Doğal Logaritma ve Doğal Üstel Fonksiyon

★ $\ln x = \int_1^x \frac{dt}{t}$ $(x>0)$ denklemi ile verilen $f(x)=\ln x$ fonksiyonuna "Doğal Logaritma" denir.

★ $(\ln^{-1} x) = e^x$ ile tanımlı ($\ln x$ fonksiyonunun ters fonksiyonu) $f(x)=e^x$ fonksiyonuna "Doğal Üstel fonksiyon" denir.

★ $f(x)=a^x$ $(a>0,\ a\neq 1)$ ile tanımlı fonksiyona "Genel üstel fonksiyon" denir.

★ $f(x)=\log_a x$ fonksiyonuna (a^x fonksiyonunun tersidir) "Genel logaritmik fonksiyon" denir. ($\ln x = \log_e x$ dir)

Özellikleri

① $\ln e = 1$

② $\ln xy = \ln x + \ln y$

③ $\ln \frac{x}{y} = \ln x - \ln y$

④ $\ln x^r = r \ln x$
$\log_a x^r = r \log_a x$

⑤ $\log_a a^x = x$
$\ln e^x = x$

⑥ $e^{\ln x} = x$
$a^{\log_a x} = x$

⑦ $a^x = e^{x \ln a}$
$x^n = e^{n \ln x}$

⑧ $\log_a x = \frac{\ln x}{\ln a}$

⑨ $\ln 1 = 0$
$\log_a 1 = 0$

⑩ $\log_a xy = \log_a x + \log_a y$

⑪ $\log_a \frac{x}{y} = \log_a x - \log_a y$

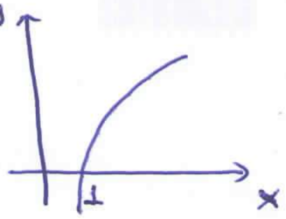
⑫ $\ln \frac{1}{x} = -\ln x$
$\log_a \frac{1}{x} = -\log_a x$

⑬ $\log_a a = 1$

⑭ $\log_a x = y \iff x = a^y$
$\ln x = y \iff x = e^y$

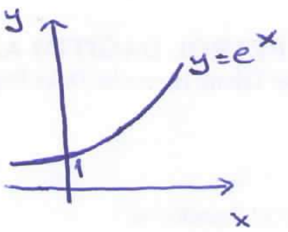
Grafikleri

★ $y = \ln x$



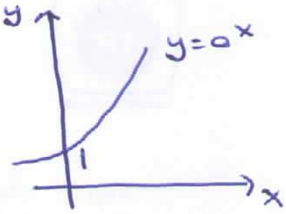
Fonk. Artandır

★ $y = e^x$



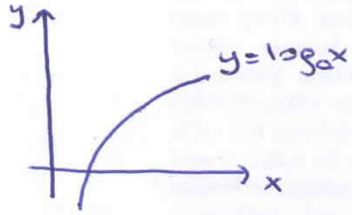
Fonk. Artandır

★ $y = a^x \quad (a>1)$



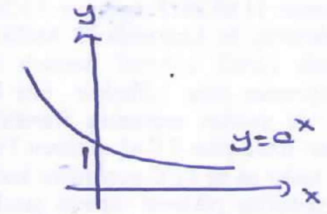
Fonk. Artandır

★ $y = \log_a x \quad (a>1)$



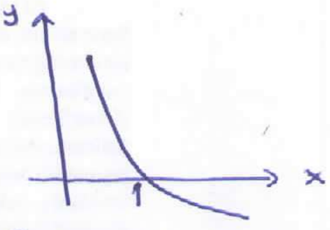
Fonk. Artandır

★ $y = a^x \quad (0<a<1)$



Fonk. Azalandır

★ $y = \log_a x \quad (0<a<1)$



Fonk. Azalandır

Limitleri

① $\lim\limits_{x\to\infty} \ln x = \infty$ ② $\lim\limits_{x\to 0^+} \ln x = -\infty$ ③ $\lim\limits_{x\to\infty} e^x = \infty$

④ $\lim\limits_{x\to -\infty} e^x = 0$ ⑤ $\lim\limits_{x\to\infty} a^x = 0 \quad (0<a<1)$ ⑥ $\lim\limits_{x\to\infty} a^x = \infty \quad (a>1)$

⑦ $\lim\limits_{x\to -\infty} a^x = \infty \quad (0<a<1)$ ⑧ $\lim\limits_{x\to -\infty} a^x = 0 \quad (a>1)$

⑨ $\lim\limits_{x\to\infty} \left(1+\frac{a}{x}\right)^x = e^a$ ⑩ $\lim\limits_{x\to 0^+} (1+ax)^{1/x} = e^a$ ⑪ $\lim\limits_{x\to 0} \frac{e^x-1}{x} = 1$

Türevleri

① $(\ln x)' = \frac{1}{x}$ ② $(\ln f(x))' = \frac{f'(x)}{f(x)}$ ③ $(e^x)' = e^x$

④ $(e^{f(x)})' = f'(x).e^{f(x)}$ ⑤ $(a^{f(x)})' = f'(x).\,a^{f(x)}\ln a$ ⑥ $(\log_a f(x))' = \frac{f'(x)}{f(x).\ln a}$

Ters Trigonometrik Fonksiyonlar

Altı temel trigonometrik fonksiyon bire-bir değillerdir; fakat tanım kümelerini bire-bir oldukları aralıklara kısıtlayabiliriz. Bu kısıtlanmış fonksiyonlar artık bire-bir oldukları için tersleri vardır ve aşağıdaki şekilde gösterilirler:

Ters Trig. Fonksiyon	Tanım Kümesi	Görüntü K.
① $f(x) = \text{ArcSin}x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
② $f(x) = \text{ArcCos}x$	$-1 \le x \le 1$	$0 \le y \le \pi$
③ $f(x) = \text{ArcTan}x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
④ $f(x) = \text{ArcCot}x$	$-\infty < x < \infty$	$0 < y < \pi$
⑤ $f(x) = \text{ArcSec}x$	$x \le -1$ veya $x \ge 1$	$0 \le y \le \pi,\ y \ne \frac{\pi}{2}$
⑥ $f(x) = \text{ArcCosec}x$	$x \le -1$ veya $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$

✯ $y = \text{ArcSin}x \Rightarrow x = \text{Sin}y$ ✯ $y = \text{ArcCos}x \Rightarrow x = \text{Cos}y$

✯ $y = \text{ArcTan}x \Rightarrow x = \text{Tan}y$ ✯ $y = \text{ArcCot}x \Rightarrow x = \text{Cot}y$

✯ $y = \text{ArcSec}x \Rightarrow x = \text{Sec}y$ ✯ $y = \text{ArcCosec}x \Rightarrow x = \text{Cosec}y$

Örnek:

x	$y = \text{ArcSin}x$	$y = \text{ArcCos}x$
$\frac{1}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$
0	0	$\pi/2$
$\frac{\sqrt{3}}{2}$	$\pi/3$	$\pi/6$

x	$y = \text{ArcSin}x$	$y = \text{ArcCos}x$
$-1/2$	$-\frac{\pi}{6}$	$\frac{2\pi}{3}$
$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$	$\frac{3\pi}{4}$
$-\frac{\sqrt{3}}{2}$	$-\frac{\pi}{3}$	$\frac{5\pi}{6}$

Örnek:

x	$y = ArcTan\,x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
0	0
$\frac{\sqrt{3}}{3}$	$\pi/6$

x	$y = ArcTan\,x$
$-\sqrt{3}$	$-\pi/3$
-1	$-\pi/4$
$-\frac{\sqrt{3}}{3}$	$-\frac{\pi}{6}$

Özdeşlikler

① $ArcCos\,x + ArcCos(-x) = \pi$

② $ArcSin\,x + ArcCos\,x = \frac{\pi}{2}$

③ $ArcTan\,x + ArcCot\,x = \frac{\pi}{2}$

④ $ArcCosec\,x + ArcSec\,x = \frac{\pi}{2}$

Türevleri

① $y = ArcSin\,x \Rightarrow y' = \frac{1}{\sqrt{1-x^2}}$ | $y = ArcSin\,f(x) \Rightarrow y' = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$

② $y = ArcCos\,x \Rightarrow y' = \frac{-1}{\sqrt{1-x^2}}$ | $y = ArcCos\,f(x) \Rightarrow y' = -\frac{f'(x)}{\sqrt{1-(f(x))^2}}$

③ $y = ArcTan\,x \Rightarrow y' = \frac{1}{1+x^2}$ | $y = ArcTan\,f(x) \Rightarrow y' = \frac{f'(x)}{1+(f(x))^2}$

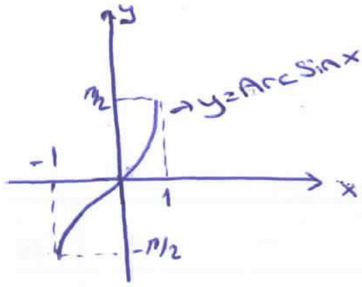
④ $y = ArcCot\,x \Rightarrow y' = \frac{-1}{1+x^2}$ | $y = ArcCot\,f(x) \Rightarrow y' = \frac{-f'(x)}{1+(f(x))^2}$

⑤ $y = ArcSec\,x \Rightarrow y' = \frac{1}{|x|.\sqrt{x^2-1}}$ | $y = ArcSec\,f(x) \Rightarrow y' = \frac{f'(x)}{|f(x)|\sqrt{(f(x))^2-1}}$

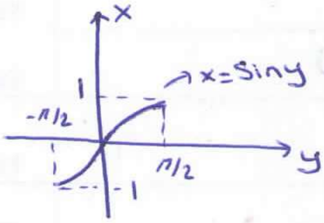
⑥ $y = ArcCosec\,x \Rightarrow y' = \frac{-1}{|x|\sqrt{x^2-1}}$ | $y = ArcCosec(f(x)) \Rightarrow y' = \frac{-f'(x)}{|f(x)|\sqrt{f^2-1}}$

Grafikleri

① $y = \operatorname{ArcSin} x$ T.K: $[-1,1]$ G.K: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



II. Yol $y = \operatorname{ArcSin} x \Rightarrow x = \operatorname{Sin} y$ olduğunu kullanırsak:



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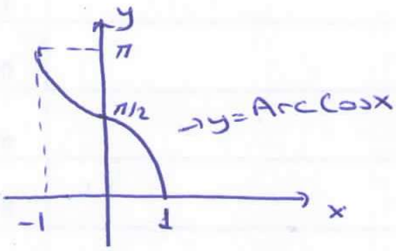
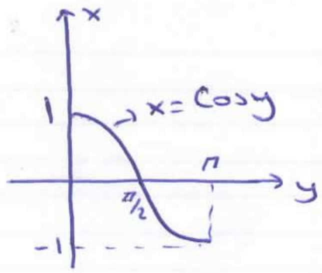
② $y = \operatorname{ArcCos} x$ T.K: $[-1,1]$ G.K: $[0,\pi]$



veya: $y = \operatorname{ArcCos} x \Rightarrow x = \operatorname{Cos} y$ olduğunu kullanırsak:



Hiperbolik fonksiyonlar

* iki üstel fonksiyon e^x ve e^{-x} in birleşimi ile oluşan fonksiyonlardır.

$$\boxed{\begin{array}{l} \text{Sinüs hiperbolik fonksiyonu: } \sinh x = \dfrac{e^x - e^{-x}}{2} \\ \text{Cosinüs hiperbolik fonksiyonu: } \cosh x = \dfrac{e^x + e^{-x}}{2} \end{array}}$$

* Bu temel çiftten hareketle Tanjant, Cotanjant, Secant ve Cosecant Hiperbolik fonk. tanımlanır.

* Hiperbolik Fonksiyonlar isimlerini oldukları Trigonometrik fonksiyonlar ile birçok benzerlik gösterirler.

Özdeşlikler

① $\tanh x = \dfrac{\sinh x}{\cosh x}$	② $\coth x = \dfrac{\cosh x}{\sinh x}$	③ $\operatorname{sech} x = \dfrac{1}{\cosh x}$
④ $\operatorname{cosech} x = \dfrac{1}{\sinh x}$	⑤ $\sinh 2x = 2\sinh x \cosh x$	⑥ $\cosh^2 x - \sinh^2 x = 1$

⑦ $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$

⑧ $\tanh^2 x = 1 - \operatorname{sech}^2 x$	⑨ $\coth^2 x = 1 + \operatorname{cosech}^2 x$

Türevleri

① $y = \sinh x \longrightarrow y' = \cosh x$	$y = \sinh f(x) \rightarrow y' = f'(x) . \cosh f(x)$
② $y = \cosh x \rightarrow y' = \sinh x$	$y = \cosh f(x) \rightarrow y' = f'(x) \sinh f(x)$
③ $y = \tanh x \rightarrow y' = \operatorname{sech}^2 x$	$y = \tanh f(x) \rightarrow y' = f'(x) . \operatorname{sech}^2 f(x)$
④ $y = \coth x \rightarrow y' = -\operatorname{cosech}^2 x$	$y = \coth f(x) \rightarrow y' = -f'(x) \operatorname{cosech}^2 f(x)$

⑤ $y = \operatorname{sech} x \rightarrow y' = -\operatorname{sech} x \tanh x$

⑥ $y = \operatorname{cosech} x \rightarrow y' = -\coth x . \operatorname{cosech} x$

Ters Hiperbolik fonksiyonlar

- $y = \sinh^{-1}x \Rightarrow x = \sinh y$
- $y = \cosh^{-1}x \Rightarrow x = \cosh y$
- $y = \tanh^{-1}x \Rightarrow x = \tanh y$
- $y = \coth^{-1}x \Rightarrow x = \coth y$
- $y = \operatorname{cosech}^{-1}x \Rightarrow x = \operatorname{cosech} y$
- $y = \operatorname{sech}^{-1}x \Rightarrow x = \operatorname{sech} y$

Özdeşlikler

① $\operatorname{sech}^{-1}x = \cosh^{-1}\frac{1}{x}$ ② $\operatorname{cosech}^{-1}x = \sinh^{-1}\frac{1}{x}$ ③ $\coth^{-1}x = \tanh^{-1}\frac{1}{x}$

Türevleri

① $y = \sinh^{-1}x \Rightarrow y' = \frac{1}{\sqrt{1+x^2}}$

② $y = \cosh^{-1}x \Rightarrow y' = \frac{1}{\sqrt{x^2-1}}$

③ $y = \tanh^{-1}x \Rightarrow y' = \frac{1}{1-x^2}$

④ $y = \coth^{-1}x \Rightarrow y' = \frac{1}{1-x^2}$

⑤ $y = \operatorname{sech}^{-1}x \Rightarrow y' = \frac{-1}{x \cdot \sqrt{1-x^2}}$

⑥ $y = \operatorname{cosech}^{-1}x \Rightarrow y' = \frac{-1}{|x| \cdot \sqrt{1+x^2}}$

Çözümlü Sorular

① $\cosh 2x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$ olduğunu gösteriniz.

$$\cosh x = \frac{e^x + e^{-x}}{2} \Rightarrow \cosh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4} \Rightarrow$$

$$2\cosh^2 x - 1 = \frac{e^{2x} + 2 + e^{-2x}}{2} - 1 = \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x \checkmark$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \Rightarrow \sinh^2 x = \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$\Downarrow$$

$$2\sinh^2 x + 1 = \frac{e^{2x} - 2 + e^{-2x}}{2} + 1 = \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x \checkmark$$

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② $2\cosh(\ln x) = ?$ $\qquad \cosh x = \frac{e^x + e^{-x}}{2}$ olduğundan

$$2.\cosh(\ln x) = 2.\frac{e^{\ln x} + e^{-\ln x}}{2} \Leftarrow \left(\begin{array}{l} e^{\ln x} = x \\ e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x} \end{array} \right)$$

$$= 2.\frac{x + \frac{1}{x}}{2} = x + \frac{1}{x}$$

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③ $y = \ln(\sinh x) \Rightarrow y' = ?$

$$y' = \frac{(\sinh x)'}{\sinh x} = \frac{\cosh x}{\sinh x} = \coth x$$

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④ $y = \sinh(e^{\cosh x}) \Rightarrow y' = ?$

$$y' = \sinh x . e^{\cosh x} . \cosh(e^{\cosh x})$$

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⑤ $y = \operatorname{sech}(\ln(\cos x)) \Rightarrow y' = ?$

$$y' = \frac{-\sin x}{\cos x} . (-\operatorname{sech}(\ln \cos x) . \tanh(\ln \cos x))$$

F9

6) $\sin\left(\operatorname{ArcCos}\frac{\sqrt{2}}{2}\right) = ?$ $\Rightarrow$ $\sin\left(\operatorname{ArcCos}\frac{\sqrt{2}}{2}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

7) $\sec\left(\operatorname{ArcCos}\frac{1}{2}\right) = ?$ $\Rightarrow$ $\sec\left(\operatorname{ArcCos}\frac{1}{2}\right) = \sec\frac{\pi}{3} = 2$

8) $\tan\left(\operatorname{ArcSin}\left(-\frac{1}{2}\right)\right) = ?$ $\Rightarrow$ $\tan\left(\operatorname{ArcSin}\frac{-1}{2}\right) = \tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

9) $\lim\limits_{x\to 1^-} \operatorname{ArcSin}x = \frac{\pi}{2}$ $\quad \lim\limits_{x\to\infty} \operatorname{ArcTan}x = \frac{\pi}{2}$ $\quad \lim\limits_{x\to\infty} \operatorname{ArcCosec}x = 0$

10) $y = \operatorname{ArcCos}\frac{1}{x}$ $\Rightarrow$ $y' = -\dfrac{-\frac{1}{x^2}}{\sqrt{1-\frac{1}{x^2}}}$

11) $y = \operatorname{ArcTan}(\ln x)$ $\Rightarrow$ $y' = \dfrac{\frac{1}{x}}{1+\ln^2 x}$

12) $y = \operatorname{ArcSin}e^x$ $\Rightarrow$ $y' = \dfrac{e^x}{\sqrt{1-e^{2x}}}$

13) $y = \operatorname{ArcCos}(\ln \sin x)$ $\Rightarrow$ $y' = -\dfrac{\frac{\cos x}{\sin x}}{\sqrt{1-(\ln\sin x)^2}} = -\dfrac{\cot x}{\sqrt{1-\ln^2 \sin x}}$

14) $y = \ln(\operatorname{Arctan}x)$ $\Rightarrow$ $y' = \dfrac{\frac{1}{1+x^2}}{\operatorname{Arctan}x} = \dfrac{1}{(1+x^2)\operatorname{Arctan}x}$

(15) $\lim\limits_{x\to 0} \dfrac{\text{ArcSin}x}{x} = ?$

$$\left.\begin{array}{l} \text{ArcSin}x = y \\ x\to 0 \quad y \to 0 \\ x = \text{Sin}y \end{array}\right\} \Rightarrow \lim_{x\to 0}\frac{\text{ArcSin}x}{x} = \lim_{y\to 0}\frac{y}{\text{Sin}y} = 1$$

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(16) $\lim\limits_{x\to 0^+} \dfrac{(\text{Arctan}\sqrt{x})^2}{x\sqrt{x+1}} = ?$

$$\left.\begin{array}{l} \text{Arctan}\sqrt{x} = y \\ \Downarrow \\ \text{Tan}\, y = \sqrt{x} \\ x\to 0^+ \Rightarrow y \to 0^+ \\ 1+\text{Tan}^2 y = \text{Sec}^2 y \end{array}\right\} \lim_{x\to 0^+}\frac{(\text{Arctan}\sqrt{x})^2}{x\sqrt{x+1}} = \lim_{y\to 0^+}\frac{y^2}{(\text{Tan}y)^2 \cdot \sqrt{1+\text{Tan}^2 y}}$$

$$= \lim_{y\to 0^+}\underbrace{\left(\frac{y}{\text{Tan}y}\right)^2}_{1} \cdot \underbrace{\frac{1}{\text{Sec}y}}_{1} = 1$$

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(17) $\lim\limits_{x\to\infty}\left(\dfrac{x+7}{x+3}\right)^{2x+3} = ?$

$$\lim_{x\to\infty}\left(\frac{x+7}{x+3}\right)^{2x+3} = \lim_{x\to\infty}\Bigg[\underbrace{\left(1+\frac{4}{x+3}\right)^{x+3}}_{e^4}\Bigg]^{\underbrace{\frac{2x+3}{x+3}}_{2}} = (e^4)^2 = e^8$$

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(18) $\lim\limits_{x\to\infty}\left(\dfrac{x}{x+2}\right)^{3x} = ?$

$$\lim_{x\to\infty}\left(\frac{x}{x+2}\right)^{3x} = \lim_{x\to\infty}\frac{1}{\Big(\underbrace{\left(1+\frac{2}{x}\right)^{x}}_{e^2}\Big)^3} = \frac{1}{e^6}$$

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(19) $f(x) =$ ArcSinx 'in türevinin $= \dfrac{1}{\sqrt{1-x^2}}$ olduğunu gösteriniz.

$f(x) = \text{Sin}x$ olsun. $f^{-1}(x) = \text{ArcSin}x$ olur.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\text{Cos}(\underbrace{\text{ArcSin}x}_{a})} = \frac{1}{\text{Cos}a} = \frac{1}{\sqrt{1-x^2}} \leftarrow \left\{\begin{array}{l} \text{ArcSin}x = a \text{ olsun.} \\ \Downarrow \\ x = \text{Sin}a \\ \text{Cos}a = \sqrt{1-x^2} \end{array}\right.$$