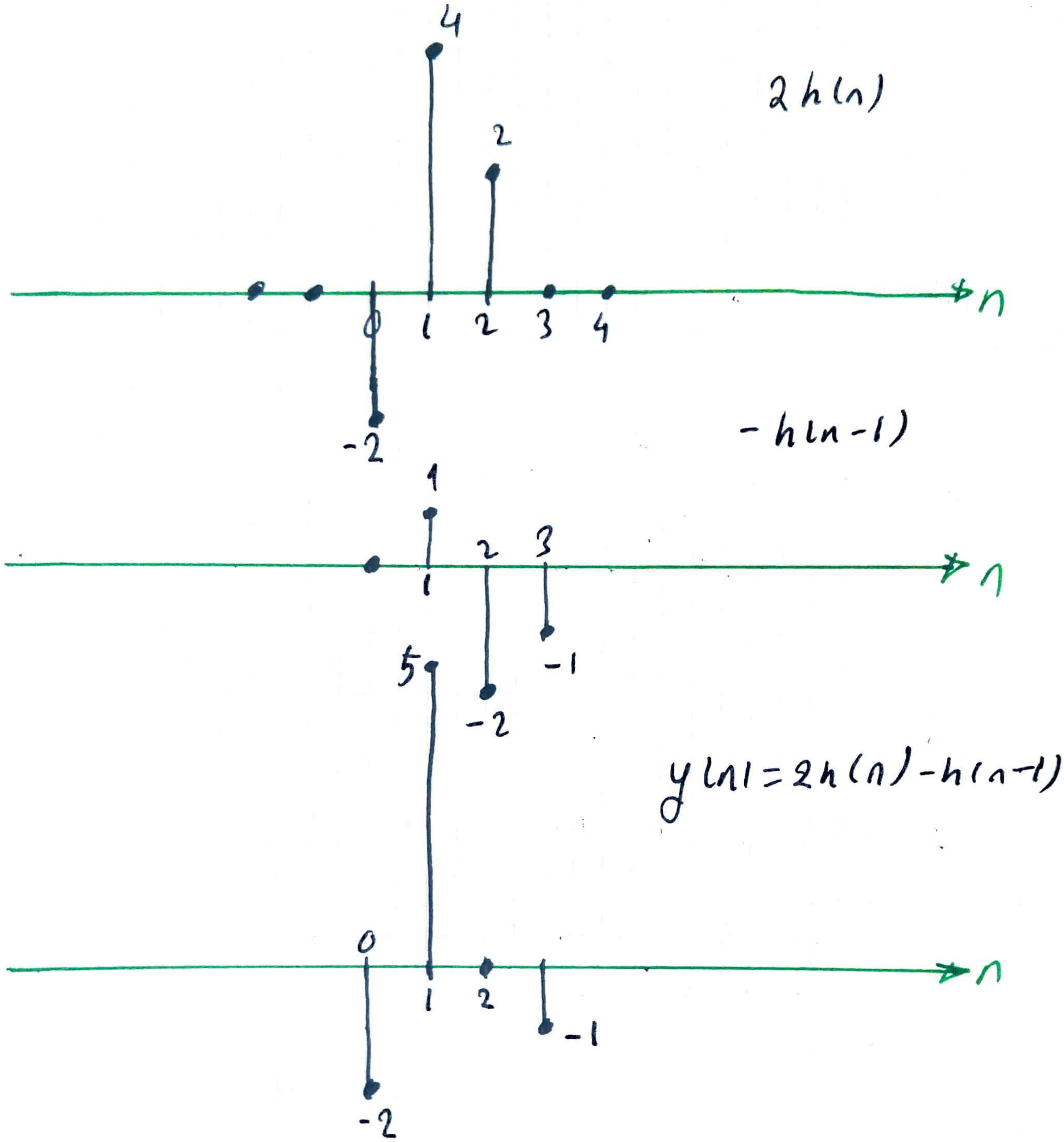
b.





5 b

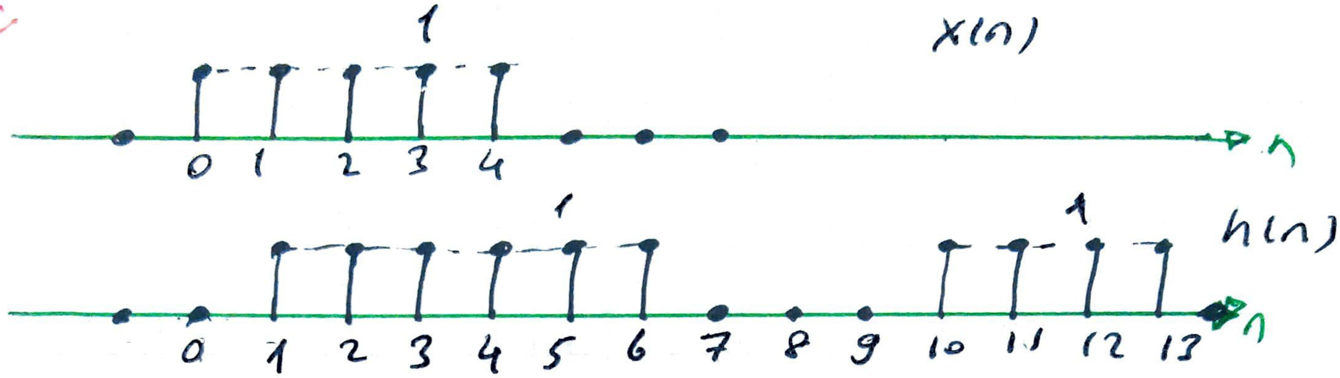
$$x(n) = 2\delta(n) - \delta(n-1)$$
$$y(n) = [2\delta(n) - \delta(n-1)] * h(n)$$
$$= 2h(n) - h(n-1)$$



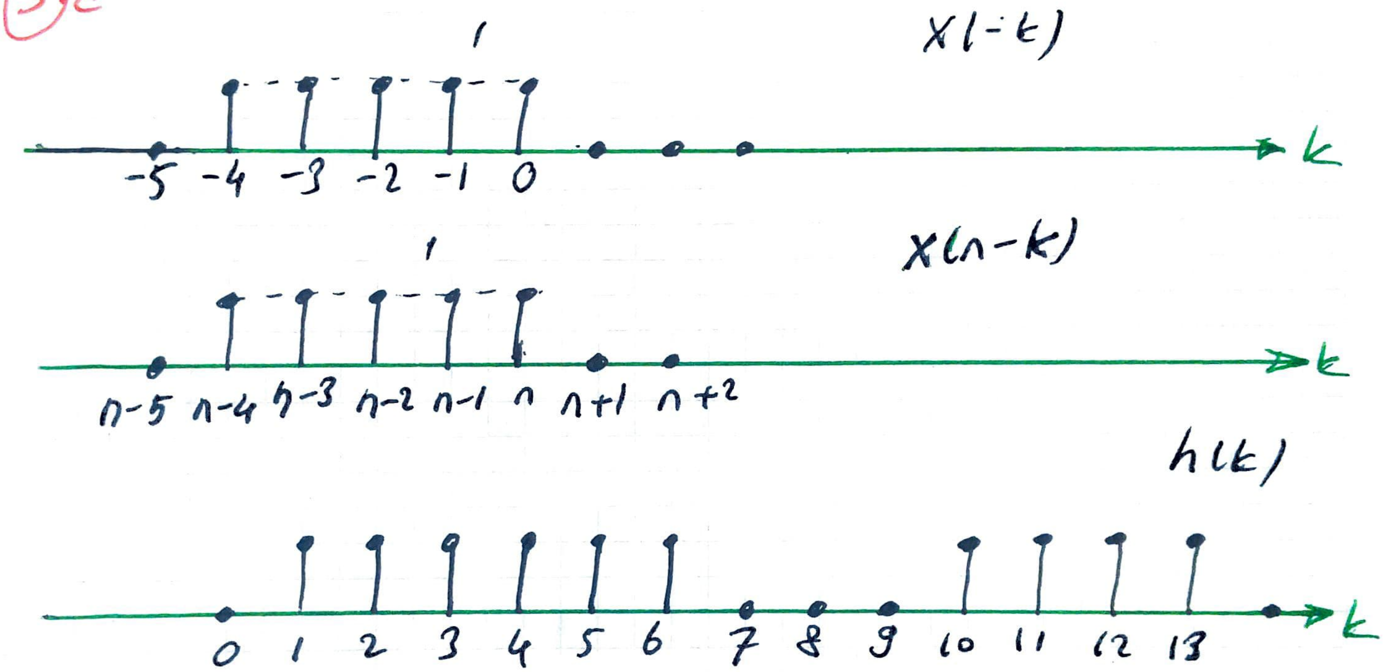
$$\boxed{\text{CONVOLUTION PROPERTY}}$$

$$\boxed{x(n) * \delta(n-n_0) = x(n-n_0)}$$

c



5) c



$$y(n) = \sum_{n=-\infty}^{\infty} h(k)\, x(n-k)$$

$n \leq 0$, $y(n) = 0$

$n = 1$, $y(n) = 1 \cdot 1 = 1$

$n = 2$, $y(n) = 1 \cdot 1 + 1 \cdot 1 = 2$

$n = 3$, $y(n) = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 3$

$n = 4$, $y(n) = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 4$

$n = 5$, $y(n) = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 5$

$n=6$, $y(n) = 1\cdot1+1\cdot1+1\cdot1+1\cdot1+1\cdot1 = 5$

$n=7$, $y(n) = 1\cdot1+1\cdot1+1\cdot1+1\cdot1+1\cdot0 = 4$

$n=8$, $y(n) = 1\cdot1+1\cdot1+1\cdot1+0\cdot1+0\cdot1 = 3$

$n=9$, $y(n) = 1\cdot1+1\cdot1+0\cdot1+0\cdot1+0\cdot1 = 2$

$n=10$, $y(n) = 1\cdot1+0\cdot1+0\cdot1+0\cdot1+1\cdot1 = 2$

$n=11$, $y(n) = 0\cdot1+0\cdot1+0\cdot1+1\cdot1+1\cdot1 = 2$

$n=12$, $y(n) = 0\cdot1+0\cdot1+1\cdot1+1\cdot1+1\cdot1 = 3$

$n=13$, $y(n) = 0\cdot1+1\cdot1+1\cdot1+1\cdot1+1\cdot1 = 4$

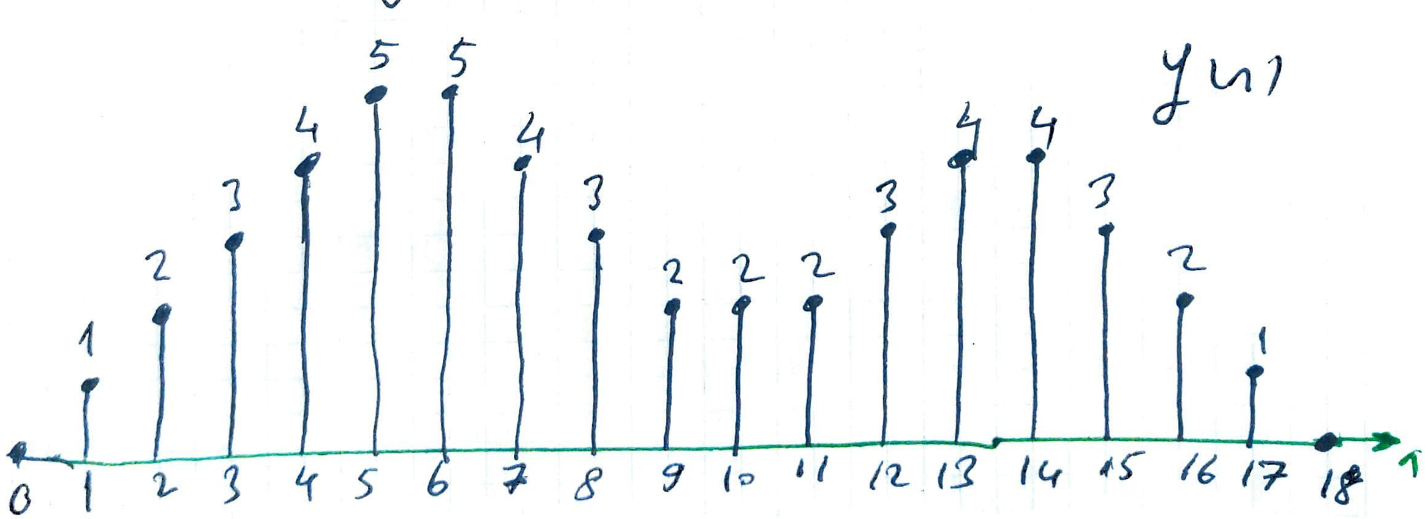
$n=14$, $y(n) = 1\cdot1+1\cdot1+1\cdot1+1\cdot1+1\cdot0 = 4$

$n=15$, $y(n) = 1\cdot1+1\cdot1+1\cdot1+0\cdot1+0\cdot1 = 3$

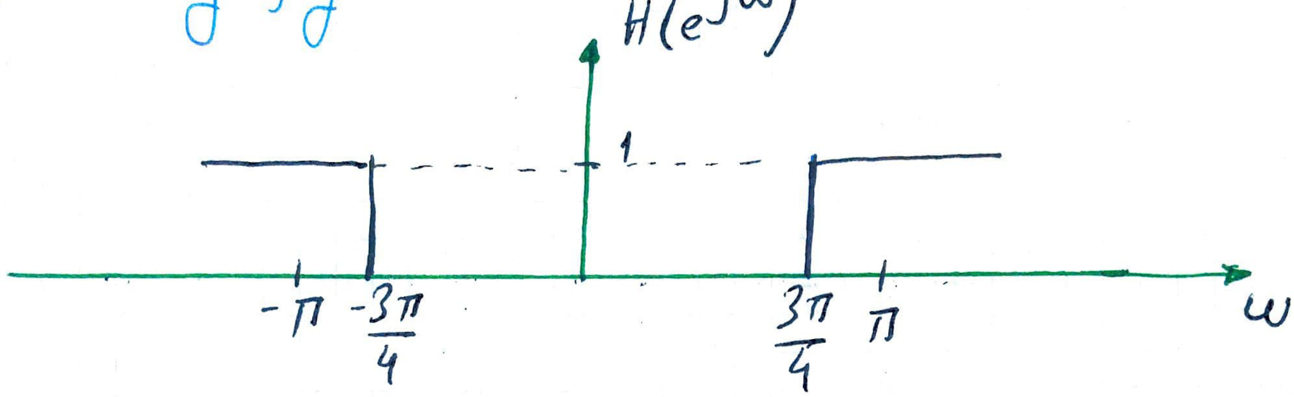
$n=16$, $y(n) = 1\cdot1+1\cdot1+0\cdot1+0\cdot1+0\cdot1 = 2$

$n=17$, $y(n) = 1\cdot1+0\cdot1+0\cdot1+0\cdot1+0\cdot1 = 1$

$n>17$, $y(n) = 0$



Q3: Consider the high-pass filter that has a cut-off frequency $\omega_c = 3\pi/4$ as shown in the following figure.



i. Find the unit sample response, $h(n)$

ii. A new system is defined so that its unit sample is $h_1(n) = h(2n)$. Sketch the frequency response $H_1(e^{j\omega})$, of this system.

S3. i. The unit sample response may be found two different ways. The first is to use the inverse DTFT formula and perform the integration. The second approach is to use the modulation property and note that if

$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & , \ |\omega| \leq \pi/4 \\ 0 & , \ \text{else} \end{cases}$$

$H(e^{j\omega})$ may be written as

$$H(e^{j\omega}) = H_{LP}(e^{j(\omega-\pi)})$$

Therefore, it follows from the modulation property that

$$h(n) = e^{jn\pi} h_{LP}(n) = (-1)^n h_{LP}(n)$$

$$h_{lp}(n) = \frac{\sin(n\pi/4)}{n\pi}$$

we have

$$h(n) = (-1)^n \frac{\sin(n\pi/4)}{n\pi}$$

ii. The frequency response of the system that has a unit sample response $h_1(n) = h(2n)$ may be found by evaluating the DTFT sum directly

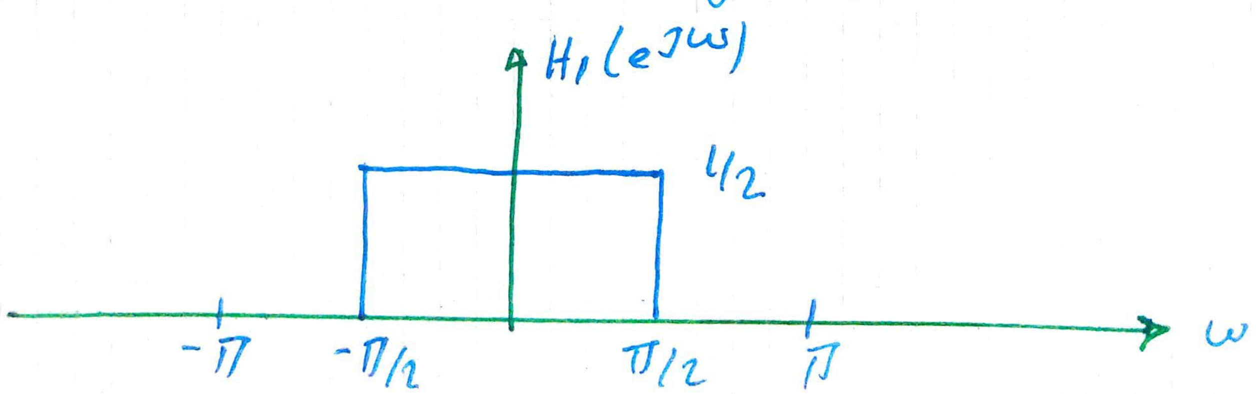
$$H_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_1(n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} h(2n) e^{-j\omega n}$$

$$= \sum_{n \text{ even}} h(n) e^{-j\omega n/2}$$

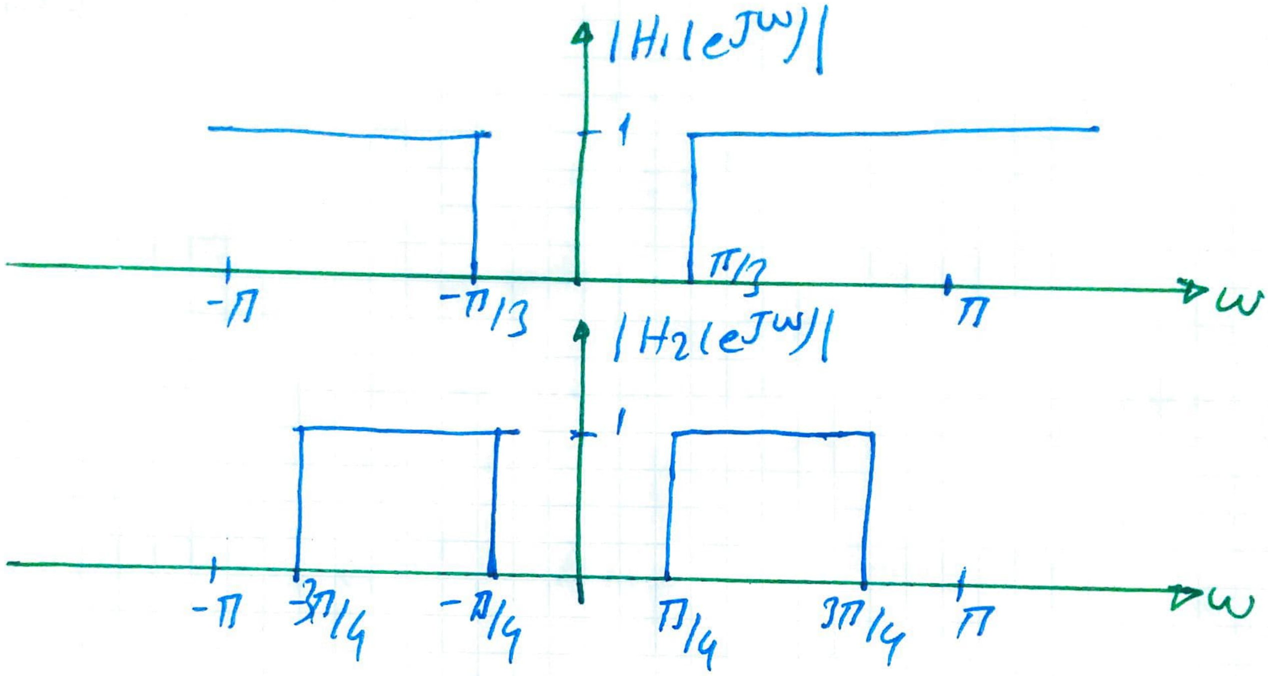
However, an easier approach is to note that

$$h_1(n) = h(2n) = (-1)^{2n} \frac{\sin(2n\pi/4)}{2n\pi} = \frac{\sin(n\pi/2)}{2n\pi}$$

which is a low pass filter with a cutoff frequency of $\pi/2$ and a gain of $1/2$. A plot of $H_1(e^{j\omega})$ is shown in the following figure.



Q4: The ideals filters that have frequency responses as shown in the figure below are connected in cascade:



For an arbitrary input $x(n)$, find the range of frequencies that can be present in the output $y(n)$. Repeat for the case in which the two systems are connected in parallel.

S4: If these two filters are connected in cascade, the frequency of the cascade is

$$H(e^{j\omega}) = H_1(e^{j\omega})\, H_2(e^{j\omega})$$

Therefore any frequencies in the output, $y(n)$, must be passed by both filters.

Because the passband for $H_1(e^{j\omega})$ is $|\omega| > \pi/3$, and the passband for $H_2(e^{j\omega})$ is $\pi/4 < |\omega| < 3\pi/4$.

The passband for the cascade (the frequencies for which both $|H_1(e^{j\omega})|$ and $|H_2(e^{j\omega})|$ are equal to 1) is

$$\boxed{\frac{\pi}{3} \leq |\omega| \leq \frac{3\pi}{4}}$$

With a parallel connection, the overall frequency response is

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

Therefore, the frequencies that are contained in the output are those that are passed by either filter, or

$$\boxed{|\omega| > \frac{\pi}{4}}$$

Q5: Find the DTFT of the following sequences:

i. $x(n) = \delta(n-3)$

ii. $x(n) = 1/2\, \delta(n+1) + \delta(n) + 1/2\, \delta(n-1)$

iii. $x(n) = u(n+3) - u(n-4)$

S5.

i. $$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} \delta(n-3)\, e^{-jwn} = \boxed{e^{-jw3}}$$

ii. $$X(e^{jw}) = \sum_{n=-\infty}^{\infty} \left[1/2\, \delta(n+1) + \delta(n) + \frac{1}{2}\delta(n-1) \right] e^{-jwn}$$

$$= \frac{1}{2} e^{jw} + 1 + \frac{1}{2} e^{-jw}$$

$$\boxed{X(e^{jw}) = 1 + \cos w}$$

iii. $$x(n) = u(n+3) - u(n-4) = \begin{cases} 0 & , n < -3 \text{ and } n > 3 \\ 1 & , -3 \le n \le 3 \end{cases}$$

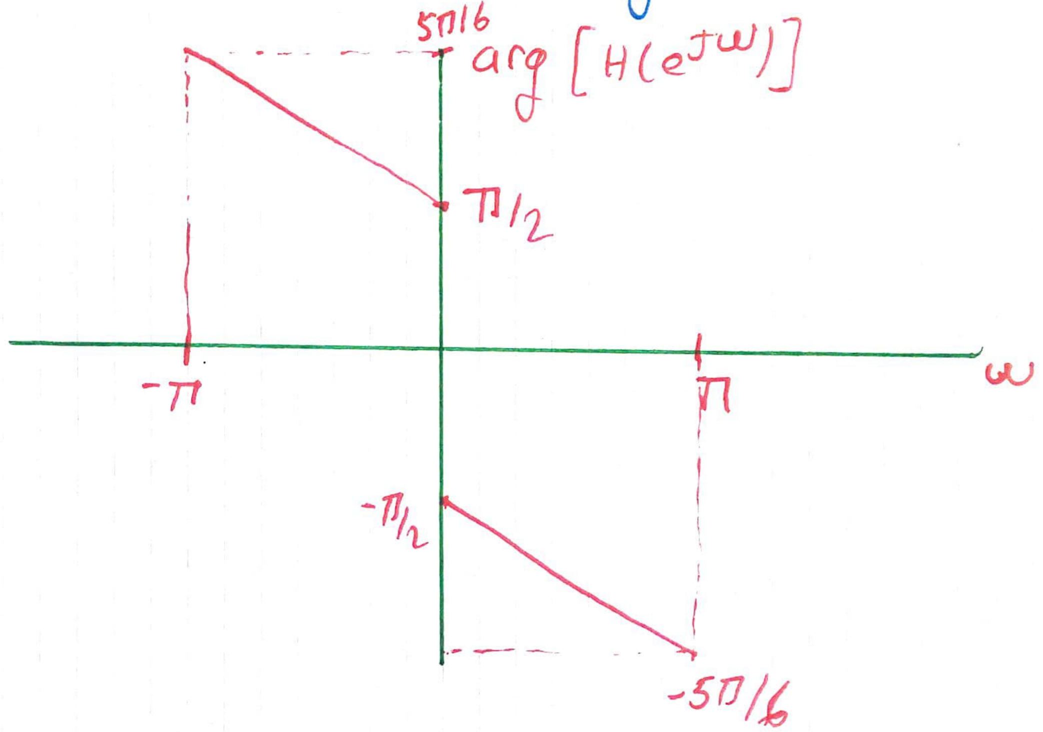
$$X(e^{jw}) = \sum_{n=-3}^{3} e^{-jwn} = e^{j3w} \sum_{n=0}^{6} e^{-jwn}$$

$$\boxed{X(e^{jw}) = \frac{\sin\left(\frac{7w}{2}\right)}{\sin\left(\frac{w}{2}\right)}}$$

Q1 Consider an LTI system with $|H(e^{j\omega})| = 1$, and $\arg[H(e^{j\omega})]$ be as shown in the following figure. If the input is

$$x(n) = \cos\left(\frac{3\pi}{2} n + \frac{\pi}{4}\right)$$

determine the output $y(n)$



S1. For systems having real-valued impulse response, If input signal $x(n)$ is

$$x(n) = A\cos(\omega_0 n + \phi)$$

the system's output is

$$y(n) = A\,|H(e^{j\omega_0})|\cos(\omega_0 n + \phi + \theta)$$

where $\theta = \measuredangle H(e^{j\omega_0})$ is the phase of the system function at frequency ω_0

Our system has real-valued frequency response,

$$H(e^{-j\omega}) = H^*(e^{j\omega})$$

Hence, we can apply the obtained results

$$y[n] = |He^{j3\pi/2}| \cos\left(\frac{3\pi}{2}n + \frac{\pi}{4} + \theta\right)$$

$$\theta = \measuredangle H(e^{j3\pi/2})$$

To find $H(e^{j\omega})\Big|_{\omega=3\pi/2}$, we use the fact that $H(e^{j\omega})$ is periodic over 2π, so

$$H(e^{j3\pi/2}) = H(e^{-j\pi/2})$$
$$= e^{j2\pi/3}$$

$|H(e^{j3\pi/2})| = 1$

$\measuredangle H(e^{j3\pi/2}) = \frac{2\pi}{3}$

Therefore,

$$y[n] = 1 \cdot \cos\left(\frac{3\pi}{2}n + \frac{\pi}{4} + \frac{2\pi}{3}\right)$$

$$\boxed{y(n) = \cos\left(\frac{3\pi}{2}n + \frac{11\pi}{12}\right)}$$

Q2: For $X(e^{jw}) = \frac{1}{1-ae^{-jw}}$, with $-1<a<0$,

determine and sketch the following as a function of w:

i. $Re[X(e^{jw})]$

ii. $Im[X(e^{jw})]$

iii. $|X(e^{jw})|$

iv. $\measuredangle X(e^{jw})$

5.2 $Re[x(n)] = \frac{1}{2}(x(n) + x^*(n))$

i

$$Re[X(e^{jw})] = \frac{1}{2}\left(X(e^{jw}) + X^*(e^{jw})\right)$$

$$= \frac{1}{2}\left[\frac{1}{1-ae^{-jw}} + \frac{1}{1-ae^{jw}}\right]$$

$$= \frac{1}{2}\left[\frac{1-ae^{jw}+1-ae^{-jw}}{1-ae^{-jw}-ae^{jw}+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2-2a\cos(w)}{1-2a\cos(w)+a^2}\right]$$

$$= \frac{1-a\cos(w)}{1-2a\cos(w)+a^2}$$

ii

$$Im[X(e^{jw})] = \frac{1}{2j}\left[X(e^{jw}) - X^*(e^{jw})\right]$$

$$Im[X(e^{jw})] = \frac{1}{2j}\left[\frac{1}{1-ae^{-jw}} - \frac{1}{1-ae^{jw}}\right]$$

$$Im\,(X(e^{J\omega})) = \frac{-a\sin(\omega)}{1-2a\cos(\omega)+a^2}$$

iii

$$|X(e^{J\omega})| = |X(e^{J\omega})\,X^*(e^{J\omega})|^{1/2}$$

$$= \sqrt{|X(e^{J\omega})\,X^*(e^{J\omega})|}$$

$$= \left|\left(\frac{1}{1-ae^{-J\omega}}\right)\left(\frac{1}{1-ae^{J\omega}}\right)\right|^{1/2}$$

$$= \left(\frac{1}{1-2a\cos(\omega)+a^2}\right)^{1/2}$$

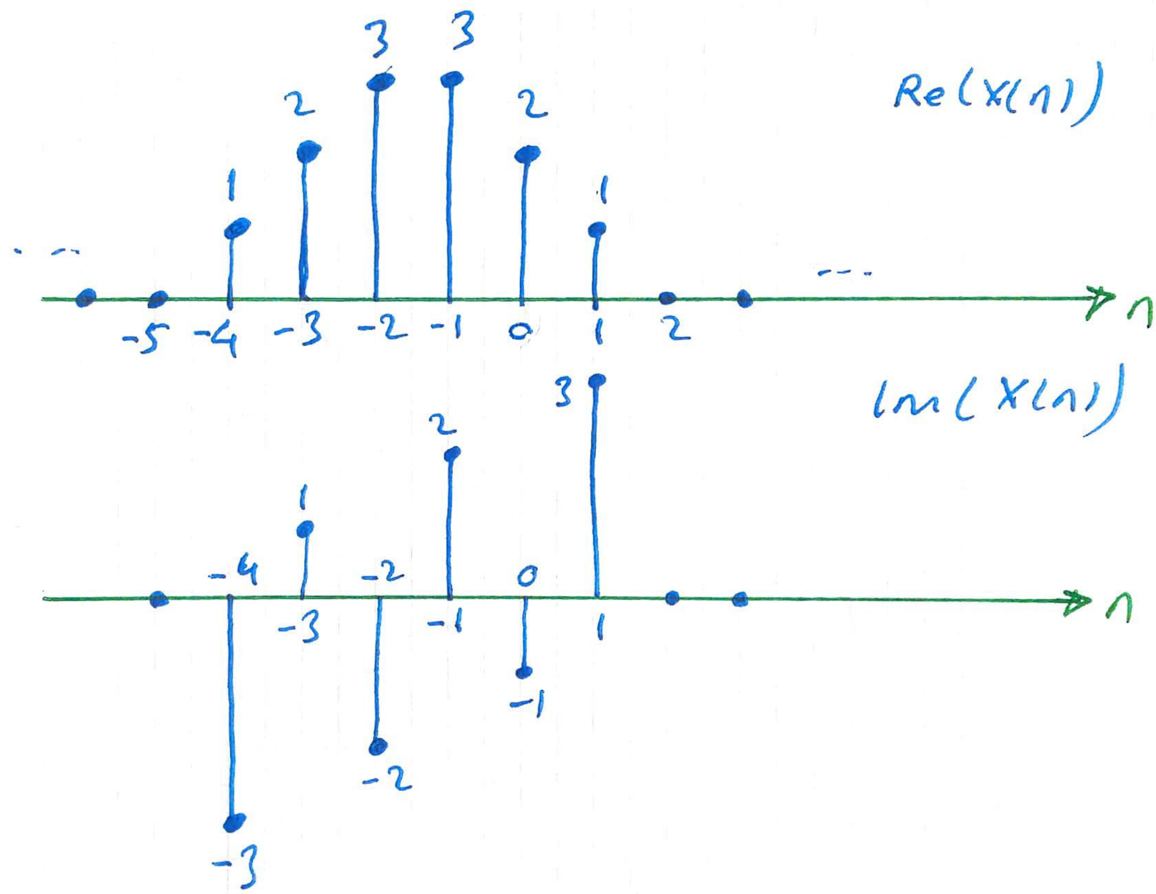
iv

$$\measuredangle X(e^{J\omega}) = \tan^{-1}\left[\frac{Im[X(e^{J\omega})]}{Re[X(e^{J\omega})]}\right]$$

$$= \tan^{-1}\left[\left(\frac{-a\sin(\omega)}{1-2a\cos(\omega)+a^2}\right)\Big/\left(\frac{1-a\cos(\omega)}{1-2a\cos(\omega)+a^2}\right)\right]$$

$$= \tan^{-1}\left(\frac{-a\sin(\omega)}{1-a\cos(\omega)}\right)$$

Q3: $X(e^{j\omega})$ denotes the Fourier Transform of the complex-valued signal $x(n)$, where the real and imaginary parts of $x(n)$ are given in the following figure (Note: The sequence is zero outside the interval shown).



Perform the following calculations without explicitly evaluating $X(e^{j\omega})$

i. $X(e^{j\omega})|_{\omega=0}$

ii. $X(e^{j\omega})|_{\omega=\pi}$

iii. $\int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega$

iv. Determine the signal whose Fourier Transform is $X(e^{-j\omega})$

v. Determine the signal whose Fourier Transform is $j\,Im(X(e^{j\omega}))$

53.

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$$

i.

$$X(e^{jw})\Big|_{w=0} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(0)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)$$

$$= 1+2+3+3+2+1 + j(-3+1-2+2-1+3) = \boxed{12}$$

ii.

$$X(e^{jw})\Big|_{w=\pi} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\pi n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\,(\cos(\pi n) - \underbrace{j\sin(\pi n)}_{0})$$

$$= \sum_{n=-\infty}^{\infty} x(n)\,(-1)^n$$

$$= (1-3j) - (2+j) + (3-2j) - (3+2j) + (2-j) - (1+3j)$$

$$= \boxed{-12j}$$

(iii) Recall that inverse Fourier Transform formula

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn}\, dw$$

$$\int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn}\, dw = 2\pi\, x(n)$$

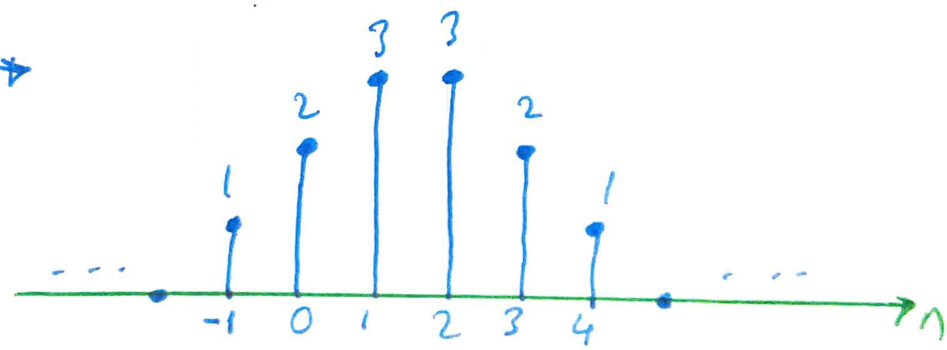
$$\int_{-\pi}^{\pi} X(e^{jw})\, \underbrace{e^{jw(0)}}_{1}\, dw = 2\pi\, x(0)$$

$$\int_{-\pi}^{\pi} X(e^{jw})\, dw = \boxed{2\pi(2-j)}$$

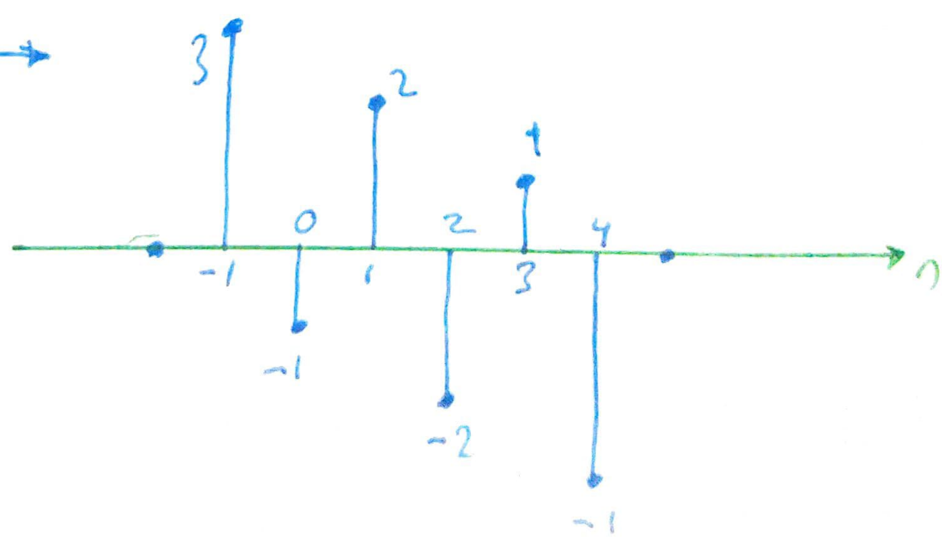
(iv) If $x(n) \xleftrightarrow{F} X(e^{jw})$

then $x(-n) \xleftrightarrow{F} X(e^{-jw})$

Re $(x(-n)) \rightarrow$



Im $(x(-n)) \rightarrow$



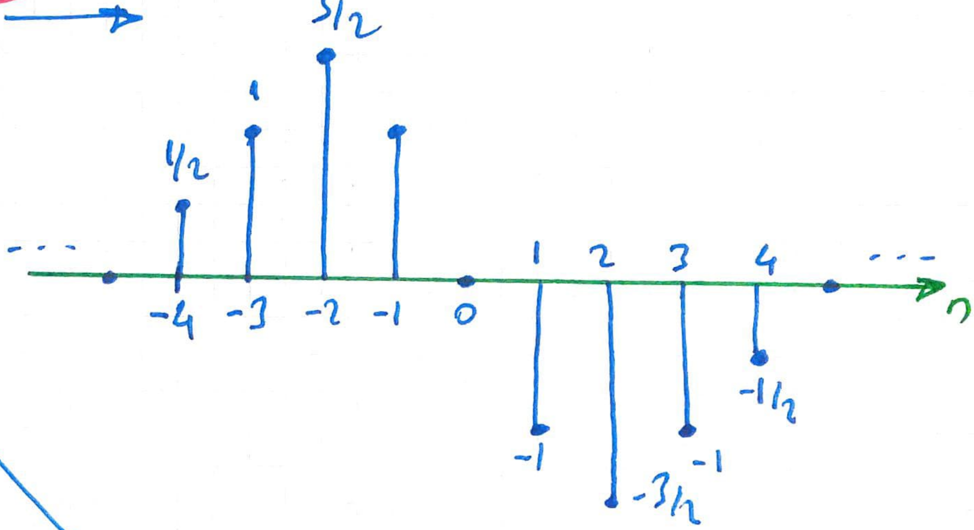
v

If $x(n) \xleftrightarrow{\tilde{F}} X(e^{jw})$

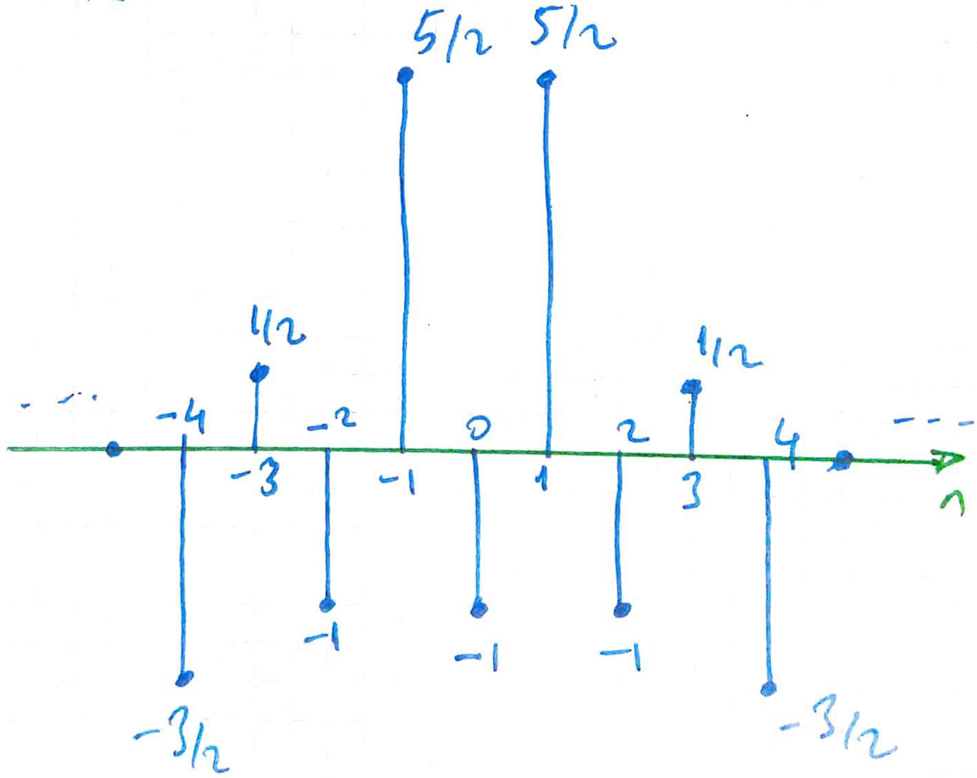
then $x_o(n) \xleftrightarrow{\tilde{F}} j\,\mathrm{Im}(X(e^{jw}))$

where $x_o(n) = \frac{1}{2}\left[\,x(n) - x^*(-n)\right]$

$\mathrm{Re}(x_o(n))$ →



$\mathrm{Im}(x_o(n))$



Q4: An LTI system has the frequency response

$$H(e^{jw}) = \frac{1-1{,}25e^{-jw}}{1-0{,}8e^{-jw}} = 1 - \frac{0{,}45e^{-jw}}{1-0{,}8e^{-jw}}$$

i. Specify the difference equation that is satisfied by the input $x(n)$ and the output $y(n)$

ii. Use one of the above forms of the frequency response to determine the impulse response. $h(n)$.

iii. Show that

$$|H(e^{jw})|^2 = G$$

where G is a constant. Determine the constant G. ($\longrightarrow$ "allpass filter" example)

iv. If the input to the above system is

$x(n) = \cos(0{,}2\pi n)$

the output should be of the form

$y(n) = A\cos(0{,}2\pi n + \theta)$. What are A and θ?

S4. i

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{1-1{,}25e^{-jw}}{1-0{,}8e^{-jw}}$$

$$\mathcal{F}^{-1}\left[Y(e^{jw}) - 0{,}8Y(e^{jw})e^{-jw} = X(e^{jw}) - 1{,}25X(e^{jw})e^{-jw}\right]$$

$$\mathcal{F}^{-1}[Y(e^{jw})] - \mathcal{F}^{-1}[0{,}8Y(e^{jw})e^{-jw}] = \mathcal{F}^{-1}[X(e^{jw})] - \mathcal{F}^{-1}[1{,}25X(e^{jw})e^{-jw}]$$

$$\boxed{y(n) - 0{,}8\,y(n-1) = x(n) - 1{,}25\,x(n-1)}$$

ii) $$H(e^{jw}) = 1 - \frac{0{,}45e^{-jw}}{1-0{,}8e^{-jw}}$$

$$h(n) = \mathcal{F}^{-1}\left[H(e^{jw})\right] = \mathcal{F}^{-1}[1] - \mathcal{F}^{-1}\left[\frac{0{,}45e^{-jw}}{1-0{,}8e^{-jw}}\right]$$

$$h(n) = \delta(n) - 0{,}45\,(0{,}8)^{n-1}\,u(n-1)$$

(*) When performing calculations, we made use of Fourier Transform Table.

iii) $$|H(e^{jw})|^2 = \left(H(e^{jw})\right)\left(H^*(e^{jw})\right)$$

$$= \left(\frac{1-1{,}25e^{-jw}}{1-0{,}8e^{-jw}}\right)\left(\frac{1-1{,}25e^{jw}}{1-0{,}8e^{jw}}\right)$$

$$= \frac{1-1{,}25e^{jw}-1{,}25e^{-jw}+(1{,}25)^2}{1-0{,}8e^{jw}-0{,}8e^{-jw}+(0{,}8)^2}$$

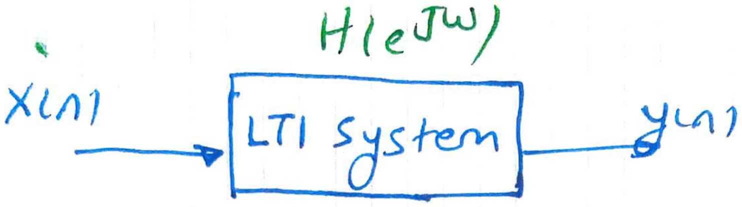
$$= \frac{1-2{,}5\cos(w)+1{,}5625}{1-1{,}6\cos(w)+0{,}64}$$

$$= \frac{2{,}5625-2{,}5\cos(w)}{1{,}64-1{,}6\cos(w)}$$

$$= \frac{2{,}5625\,(1-0{,}9756\cos(w))}{1{,}64\,(1-0{,}9756\cos(w))}$$

$$= \frac{2{,}5625}{1{,}64} = (1{,}25)^2 \longrightarrow \boxed{G = 1{,}25}$$

iv)



If $x(n)$ is $B\cos(w_0 n)$, then the output sequence

$$y(n) = B\,|H(e^{jw_0})|\,\cos(w_0 n + \angle H(e^{jw_0}))$$

We have an input signal as follows

$$x(n) = \cos(0{,}2\pi n)$$

$$\boxed{\begin{aligned} B &= 1 \\ w_0 &= 0{,}2\pi \end{aligned}}$$

$$y(n) = |H(e^{j0{,}2\pi})|\,\cos(0{,}2\pi n + \angle H(e^{j0{,}2\pi}))$$

$$H(e^{j0{,}2\pi}) = \frac{1 - 1{,}25\,e^{-j0{,}2\pi}}{1 - 0{,}8\,e^{-j0{,}2\pi}}$$

$$H(e^{j0{,}2\pi}) = 1{,}25\,e^{j0{,}210\pi}$$

$$\boxed{y(n) = 1{,}25\cos(0{,}2\pi n + 0{,}210\pi)}$$

Q1 Find the z-transform of each of the following sequences

i. $x(n) = 2^n u(n) + 3\left(\frac{1}{2}\right)^n u(n)$

ii. $x(n) = \cos(w_0 n)\, u(n)$

S1:

i $$x(n) = \underbrace{2^n u(n)}_{x_1(n)} + \underbrace{3\left(\frac{1}{2}\right)^n u(n)}_{x_2(n)}$$

! $$\boxed{a^n u(n) \xleftrightarrow{\mathcal{Z}} \frac{1}{1-az^{-1}}, \quad |z|>|a|}$$

$$X_1(z) = \frac{1}{1-2z^{-1}}, \quad |z|>2$$

$$X_2(z) = \frac{3}{1-\frac{1}{2}z^{-1}}, \quad |z|>\frac{1}{2}$$

$$X(z) = X_1(z) + X_2(z), \quad ROC_{x_1} \cap ROC_{x_2}$$

$$X(z) = \frac{1}{1-2z^{-1}} + \frac{3}{1-\frac{1}{2}z^{-1}}, \quad z>2$$

ii $$x(n) = \cos(w_0 n)u(n) = \frac{1}{2}e^{jw_0 n} + \frac{1}{2}e^{-jw_0 n}$$

! $$\boxed{a^n u(n) \xleftrightarrow{\mathcal{Z}} \frac{1}{1-az^{-1}}, \quad |z|>|a|}$$

$$X(z) = \frac{1}{2}\,\frac{1}{1-e^{jw_0}z^{-1}} + \frac{1}{2}\,\frac{1}{1-e^{-jw_0}z^{-1}}$$

$$X(z) = \frac{1-(\cos w_0)z^{-1}}{1-2(\cos w_0)z^{-1}+z^{-2}}, \quad |z|>1$$

Q2: Find the z transform of each of the following sequences. Whenever convenient, use the properties of the z-transform to make the solution easier.

i. $x(n) = \left(\frac{1}{3}\right)^n u(-n)$

ii. $x(n) = \left(\frac{1}{2}\right)^n u(n+2) + 3^n u(-n-1)$

iii. $x(n) = \left(\frac{1}{3}\right)^n \cos(n\omega_0)\, u(n)$

iv. $x(n) = a^{|n|}$

S2: i

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^n u(-n) z^{-n}$$

$$= \sum_{n=-\infty}^{0} \left(\frac{1}{3}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} 3^n z^n$$

$$= \sum_{n=0}^{\infty} (3z)^n = \frac{1}{1-3z}, \quad |3z| < 1$$

$$= \boxed{\frac{1}{1-3z}, \quad |z| < \frac{1}{3}}$$

Alternatively, the time reversed sequence $y(n) = x(-n) = \left(\frac{1}{3}\right)^{-n} u(n) = 3^n u(n)$ has a z-transform given by

$$Y(z) = \frac{1}{1-3z^{-1}}$$

with a region of convergence given by $|z|>3$. Therefore, using the time-reversal property of z-transform, $Y(z)=X(z^{-1})$, we obtain the same result.

(ii) $x(n)=\underbrace{\left(\frac{1}{2}\right)^n u(n+2)}_{x_1(n)}+\underbrace{3^n u(n-1)}_{x_2(n)}$

$$X_1(z)=\sum_{n=-\infty}^{\infty} x_1(n)\, z^{-n}$$

$$=\sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\right)^n u(n+2)\, z^{-n}$$

$$=\sum_{n=-2}^{\infty}\left(\frac{1}{2}\right)^n z^{-n}$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n-2} z^{-(n-2)}$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n(4)(z^{-n})(z^2)$$

$$=\sum_{n=0}^{\infty}4z^2\left(\frac{1}{2}\right)^n z^{-n}$$

$$=4z^2\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n}$$

$$=4z^2\cdot\frac{1}{1-\frac{1}{2}z^{-1}}\ ,\quad |z|>\frac{1}{2}$$

$$\boxed{=\frac{4z^2}{1-\frac{1}{2}z^{-1}}\ ,\ |z|>\frac{1}{2}}$$

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} 3^n u(-n-1) z^{-n}$$

$$= \sum_{n=-\infty}^{1} 3^n z^{-n}$$

$$= \sum_{n=1}^{\infty} 3^{-n} z^n$$

$$= \sum_{n=0}^{\infty} 3^{-(n+1)} z^{n+1}$$

$$= \sum_{n=0}^{\infty} 3^{-n} (3^{-1}) (z^n)(z)$$

$$= \frac{1}{3} z \sum_{n=0}^{\infty} 3^{-n} z^n$$

$$= \frac{1}{3} z \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n$$

$$= \frac{1}{3} z \frac{1}{1-\left(\frac{z}{3}\right)}, \quad \left|\frac{z}{3}\right| < 1$$

$$= \frac{1}{3} z \frac{3}{3-z}, \quad \left|\frac{z}{3}\right| < 1$$

$$= \boxed{-\frac{1}{1-3z^{-1}}, \quad |z| < 3}$$

$$X(z) = X_1(z) + X_2(z), \quad ROC_{X_1} \cap ROC_{X_2}$$

$$= \boxed{\frac{4z^2}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-3z^{-1}}, \quad \frac{1}{2} < |z| < 3}$$

(iii) The z-transform of $\cos(n\omega_0)u(n)$ is

$$\cos(n\omega_0)u(n) \xleftrightarrow{\mathcal{Z}} \frac{1-(\cos\omega_0)z^{-1}}{1-(\cos\omega_0)z^{-1}+z^{2}}, \quad |z|>1$$

By using the exponentiation property of z-transform

$$a^n x(n) \xleftrightarrow{\mathcal{Z}} X(a^{-1}z)$$

We have

$$\left(\frac{1}{3}\right)^n \cos(n\omega_0)u(n) \xleftrightarrow{\mathcal{Z}} \frac{1-\frac{1}{3}(\cos\omega_0)z^{-1}}{1-\frac{2}{3}(\cos\omega_0)z^{-1}+\frac{1}{9}z^{-2}}$$

$$|z|>\frac{1}{3}$$

(iv)

$$x(n)=a^{|n|}=a^n u(n)+a^{-n}u(-n)-\delta(n)$$

We may use the linearity and time-reversal properties

$$X(z)=\frac{1}{1-az^{-1}}+\frac{1}{1-az}-1, \quad \frac{1}{a}<|z|<a$$

which may be simplified to

$$X(z)=\frac{1-a^2}{(1-az^{-1})(1-az)}, \quad \frac{1}{a}<|z|<a$$

Q3: Consider the causal LTI system whose difference equation is

$$y(n) = x(n) - \frac{1}{3}x(n-1) + \frac{1}{6}x(n-2) + x(n-3)$$

a. Determine the impulse response of the system

b. Determine the transfer function $H(z)$ of " "

c. Draw the block diagram of the system

d. Determine the stability of the system.

S3. (a)

$$y(n) = x(n) * h(n)$$

$$= x(n) * \left[\delta(n) - \frac{1}{3}\delta(n-1) + \frac{1}{6}\delta(n-2) + \delta(n-3)\right]$$

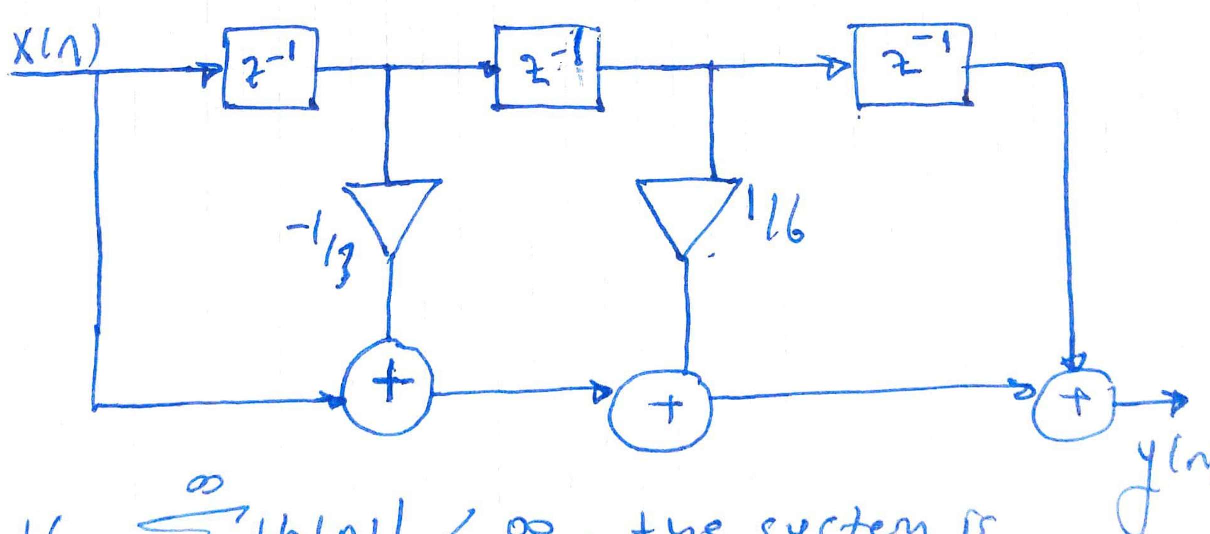
(b)

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\delta(n) - \frac{1}{3}\delta(n-1) + \frac{1}{6}\delta(n-2) + \delta(n-3)\right] z^{-n}$$

$$= 1 - \frac{1}{3}z^{-1} + \frac{1}{6}z^{-2} + z^{-3}$$

(c)



(d) If $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$, the system is called stable. System of interest FIR-

FIR systems are always stable.

Q4: When the input to an LTI system is

$$x(n) = \left(\frac{1}{2}\right)^n u(n) + 2^n u(-n-1)$$

the output is

$$y(n) = 6\left(\frac{1}{2}\right)^n u(n) - 6\left(\frac{3}{4}\right)^n u(n)$$

a. Find the system function / transfer function $H(z)$ of the system. Plot the poles and zeros of $H(z)$, and indicate the ROC.

b. Find the impulse response $h(n)$ of the system for all values of n.

c. Write the difference equation that characterizes the system.

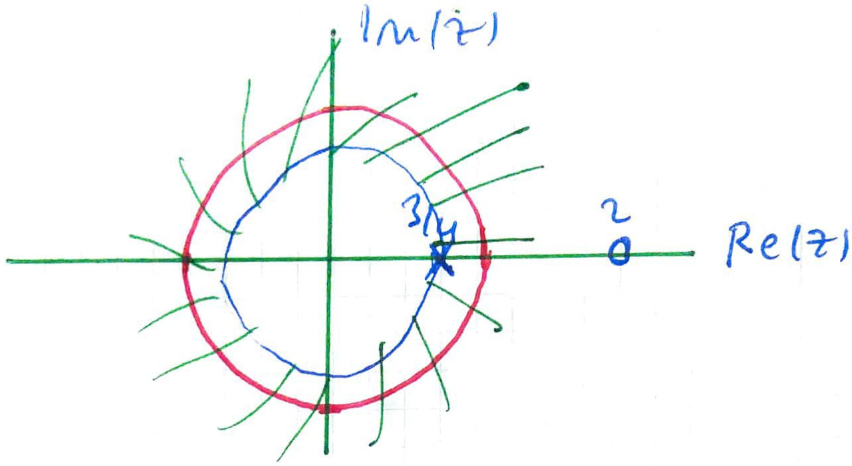
d. Is the system stable? Is it causal?

S4. a)

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z} \quad \frac{1}{2} < |z| < 2$$

$$Y(z) = \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{6}{1-\frac{3}{4}z^{-1}} \quad , |z| > \frac{3}{4}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}} \quad , \quad |z| > \frac{3}{4}$$



(b) $$H(z) = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}, \quad |z| > \frac{3}{4}$$

$$H(z) = \frac{1}{1-\frac{3}{4}z^{-1}} - \frac{2z^{-1}}{1-\frac{3}{4}z^{-1}}, \quad |z| > \frac{3}{4}$$

$$\mathcal{Z}^{-1} \updownarrow \qquad \mathcal{Z}^{-1} \qquad \mathcal{Z}^{-1}$$

$$h(n) = \left(\frac{3}{4}\right)^n u(n) - 2\left(\frac{3}{4}\right)^{n-1} u(n-1)$$

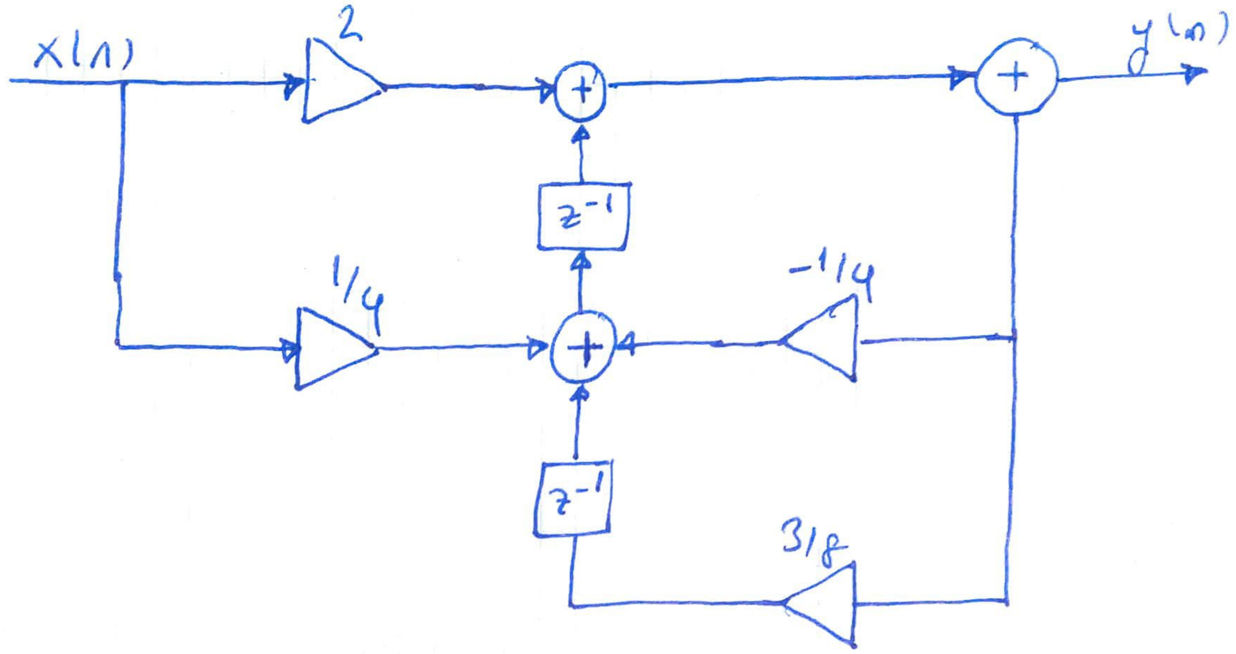
(c) $$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}$$

$$Y(z) - \frac{3}{4}z^{-1}Y(z) = X(z) - 2z^{-1}X(z)$$

$$y(n) - \frac{3}{4}y(n-1) = x(n) - 2x(n-1)$$

(d) The RoC is outside $|z| > \frac{3}{4}$, which contains the unit circle. Therefore the system is stable. The $h(n)$ we found in part (b) tells us the system is causal.

Q5: The block diagram of a system is the following:



i. Find the difference equation that is satisfied by the input and output sequence.

ii. Find the transfer function $H(z)$ of the system.

S5. (i) $2x(n) + \frac{1}{4}x(n-1) + \frac{3}{8}y(n-2) - \frac{1}{4}y(n-1) = y(n)$

(ii) $2X(z) + \frac{1}{4}z^{-1}X(z) = Y(z) + \frac{1}{4}z^{-1}Y(z) - \frac{3}{8}z^{-2}Y(z)$

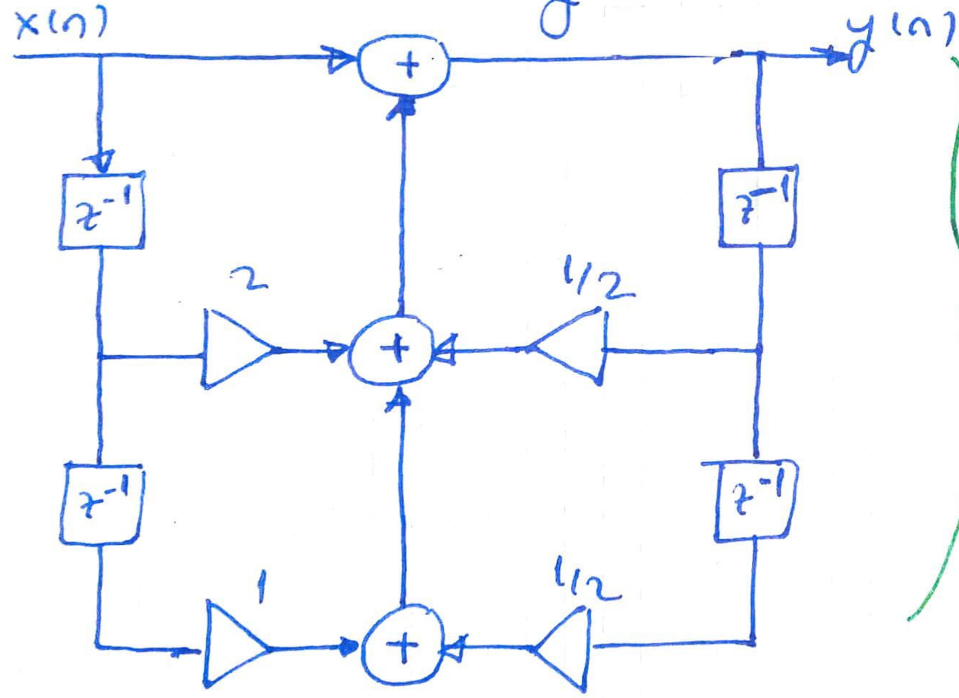
$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + \frac{1}{4}z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}$$

Q6: Consider the difference equation of the system

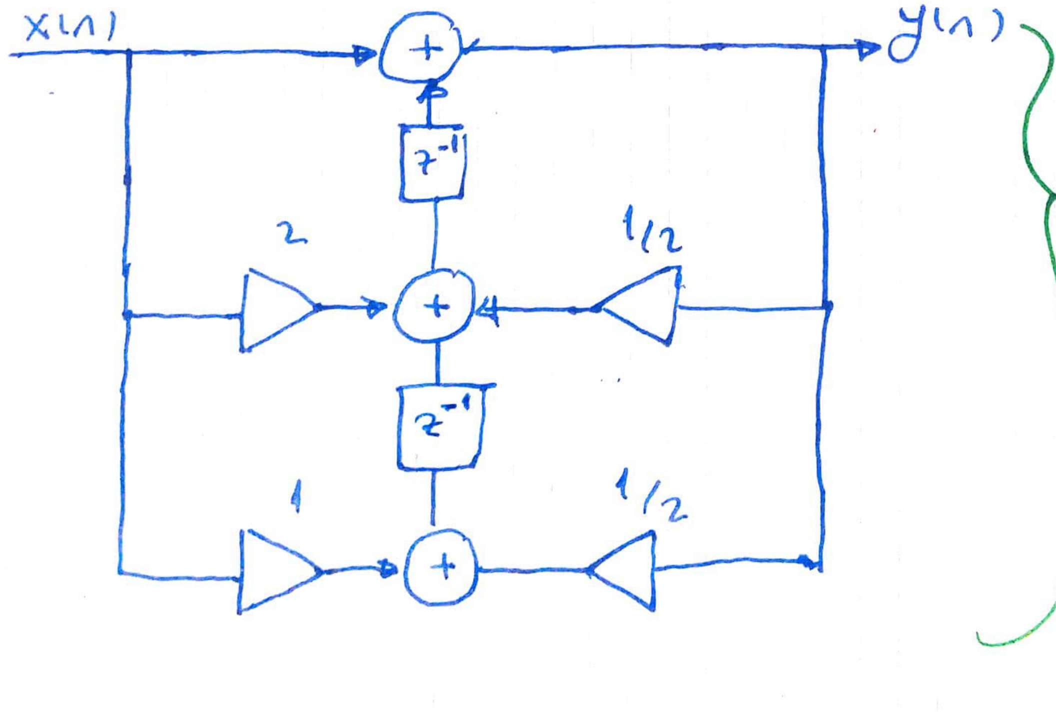
$$x(n) + 2x(n-1) + x(n-2) = y(n) - \frac{1}{2}y(n-1) - \frac{1}{2}y(n-2)$$

i. Draw the block diagram of the system.
ii. Determine the transfer function $H(z)$ of the system.
iii. Is the system is stable? Explain.

S6 i: The block diagram can be drawn in two ways.



Block Diagram 1



Block Diagram 2

(ii) $x(n)+2x(n-1)+x(n-2)=y(n)-\frac{1}{2}y(n-1)-\frac{1}{2}y(n-2)$

$$X(z)+2z^{-1}X(z)+z^{-2}X(z)=Y(z)-\frac{1}{2}z^{-1}Y(z)-\frac{1}{2}z^{-2}Y(z)$$

$$X(z)\left(1+2z^{-1}+z^{-2}\right)=Y(z)\left(1-\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2}\right)$$

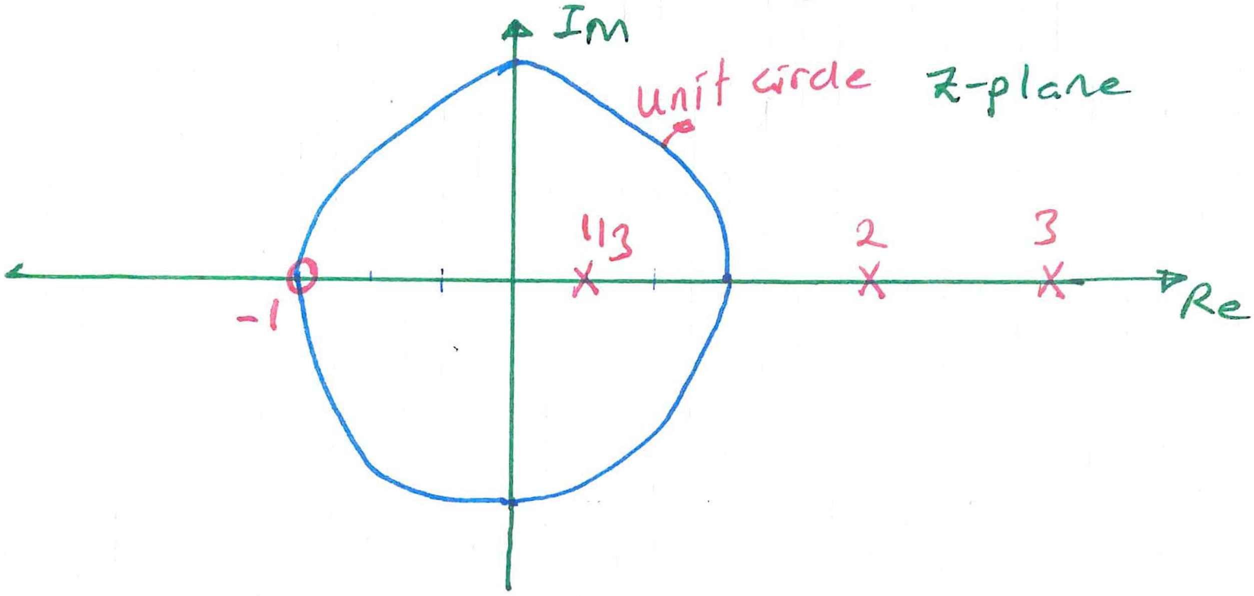
$$H(z)=\frac{Y(z)}{X(z)}=\frac{1+2z^{-1}+z^{-2}}{1-\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2}}$$

(iii) $$H(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2}}$$

$$H(z)=\frac{(1+z^{-1})(1+z^{-1})}{(1+\frac{1}{2}z^{-1})(1-z^{-1})}$$

The system function has poles at $z=-\frac{1}{2}$ and $z=1$. Since the second pole is on the unit circle, the system is not stable.

Q1 Consider the z-transform $X(z)$ whose pole-zero plot is as shown in the figure below.



i. Determine the ROC of $X(z)$ if it is known that Fourier Transform exists. For this case, determine whether the corresponding sequence $x(n)$ is right-sided, left-sided, or two sided.

ii. How many possible two-sided sequences have the pole-zero plot shown in the figure.

iii. Is it possible for the pole-zero plot in the figure to be associated with a sequence that is both stable and causal? If so, give the appropriate ROC.

S1: i For the Fourier Transform of $x(n)$ to exist, the z-transform of $x(n)$ must have an ROC which includes the unit circle, therefore $\left|\frac{1}{3}\right| < |z| < |2|$

Since this ROC lies outside $\frac{1}{3}$, this pole contributes a right-sided sequence. Since the ROC lies inside 2 and 3, these poles contribute left-sided sequences. The overall $x[n]$ is therefore two-sided.

ii) Two sided have ROC's which look like annulus/washer. There are two possibilities. The ROC's corresponding to these are:

- $\left|\frac{1}{3}\right| < |z| < |2|$
- $|2| < |z| < 3$

iii) The ROC must be a connected region. For stability, the ROC must contain the unit circle. For causality the ROC must be outside the outermost pole. These conditions can not be met by any of the possible ROC's of this pole-zero plot.

Q2: Following are several z-transforms. For each, determine the inverse z-transform using both partial fraction method and power series expansion method. In addition, indicate in each case whether the fourier Transform exists

i. $X(z) = \dfrac{1}{1 + \frac{1}{2}z^{-1}}$, $|z| > \frac{1}{2}$

ii. $X(z) = \dfrac{1}{1 + \frac{1}{2}z^{-1}}$, $|z| < \frac{1}{2}$

iii. $X(z) = \dfrac{1 - \frac{1}{2}z^{-1}}{1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$, $|z| > \frac{1}{2}$

S2. i) • This z-transform has one pole. Because of this, we can calculate the inverse z-transform by inspection.

$$x(n) = \left(-\frac{1}{2}\right)^n u(n)$$

$$
\begin{array}{r|l}
 & 1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \dots \\
\hline
1 + \frac{1}{2}z^{-1} & 1 \\
 & -\left(1 + \frac{1}{2}z^{-1}\right) \\
\hline
 & -\frac{1}{2}z^{-1} \\
 & -\left(-\frac{1}{2}z^{-1} - \frac{1}{4}z^{-2}\right) \\
\hline
 & \frac{1}{4}z^{-2} \\
 & \frac{1}{4}z^{-2} + \frac{1}{8}z^{-3}
\end{array}
$$

$$X(z) = 1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} - \frac{1}{8}z^{-3} + \cdots$$

$$x(n) = \delta(n) - \frac{1}{2}\delta(n-1) + \frac{1}{4}\delta(n-2) - \frac{1}{8}\delta(n-3) + \cdots$$

$$x(n) = \left(-\frac{1}{2}\right)^n u(n) \longrightarrow$$ Fourier Transform for this sequence exists

(ii) • Since the z-transform has one pole, we can find the inverse z-transform by inspection. Notice that the ROC has circle-shape, The sequence is left-sided.

$$x(n) = -\left(-\frac{1}{2}\right)^n u(-n-1)$$

• Long division: $1 \div \left(1 + \frac{1}{2}z^{-1}\right)$

$$\begin{array}{r|l} & 2z - 4z^2 + 8z^3 + \cdots \\ 1 + \frac{1}{2}z^{-1} & 1 \\ & -(2z + 1) \\ \hline & -2z \\ & -(-4z^2 - 2z) \\ \hline & 4z^2 \\ & -(8z^3 + 4z^2) \\ \hline & -8z^3 \cdots \end{array}$$

$$X(z) = 2z - 4z^2 + 8z^3 + \cdots$$

$$= -\left(-\frac{1}{2}\right)^n u(-n-1)$$

Fourier Transform for this sequence doesn't exist iii)

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \quad , |z| > \frac{1}{2}$$

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{\left(1 + \frac{1}{4}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)} \quad , |z| > \frac{1}{2}$$

$$X(z) = \frac{A_1}{\left(1 + \frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1 + \frac{1}{2}z^{-1}\right)}$$

$$A_1 = \left(\left(1 + \frac{1}{4}z^{-1}\right) X(z)\right)\Big|_{z=-\frac{1}{4}}$$

$$A_1 = \frac{1 - \frac{1}{2}(-4)}{1 + \frac{1}{2}(-4)} = -3$$

$$A_2 = \left(\left(1 + \frac{1}{2}z^{-1}\right) X(z)\right)\Big|_{z=-\frac{1}{2}}$$

$$A_2 = \frac{1 - \frac{1}{2}(-2)}{1 + \frac{1}{4}(-2)} = \frac{2}{\frac{1}{2}} = 4$$

$$X(z) = \frac{-3}{1 + \frac{1}{4}z^{-1}} + \frac{4}{1 + \frac{1}{2}z^{-1}} \quad , |z| > \frac{1}{2}$$

$$x(n) = -3\left(-\frac{1}{4}\right)^n u(n) + 4\left(-\frac{1}{2}\right)^n u(n)$$

$$x(n) = \left[-3\left(-\frac{1}{4}\right)^n + 4\left(-\frac{1}{2}\right)^n\right] u(n)$$

$$
\begin{array}{r|l}
 & 1 + \left(-\frac{5}{4}z^{-1}\right) + \left(+\frac{11}{16}z^{-2}\right) \\
\hline
1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} & 1 - \frac{1}{2}z^{-1} \\
 & -\quad 1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} \\
\hline
 & -\frac{5}{4}z^{-1} - \frac{1}{8}z^{-2} \\
 & -\quad -\frac{5}{4}z^{-1} - \frac{15}{16}z^{-2} - \frac{5}{32}z^{-3} \\
\hline
 & +\frac{11}{16}z^{-2} + \frac{5}{32}z^{-3}
\end{array}
$$

$$x(n) = \left[-3\left(\left(-\frac{1}{4}\right)^{n} + \left(-\frac{1}{2}\right)^{n-2}\right)\right] u(n)$$

Fourier Transform of this sequence exists.

Q3: Following are four z-transform, Determine which ones could be the z-transform of a causal sequence. Do not evaluate the z-transform. Clearly state your reasons in each case.

i. $$\frac{(1-z^{-1})^2}{(1-\frac{1}{2}z^{-1})}$$

ii. $$\frac{(z-1)^2}{(z-\frac{1}{2})}$$

iii. $$\frac{(z-\frac{1}{4})^5}{(z-\frac{1}{2})^6}$$

iv. $$\frac{(z-\frac{1}{4})^6}{(z-\frac{1}{2})^5}$$

S3. $x(n)$ causal $\rightarrow X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

which means this summation will include no positive powers of z. This means that the closed form of $X(z)$ must converge at $z=\infty$, i.e., $z=\infty$ must be in the ROC of $X(z)$, or

$$\lim_{z \to \infty} X(z) \neq \infty$$

(i) $\lim_{z \to \infty} \frac{(1-z^{-1})^2}{(1-\frac{1}{2}z^{-1})} = \frac{1-0}{1} = 1$, COULD BE CAUSAL

(ii) $\lim_{z \to \infty} \frac{(z-1)^2}{(z-\frac{1}{2})} = \infty$, COULD NOT BE CAUSAL.

(iii) $\lim_{z \to \infty} \frac{(z-\frac{1}{4})^5}{(z-\frac{1}{2})^6} = 0$, COULD BE CAUSAL

(iv) $\lim_{z \to \infty} \frac{(z-\frac{1}{4})^6}{(z-\frac{1}{2})^5} = \infty$, COULD NOT BE CAUSAL

Q4: For each of following pairs of input z-transform $X(z)$ and system function $H(z)$, determine the ROC for the output z-transform $Y(z)$

i. $$X(z) = \frac{1}{1+\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$H(z) = \frac{1}{1-\frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}$$

ii. $$X(z) = \frac{1}{1-2z^{-1}}, \quad |z| < 2$$

$$H(z) = \frac{1}{1-\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

iii. $$X(z) = \frac{1}{\left(1-\frac{1}{5}z^{-1}\right)\left(1+3z^{-1}\right)}, \quad \frac{1}{5} < |z| < 3$$

$$H(z) = \frac{1+3z^{-1}}{1+\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

S4. i

$$Y(z) = X(z)H(z)$$

$$= \frac{1}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

The ROC($Y(z)$) includes the intersection of ROC($H(z)$) and ROC($X(z)$) $\rightarrow$ ROC($Y(z)$): $|z| > \frac{1}{2}$

(ii) $$Y(z) = \frac{1}{(1-2z^{-1})(1-\frac{1}{3}z^{-1})}$$

ROC(Y(z)) is the intersection of ROCs of H(z) and X(z)

ROC(Y(z)) : $\frac{1}{3} < |z| < 2$

(iii) $$Y(z) = \frac{1}{(1-\frac{1}{5}z^{-1})(1+\frac{1}{3}z^{-1})}$$

ROC(Y(z)) : $|z| > \frac{1}{3}$

Q5: For each of the following sequences, determine the z-transform and ROC, and sketch the pole-zero diagram.

i. $x[n] = a^n u[n] + b^n u[n] + c^n u[n-1]$, $|a|<|b|<|c|$

ii. $x[n] = n^2 a^n u[n]$

iii. $x[n] = e^{n^4}\left[\cos\left(\frac{\pi}{12} n\right)\right] u[n] - e^{n^4}\left[\cos\left(\frac{\pi}{12} n\right)\right] u[n-1]$

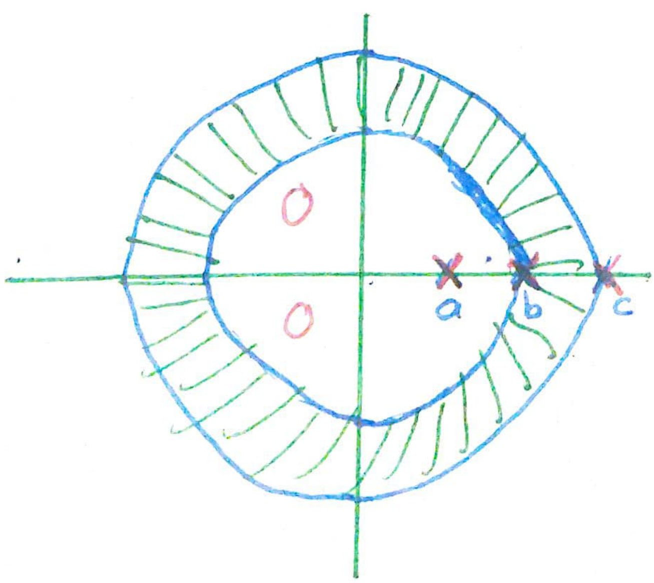
S5. i

$$X(z) = \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}} - \frac{1}{1-cz^{-1}}$$

$$X(z) = \frac{1-2cz^{-1}+(bc+ac-ab)z^{-2}}{(1-az^{-1})(1-bz^{-1})(1-cz^{-1})} \quad , |b|<|z|<|c|$$

Poles : a, b, c

Zeros : z_1, z_2, ∞ where z_1 and z_2 are roots of the numerator quadratic



ii) $x(n) = n^2 \underbrace{a^n u(n)}_{x_1(n)}$

$x_1(n) = a^n u(n)$

$X_1(z) = \dfrac{1}{1 - az^{-1}}$, $|z| > a$

$x_2(n) = n a^n u(n) = n x_1(n)$

$X_2(z) = -z \dfrac{d}{dz} X_1(z)$

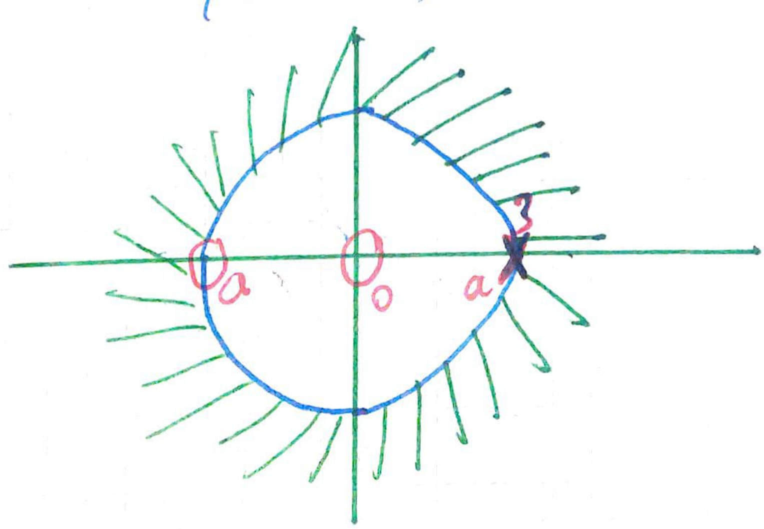
$= \dfrac{az^{-1}}{(1 - az^{-1})^2}$, $|z| > a$

$x(n) = n^2 a^n u(n) = n x_2(n)$

$X(z) = -z \dfrac{d}{dz} X_2(z)$

$= -z \dfrac{d}{dz} \left(\dfrac{az^{-1}}{(1 - az^{-1})^2} \right)$, $|z| > a$

$X(z) = \dfrac{-az^{-1}(1 + az^{-1})}{(1 - az^{-1})^3}$, $|z| > a$



(iii) $x(n) = e^{n^4}\left(\cos\left(\frac{\pi}{12}n\right)\right)u(n) - e^{n^4}\left(\cos\left(\frac{\pi}{12}n\right)\right)u(n-1)$

$$x(n) = \cos\left(\frac{\pi}{12}n\right) e^{n^4} \underbrace{\left[u(n) - u(n-1) \right]}_{\delta(n)}$$

$$x(n) = \cos(0)\, e^0\, \delta(n)$$

$$x(n) = \delta(n)$$

Therefore, $X(z) = 1$ for all $|z|$

Q1 The signal

$$x_c(t) = \sin(2\pi(100)t)$$

was sampled with a sampling period $T = 1/400$ second to obtain a discrete-time signal $x(n)$. What is the resulting sequence $(x(n))$?

S1. $x(n) = x_c(nT)$

$$x_c(t) = \sin(2\pi 100 t)$$

$$x(n) = \sin\left(2\pi 100 \frac{1}{400}\right)$$

$$= \sin\left(\frac{\pi}{2} n\right)$$

Q2 The sequence $x(n) = \cos\left(\frac{\pi}{4} n\right)$, $-\infty < n < \infty$, was obtained by sampling the continuous-time signal $x_c(t) = \cos(\Omega_0 t)$, $-\infty < t < \infty$ at a sampling rate of 1000 samples/s. What are two possible positive values of Ω_0 that could have resulted in the sequence $x(n)$?

S2. Since $w = \Omega T$ and $T = \frac{1}{1000}$ seconds, the signal frequency could be

$$\Omega_0 = \frac{\pi}{4} \cdot 1000 = 250\pi$$

or possibly

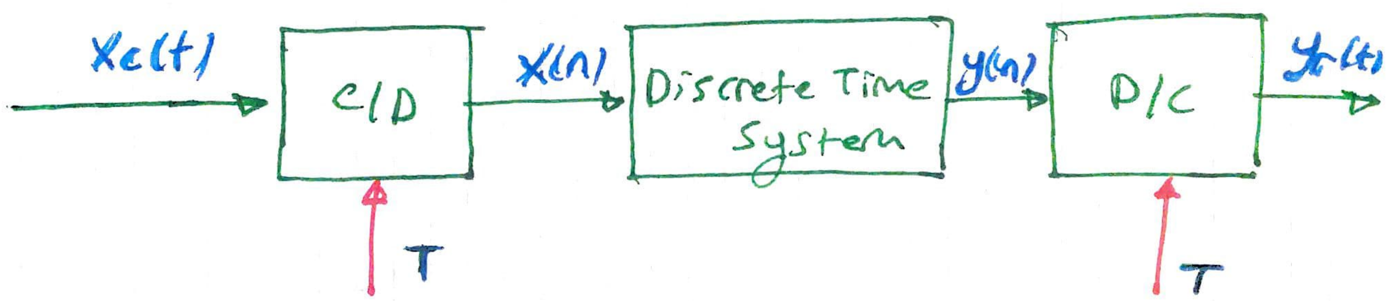
$$\Omega_0 = \left(2\pi + \frac{\pi}{4}\right) \cdot 1000 = 2250\pi$$

Q3: Consider the system of the figure below, with the discrete-time system an ideal Low pass Filter with cutoff frequency $\pi/8$ radian/s

(i) If $x_c(t)$ is band limited to 5 kHz, what is the maximum value of T that will avoid aliasing in the C/D converter?

(ii) If $1/T = 10$ kHz, what will the cutoff frequency of the effective continuous time filter be?

(iii) Repeat part (b) for $1/T = 20$ kHz.



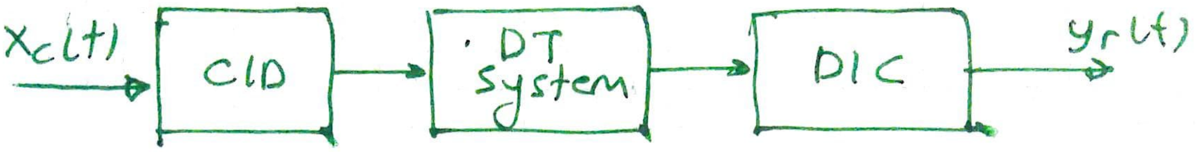
S3 (i) $X_c(j\Omega) = 0, \quad |\Omega| \geq \underbrace{2\pi \cdot 5000}_{\Omega_N}$

$\Omega_s \geq 2\Omega_N$

$\Omega_s \geq 20000\pi$

$T \leq \dfrac{1}{10000} \rightarrow \boxed{T = \dfrac{1}{10000}\ s}$

ii



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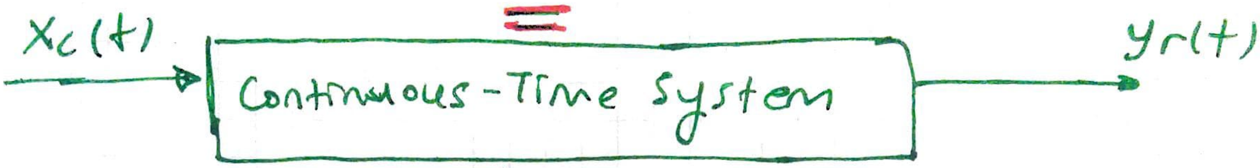


$H_c(j\Omega)$

$$H_c(j\Omega) = \begin{cases} H(e^{j\Omega T}) , & |\Omega| < \frac{\pi}{T} \\ 0 , & \text{else} \end{cases}$$

$$w = \Omega T$$

$$\frac{\pi}{8} = \frac{1}{10000} \Omega_1 \longrightarrow \Omega_1 = 2\pi(625) \text{ rad/s}$$

$$f_1 = 625 \text{ Hz}$$

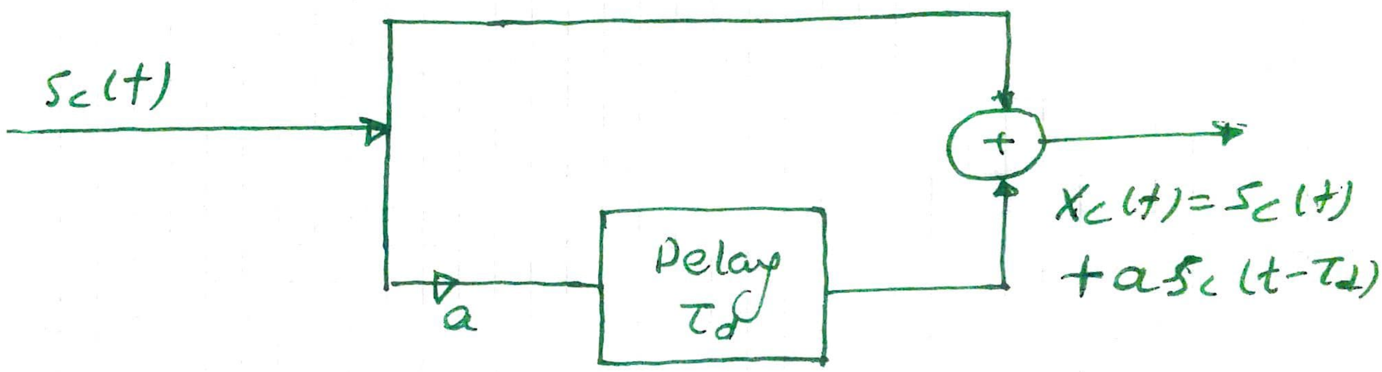
iii

$$w = \Omega T$$

$$\frac{\pi}{8} = \frac{1}{20000} \Omega_2 \longrightarrow \Omega_2 = 2\pi(1250) \text{ rad/s}$$

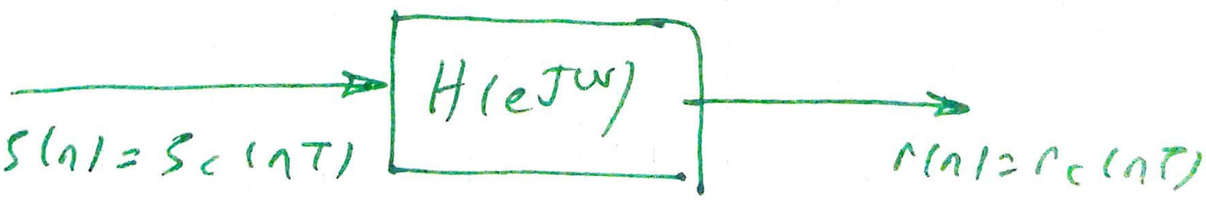
$$f_2 = 1250 \text{ Hz}$$

Q4: A simple model of a multipath communication channel is indicated the following figure Assume that $s_c(t)$ is bandlimited such that $S_c(\Omega) = 0$ for $|\Omega| \geq \pi/T$ and that $x_c(t)$ is sampled with a sampling period T to obtain the sequence. $x[n] = x_c(nT)$

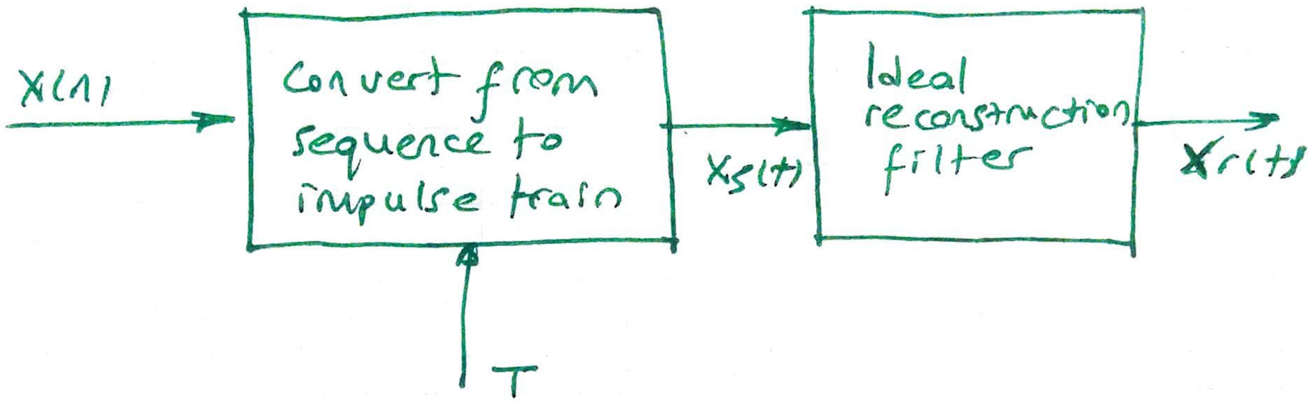


(i) Determine the Fourier Transform of $x_c(t)$ and the Fourier transform of $x[n]$ in terms of $S_c(j\Omega)$

(ii) We want to simulate the multipath system with a discrete-time system by choosing $H(e^{j\omega})$ in the figure below so that the output $r[n] = x_c(nT)$ when the input is $s[n] = s_c(nT)$. Determine $H(e^{j\omega})$ in terms of T and τ_d.



(iii) Determine the impulse response $h(n)$ in the figure below. when (i) $T_d = T$ and (ii) $T_d = T/2$



S4) (i) $x_c(t) = s_c(t) + a s_c(t - T_d)$

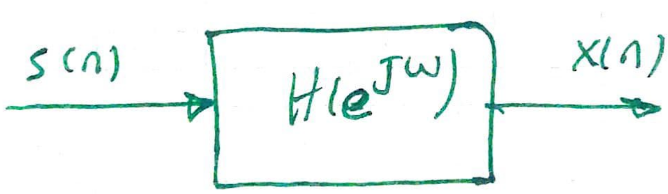
$$X_c(j\Omega) = S_c(j\Omega) + a S_c(j\Omega) e^{-j\Omega T_d}$$

$$X_c(j\Omega) = (1 + a e^{-j\Omega T_d}) S_c(j\Omega)$$

$$X(e^{jw}) = \frac{1}{T} \sum_k X_c\left[j\left(\frac{w}{T} - \frac{2\pi k}{T}\right)\right]$$

$$= \frac{1}{T} \sum_k \left(S_c\left[j\left(\frac{w}{T} - \frac{2\pi k}{T}\right)\right]\right)\left(1 + a e^{-jwT_d/T}\right)$$

ii)



$$X(j\Omega) = \underbrace{H(j\Omega)}_{(1 + a e^{-j\Omega T_d})} S(j\Omega)$$

$$H(j\Omega) = \begin{cases} 1 + a e^{-j\Omega \tau_d} & , |\Omega| < \frac{\pi}{T} \\ 0 & , \text{else} \end{cases}$$

Using $w = \Omega T$, we obtain a discrete-time system simulating the above response

$$H(e^{jw}) = 1 + a e^{-j\tau_d w/T}$$

iii $\quad h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{jw}) e^{jwn} dw$

- for $\tau_d = T$, $h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + a e^{-jwT/T}) e^{jwn} dw$

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{jwn} + a e^{jw(n-1)}) dw$$

$$= \frac{\sin(\pi n)}{\pi n} + a \frac{\sin(\pi(n-1))}{\pi(n-1)}$$

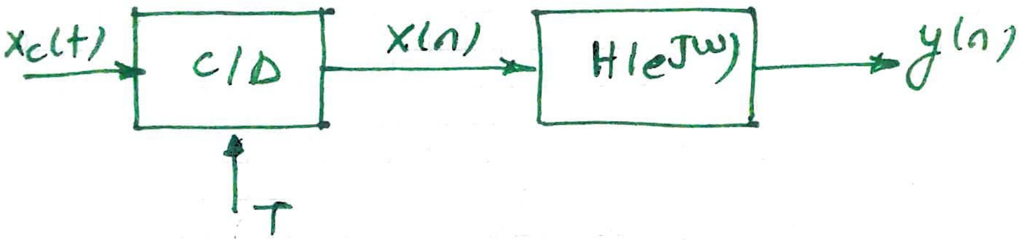
- for $\tau_d = T/2$

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + a e^{-jwT/2/T}) e^{jwn} dw$$

$$= \frac{\sin(\pi n)}{\pi n} + a \frac{\sin(\pi(n-1/2))}{\pi(n-\frac{1}{2})}$$

Q5: consider the system in the following figure. with the following relations:

- $X_c(j\Omega) = 0 \quad , \quad |\Omega| \geq 2\pi \times 10^4$
- $x(n) = x_c(nT)$
- $y(n) = T \sum_{k=-\infty}^{n} x(k)$



(i) For this system, what is the maximum allowable value of T if aliasing is to be avoid, i.e., so that $x_c(t)$ can be recovered from $x(n)$

(ii) Determine $h(n)$

(iii) In terms of $X(e^{j\omega})$, what is the value of $y(n)$ for $n \to \infty$?

(iv) Determine whether there is any value of T for which

$$y(n)\Big|_{n=\infty} = \int_{-\infty}^{\infty} x_c(t)\,dt \qquad \cdots (1)$$

If there is such a value of T, determine the maximum value. If there is not, explain and specify how T would be chosen so that the equality in Eq(1) is best approximated.

55

i For $x_c(t)$ to be recoverable from $x(n)$, aliasing distortion must not occur → Nyquist criteria

$$\Omega_s \geq 2\Omega_N \quad , \quad \Omega_N = 2\pi \times 10^4$$

$$\Omega_s \geq 2 \cdot 2\pi 10^4$$

$$\frac{2\pi}{T} \geq 4\pi 10^4$$

$$T \leq \frac{1}{20000}$$

ii $X(e^{j\omega}) = H(e^{j\omega}) Y(e^{j\omega})$

$$Y(e^{j\omega}) = \tilde{F}\left[T \sum_{k=-\infty}^{n} x(k) \right]$$

$$= T\left[\tilde{F}(x(n)) + \tilde{F}(x(n-1)) + \tilde{F}(x(n-2)) + \cdots \right]$$

$$= T(1 + e^{-j\omega} + e^{-j2\omega} + \cdots) X(e^{j\omega})$$

$$X(e^{j\omega}) = H(e^{j\omega}) Y(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$= \frac{T(1 + e^{-j\omega} + e^{-j2\omega} + \cdots) X(e^{j\omega})}{X(e^{j\omega})}$$

$$H(e^{j\omega}) = T(1 + e^{-j\omega} + e^{-j2\omega} + \cdots)$$

$$h(n) = T[\delta(n) + \delta(n-1) + \delta(n-2) + \cdots)$$

$$h(n) = T u(n)$$

(iii) $$y[n] = T \sum_{k=-\infty}^{n} x(k)$$

$$\lim_{n\to\infty} y[n] = \lim_{n\to\infty} T \sum_{k=-\infty}^{n} x(k)$$

$$= T \sum_{k=-\infty}^{\infty} x(k)$$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jwn}$$

$$X(e^{jw})\Big|_{w=0} = \sum_{n=\infty}^{\infty} x(n)$$

$$\lim_{n\to\infty} y[n] = T X(e^{jw})\Big|_{w=0}$$

(iv) $$X(e^{jw}) = \frac{1}{T} \sum_{r=-\infty}^{\infty} X_c\left(j\frac{w}{T} + j\frac{2\pi r}{T}\right)$$

From part (iii), we have

$$T X(e^{jw})\Big|_{w=0} = \sum_{r=-\infty}^{\infty} X_c\left(j\frac{2\pi r}{T}\right)$$

From the given information, we seek a value of T such that

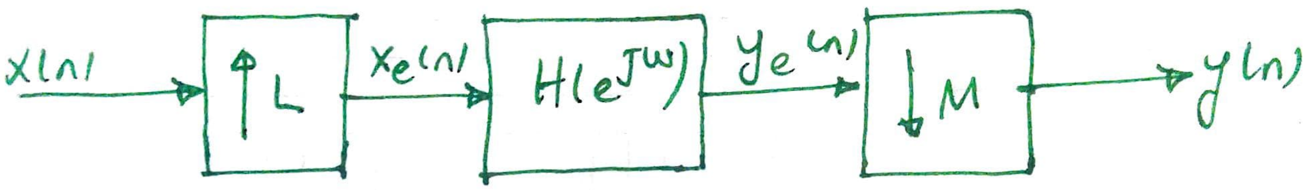
$$\sum_{r=-\infty}^{\infty} X_c\left(j\frac{2\pi r}{T}\right) = \int_{-\infty}^{\infty} x_c(t)\,dt$$

$$= X_c(j\Omega)\Big|_{\Omega=0}$$

For the final equality to be true, there must be no contribution from the terms for which $r \neq 0$. That means, we require no aliasing at $\Omega = 0$. Since we are only interested in preserving the spectral component at $\Omega = 0$, we may sample at a rate which is lower than the Nyquist rate. The maximum value of T to satisfy these conditions is

$$T \leq \frac{1}{104} \ [s]$$

Q1 Consider the discrete-time system shown in the figure.



where

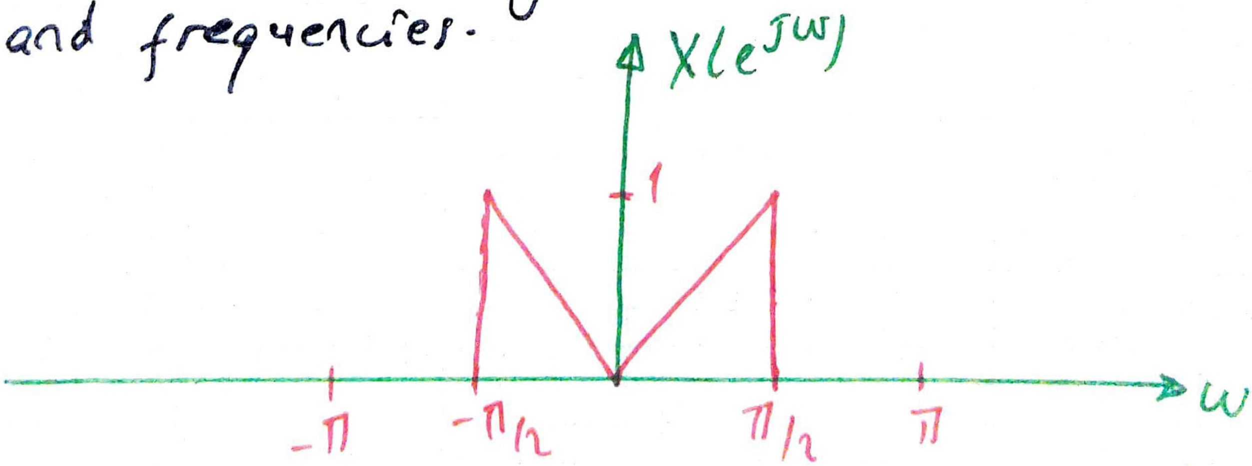
L and M are positive integers.

$$x_e(n) = \begin{cases} x(n/L) & , n = kL,\ k \text{ is any integer} \\ 0 & , \text{otherwise} \end{cases}$$

$$y(n) = y_e(nM)$$

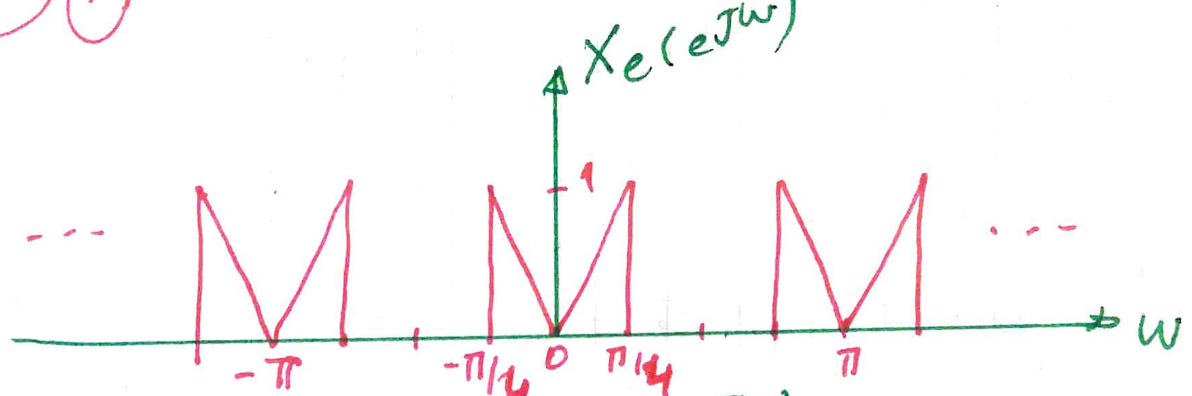
$$H(e^{j\omega}) = \begin{cases} M & , \quad |\omega| \le \pi/4 \\ 0 & , \quad \frac{\pi}{4} < |\omega| \le \pi \end{cases}$$

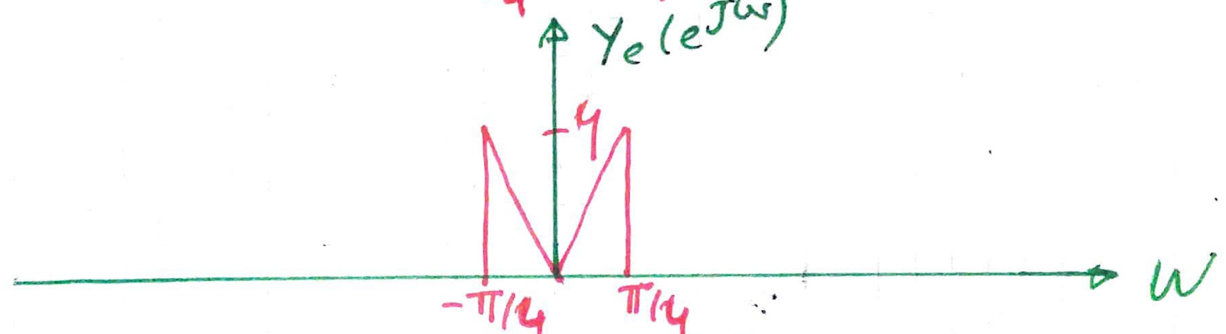
i. Assume that $L=2$ and $M=4$, and that $X(e^{j\omega})$, the DTFT of $x(n)$ is real and is as shown in the figure below. Make an appropriately labeled sketch of $X_e(e^{j\omega})$, $Y_e(e^{j\omega})$ and $Y(e^{j\omega})$, the DTFT's of $x_e(n)$, $y_e(n)$, and $y(n)$ respectively. Be sure to clearly label salient amplitudes and frequencies.

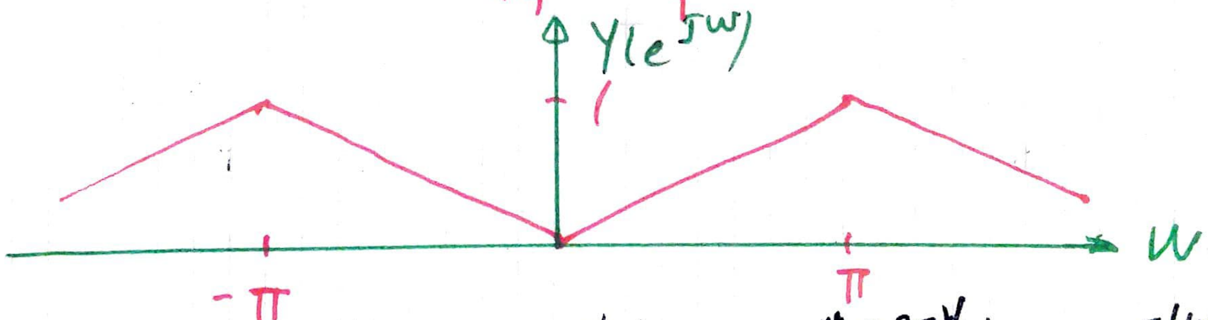


ii. Now assume $L=2$ and $M=8$. Determine $y[n]$ in this case (Hint: See which diagrams in your answer to part (a) change)

S1) i)

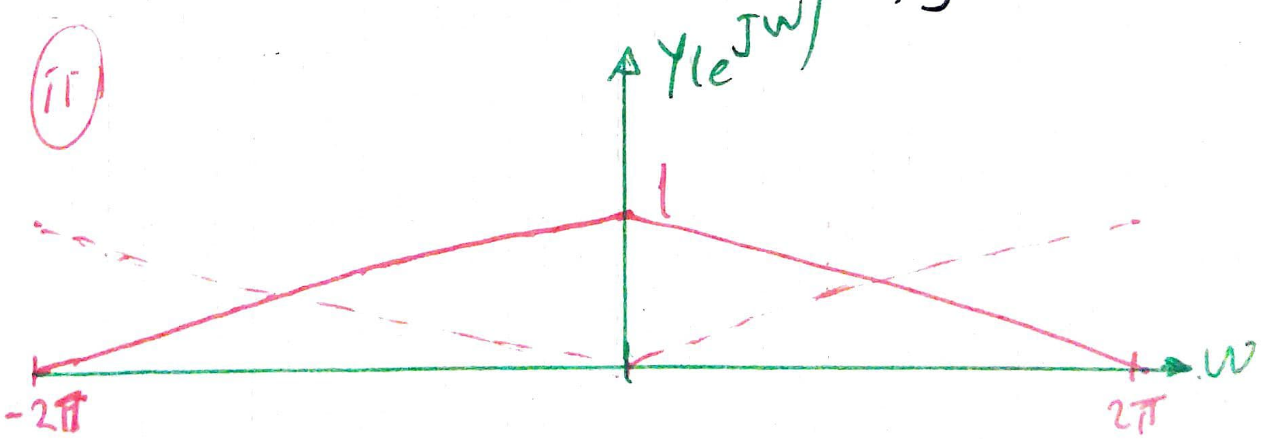


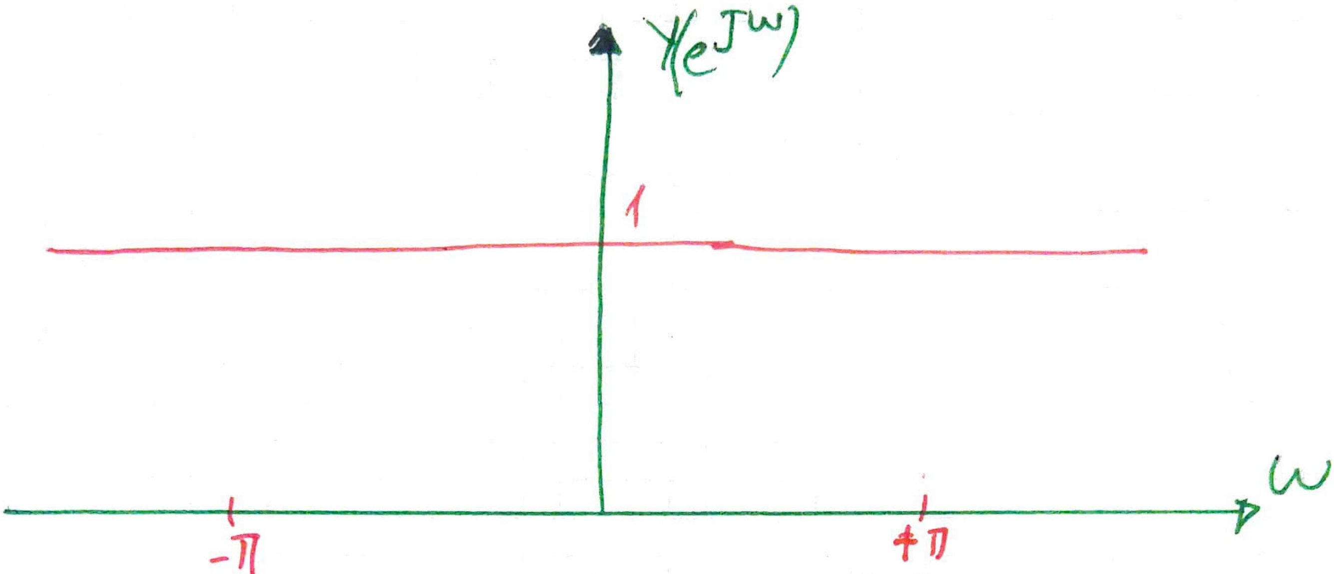




$$Y(e^{jw}) = \frac{1}{4}\left[Ye(e^{j(w/4)}) + Ye(e^{j(w-2\pi)/4}) + Ye(e^{j(w-4\pi)/4}) + Ye(e^{j(w-6\pi)/4})\right]$$

ii)

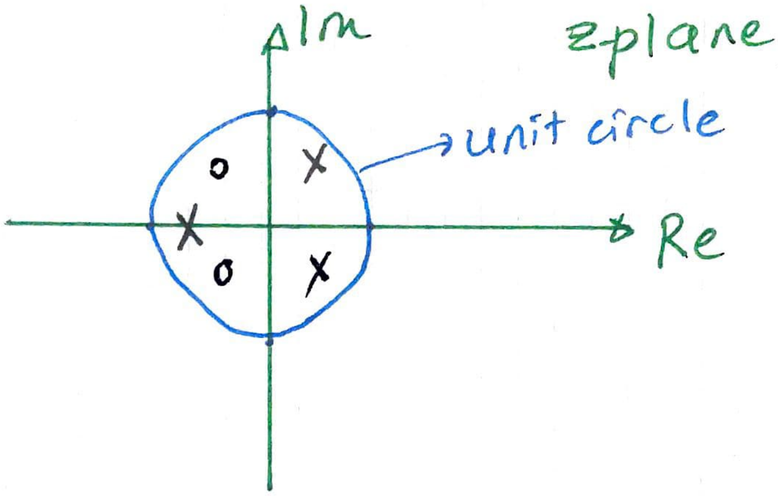




With $L=2$ and $M=8$, $X_e(e^{J\omega})$ and $Y_e(e^{J\omega})$ remain as in part (i), except that $Y_e(e^{J\omega})$ now has a peak value of 8. After expanding we have the above frequency response.

We see that $Y(e^{J\omega})=1$ for all ω. Inverse transforming gives $y(n)=\delta(n)$ in this case.

Q3: If the system function $H(z)$ of an LTI system has a pole-zero diagram as shown in the figure, and the system is causal, can the inverse system $H_i(z)$, where $H(z)H_i(z)=1$, be both causal and stable? Clearly justify your answer.



S3

Pole zero diagram shows that the system has 2 zeros and 3 poles in the unit circle. Since the number of poles must equal to the number of zeros, there must be an additional zero at $z=\infty$.

$H(z)$ is causal, so the ROC lies outside the largest pole and includes the unit circle. Therefore the system is stable.

The inverse system switches the poles and zeros. The inverse system could have a ROC that includes $|z|=1$ making it stable. However, the zero at $z=\infty$ of $H(z)$ is a pole for $H_i(z)$, so the system ~~$H_i(z)$~~ cannot be causal.

Q4: A discrete time causal LTI system has the system function

$$H(z) = \frac{(1+0{,}2z^{-1})(1-9z^{-2})}{(1+0{,}81z^{-2})}$$

i: Is the system is stable?

ii: Determine expressions for a minimum-phase system $H_1(z)$ and an all-pass system $H_{ap}(z)$ such that

$$H(z) = H_1(z)\,H_{ap}(z)$$

S4 i) Yes. The poles of $H(z)$ is

$p_1 = +0{,}9i$
$p_2 = -0{,}9i$
$p_3 = 0$

All of the poles are inside the unit circle. Therefore the system is stable.

ii)

$$H(z) = \underbrace{\frac{(1+0{,}2z^{-1})}{(1+0{,}81z^{-2})}}_{\text{minimum-phase system}} \underbrace{(1-9z^{-2})}_{\text{poles-zeros outside the unit circle.}}$$

All pass systems have poles and zeros that occur in conjugate reciprocal pairs. If we include the factor $(1-\frac{1}{9}z^{-2})$ in both parts of the equation above, the first part will remain minimum-phase system and the second part will become allpass system.

$$H(z) = \frac{(1+0{,}2z^{-1})(1-\frac{1}{9}z^{-2})}{(1+0{,}81z^{-2})} \frac{(1-9z^{-2})}{(1-\frac{1}{9}z^{-2})}$$

$H_1(z)$: minimum-phase system

$Hap(z)$: all-pass system

$$H_1(z) = \frac{(1+0{,}2z^{-1})(1-\frac{1}{9}z^{-2})}{(1+0{,}81z^{-2})}$$

$$Hap(z) = \frac{(1-9z^{-2})}{(1-\frac{1}{9}z^{-2})}$$

Q5: Determine the group delay with given each system.

i. $h_1[n] = \delta(n) + \delta(n-4)$

ii. $h_2[n] = -\delta(n+1) + \delta(n) + 2\delta(n-1) + 2\delta(n-2) + \delta(n-3) - \delta(n-4)$

iii. $h_3[n] = \delta(n-1) - \delta(n-3)$

iv. $h_4[n] = u(n) - u(n-8)$

S5) Due to the symmetry of the impulse responses, all the systems have generalized linear phase of

$$\arg[H(e^{jw})] = \beta - n_0 w$$

where n_0 is the point of symmetry of the impulse responses. The group delay is

$$grd[H_i(e^{jw})] = \frac{-d}{dw}\left[\arg[H_i(e^{jw})]\right] = n_0$$

To find each system's group delay we need only find the point of symmetry n_0 in each system's impulse response.

i) $grd[H_1(e^{jw})] = 2$

ii) $grd[H_2(e^{jw})] = 1{,}5$

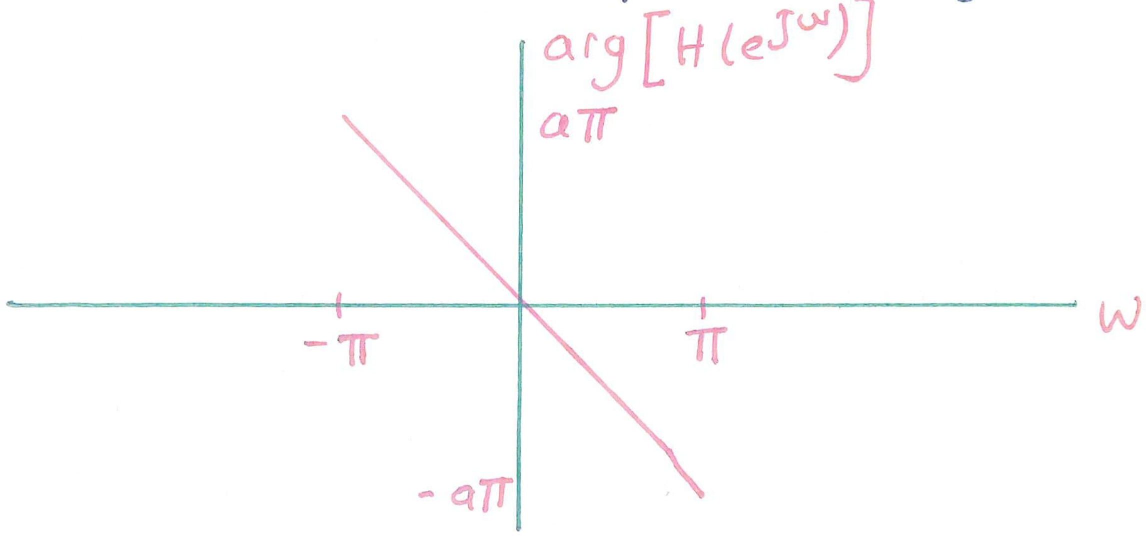
iii) $grd[H_3(e^{jw})] = 2$

iv) $grd[H_4(e^{jw})] = 3{,}5$

Q1 The following figure shows the continuous phase $\arg[H(e^{jw})]$ for the frequency response of a specific LTI system, where

$$\arg[H(e^{jw})] = -aw$$

for $|w| < \pi$ and a is a positive integer



Is the impulse response $h[n]$ of the system a causal sequence. If the system is definitely causal, or if it is not, give a proof. If the causality of the system cannot be determined from the figure give examples of a noncausal sequence and a causal sequence that both have the foregoing phase response $\arg[H(e^{jw})]$

S1. The causality of the system cannot be determined from the figure. A causal system $h_1[n]$ having a linear phase response $\angle H(e^{jw}) = -aw$ is

$$h_1(n) = \delta(n) + 2\delta(n-1) + \delta(n-2)$$

$$H_1(e^{jw}) = 1 + 2e^{-jw} + e^{-j2w}$$

$$= e^{-jw}(e^{jw} + 2 + e^{-jw})$$

$$= e^{-jw}(2 + 2\cos w)$$

$$|H_1(e^{jw})| = 2 + 2\cos w$$

$$\measuredangle H_1(e^{jw}) = -w$$

An example of a noncausal system with the same phase response is

$$h_2(n) = \delta(n+1) + \delta(n) + 4\delta(n-1) + \delta(n-2) + \delta(n-3)$$

$$H_2(e^{jw}) = e^{-jw}(4 + 2\cos w + 2\cos 2w)$$

$$|H_2(e^{jw})| = 4 + 2\cos w + 2\cos 2w$$

$$\measuredangle H_2(e^{jw}) = -w$$

Thus, both the causal sequence $h_1(n)$ and the non-causal sequence $h_2(n)$ have a linear phase response $\measuredangle H(e^{jw}) = -aw$, where $a = 1$

Q2 For each of the following system functions $H_k(z)$, specify a minimum-phase system function $H_{min}(z)$ such that the frequency response magnitudes of the two systems are equal, i.e, $|H_k(e^{j\omega})| = |H_{min}(e^{j\omega})|$

a. $$H_1(z) = \frac{1-2z^{-1}}{1+\frac{1}{3}z^{-1}}$$

b. $$H_2(z) = \frac{(1+3z^{-1})(1-\frac{1}{2}z^{-1})}{z^{-1}(1+\frac{1}{3}z^{-1})}$$

c. $$H_3(z) = \frac{(1-3z^{-1})(1-\frac{1}{4}z^{-1})}{(1-\frac{3}{4}z^{-1})(1-\frac{4}{3}z^{-1})}$$

S2. A minimum phase system with an equivalent magnitude spectrum can be found by analyzing the system function, and reflecting all poles are zeros that are outside the unit circle to their conjugate reciprocal locations. This will move them inside the unit circle. Then, all poles and zeros for $H_{min}(z)$ will be inside the unit circle. Note that a scalefactor may be introduced when the pole or zero is reflected inside the unit circle.

a. Simply reflect the zero at $z=2$ to conjugate reciprocal location at $z=\frac{1}{2}$. Then, determine the scale factor

$$H_{min}(z)=2\frac{(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{3}z^{-1})}$$

b. First, simply reflect the zero at $z=-3$ to its conjugate reciprocal location at $z=\frac{-1}{3}$. Then, determine the scale factor. This results in

$$H_{min}(z)=3\frac{(1+\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}{z^{-1}(1+\frac{1}{3}z^{-1})}$$

The $(1+\frac{1}{3}z^{-1})$ terms cancel, leaving

$$H_{min}(z)=3\frac{(1-\frac{1}{2}z^{-1})}{z^{-1}}$$

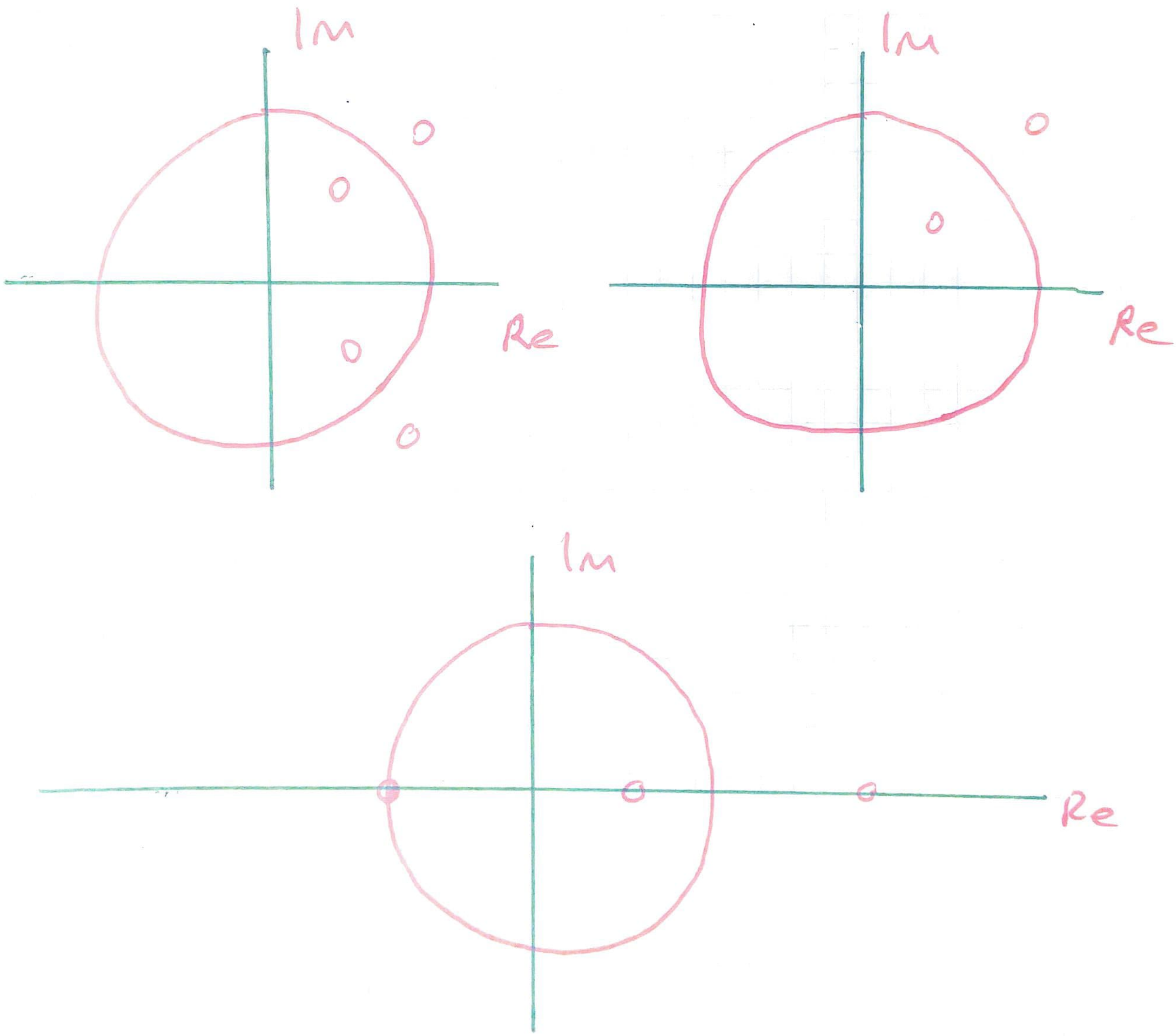
Note that the term $\frac{1}{z^{-1}}$ does not affect the frequency response magnitude of the system. Consequently, it can be removed. Thus the remaining term has a zero inside the unit circle. and is therefore minimum phase. As a result we are left with the system

$$H_{min}(z)=3(1-\frac{1}{2}z^{-1})$$

c. Simply reflect the zero at $z=3$ to its conjugate reciprocal location at $z=\frac{1}{3}$ and reflect the pole at $\frac{4}{3}$ to its conjugate reciprocal location at $z=\frac{3}{4}$. Then, determine the scale factor.

$$H_{min}(z) = \frac{9}{4} \frac{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}{\left(1-\frac{3}{4}z^{-1}\right)^2}$$

Q3 The following figure shows just the zero locations for several different system functions. For each plot, state whether the system function could be a generalized linear-phase system implemented by a linear constant-coefficient difference equation with real coefficients.



53. a. YES. The zeros come in a set of four: a zero, its conjugate, and the two conjugate reciprocals. The pole-zero plot could correspond to a type I FIR linear phase system.

b. No. This system function could not be a generalized linear phase system implemented by a LCCDE with real coefficients. Since the LCCDE has real coefficients, its poles and zeros must come in conjugate pairs. However, the zeros in this pole-zero plot do not have conjugate zeros.

c. YES. The system function could be a generalized linear phase system implemented by a LCCDE with real coefficients. The pole-zero plot could correspond to a Type II Linear phase FIR system.

Q4. In this problem, we are dealing with only causal systems. Clearly describe the z-plane characteristic corresponding to each of the following properties.

a. Real valued impulse response
b. Finite impulse response
c. $h(n) = h(2a-n)$ where $2a$ is an integer
d. Minimum phase
e. All pass

S4. a. Poles not being real must be in complex conjugate pairs. Zeros that aren't real must be in complex conjugate pairs.

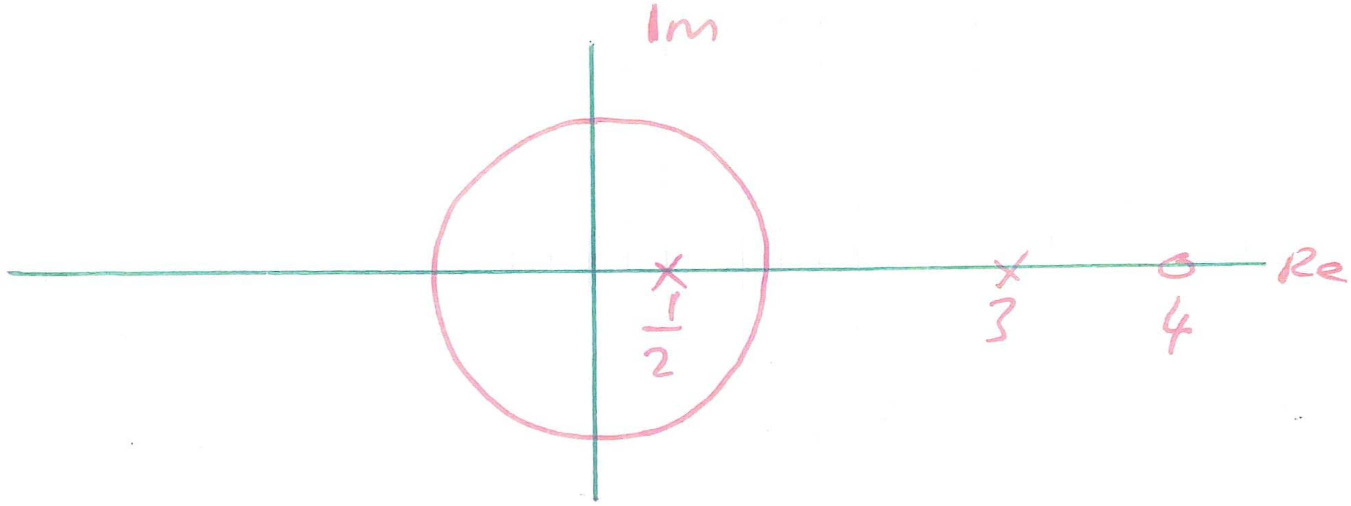
b. All poles are at the origin. The ROC is the entire z-plane, except possibly at $z=0$

c. Causality combined with the given symmetry property implies a finite length $h(n)$ that can only be a nonzero between time zero and time $2a$. Thus we must have all poles at the origin. and at most $2a$ zeros. $\mathcal{Z}[h(2a-n)] = z^{-2a} H(1/z)$. So any zero of $H(z)$ at $c \neq 0$ must be paired with a zero at $1/c$.

d. All poles and zeros are inside the unit circle (so that the inverse can be stable and causal)

e. Each pole is paired with a zero at the conjugate reciprocal location.

Q5. A stable system with system function $H(z)$ has the pole-zero diagram shown in the figure. It can be represented as the cascade of a stable minimum-phase system $H_{min}(z)$ and a stable all-pass system $H_{ap}(z)$



Determine a choice for $H_{min}(z)$ and $H_{ap}(z)$ (up to a scale factor) and draw their corresponding pole-zero plots. Indicate whether your decomposition is unique up to a scale factor.

S5. We find a minimum phase system $H_{min}(z)$ that has the same frequency response magnitude as $H(z)$ up to a scale factor. Poles and zeros that were outside the unit circle are moved to their conjugate reciprocal locations (3 to $\frac{1}{3}$, 4 to $\frac{1}{4}$, ∞ to 0)

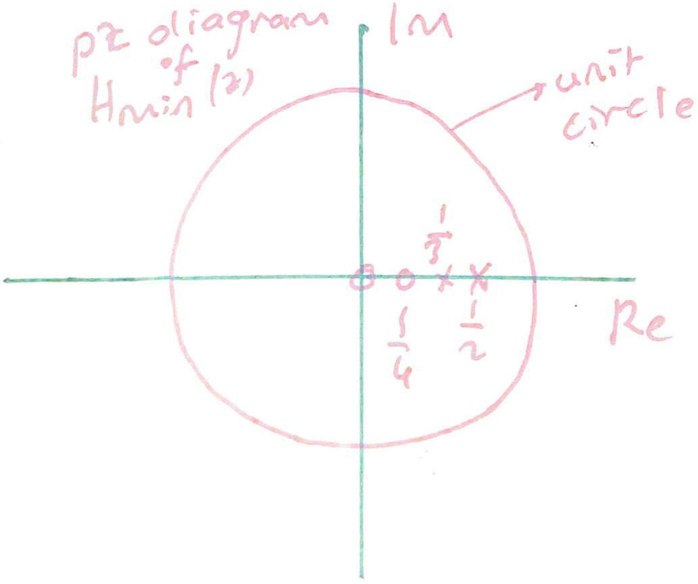
$$H_{min}(z) = K_1 \frac{(1-\frac{1}{4}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$$

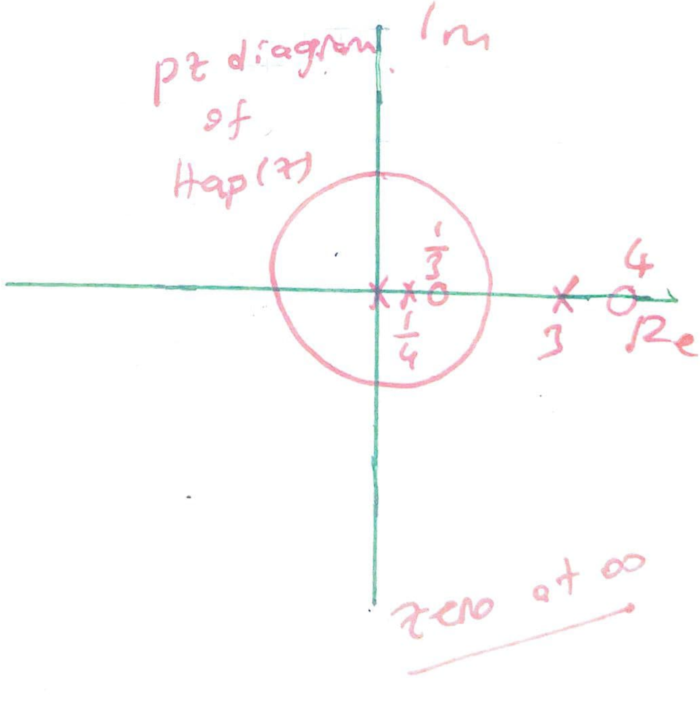
There is no need to include an explicit z term to account for the zero at the origin.

We now include allpass terms in $H_{ap}(z)$ to move poles and zeros back to their original locations in $H(z)$. The term $\frac{z^{-1}-3}{1-3z^{-1}}$ moves the pole at $\frac{1}{3}$ back to 3, the term z^{-1} moves the zero from 0 to ∞ and so on:

$$H_{ap}(z) = z^{-1}\left(\frac{z^{-1}-3}{1-3z^{-1}}\right)\left(\frac{z^{-1}-\frac{1}{4}}{1-\frac{1}{4}z^{-1}}\right)$$

The decomposition is unique up to a scale factor. We cannot include additional allpass terms in $H_{ap}(z)$, since it is not possible for $H_{min}(z)$ to cancel the resulting poles and zeros outside the unit circle



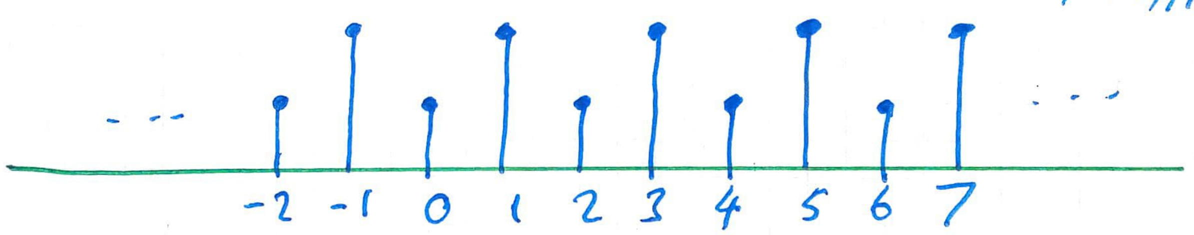


Q6: Suppose $\tilde{x}(n)$ is a periodic sequence with period N. Then $\tilde{x}(n)$ is also periodic with period $3N$. Let $\tilde{X}(k)$ denote the DFS coefficients of $x(n)$ considered as a periodic sequence with period N, and let $\tilde{X}_3(k)$ denote the DFS coefficients of $\tilde{x}(n)$ considered as a periodic sequence with period $3N$.

i. Express $\tilde{X}_3(k)$ in terms of $\tilde{X}(k)$

ii. By explicitly calculating $\tilde{X}(k)$ and $\tilde{X}_3(k)$, verify your result in part (i) when $\tilde{x}(n)$ is as given in the following figure.

$x(n), N=2$



S6.

$$\tilde{X}(k) \triangleq \sum_{n=0}^{N-1} \tilde{x}(n) W_N^{kn} \rightarrow \text{for } \tilde{x}(n) \text{ with period } N$$

$$\tilde{X}_3(k) = \sum_{n=0}^{3N-1} \tilde{x}(n) W_{3N}^{kn}$$

$$= \sum_{n=0}^{N-1} \tilde{x}(n) W_{3N}^{kn} + \sum_{n=N}^{2N-1} \tilde{x}(n) W_{3N}^{kn}$$

$$+ \sum_{n=2N}^{3N-1} \tilde{x}(n) W_{3N}^{kn}$$

Performing a change of variables in the second and third summations of $X_3(k)$,

$$\tilde{X}_3(k) = \sum_{n=0}^{N-1} \tilde{x}(n) W_{3N}^{kn} + W_{3N}^{kN} \sum_{n=0}^{N-1} \tilde{x}(n+N) W_{3N}^{kn}$$

$$+ W_{3N}^{2kN} \sum_{n=0}^{N-1} \tilde{x}(n+2N) W_{3N}^{kn}$$

Since $x(n)$ is periodic with period N, and

$$W_{3N}^{kn} = W_N^{k/3\, n}$$

$$\tilde{X}_3(k) = \left[1 + e^{-j2\pi(k/3)} + e^{-j2\pi(2k/3)}\right] \underbrace{\sum_{n=0}^{N-1} x(n) W_N^{k/3}}$$

$$\tilde{X}_3(k) = (1 + e^{-j2\pi(k/3)} + e^{-j2\pi(2k/3)}) \cdot \tilde{X}(k)$$

$$\tilde{X}_3(k) = \begin{cases} 3\tilde{X}(k/3) & , k = 3\ell \\ 0 & , \text{otherwise} \end{cases}$$

ii.

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) W_N^{kn}$$

$$= \sum_{n=0}^{1} \tilde{x}(n) e^{-j2\pi/2\, kn}$$

$$= \tilde{x}(0) + \tilde{x}(1) e^{-j\pi k}$$

$$= 1 + 2(-1)^k$$

$$\tilde{X}(k) = \begin{cases} 3 & , \quad k=0 \\ -1 & , \quad k=1 \end{cases}$$

Observe from the given figure that $x(n)$ is also periodic with period $3N=6$

$$\tilde{X}_3(k) = \sum_{n=0}^{3N-1} \tilde{x}(n) W_{3N}^{kn}$$

$$= \sum_{n=0}^{5} \tilde{x}(n) e^{-j2\pi/3 \, kn}$$

$$= \tilde{x}(0) + \tilde{x}(1) e^{-j2\pi/3 k} + \tilde{x}(2) e^{-j4\pi/3 k} \ldots$$

$$+ \tilde{x}(3) e^{-j2\pi k} + \tilde{x}(4) e^{-j8\pi/3 k}$$

$$+ \tilde{x}(5) e^{-j10\pi/3 k}$$

$$\tilde{X}_3(k) = \begin{cases} 9 & , \quad k=0 \\ -3 & , \quad k=3 \\ 0 & , \quad k=1,2,4,5 \end{cases}$$

Q7. Consider the sequence $x(n)$ given by

$$x(n) = a^n u(n) \quad , \quad |a| < 1$$

A periodic sequence $\tilde{x}(n)$ is constructed from $x(n)$ in the following way:

$$\tilde{x}(n) = \sum_{r=-\infty}^{\infty} x(n + rN)$$

i. Determine the Fourier Transform $X(e^{j\omega})$ of $x(n)$

ii. Determine the DFS coefficients $\tilde{X}(k)$ for $\tilde{x}(n)$

iii. How is $\tilde{X}(k)$ related to $X(e^{j\omega})$?

S7. i.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \frac{1}{1 - a e^{-j\omega}} \quad , \quad |a| < 1$$

ii.

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \sum_{r=-\infty}^{\infty} x(n + rN) W_N^{kn}$$

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \sum_{r=-\infty}^{\infty} a^{n+rN} u(n+rN) W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \sum_{r=0}^{\infty} a^{n+rN} W_N^{kn}$$

$$= \sum_{r=0}^{\infty} a^{rN} \sum_{n=0}^{N-1} a^n W_N^{kn}$$

$$= \sum_{r=0}^{\infty} a^{rN} \left(\frac{1-a^N e^{-j2\pi k}}{1-ae^{-j2\pi k/N}} \right) , |a|<1$$

$$= \frac{1}{1-a^N} \left(\frac{1-a^N e^{-j2\pi k}}{1-ae^{-j2\pi k/N}} \right) , |a|<1$$

$$\tilde{X}(k) = \frac{1}{1-ae^{-j(2\pi k/N)}} , |a|<1$$

iii. $\tilde{X}(k) = X(e^{j\omega}) \Big|_{\omega = 2\pi k/N}$

Q1. Compute the DFT of each of the following sequences considered to be of length N (where N is even)

a. $x(n) = \delta(n)$

b. $x(n) = \delta(n-n_0)$, $0 \leq n_0 \leq N-1$

c. $x(n) = \begin{cases} 1, & n \text{ even} \quad 0 \leq n \leq N-1 \\ 0, & n \text{ odd} \quad 0 \leq n \leq N-1 \end{cases}$

d. $x(n) = \begin{cases} 1, & 0 \leq n \leq N/2 - 1 \\ 0, & N/2 \leq n \leq N-1 \end{cases}$

e. $x(n) = \begin{cases} a^n, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$

S1. a.

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

$$= \sum_{n=0}^{N-1} \delta(n) W_N^{nk} = 1$$

b.

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

$$= \sum_{n=0}^{N-1} \delta(n-n_0) W_N^{nk}$$

$$= W_N^{n_0 k}$$

c. $$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

$$= \sum_{n=0}^{(N/2)-1} W_N^{2kn}$$

$$= \frac{1 - e^{-j2\pi k}}{1 - e^{-j(2\pi k/N)}}$$

$$X(k) = \begin{cases} N/2 & , k = 0, N/2 \\ 0 & , \text{otherwise} \end{cases}$$

d. $$X(k) = \sum_{n=0}^{(N/2)-1} W_N^{nk}$$

$$= \frac{1 - e^{-j\pi k}}{1 - e^{-j(2\pi k/N)}}$$

$$X(k) = \begin{cases} N/2 & , k=0 \\ \dfrac{2}{1-e^{-j(2\pi k/N)}} & , k \text{ odd} \\ 0 & , k \text{ even} \end{cases} , \quad 0 \le k \le N-1$$

e. $$X(k) = \sum_{n=0}^{N-1} a^n W_N^{nk}$$

$$= \frac{1 - a^N e^{-j2\pi k}}{1 - a e^{-j(2\pi k/N)}}$$

$$= \frac{1 - a^N}{1 - a e^{-j(2\pi k/N)}}$$

Q2. Consider $x(n)$ is given

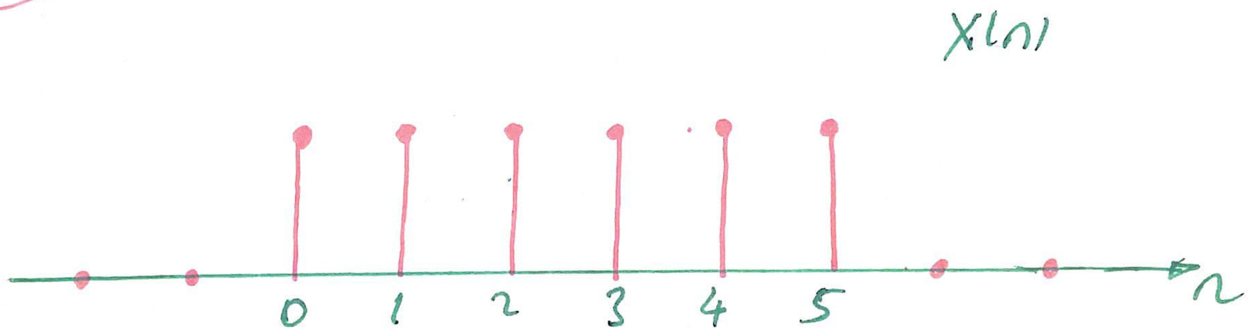
$$x(n) = u(n) - u(n-6)$$

Let $X(z)$ be the z-transform of $x(n)$. If we sample $X(z)$ at $z = e^{j(2\pi/4)k}$, $k = 0, 1, 2, 3$, we obtain

$$X_1(k) = X(z)\Big|_{z = e^{j(2\pi/4)k}}, \quad k = 0, 1, 2, 3$$

Sketch the sequence $x_1(n)$ obtained as the inverse DFT of $X_1(k)$.

52



$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(k) = X(z)\Big|_{z = e^{j(2\pi k/4)}}$$

$$X(k) = \sum_{n=0}^{5} x(n) W_4^{nk}, \quad 0 \le k \le 3$$

Note that we have taken a 4-point DFT. However original sequence $x(n)$ is of length 6

ALIASING

As a result, we can expect some aliasing when we return to the time domain via the Inverse DFT.

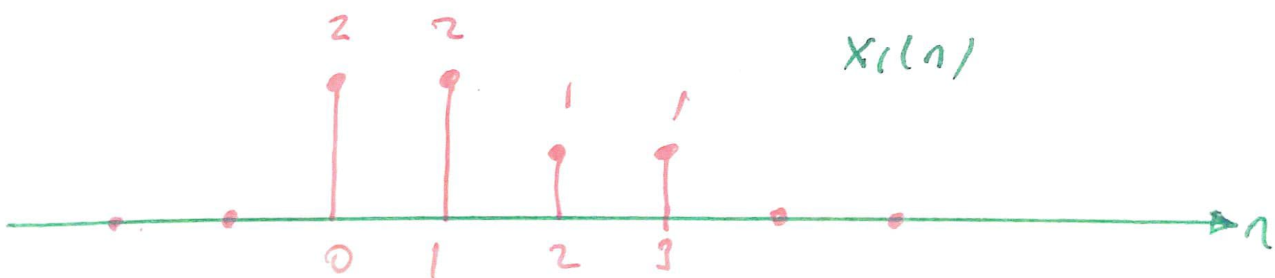
$$X(k) = W_4^{0k} + W_4^{1k} + W_4^{2k} + W_4^{3k} + W_4^{4k} + W_4^{5k}$$
$$(0 \le k \le 3)$$

$$\boxed{\begin{aligned} W_4^{4k} &= W_4^{0k} \\ W_4^{5k} &= W_4^{k} \end{aligned}}$$

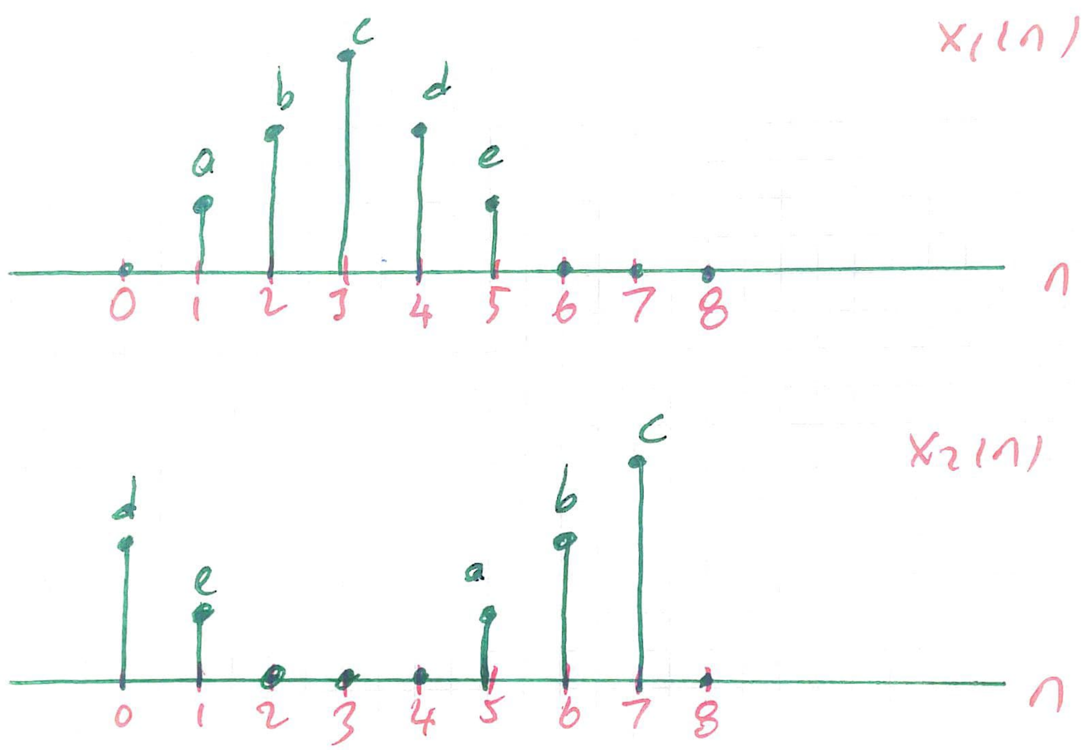
$$X_1(k) = 2W_4^{0k} + 2W_4^{1k} + W_4^{2k} + W_4^{3k}$$

The resulting time-domain signal is

$$x_1(n) = 2\delta(n) + 2\delta(n-1) + \delta(n-2) + \delta(n-3)$$



Q3. The two eight-point sequences $x_1(n)$ and $x_2(n)$ shown in the figure below have DFT's $X_1(k)$ and $X_2(k)$ respectively. Determine the relationship between $X_1(k)$ and $X_2(k)$



S3) The two sequences are related through a circular shift. Specifically,

$$x_2(n) = x_1[(n-4)_8]$$

From circular shift property of DFT,

$$X_2(k) = W_8^{4k} X_1(k)$$

$$X_2(k) = W_2^{k} X_1(k)$$

$$X_2(k) = (-1)^{k} X_1(k)$$

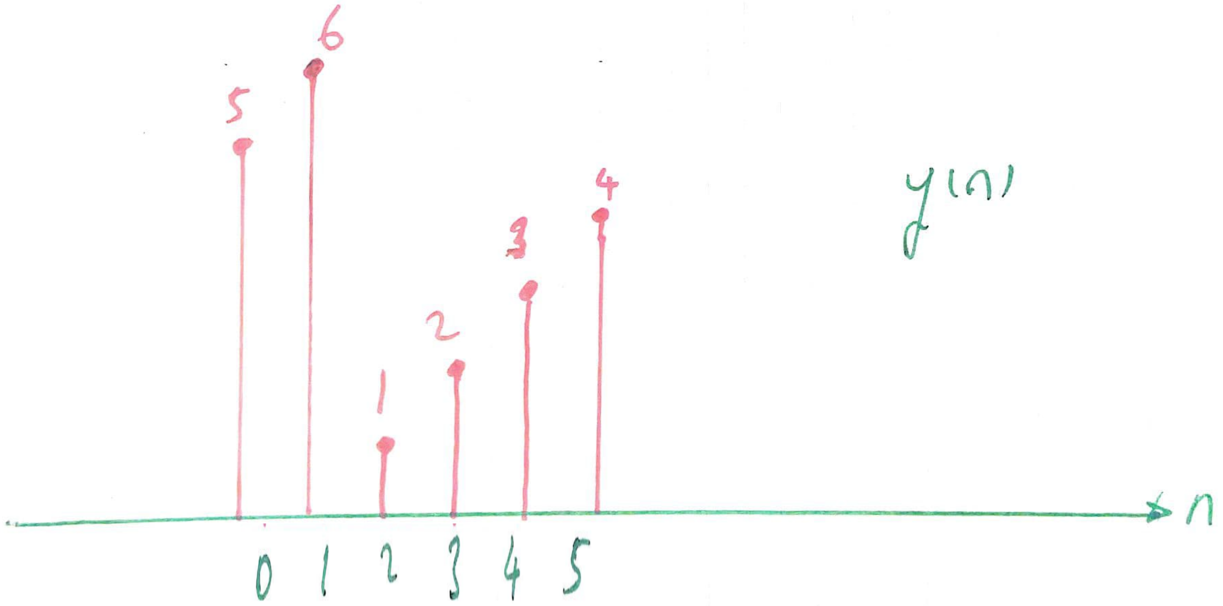
Q4. Consider two finite-length sequences $x_1(n)$ and $x_2(n)$: Find and sketch their six point circular convolution

$$x_1(n) = (n+1)[u(n) - u(n-6)]$$

$$x_2(n) = \delta(n-2)$$

S4. Since $x_2(n)$ is just a shifted impulse, the circular convolution coincides with a circular shift of $x_1(n)$ by two points.

$$y(n) = x_1[(n-2)_6]$$

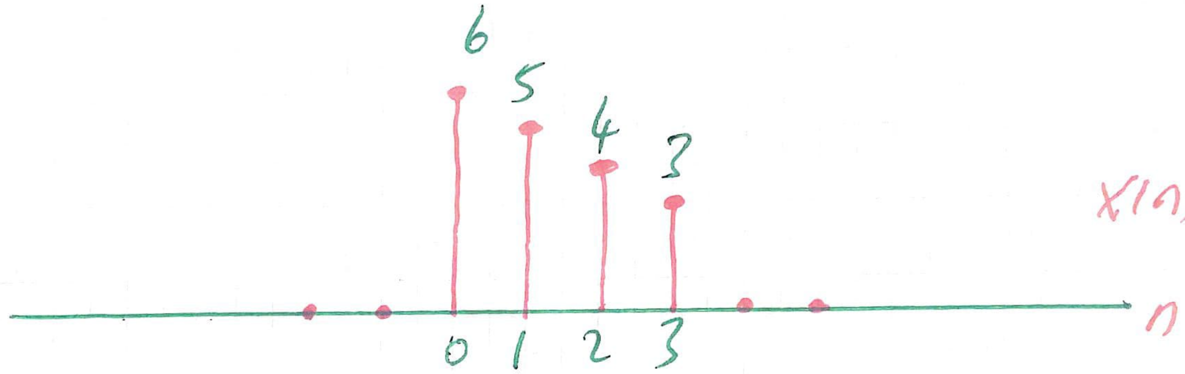


Q5. Following figure shows a finite-length sequence $x(n)$. Sketch the sequences.

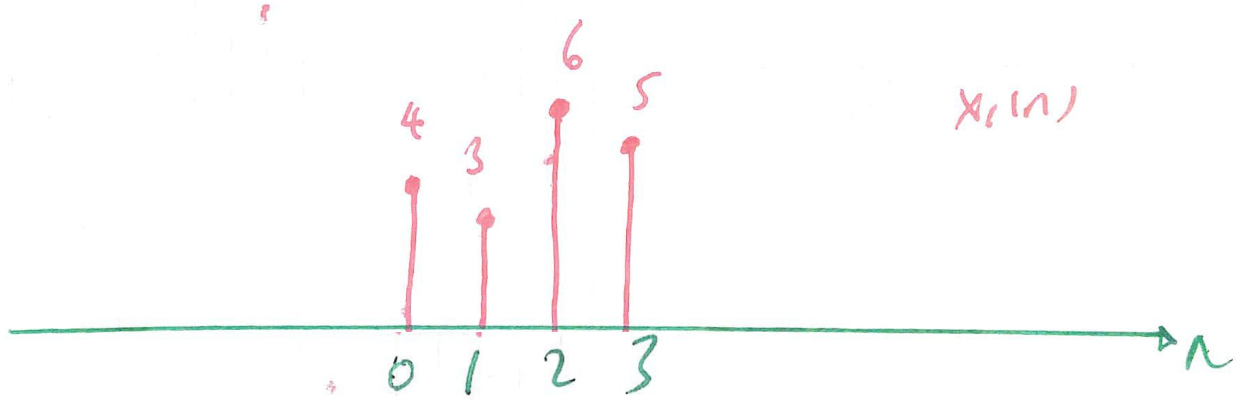
$$x_1(n) = x[(n-2)_4] \quad , 0 \le n \le 3$$

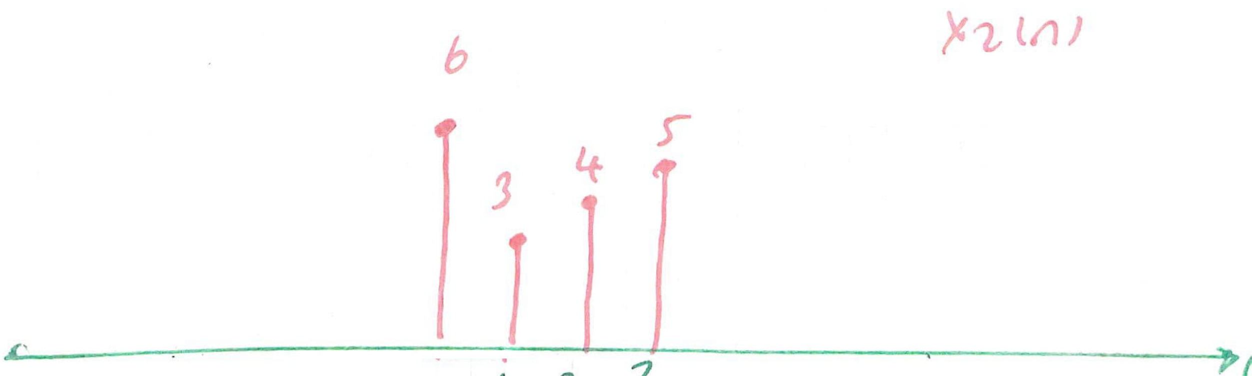
and

$$x_2(n) = x[(-n)_4] \quad , 0 \le n \le 3.$$



S5





Q6. Consider the six-point sequence

$$x(n) = (6-n)[u(n) - u(n-6)]$$

a. Determine $X(k)$, the six-point DFT of $x(n)$. Express your answer in terms of W_6

b. Plot the sequence $w(n)$, $n=0,\dots,5$, that is obtained by computing the inverse six-point DFT of $W(k) = W_6^{-2k} X(k)$

c. Use any convenient method to evaluate the six-point circular convolution of $x(n)$ with the sequence $h(n) = \delta(n) + \delta(n-1) + \delta(n-2)$

d. If we convolve the given $x(n)$ with the given $h(n)$ by N-point circular convolution, how should N be chosen so that the result of the circular convolution is identical to the result of linear convolution?

S6 a.
$$x(n) = 6\delta(n) + 5\delta(n-1) + 4\delta(n-2) + 3\delta(n-3) + 2\delta(n-4) + 1\delta(n-5)$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_6^{nk}$$

$$= 6W_6^0 + 5W_6^k + 4W_6^{2k} + 3W_6^{3k} + 2W_6^{4k} + W_6^{5k}$$

(b) $W(k) = W_6^{-2k} X(k)$

$$= 6W_6^{-2k} + 5W_6^{-k} + 4 + 3W_6^{k} + 2W_6^{2k} + W_6^{3k}$$

$$\boxed{\begin{aligned} W_6^{-2k} &= W_6^{4k} \\ W_6^{-k} &= W_6^{5k} \end{aligned}}$$

$$W(k) = 4 + 3W_6^{k} + 2W_6^{2k} + W_6^{3k} + 6W_6^{4k} + 5W_6^{5k}$$

$$w(n) = 4\delta(n) + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3) + 6\delta(n-4) + 5\delta(n-5)$$

(c) $X(k) = 6W_6^{0} + 5W_6^{k} + 4W_6^{2k} + 3W_6^{3k} + 2W_6^{4k} + W_6^{5k}$

$$H(k) = 1 + W_6^{k} + W_6^{2k}$$

$$Y(k) = X(k)H(k)$$

$$Y(k) = 9 + 12W_6^{k} + 15W_6^{2k} + 12W_6^{3k} + 9W_6^{4k} + 6W_6^{5k}$$

$$y(n) = 9\delta(n) + 12\delta(n-1) + 15\delta(n-2) + 12\delta(n-3) + 9\delta(n-4) + 6\delta(n-5)$$

(d) $N \geq 6 + 3 - 1$

$$N \geq 8$$

DSP TUTORİAL 11

13.01.2021

1 Consider a causal continuous-time system with impulse response $h_c(t)$ and system function

$$H_c(s) = \frac{s+a}{(s+a)^2+b^2}$$

Use impulse invariance to determine $H_1(z)$ for a discrete-time system such that $h_1[n] = h_c(nT)$

S1 Using the partial technique,

$$H_c(s) = \frac{s+a}{(s+a)^2+b^2} = \frac{0,5}{s+a+jb} + \frac{0,5}{s+a-jb}$$

Recall that the following Laplace Transform pair

$$e^{-at}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+a}$$

By using the pair, we obtain

$$h_c(t) = \frac{1}{2}\left[e^{-(a+jb)t} + e^{-(a-jb)t}\right]u(t)$$

$$h_1(n) = h_c(nT) = \frac{1}{2}\left[e^{-(a+jb)nT} + e^{-(a-jb)nT}\right]u(n)$$

$$H_1(z) = \frac{0,5}{1-e^{-(a+jb)T}z^{-1}} + \frac{0,5}{1-e^{-(a-jb)T}z^{-1}},$$

$$|z| > e^{-aT}$$

2 Use the bilinear transformation to design a first-order low-pass Butterworth filter that has a 3- dB cutoff frequency $\omega_c = 0.2\pi$

S2 If a digital LPF is to have a 3dB cut-off frequency at $w_c = 0{,}2\pi$, the analog Butterworth filter should have a 3-dB cutoff frequency.

$$\Omega_c = \tan\left(\frac{w_c}{2}\right) = \tan(0{,}1\pi) = 0{,}3249$$

For a first order Butterworth filter

$$H_a(s)\, H_a(-s) = \frac{1}{1+(s/j\Omega_c)^2}$$

$$= \frac{\Omega_c^2}{\Omega_c^2 - s^2} = \frac{\Omega_c}{\Omega_c + s} \quad \frac{\Omega_c}{\Omega_c - s}$$

$$H_a(s) = \frac{\Omega_c}{\Omega_c + s}$$

$$s = \frac{1-z^{-1}}{1+z^{-1}}$$

$$H_z = \frac{\Omega_c}{s+\Omega_c} = \frac{0{,}3249}{\frac{1-z^{-1}}{1+z^{-1}} + 0{,}3249}$$

$$= \frac{0{,}2452(1+z^{-1})}{(1-0{,}509z^{-1})}$$

3 Use the window design method to design a linear phase FIR filter of order $N = 24$ to approximate the following ideal frequency response magnitude:

$$|H_d(e^{j\omega})| = \begin{cases} 1 & |\omega| \leq 0.2\pi \\ 0 & 0.2\pi < |\omega| \leq \pi \end{cases}$$

S3

The ideal filter to be approximated is a LPP with $\omega_p = 0{,}2\pi$. With $N=24$, the spectrum of the filter to be designed has the form

$$H(e^{j\omega}) = \sum_{n=0}^{24} h(n)\, e^{-j\omega n}$$

The delay of $h(n)$ is $\alpha = N/2 = 24/2 = 12$, and the ideal unit sample response to be windowed is

$$h_d(n) = \frac{\sin[0{,}2\pi(n-12)]}{(n-12)\pi}$$

All that is left to do in the design is to select a window. $\longrightarrow$ tradeoff between the width of the transition band and the amplitude of the passband and stopband ripple

For rectangular window, the smallest transition band

$$\Delta\omega = 2\pi \cdot \frac{0{,}9}{N} = 2\pi\,\frac{0{,}9}{24} = 0{,}075\pi$$

and the filter

$$h[n] = \begin{cases} \dfrac{\sin[0{,}2\pi(n-12)]}{(n-12)\pi} & , \quad 0 \leq n \leq 24 \\ 0 & , \quad \text{otherwise} \end{cases}$$

However, the stopband attenuation is only 21 dB, which is equivalent to a ripple of $\delta_s = 0{,}089$. With a Hamming window,

$$h[n] = \left[0{,}54 - 0{,}46 \cos\left(\frac{2\pi n}{24}\right)\right] \frac{\sin[0{,}2\pi(n-12)]}{(n-12)\pi} ,$$

$$0 \leq n \leq 24$$

and the stopband attenuation is 53 dB, or $\delta_s = 0{,}0022$. However, the width of the transition band increases to

$$\Delta w = 2\pi \cdot \frac{3{,}3}{24} = 0{,}275\pi$$

which, for most designs, would be too wide

4 Use the window design method to design a minimum-order high-pass filter with a stopband cutoff frequency $\omega_s = 0.22\pi$, a passband cutoff frequency $\omega_p = 0.28\pi$, and a stopband ripple $\delta_s = 0.003$.

S4 A stopband ripple of $\delta_s = 0{,}003$ corresponds to a stopband attenuation of $a_s = -20 \log \delta_s = 50{,}46$.

For the minimum order filter, we use a Kaiser window with

$$\beta = 0{,}1102(a_s - 8{,}7) = 4{,}6$$

Because the transition width is $\Delta\omega = 0{,}06\pi$ or, $\Delta f = 0{,}03$, the required window length is

$$N = \frac{a_s - 7{,}95}{14{,}36 \cdot \Delta f} = 98{,}67 \xrightarrow{\lceil \cdot \rceil} \underline{\underline{99}}$$ (rounding up)

Type II Linear Phase FIR filter

Type II linear phase FIR filter have a zero in its system function at $z=-1$. Because this produces a null in the frequency response at $\omega = \pi$, This is not acceptable.

Therefore, we increase the order by 1 to obtain a type I linear phase FIR filter with $N=100$.

In order to have a transition band that extends from $w_s = 0{,}22\pi$ to $w_p = 0{,}28\pi$, we set the cutoff frequency of the ideal high-pass filter equal to the midpoint

$$w_c = \frac{w_p + w_s}{2} = 0{,}25\pi$$

The unit sample response of an ideal zero-phase high pass filter with a cutoff frequency $w_c = 0{,}25\pi$ is

$$h_{hp}(n) = \delta(n) - \frac{\sin(0{,}25\pi n)}{n\pi}$$

where the second term is a low-pass filter with a cutoff frequency $w_c = 0{,}25\pi$. Delaying $h_{hp}(n)$ by $N/2 = 50$, we have

$$h_d(n) = \delta(n-50) - \frac{\sin(0{,}25\pi(n-50))}{(n-50)\pi}$$

and the resulting FIR high-pass filter is

$$h(n) = h_d(n) \cdot \underbrace{w(n)}_{\substack{\text{Kaiser Window} \\ \text{with } N=100 \text{ \& } \beta=4{,}6}}$$

5 Consider the following specifications for a low-pass filter:

$$0.99 \leq \left|H\left(e^{j\omega}\right)\right| \leq 1.01 \quad 0 \leq |\omega| \leq 0.3\pi$$
$$\left|H\left(e^{j\omega}\right)\right| \leq 0.01 \quad 0.35\pi \leq |\omega| \leq \pi$$

Design a linear phase FIR filter to meet these specifications using the window design method.

55 Designing a low-pass filter with window design method generally produces a filter with ripples of the same amplitude in the passband and stopband. Therefore, because the two ripples in the filter specs. are the same, we only need to be concerned about the stopband ripple requirement. A stopband ripple of $\delta_s = 0{,}01$ corresponds to a stopband attenuation of -40 dB. Therefore, from the windows table, it follows that we may use Hanning (Hann) window, which provides an attenuation of approximately 44dB. The specification on the transition band is that $\Delta\omega = 0{,}05\pi$ or $\Delta f = 0{,}025$. Therefore, the required filter order is

$$N = \frac{3.1}{\Delta f} = 124$$

and we have

$$w(n) = 0{,}5 - 0{,}5\cos\left(\frac{2\pi n}{124}\right), \quad 0 \le n \le 124$$

With an ideal LPF having cutoff frequency of $w_c = 0{,}325$ (the midpoint of the transition band), and a delay of $N/2 = 62$ so that $h_d(n)$ is placed symmetrically within the interval $[0, 124]$, we have

$$h_d(n) = \frac{\sin[0{,}325\pi(n-62)]}{\pi(n-62)}$$

Therefore, the filter is

$$h(n) = \left[0.5 - 0{,}5\cos\left(\frac{2\pi n}{124}\right)\right]\frac{\sin(0{,}325\pi(n-62))}{\pi(n-62)},$$

$$0 \le n \le 124$$

Note that if we were to use Hamming and Blackman window instead of Hanning window, the stopband and passband ripple requirements would have been exceeded, and the required filter order would have been larger. With a Blackman window, for example, the filter order required to meet the transition band requirement is

$$N = \frac{5{,}5}{0{,}025} = 220$$

6 Consider the following specifications for a bandpass filter:

$$\begin{array}{rll} & \left|H\left(e^{j\omega}\right)\right| \le 0.01 & 0 \le |\omega| \le 0.2\pi \\ 0.95 \le & \left|H\left(e^{j\omega}\right)\right| \le 1.05 & 0.3\pi \le |\omega| \le 0.7\pi \\ & \left|H\left(e^{j\omega}\right)\right| \le 0.02 & 0.8\pi \le |\omega| \le \pi \end{array}$$

(a) Design a linear phase FIR filter to meet these specifications using a Blackman window.

(b) Repeat part (a) using a Kaiser window.

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(a) For this filter, the width of each transition band is $\Delta\omega = 0.1\pi$. The ripples in the lower stop-band, pass band, and upper stopband are $\delta_1 = 0.01$, $\delta_2 = 0.05$, and $\delta_3 = 0.027$ respectively, and are all different. Because the ripples produced with the window design method will be approximately the same in all three bands, the filter must be designed so that it has a maximum ripple of $\delta_1 = 0.01$ in all three bands.

With

$$-20\log\delta_1 = -40$$

it follows that the Blackman window will satisfy this requirement. An estimate of the filter order necessary to meet the transition bandwidth requirement of $\Delta f = 0.05$ with a Blackman window is

$$N = \frac{5.5}{\Delta f} = 110$$

Finally, for the unit sample response of the ideal that is to be windowed, we have

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{jn\omega} \, d\omega$$

where $H_d(e^{j\omega})$ is the frequency response of an ideal Bandpass filter. For the cutoff frequencies of $H_d(e^{j\omega})$, we choose the midpoints of the transition bands of $H(e^{j\omega})$. Therefore,

$$|H_d(e^{j\omega})| = \begin{cases} 1 & , \quad 0{,}25\pi \le |\omega| \le 0{,}75\pi \\ 0 & , \quad \text{otherwise} \end{cases}$$

Thus, the unit sample response of the ideal band-pass filter with zerophase is

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \int_{0{,}25\pi}^{0{,}75\pi} e^{jn\omega} \, d\omega + \frac{1}{2\pi} \int_{-0{,}75\pi}^{-0{,}25\pi} e^{jn\omega} \, d\omega$$

$$= \frac{\sin(0{,}75\pi n)}{\pi n} - \frac{\sin(0{,}25\pi n)}{\pi n}$$

However, we want to delay this filter so that it is centered at $\frac{N}{2} = 55$. Therefore, the unit sample response of the filter to be windowed should be

$$h_d(n) = \frac{\sin(0{,}75\pi(n-55))}{(n-55)\pi} - \frac{\sin(0{,}25\pi(n-55))}{(n-55)\pi}$$

(b) For a Kaiser window design, the order of the filter required is

$$N = \frac{-20\log(0{,}01) - 7{,}95}{14{,}36\,(0{,}05)} = 44{,}64$$

Therefore, we set $N = 45$. Next, for the Kaiser window parameter, with an attenuation of 40dB, we have

$$\beta = 0{,}5842\,(40-21)^{0{,}4} + 0{,}07886\,(40-21) = 3{,}3953$$

Therefore, the filter is

$$h(n) = w(n)\,h_d(n)$$

where

$$h_d(n) = \frac{\sin[0{,}75\pi(n-22{,}5)]}{(n-22{,}5)\pi} - \frac{\sin[0{,}25\pi(n-22{,}5)]}{(n-22{,}5)\pi}$$

EHB 315E Digital Signal Processing
Fall 2020

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DSP TUTORIAL 12

20.01.2021

1 Suppose we are given a linear shift-invariant system having a system function

$$H(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

that is excited by zero mean exponentially correlated noise $x(n)$ with an autocorrelation sequence

$$r_x(k) = \left(\frac{1}{2}\right)^{|k|}$$

Let $y(n)$ be the output process, $y(n) = x(n) * h(n)$

(a) Find the power spectrum, $P_y(z)$, of $y(n)$.

(b) Find the autocorrelation sequence, $r_y(k)$, of $y(n)$.

(c) Find the cross-correlation, $r_{xy}(k)$, between $x(n)$ and $y(n)$.

(d) Find the cross-power spectral density, $P_{xy}(z)$, which is the z transform of the crosscorrelation $r_{xy}(k)$

S1 a $P_y(z) \triangleq \sum_{k=-\infty}^{\infty} r_y(k) z^{-k}$

$$P_x(z) = \sum_{n=-\infty}^{\infty} r_x(k) z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|k|} z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} z^{-k} + \sum_{k=-\infty}^{0} \left(\frac{1}{2}\right)^{-k} z^{-k} - 1$$

$$P_x(z) = \frac{1-\left(\frac{1}{2}\right)^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z\right)}$$

$$P_y(z) = H(z)\,H(z^{-1})\,P_x(z)$$

$$= \left(\frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{3}z^{-1}}\right)\left(\frac{1-\frac{1}{2}z}{1-\frac{1}{3}z}\right)\frac{\frac{3}{4}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z\right)}$$

$$= \frac{3/4}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{3}z\right)}$$

(b) The autocorrelation sequence for $y(n)$ may be easily found using the z-transform pair

$$a^{|k|} \overset{Z}{\longleftrightarrow} \frac{1-a^2}{\left(1-az^{-1}\right)\left(1-az\right)}$$

$$\left(\frac{1}{3}\right)^{|k|} \overset{Z}{\longleftrightarrow} \frac{8/9}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{3}z\right)}$$

$$P_y(z) = \frac{27/32 \cdot 8/9}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{3}z\right)}$$

$$r_y(k) = \frac{27}{32}\left(\frac{1}{3}\right)^{|k|}$$

(c)

$r_{xy}(k) = r_x(k) * h(-k)$

This may be computed using z-transforms as follows:

$$P_{xy}(z) = P_x(z)\,H(z^{-1})$$

$$= \frac{3/4}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)} \cdot \frac{(1-\frac{1}{2}z)}{(1-\frac{1}{3}z)}$$

$$= \frac{3/4}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z)}$$

If we perform partial fraction expansion to the $P_{xy}(z)$ we obtain

$$P_{xy}(z) = \frac{9/10}{(1-\frac{1}{2}z^{-1})} + \frac{3/10}{(z^{-1}-\frac{1}{3})}$$

Inverse z-transform gives

$$r_{xy}(k) = \frac{9}{10}\left(\frac{1}{2}\right)^k u(k) + \left(\frac{9}{10}\right)(3)^{-k} u(-k-1)$$

(d)

$$P_{xy}(z) = \frac{3/4}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}$$

2 Find the power spectrum for each of the following wide-sense stationary random processes that have the given autocorrelation sequences.

(a) $r_x(k) = 2\delta(k) + j\delta(k-1) - j\delta(k+1)$

(b) $r_x(k) = \delta(k) + 2(0.5)^{|k|}$

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(a) $P_x(e^{j\omega}) = \tilde{F}[r_x(k)]$

$$= 2 + je^{-j\omega} - je^{+j\omega}$$

$$= 2 + 2\left[\frac{e^{j\omega} - e^{-j\omega}}{2j}\right]$$

$$= 2 + 2\sin\omega$$

(b) With a real, using the DTFT pair

$$a^{|k|} \overset{\tilde{F}}{\longleftrightarrow} \frac{1-a^2}{|1-ae^{-j\omega}|^2}$$

we have

$$P_x(e^{j\omega}) = 1 + 2\,\frac{1-\left(\frac{1}{2}\right)^2}{|1-\frac{1}{2}e^{-j\omega}|^2}$$

$$= 1 + \frac{3/2}{\frac{5}{4} - \cos\omega} = \frac{11 - 4\cos\omega}{5 - 4\cos\omega}$$

3 Let $d(n)$ be an AR(1) process with an autocorrelation sequence

$$r_d(k) = \alpha^{|k|}$$

with $0 < \alpha < 1$, and suppose that $d(n)$ is observed in the presence of uncorrelated white noise, $v(n)$, that has a variance of σ_v^2

$$x(n) = d(n) + v(n)$$

Let us design a first-order FIR Wiener filter to reduce the noise in $x(n)$. With

$$W(z) = w(0) + w(1)z^{-1}$$

S3

the Wiener-Hopf Equations are

$$\underbrace{\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix}}_{\underline{R}} \underbrace{\begin{bmatrix} w(0) \\ w(1) \end{bmatrix}}_{\underline{w}} = \underbrace{\begin{bmatrix} r_{dx}(0) \\ r_{dx}(1) \end{bmatrix}}_{\underline{P}}$$

Since $d(n)$ and $v(n)$ are assumed to be uncorrelated,

then

$$r_{dx}(k) = r_d(k) = a^{|k|}$$

and

$$r_x(k) = r_d(k) + r_v(k) = a^{|k|} + \sigma_v^2 \delta(k)$$

Therefore, the Wiener-Hopf equations become

$$\begin{bmatrix} 1+\sigma_v^2 & a \\ a & 1+\sigma_v^2 \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix}$$

Solving for $w(0)$ and $w(1)$, we have

$$\begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \frac{1}{(1+\sigma_v^2)^2 - a^2} \begin{bmatrix} 1+\sigma_v^2 - a^2 \\ a\sigma_v^2 \end{bmatrix}$$

Therefore, the Wiener filter is

$$W(z) = \frac{1}{(1+\sigma_v^2)^2 - a^2} \left[(1+\sigma_v^2 - a^2) + a\sigma_v^2 z^{-1} \right]$$

As a specific example, let $a = 0,8$ and $\sigma_v^2 = 1$.
In this case, the Wiener filter becomes

$$W(e^{jw}) = 0,4048 + 0,2381 e^{-jw} \longrightarrow \underline{\underline{LPF}}$$

$$P_d(e^{jw}) = \frac{0,36}{1.64 - 1,6\cos w}$$

Since $P_d(e^{jw})$ decreases with increasing w and since the power spectrum of the noise is constant for all w, then the signal-to-noise ratio decreases with increasing w. For the mean-square error, we have

$$\xi_{min} = E\left[|e(n)|^2\right] = r_d(0) - w(0) r_{dx}^*(0) - w(1) r_{dx}^*(1)$$

$$= 0,4048$$

4 In this example we find the optimum linear predictor for an AR(1) process $x(n)$ that has an autocorrelation sequence given by

$$r_x(k) = \alpha^{|k|}$$

With a first-order predictor of the form

$$\hat{x}(n+1) = w(0)x(n) + w(1)x(n-1)$$

S4

the Wiener-Hopf Eq'ns. are

$$\begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \begin{bmatrix} a \\ a^2 \end{bmatrix}$$

Solving for the predictor coefficients, we find

$$\begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \frac{1}{1-a^2} \begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix} \begin{bmatrix} a \\ a^2 \end{bmatrix}$$

$$= \begin{bmatrix} a \\ 0 \end{bmatrix}$$

Therefore, the predictor for $\hat{x}(n+1) = a\,x(n)$ and the value of $x(n-1)$ is never used in the prediction of $x(n+1)$

The mean-square linear prediction error is

$$\xi_{min} = r_x(0) - w(0)r_x(1) - w(1)r_x(2)$$

$$= 1 - a \cdot a - 0 = 1 - a^2$$

5 Let us reconsider the linear prediction problem in the previous question when the measurement of $x(n)$ is noisy. Suppose that

$$y(n) = x(n) + v(n)$$

where $v(n)$ is zero mean white noise with a variance of σ_v^2.

S5 Assuming that $v(n)$ is uncorrelated with $x(n)$, the Wiener-Hopf Equations are

$$\left[\underline{R}_x + \sigma_v^2 \underline{I}\right] \underline{w} = \underline{p} \qquad \cdots \underline{p} = \underline{r}_{dy}$$

$$r_{dy}(k) = r_x(k+1)$$

If $x(n)$ is an AR(1) process with an autocorrelation sequence

$$r_x(k) = a^{|k|}$$

for a first-order predictor

$$W(z) = w(0) + w(1) z^{-1}$$

the Wiener-Hopf Equations are

$$\begin{bmatrix} 1+\sigma_v^2 & a \\ a & 1+\sigma_v^2 \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \begin{bmatrix} a \\ a^2 \end{bmatrix}$$

$$\begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \frac{a}{(1+\sigma_v^2)^2 - a^2} \begin{bmatrix} 1+\sigma_v^2 - a^2 \\ a\sigma_v^2 \end{bmatrix}$$

6 Let $x(n)$ be a second-order autoregressive process that is generated according to the difference equation

$$x(n) = 1.2728x(n-1) - 0.81x(n-2) + v(n)$$

where $v(n)$ is unit variance white noise. Let us find and examine adaptive linear predictor by using the LMS filter.

56 Normally, in order to design this predictor, it is necessary to know the autocorrelation sequence of $x(n)$. Therefore, suppose we consider an adaptive linear predictor of the form:

$$\hat{x}(n) = w_n(1)\, x(n-1) + w_n(2)\, x(n-2)$$

With the LMS algorithm, the predictor coefficients $w_n(k)$ are updated as follows:

$$w_{n+1}(k) = w_n(k) + \mu e(n) x^*(n-k)$$

If the stepsize μ is sufficiently small, then the coefficients $w_n(1)$ and $w_n(2)$ will converge in the mean to their optimum values, $w(1) = 1,2728$ and $w(2) = -0,81$ (from Wiener-Hopf Equations), respectively.

$$e(n) = x(n) - \hat{x}(n)$$
$$= [1,2728 - w_n(1)]\, x(n-1) + [-0,81 - w_n(2)]\, x(n-2) + v(n)$$

7 Let $x(n)$ be the AR(1) process

$$x(n) = 0.8x(n-1) + w(n)$$

where $w(n)$ is white noise with a variance $\sigma_w^2 = 0.36$, and let

$$y(n) = x(n) + v(n)$$

be noisy measurements of $x(n)$ where $v(n)$ is unit variance white noise that is uncorrelated with $w(n)$. Let us estimate an AR(1) process by using Kalman Filter

S7

$\left.\begin{array}{l} \underline{A}(n) = 0.8 \\ \underline{c}(n) = 1 \end{array}\right\}$ Kalman Filter state estimation equation is

$$\hat{x}(n) = 0.8\hat{x}(n-1) + K(n)\left[y(n) - 0.8\hat{x}(n-1)\right]$$

Since the state vector is a scalar, the equations for computing the Kalman gain are scalar equations

$$P(n|n-1) = (0.8)^2 P(n-1|n-1) + 0.36$$

$$K(n) = P(n|n-1)\left[P(n|n-1) + 1\right]^{-1}$$

$$P(n|n) = \left[1 - K(n)\right] P(n|n-1)$$

With $\hat{x}(0) = E[x(0)] = 0$

$p(0|0) = E\left[|x(0)|^2\right] = 1$, the kalman gain and the error covariances for the first few values of n are shown in the table. Note that after a few iterations, the kalman filter settles down into its steady state solution

$$\hat{d}(n) = 0{,}8\,\hat{d}(n-1) + 0{,}375\left[x(n) - 0{,}8\,\hat{d}(n-1)\right]$$

with a final mean-square error of $\xi = 0{,}375$, which is identical to the causal wiener filter.

n	$P(n\|n-1)$	$K(n)$	$P(n\|n)$
1	1.0000	0,5000	0,5000
2	0.6800	0,4048	0,4048
3	0,6190	0,3824	0,3824
⋮	⋮	⋮	⋮
∞	0,6000	0,3750	0,3750

Table 1. The Kalman Gain and Error Covariances