PROBABILITY AND STATISTICS (homework)

1. Suppose $S = \{1, 2, 3\}$, and we try to define P by $P(\{1, 2, 3\}) = 1$, $P(\{1, 2\}) = 0.7$, $P(\{1, 3\}) = 0.5$, $P(\{2, 3\}) = 0.7$, $P(\{1\}) = 0.2$, $P(\{2\}) = 0.5$, $P(\{3\}) = 0.3$. Is P a valid probability measure? Why or why not?

2. Suppose that an employee arrives late 10% of the time, leaves early 20% of the time, and both arrives late and leaves early 5% of the time. What is the probability that on a given day that employee will either arrive late or leave early (or both)?

$$P(A) = \frac{10}{100} = 0.1$$

$$P(B) = \frac{20}{100} = 0.2$$

$$P(A \cap B) = \frac{5}{100} = 0.05$$

the probability that the employee will be either leaves early or arrives late is $P(A \cup B)$

Hence $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 0.1 + 0.2 - 0.05$$

$$= 0.25$$

3. Suppose we choose a positive integer at random, according to some unknown probability distribution. Suppose we know that P ({1, 2, 3, 4, 5}) = 0.3, P ({4, 5, 6}) = 0.4 and P ({1}) = 0.1. What are the largest and smallest possible values of P ({2})?

$P(\{1,2,3,4,5\}) = 0.3$ and $P(\{1\}) = 0.1$

This implies that

$$P(\{2,3,4,5\}) = P(\{1,2,3,4,5\}) - P(\{1\})$$
$$= 0.3 - 0.1 = 0.2$$

Since $\{2\} \subset \{2,3,4,5\}$, we have

$$P(\{2\}) \leq P(\{2,3,4,5\}) = 0.2$$

This gives an upper bound for $P(\{2\})$

Now, consider the probability distribution

$P(\{1\}) = 0.1$ $\quad P(\{3\}) = P(\{4\}) = P(\{5\}) = 0$

$P(\{2\}) = 0.2$

$P(\{6\}) = 0.4$

$P(\{7\}) = 0.3$

For other integer x, $P(\{x\}) = 0$

The given probability distribution satisfies →

given properties and $P(\{2\}) = 0.2$.

Hence largest value for $P(\{2\}) = 0.2$.

For all probability distribution $P(\{2\}) \geq 0$.

and so 0 is a lower bound.

Now, consider the following probability distribution

$P(\{1\}) = 0.1$

$P(\{2\}) = 0$

$P(\{3\}) = 0.2$

$P(\{4\}) = P(\{5\}) = 0$

$P(\{6\}) = 0.4$

$P(\{7\}) = 0.3$

for other integers $P(\{x\}) = 0$.

therefore, 0 is the smallest value can $P(\{2\})$ be.

4. In how many ways can 5 people be seated on a sofa if there are only 3 seats available?

the number of ways 5 people can be seated 3 seats is

$5 \cdot 4 \cdot 3 = 60$.

5. How many numbers consisting of five different digits each can be made from the digits 1, 2, . . ., 9

if (a) the numbers must be odd, (b) the first two digits of each number are even?

(a) the numbers can be 1, 3, 5, 7, 9

therefore for first digits, there are 5 choices

2nd " " 4 "

3rd " " 3 "

4st " " 2

5th " " 1 "

In total, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ many numbers

(b)

Even digits are 2, 4, 6, 8

odd " " 1, 3, 5, 7, 9

there are 4 choices for the first digit and 3 remaining choice for the second. $4 \times 3 = 12$.

Now, the last 3 digits can be remaing 7 digits

Therefore, there are $4 \cdot 3 \cdot 7 \cdot 6 \cdot 5 = 2520$ many numbers

6. From 5 statisticians and 6 economists, a committee consisting of 3 statisticians and 2 economists is to be formed. How many different committees can be formed if:
(a) no restrictions are imposed,
(b) 2 particular statisticians must be on the committee,

(c) 1 particular economist cannot be on the committee?

(a) If there is no restriction, can choose $\binom{5}{3}$ meny statician and $\binom{6}{2}$ meny economist.
therefore, the committee can be formed in $\binom{6}{2}\binom{5}{3} = 15 \cdot 10 = 150$ meny ways

(b) we already chosen 2 staticians and so we will choose 1 statistician among the remaining 3 staticians.
there are $\binom{3}{1}$ meny statician that we choose and there are $\binom{6}{2}$ meny economist that we can choose.
In total, the committee can be formed in $\binom{3}{1}\binom{6}{2} = 3 \cdot 15 = 45$ meny different ways.

(c)

1 particular economist cannot be on the committee and so we choose 2 economists among 5 economists.

There are $\binom{5}{2}$ many choice for economist

There are $\binom{5}{3}$ many choice for staticians.

Hence, there are $\binom{5}{2}\binom{5}{3} = 10 \cdot 10 = 100$ many different to form the committee.

7. Suppose we are dealt five cards from an ordinary 52-card deck. What is the probability that (a) we get all four aces, plus the king of spades?
(b) all five cards are spades?
(c) we get no pairs (i.e., all five cards are different values)?

(d) we get a full house (i.e., three cards of a kind, plus a different pair)?

(a) In total we can choose $\binom{52}{5}$ many cards.

There are 4 ace in the deck then

$\binom{4}{4}$ many choice for ace.

There is only 1 king of spades

Hence, the probability that we get all four aces, plus the king of spades is

$$\frac{\binom{4}{4}\binom{1}{1}}{\binom{52}{5}} = \frac{1}{2598960}$$

(b) There are 13 spades in the standard deck. The probability that all five cards are spades is

$$\frac{\binom{13}{5}}{\binom{52}{5}} = \frac{33}{66640}$$

(c) we must select 5 ranks among the 13 ranks. Then for each selected rank, there are 4 suits. (clubs, diamonds, hearts, spades).

Therefore if there is no pair, there are

$$\binom{13}{5}4^5$$

many selection. The probability that ~~we have~~ there is no pair is

$$\frac{\binom{13}{5}4^5}{\binom{52}{5}} = \frac{2112}{4165}$$

(d)

There are 13 ways to select the rank for a triple and $\binom{4}{3}$ ways to select three of four cards of that rank.

the number of three cards of a kind is

$$\binom{13}{1}\binom{4}{3}$$

there remain 12 rank and $\binom{4}{2}$ ways to select two cards of that rank.

$$\binom{12}{1}\binom{4}{2}$$

Therefore, the number of full house is

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$$

Hence, the probability of getting a full house is

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} = \frac{6}{4165}$$

8. The probability that a man will hit a target is 2/3. If he shoots at the target until he hits it for the first time, find the probability that it will take him 5 shots to hit the target.

the probability that he cannot hit the target is $1-\frac{2}{3}=\frac{1}{3}$.

the four shots are missing.

the probability that he hit 5^{th} time first is

$$\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\cdot\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\cdot\left(\frac{2}{3}\right)$$

$$\frac{1}{3^4}\cdot\frac{2}{3}=\frac{2}{3^5}=\frac{2}{243}$$

9. Suppose we pick two cards at random from an ordinary 52-card deck. What is the probability that the sum of the values of the two cards (where we count jacks, queens, and kings as 10, and count aces as 1) is at least 4?

Let X be the sum of picked two cards.

$$P(X \geq 4) = 1 - P(X < 4)$$

and

$$P(X<4) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

Since the least value for X is 2

$$P(X=0) = P(X=1) = 0.$$

$$P(X=2) = \frac{\binom{4}{2}}{\binom{52}{2}} = \frac{6}{1326} = \frac{1}{221}$$

$P(X=3)$ we need two pick one ace and one 2. There are $\binom{4}{1}\cdot\binom{4}{1}$ many ways

Then $$P(X=3) = \frac{\binom{4}{1}\binom{4}{1}}{\binom{52}{2}} = \frac{16}{1326}$$

Hence $$P(X \geq 4) = 1 - P(X=2) - P(X=3)$$

$$= 1 - \frac{6}{1326} - \frac{16}{1326} = \frac{1304}{1326}$$

10. Suppose we roll three fair six-sided dice. What is the probability that two of them show the same value, but the third one does not?

$$\binom{3}{2} \cdot \frac{6}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{12}$$

11. Suppose we roll one fair six-sided die, and flip six coins. What is the probability that the number of heads is equal to the number showing on the die?

$$\frac{\binom{6}{1} + \binom{6}{2} + \cdots + \binom{6}{6}}{6 \cdot 2^6} = \frac{2^6 - 1}{6 \cdot 2^6} = \frac{63}{6 \cdot 2^6} = \frac{21}{2^7} = \frac{21}{128}$$

12. Suppose the probability of snow is 20%, and the probability of a traffic accident is 10%. Suppose further that the conditional probability of an accident, given that it snows, is 40%. What is the conditional probability that it snows, given that there is an accident?

$$P(A) = \frac{20}{100} = 0.2$$

$$P(B) = \frac{10}{100} = 0.1$$

$$P(B|A) = \frac{40}{100} = 0.4.$$

we need to find $P(A|B)$.

Now,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A) \cdot P(A)}{P(B)}$$

$$= \frac{\frac{4}{10} \cdot \frac{2}{10}}{\frac{1}{10}} = \frac{8}{10}$$

13. Each of three identical jewelry boxes has two drawers. In each drawer of the first box there is a gold watch. In each drawer of the second box there is a silver watch. In one drawer of the third box there is a gold watch while in the other there is a silver watch. If we select a box at random, open one of the drawers and find it to contain a silver watch, what is the probability that the other drawer has the gold watch?

Let A_1, A_2, A_3 be the boxes,

$A_1 \to$ Gold, Gold

$A_2 \to$ Gold, Silver

$A_3 \to$ Silver, Silver

Let X_i be the event that we choose box A_i.

Then we have for $i = 1, 2, 3$

$$P(X_i) = \frac{1}{3}.$$

Now, let Y be the event we select ~~p6~~ a drawer and found silver watch in it.

Then

$$P(Y \mid X_1) = 0$$

$$P(Y \mid X_2) = \frac{1}{2}$$

$$P(Y \mid X_3) = 1.$$

we need to find that the probability that the other drawer has gold watch.

In other words find the probability to choose the box A_2 if we are given that we find silver watch.

The probability is $P(X_2 \mid Y)$.

$$P(X_2 \mid Y) = \frac{P(Y \mid X_2) \cdot P(X_2)}{P(Y \mid X_1) P(X_1) + P(Y \mid X_2) P(X_2) + P(Y \mid X_3) P(X_3)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{3}}{0 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{3}}{\left(1 + \frac{1}{2}\right) \cdot \frac{1}{3}} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$$

14. An urn contains 4 black balls and 5 white balls. After a thorough mixing, a ball is drawn from the urn, its color is noted, and the ball is returned to the urn.
(a) What is the probability that 5 black balls are observed in 15 such draws?
(b) What is the probability that 15 draws are required until the first black ball is observed?

(c) What is the probability that 15 draws are required until the fifth black ball is observed?

(a) the probability that we observed a black ball is $\frac{4}{9}$ and white ball is $\frac{5}{9}$

the probability that, we observed 5 black balls among 15 drawings is

$$\binom{15}{5} \cdot \left(\frac{4}{9}\right)^5 \cdot \left(\frac{5}{9}\right)^{15-5} = \binom{15}{5} \frac{4^5 \cdot 5^{10}}{9^{15}}$$

(b) we draw 14 white balls and 15^{th} drawings is black. the probability is $\left(\frac{5}{9}\right)^{14} \cdot \left(\frac{4}{9}\right)$

(c) the probability has negative-Binomial distribution and so probability that 15 draws are required until the 5^{th} black ball is observed is $\binom{5-1+10}{10}\left(\frac{4}{9}\right)^5 \cdot \left(\frac{5}{9}\right)^{10} = \binom{14}{10} \cdot \left(\frac{4}{9}\right)^5 \left(\frac{5}{9}\right)^{10}$

15. A coin is tossed six times. If X is a random variable giving the number of heads that arise, construct

a table showing the probability distribution of X.

X can be 0, 1, 2, 3, 4, 5, 6.

$$P(X=0) = \binom{6}{0} \cdot \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^6$$

$$= \frac{1}{2^6}$$

$$P(X=1) = \binom{6}{1} \cdot \left(\frac{1}{2}\right)^1 \cdot \left(\frac{1}{2}\right)^5 = 6 \cdot \frac{1}{2^6}$$

$$P(X=2) = \binom{6}{2} \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^4 = 15 \cdot \frac{1}{2^6}$$

$$P(X=3) = \binom{6}{3} \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^3 = 20 \cdot \frac{1}{2^6}$$

$$P(X=4) = \binom{6}{4} \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^2 = \frac{15}{2^6}$$

$$P(X=5) = \binom{6}{5} \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^1 = \frac{6}{2^6}$$

$$P(X=6) = \binom{6}{6} \left(\frac{1}{2}\right)^6 = \frac{1}{2^6}$$

16. An urn holds 5 white and 7 black marbles. If 4 marbles are to be drawn at random without

replacement and X denotes the number of white marbles, find the probability mass function of X.

the possible outcomes are 0, 1, 2, 3, 4.

$P(X=0)$

There are 12 marbles and we draw 4 marbles. We can draw 4 marbles among 12 marbles without replacement in $\binom{12}{4}$ many ways. Since $X=0$.

all drawn ball must be black and the number of ways is $\binom{7}{4}$.

Therefore $P(X=0) = \dfrac{\binom{7}{4}}{\binom{12}{4}}$

Similarly

$$P(X=1) = \frac{\binom{5}{1}\binom{7}{3}}{\binom{12}{4}}$$ 1 white, 3 black

$$P(X=2) = \frac{\binom{5}{2}\binom{7}{2}}{\binom{12}{4}}$$ 2 white, 2 black

$$P(X=3) = \frac{\binom{5}{3}\binom{7}{1}}{\binom{12}{4}}$$

$$P(X=4) = \frac{\binom{5}{4}\binom{7}{0}}{\binom{12}{4}}$$

$$P(X=0) = \frac{35}{495} = \frac{7}{99}$$

$$P(X=1) = \frac{5 \cdot 35}{495} = \frac{35}{99}$$

$$P(X=2) = \frac{10 \cdot 21}{495} = \frac{14}{33}$$

$$P(X=3) = \frac{10 . 37}{495} = \frac{14}{99}$$

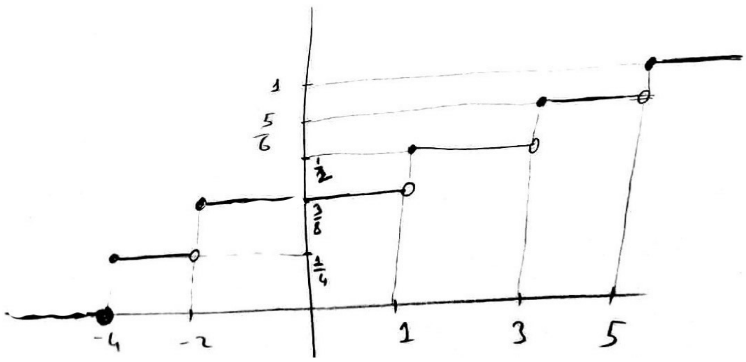
$$P(X=4) = \frac{5}{495} = \frac{1}{99}$$

17. Let X be a discrete random variable with probability mass function p given by:

a	-4	-2	1	3	5
p(a)	1/4	1/8	1/8	1/3	1/6

(a) Determine and graph the probability distribution function of X; (b) Calculate $P(-3 \le X \le 2)$.

$$F(a) = \begin{cases} 0 & a < -4 \\ \frac{1}{4} & -4 \le a < -2 \\ \frac{3}{8} & -2 \le a < 1 \\ \frac{1}{2} & 1 \le a < 3 \\ \frac{5}{6} & 3 \le a < 5 \\ 1 & 5 \le a \end{cases}$$



$$P(-3 \le X \le 2) = P(X \le 2) - P(X < -3)$$
$$= P(X \le 2) - P(X \le -4)$$
$$= F(2) - F(-4)$$
$$= \frac{1}{2} - \frac{1}{4}$$
$$= \frac{1}{4}$$

18. Let X be a discrete random variable with probability mass function p given by:

a	-7	-4	-1	2	5
p(a)	0.2	0.35	0.25	0.1	0.1

Find $E(X), E(5X-3)$, $Var(X)$ and σ_X.

$$E(x) = (-7)\cdot p(-7) + (-4)\, p(-4) + (-1)\, p(-1) + 2\cdot p(2) + 5\cdot p(5)$$

$$= (-7)\cdot \frac{2}{10} + (-4)\cdot \frac{35}{100} + (-1)\cdot \frac{1}{4} + 2\cdot \frac{1}{10} + 5\cdot \frac{1}{10}$$

$$= \frac{-14}{10} + \left(\frac{-35}{25}\right) - \frac{1}{4} + \frac{1}{5} + \frac{1}{2}$$

$$= \frac{-14}{5} - \frac{1}{4} + \frac{1}{5} + \frac{1}{2}$$

$$= \frac{-13}{5} + \frac{1}{4} = \frac{-47}{20} = -2.35$$

$$E(5x-3) = 5E(x) - 3$$

$$= 5\cdot\left(\frac{-47}{20}\right) - 3 = -\frac{59}{4}.$$

$$Var(X) = E(X^2) - E(X)^2$$

$$E(X^2) = (-7)^2 \cdot p(-7) + (-4)^2 \cdot p(-4) + (-1)^2 \cdot p(-1) + 2^2 \cdot p(2) + 5^2 \cdot p(5)$$

$$= 49 \cdot \frac{2}{10} + 16 \cdot \left(\frac{35}{100}\right) + \frac{25}{100} + 4 \cdot \frac{1}{10} + 25 \cdot \frac{1}{10}$$

$$= \frac{49 \cdot 2 + 4 + 25}{10} + \frac{16 \cdot 35 + 25}{100}$$

$$= \frac{127}{10} + \frac{585}{100} = \frac{1270 + 585}{100} = \frac{1855}{100} = 18.55$$

$$Var(X) = \frac{1855}{100} - \left(\frac{-47}{20}\right)^2 = \frac{5211}{400}$$

$$\sigma_X = \sqrt{Var(X)} = \sqrt{\frac{5211}{400}} = \frac{3\sqrt{579}}{20} = 3.60936.$$

19. Out of 800 families with 5 children each, how many would you expect to have (a) 3 boys, (b) 5

girls, (c) either 2 or 3 boys? Assume equal probabilities for boys and girls.

X be the number of boys in 1 family. Then X is a binomial distribution. we have X can be 0, 1, 2, 3, 4, 5

$$P(X=i) = \binom{5}{i}\left(\frac{1}{2}\right)^i \cdot \left(\frac{1}{2}\right)^{5-i}$$

$$= \binom{5}{i}\frac{1}{2^5}$$

(a) probability that 1 family has 3 boys is $P(X=3) = \binom{5}{3} \cdot \frac{1}{2^5} = \frac{5}{16}$.

Therefore, out of 800 families $800 \cdot \frac{5}{16} = 250$ have 3 boys.

(b) probability that 1 family has 5 girls is $P(X=0) = \binom{5}{0} \cdot \frac{1}{2^5} = \frac{1}{2^5} = \frac{1}{32}$

Therefore, out of 800 families we expect that $800 \cdot \frac{1}{32} = 125$ many families have girls.

C)

The probability that 1 family has 2 or 3 boys is $P(X=2)+P(X=3) =$

$$\binom{5}{2}\frac{1}{2^5}+\binom{5}{3}\frac{1}{2^5} = \frac{10}{32}+\frac{10}{32} = \frac{20}{32} = \frac{5}{8}.$$

Therefore, out of 800 families, we expect $800 \cdot \frac{5}{8} = 500$ families have 2 or 3 boys.

20. According to the National Office of Vital Statistics of the U.S. Department of Health and Human Services, the average number of accidental drownings per year in the United States is 3.0 per 100,000 population. Find the probability that in a city of population 300,000 there will be (a) 5, (b) between 6 and 10, (f) less than 4, accidental drownings per year.

X be random variable denotes the the number of accidental drownings per year. Then X is Poisson random variable with $\lambda = 6$. The probability mass function is

$$P(X=x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!} = \frac{e^{-6} \cdot 6^{x}}{x!}$$

a)

$$P(X=5) = \frac{e^{-6} \cdot 6^{5}}{5!} \cong 0.16062$$

b)

$$P(6 \leq X \leq 8) = P(X=6) + P(X=7) + P(X=8)$$

$$= e^{-6}\left[\frac{6^6}{6!} + \frac{6^7}{7!} + \frac{6^8}{8!}\right]$$

$$= e^{-6} \cdot 162 \cong 0.40155$$

c)

$$P(X \leq 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

$$= e^{-6}\left[1 + \frac{6}{1!} + \frac{6^2}{2!} + \frac{6^3}{3!} + \frac{6^4}{4!}\right]$$

$$= e^{-6} \cdot 115 \cong 0.28505$$