PHYS414 Final Project Report

Yiğit Fatih Çakıllı

Newton Part

Part(A)

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \qquad \frac{dP(r)}{dr} = -\frac{G m(r)\,\rho(r)}{r^2} \qquad P = K\rho^{1+\frac{1}{n}}$$

$$\Rightarrow \frac{dP(r)}{dr}\cdot\left(-\frac{r^2}{G\rho(r)}\right) = m(r) \Rightarrow \frac{d}{dr}\left(\frac{dP(r)}{dr}\cdot\left(-\frac{r^2}{G\rho(r)}\right)\right) = \frac{dm(r)}{dr}$$

by plugging in the given relation for the first derivative of m(r):

$$\Rightarrow \frac{d}{dr}\left(\frac{dP(r)}{dr}\cdot\left(-\frac{r^2}{G\rho(r)}\right)\right) = 4\pi r^2 \rho(r)$$

Using the relation:

$$\Rightarrow \frac{dP}{dr} = \frac{d}{dr}\left(K\rho^{1+\frac{1}{n}}\right) = \left(1+\frac{1}{n}\right)K\rho^{\frac{1}{n}}\frac{d\rho}{dr}$$ by substituting

$$\Rightarrow \frac{d}{dr}\left(\left(1+\frac{1}{n}\right)K\rho^{\frac{1}{n}}\frac{d\rho}{dr}\cdot\left(-\frac{r^2}{G\rho(r)}\right)\right) = 4\pi r^2\rho(r)$$

$$\Rightarrow 4\pi r^2\rho(r) + \frac{d}{dr}\left(\left(1+\frac{1}{n}\right)\frac{Kr^2}{G}\cdot\rho^{\frac{1}{n}-1}\frac{d\rho}{dr}\right) = 0$$

$$\Rightarrow \frac{1}{4\pi r^2}\frac{d}{dr}\left(\left(1+\frac{1}{n}\right)\frac{Kr^2}{G}\rho^{\frac{1}{n}-1}\frac{d\rho}{dr}\right) + \rho(r) = 0$$

let $r = \alpha\xi$ where $\alpha^2 = \frac{(n+1)K}{4\pi G}\rho_c^{\frac{1}{n}-1}$

let $\theta = \rho_c \rho^{\frac{1}{n}}$

$$\Rightarrow \frac{d\rho}{dr} = \underbrace{\frac{d\rho}{d\theta}}\,\frac{d\theta}{d\xi}\cdot\underbrace{\frac{d\xi}{dr}}$$

$$\rho = \rho_c\,\theta^n \Rightarrow \underbrace{\frac{d\rho}{d\theta} = \rho_c n\,\theta^{n-1}}\;;\quad \xi = \frac{r}{\alpha} \Rightarrow \underbrace{\frac{d\xi}{d\theta} = \frac{1}{\alpha}}$$

$$\Rightarrow \frac{d\rho}{dr} = \rho_c n\,\theta^{n-1}\cdot\frac{1}{\alpha}\cdot\frac{d\theta}{d\xi}$$

by substituting these relations to the equation we obtain:

$$\Rightarrow \frac{1}{\alpha^2\xi^2}\,\frac{d}{\alpha\, d\xi}\left(\left(\cancel{1+\frac{1}{n}}\right)\frac{Kr^2}{4\pi G}\left(\rho_c\cancel{\theta^n}\right)^{\frac{1}{n}-1}\cancel{\rho_c}\,\cancel{n}\,\cancel{\theta^{n-1}}\cdot\frac{1}{\alpha}\,\frac{d\theta}{d\xi}\right)+\cancel{\rho_c}\,\theta^n = 0$$

$$\Rightarrow \frac{1}{\alpha^3\xi^2}\,\frac{d}{d\xi}\left(\underbrace{\frac{(n+1)K}{4\pi G}\cdot\rho_c^{\frac{1}{n}-1}}_{\alpha^2}\cdot\frac{r^2}{\alpha}\,\frac{d\theta}{d\xi}\right)+\theta^n = 0$$

$$\Rightarrow \frac{1}{\xi^2}\,\frac{d}{d\xi}\left(\underbrace{\frac{\alpha^2 r^2}{\alpha^4 2}}_{\xi^2}\,\frac{d\theta}{d\xi}\right)+\theta^n = 0 \Rightarrow \boxed{\frac{1}{\xi^2}\,\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^n = 0}$$

In order to find the mass of the star we need to integrate the mass density over its volume. We know the that the surface of the star is at ξ. We need to find

Mathematica Part

Newton Part A

```
In[∘]:= Dsolve[{ ((1/ξ)^2) * D[ξ^2 * θ'[ξ], ξ] + θ[ξ]^n == 0 }]
```

$$\text{Dsolve}\left[\left\{\theta[\xi]^n + \frac{2\,\xi\,\theta'[\xi] + \xi^2\,\theta''[\xi]}{\xi^2} == 0\right\}\right]$$

```
In[∘]:= AsymptoticDSolveValue[{ ((1/ξ)^2) * D[ξ^2 * θ'[ξ], ξ] + θ[ξ]^n == 0, θ[0] == 1 }
         θ[ξ], {ξ, 0, 6}, n]
```

$$\text{AsymptoticDSolveValue}\left[\left\{\theta[\xi]^n + \frac{2\,\xi\,\theta'[\xi] + \xi^2\,\theta''[\xi]}{\xi^2} == 0,\ \theta[0] == 1\right\},\ \theta[\xi],\ \{\xi, 0, 6\}\right]$$

```
In[∘]:= AsymptoticDSolveValue[{ξ * θ[ξ]^n + 2 θ'[ξ] + ξ θ''[ξ] == 0, θ[0] == 1}, θ[ξ], {ξ, 0, 4}]
```

Out[∘]= $1 - \frac{\xi^2}{6} + \frac{n\,\xi^4}{120}$

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```
DSolve[{ξ * θ[ξ]^n + 2 θ'[ξ] + ξ θ''[ξ] == 0, θ'[0] == 0, θ[0] == 1}, θ[ξ], {ξ, 0, ∞}]
```

$1 - \frac{\xi^2}{6} + \frac{n\,\xi^4}{120}$

```
In[•]:= FullSimplify[DSolve[{ξ * θ[ξ]^1 + 2 * θ'[ξ] + ξ θ''[ξ] == 0, θ'[0] == 0, θ[0] == 1}, θ[ξ],
    {ξ, 0, ∞}]]
```

··· Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is $i(2ic_1 + c_2) == 0$.

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··· Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\left\{\left\{\theta[\xi] \to \frac{\mathrm{Sin}[\xi]}{\xi}\right\}\right\}$$

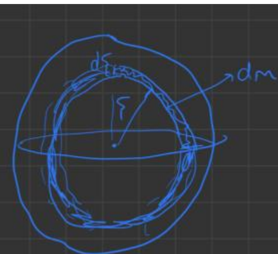
In order to find the mass of the star we need to integrate the mass density over its volume. We know the that the surface of the star is at ξ. We need to find

$$M_s = \int_{V_s} dm$$ and relate dm to $d\xi$

Given relations:

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \qquad \rho = \rho_c \theta^n \quad \& \quad r = \alpha\xi$$

$$\frac{dm}{d(\alpha\xi)} = 4\pi(\alpha\xi)^2 \rho_c \theta^n \Rightarrow dm = 4\pi\alpha^3\xi^2\rho_c\theta^n d\xi$$



$$\Rightarrow M_s = \int_{R_s} dm = \int_0^{\xi} 4\pi\alpha^3 \xi'^2 \rho_c \theta^n(\xi')\, d\xi'$$

$$= 4\pi\alpha^3\rho_c \int_0^{\xi} \underline{\xi'^2\theta^n(\xi')}\, d\xi'$$

from the equation derived above:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \Rightarrow \underline{\xi^2\theta^n} = -\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)$$

by substituting:

$$\Rightarrow M_s = 4\pi\alpha^3\rho_c\int_0^{\xi} -d\left(\xi'^2\frac{d\theta}{d\xi}\right) = -4\pi\alpha^3\rho_c\, \xi_n^2 \left(d\theta/d\xi\right)\Big|_{\xi_n} \qquad \underbrace{\xi_n = \frac{R}{\alpha}}_{\xi \text{ at the surface}}$$

$$\Rightarrow M_s = -4\pi\rho_c\alpha^3 \xi_n^3 \cdot \frac{1}{\xi^n}\theta'(\xi_n) = 4\pi\rho_c\cancel{\alpha^3}\cdot\frac{R^3}{\cancel{\alpha^3}}\cdot\left(-\frac{\theta'(\xi_n)}{\xi_n}\right)$$

$$\Rightarrow \boxed{M_s = 4\pi\rho_c R^3\left(-\frac{\theta'(\xi_n)}{\xi_n}\right)}$$

Using the relation, we can write ρ_c as:

$$\alpha^2 = \frac{(n+1)K}{4\pi G}\rho_c^{\frac{1}{n}-1} \Rightarrow \rho_c = \left(\frac{4\pi G}{(n+1)K}\cdot\alpha^2\right)^{\frac{n}{n-1}} \Rightarrow \underline{\rho_c = \left(\frac{4\pi G}{(n+1)K}\right)^{n/n-1}\cdot \alpha^{2n/n-1}}$$

by substituting into the equation and using the definition of ξ_n at R:

$$M_s = 4\pi\left(\frac{4\pi G}{(n+1)K}\right)^{n/n-1}\left(\frac{\xi_n}{R}\right)^{2n/n-1} R^3\left(-\frac{\theta'(\xi_n)}{\xi_n}\right)$$

$$\Rightarrow M_s \propto R^{\frac{3n-3-2n}{n-1}} = R^{\frac{n-3}{n-1}}$$

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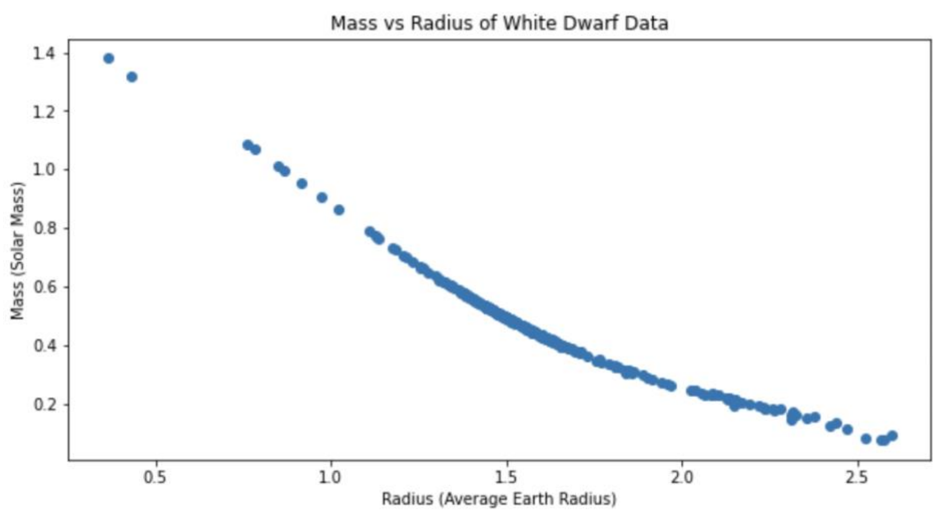
$$\Rightarrow \boxed{M_s \propto R^{\frac{3-n}{1-n}}}$$

Part(B)



Part(C)

```
In[◦]:= Assuming[{x → 0}, Series[C * (x * (2 * x^2 - 3) * (x^2 + 1)^(1/2) + 3 * ArcSinh[x]),
        {x, 0, 10}]]
```

Out[◦]= $\frac{8\,C\,x^5}{5} - \frac{4\,C\,x^7}{7} + \frac{C\,x^9}{3} + O[x]^{11}$

Using mathematica; Series for the first 3 terms by assuming $x \to 0$ $(x \ll 1)$ I obtained:

$$P = \frac{8Cx^5}{5} - \frac{4Cx^7}{7} + \frac{Cx^9}{3} + O(x^{11}) \quad ; \quad x = \left(\frac{\rho}{D}\right)^{1/q}$$

Further simplification for $x \ll 1$ - get the leading term:

$$P = \frac{8C}{5} \cdot \left(\frac{\rho}{D}\right)^{5/q} = \frac{8C}{5} \frac{1}{D^{5/q}} \rho^{\frac{5+q-q}{q}}$$

$$\Rightarrow P = \underbrace{\frac{8C}{5} \cdot \frac{1}{D^{5/q}}}_{K_*} \cdot \rho^{1 + \underbrace{\frac{5-q}{q}}_{1/n_*}}$$

$$\Rightarrow \boxed{K = \frac{8C}{5D^{5/q}} \quad , \quad n_* = \frac{q}{5-q}}$$

Using the generic relation between M and K and plugging K_* and n_*:

$$M = \underbrace{-4\pi \left(\frac{4\pi G}{(n_*+1)K_*}\right)^{\frac{n_*}{1-n_*}} \xi_{n_*}^{\frac{n_*+1}{n_*-1}} \theta'(\xi_{n_*})}_{A} R^{\overbrace{\frac{3-n_*}{1-n_*}}^{B}}$$

Introduce A & B constants to simplify the relationship between M & R

$$\Rightarrow M = AR^B$$

Where:

$$A = -4\pi \left(\frac{4\pi G}{(n_*+1)K_*}\right)^{\frac{n_*}{1-n_*}} \xi_{n_*}^{\frac{n_*+1}{n_*-1}} \theta'(\xi_{n_*}) \quad ; \quad B = \frac{3-n_*}{1-n_*}$$

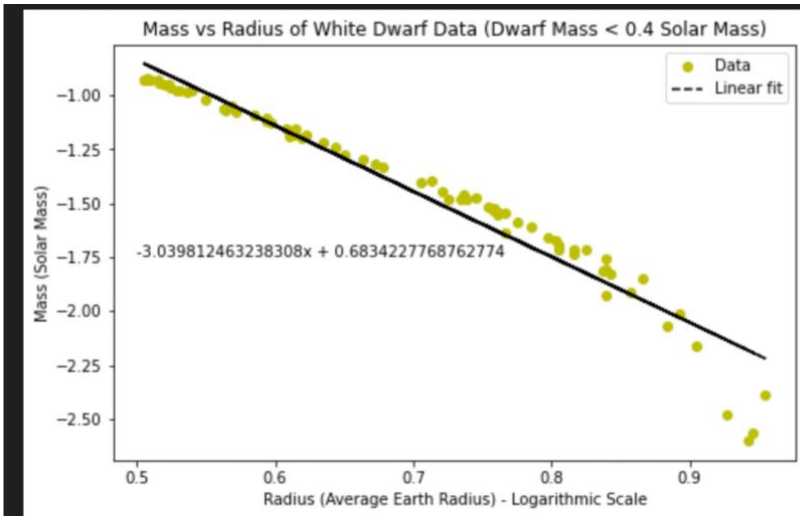
Given a power relation, as in our case $M = AR^B$ where one of the parameters appear in the exponent; it is intuitive to do a fit to the logarithm of the relation.

$$\ln M = \ln(AR^B) = \ln A + \ln R^B = \ln A + B \ln R$$

We have a linear relation between $\ln M$ and $\ln R$

$$\Rightarrow \ln M = B \ln R + \ln A$$

$\Rightarrow$ linear fit to the ln of the data points

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First order coefficient for the fit: -3.039812463238308
Zeroth order coefficient for the fit: 0.6834227768762774
Sample Size: 76
```

From the linear fit of the form $mx+b$; the obtained coefficients are:

$$m = -3.0398 \qquad b = 0.6834$$

n_* can be found directly:

$$B = \frac{3-n_*}{1-n_*} = -3.0398 \Rightarrow n_* \to 1.48507 \approx 1.5$$

For K_* we have the relation:

$$\ln A = \ln\left[-4\pi\left(\frac{4\pi G}{(n_*+1)K_*}\right)^{\frac{n_*}{1-n_*}} \xi_{n_*}^{\frac{n_*+1}{n_*-1}} \, \theta'(\xi_{n_*})\right] = 0.6834$$

where $n_* = 1.5$

So we need the values for ξ_{n*} and $\theta'(\xi_{n_*})$

Given the definition $\theta(\xi_{n_*}) = 0$; which corresponds to the surface of the star we obtain a root-finding problem to find ξ_{n_*}. But first we need to find the behaviour of θ given the differential equation.

We have the Lane - Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^n=0$$

Can be written:

$$\frac{1}{\xi^2}\left(2\xi\frac{d\theta}{d\xi}+\xi^2\frac{d^2\theta}{d\xi^2}\right)+\theta^n=0$$

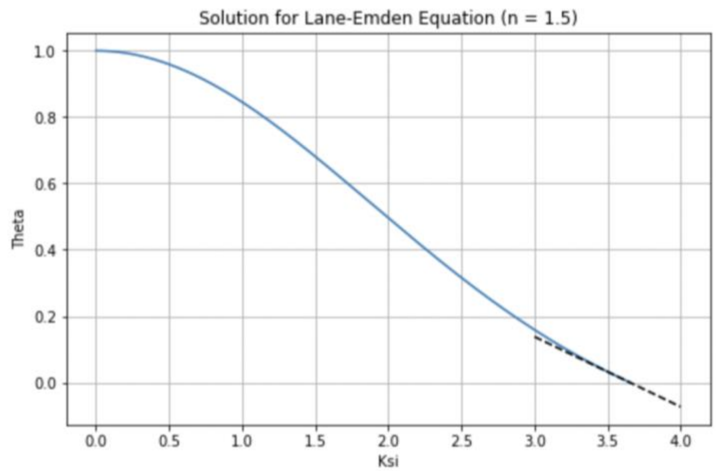
$$\Rightarrow \frac{2}{\xi}\frac{d\theta}{d\xi}+\frac{d^2\theta}{d\xi^2}+\theta^n=0$$

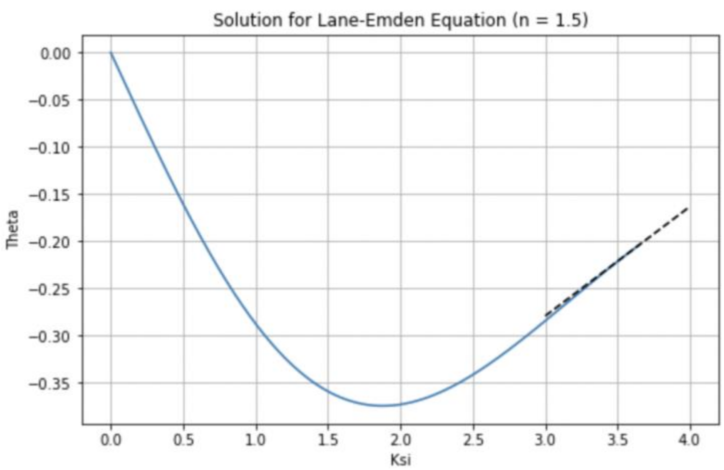
$$\Rightarrow \frac{d^2\theta}{d\xi^2}=-\underbrace{\left(\frac{2}{\xi}\frac{d\theta}{d\xi}+\theta^n\right)}_{RHS}$$

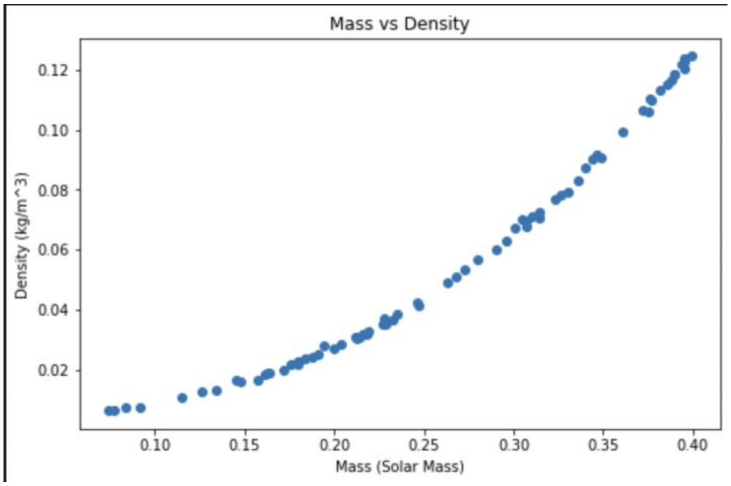
From numerics:

$\xi_n = 3.6581$ $\qquad$ $\theta'(\xi_n) = -0.2033$

$$K_* = 5.4058$$





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ksi_n = 3.653083969542544
dtheta(ksi_n) = -0.20330250655674104
K_star = 5.40582658418237
```

Part(D)

I have implemented this method partially (most functions)

In order to solve ivp for R & D we need the relation between $\frac{dP}{dr}$ and $\frac{d\rho}{dr}$

mass density relation

Use the EOS:

$$P = C\left[x(2x^2-3)\sqrt{x^2+1}\ 3\sinh^{-1}x\right] \qquad x = \left(\frac{\rho}{D}\right)^{1/q}$$

$$\Rightarrow \frac{dP}{dr} = \frac{dP}{dx} \cdot \frac{dx}{d\rho} \cdot \frac{d\rho}{dr}$$

using Mathematica:

$$\frac{dP}{dx} = \frac{8Cx^4}{\sqrt{1+x^2}}$$

$$\frac{dx}{d\rho} = \frac{1}{D^{1/q}} \cdot \frac{1}{q}\rho^{\frac{1}{q}-1}$$

the original equation:

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$$

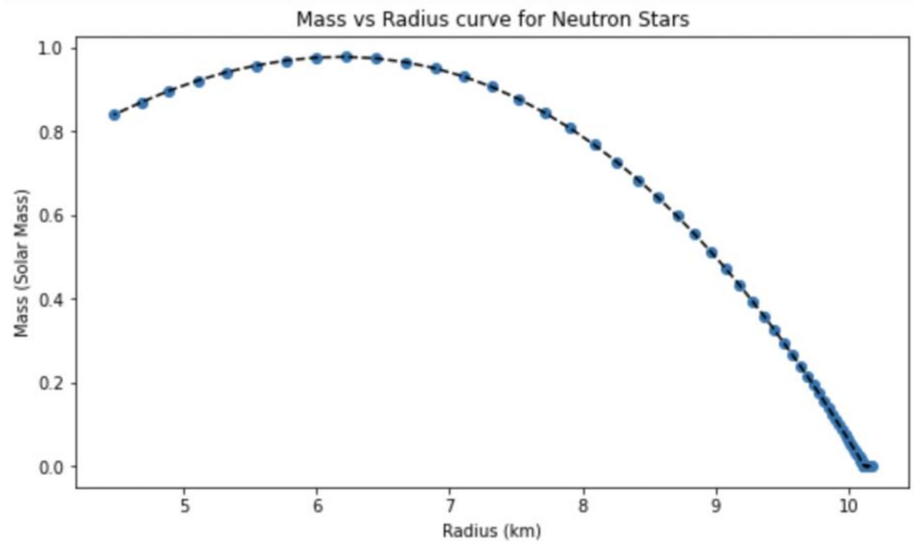
We obtain:

$$\boxed{\frac{d\rho}{dr} = \frac{\sqrt{1+x^2}}{8Cx^4} \cdot q \cdot D^{1/q} \cdot \rho^{1-\frac{1}{q}} \cdot \left(-\frac{Gm\rho}{r^2}\right)}$$

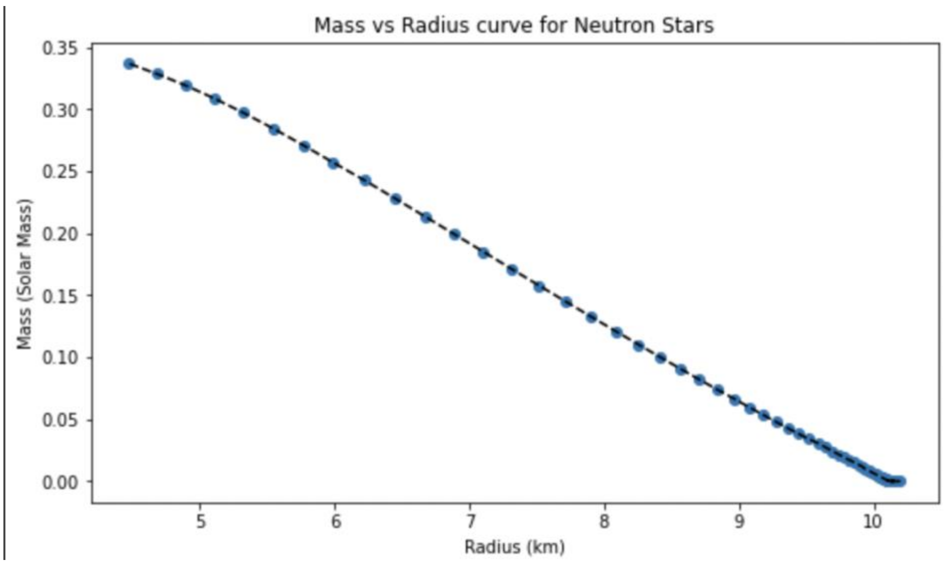
Einstein Part

I think I have scaling issue (around 2) for this part which I realized at the last part when finding valid K values. I couldn't address the reason.

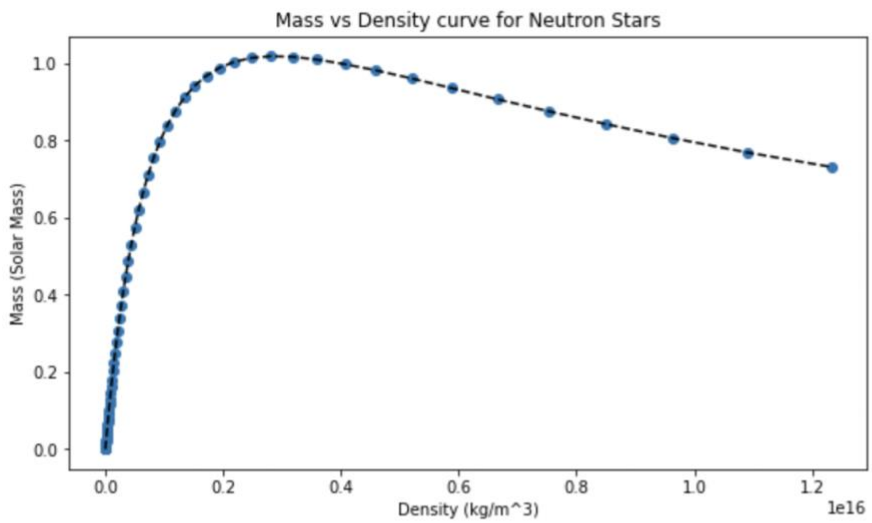
Part(A)

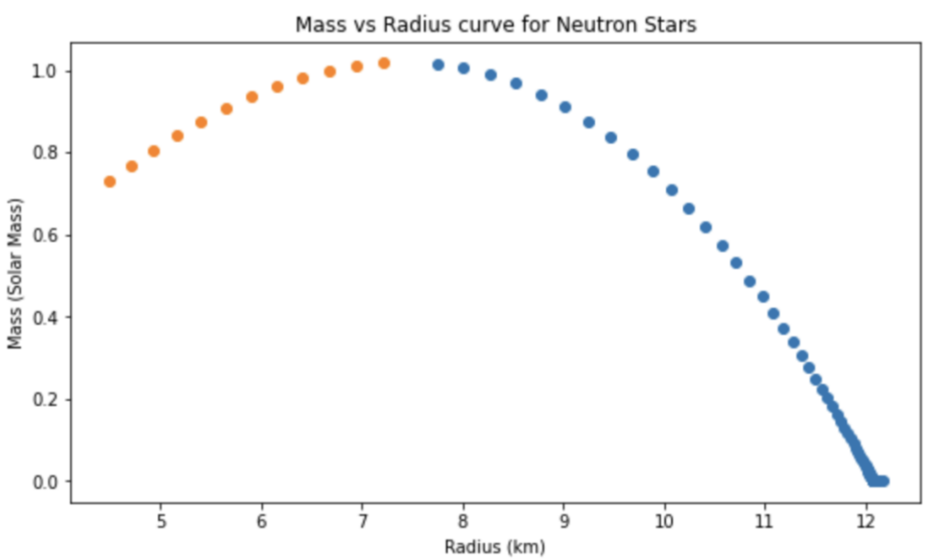


Part(B)



Part(C)






```
The maximal NS mass: 1.018979660888874 Solar Masses
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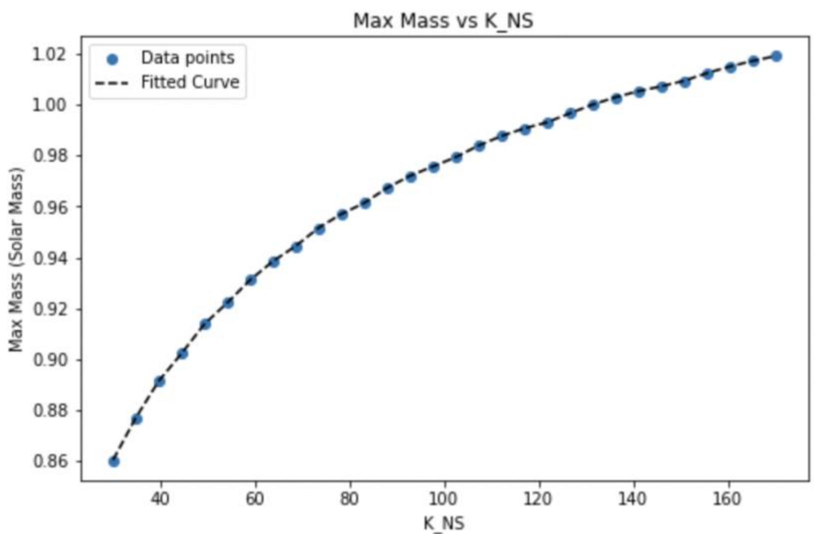
Part(D)

I scaled with my hand the observed mass value to match with my computation

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Max allowed K value: 124.87487487487488
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Part(E)

```
In[6]:= FullSimplify[DSolve[v'[r] == (2*M)/(r*(r-2*M)), v[r], r]]
Out[6]= {{v[r] → c1 - Log[r] + Log[-2 M + r]}}
```

The obtained solution by mathematica:

$$v(r) = C - \ln(r) + \ln(-2M+r)$$

$$\Rightarrow v(r) = \ln\left(\frac{-2M}{r}+1\right) + C$$

At $r = R$ we have:

$$v(R) = \ln\left(\frac{-2M}{R}+1\right) + C$$

$$\Rightarrow C = v(R) - \ln\left(\frac{-2M}{R}+1\right)$$

Then by substituting this for the generic equation with $r>R$ conditi
we obtain:

$$\boxed{v(r>R) = \ln\left(1-\frac{2M}{r}\right) - \ln\left(1-\frac{2M}{r}\right) + v(R)}$$