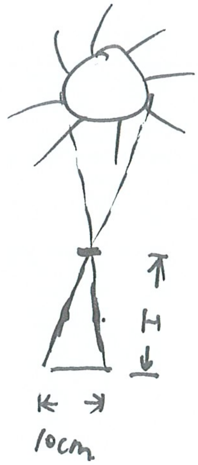
Yigong Zhang HW2

BioE 165.

1.



$$\frac{D_{sun}}{10cm} = \frac{L_{sun-earth}}{H}$$

$$H = \frac{10cm \cdot 149.6 \times 10^6 \ km}{1.391 \times 10^6 \ km}$$

$$= 1075.5 \ cm$$

$$\approx 10.75 \ m.$$

2. The general rule of convolution of $\delta(x-x_0)$.

$$\cancel{f(x) * \delta(x-x_0) = \int du \ f(u) \cdot \delta x}$$

$$g(x) \equiv \delta(x-x_0).$$

$$f(x) * \delta(x-x_0) = f(x) * g(x) = \int_{-\infty}^{\infty} du \ f(u) \cdot g(x-u) = \int_{-\infty}^{\infty} du \ f(u) \cdot \delta(x-u-x_0)$$

$$= f(x-x_0).$$

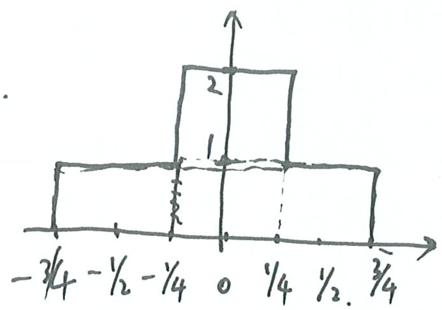
So $f(x) * \delta(x-x_0) = f(x-x_0)$

a) $\left[\delta(x-\frac{a}{2})+\delta(x+\frac{a}{2})\right] * \mathrm{rect}(x)$

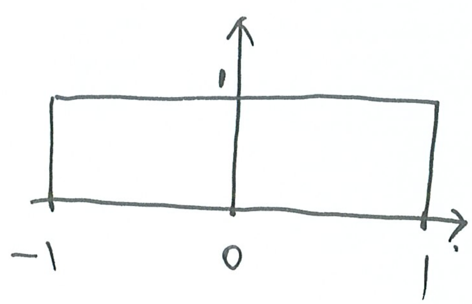
$= \mathrm{rect}(x-\frac{a}{2}) + \cancel{\delta(x+\frac{a}{2})}\ \mathrm{rect}(x+\frac{a}{2})$. where $a = \frac{1}{2}, 1, 2$.

b). ~~$\left[\delta(x-\frac{a}{2})+\delta(x+\frac{a}{2})\right]$~~

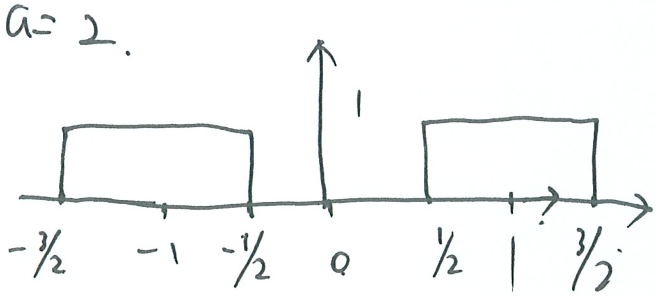
$a = \frac{1}{2}$.



$a = 1$



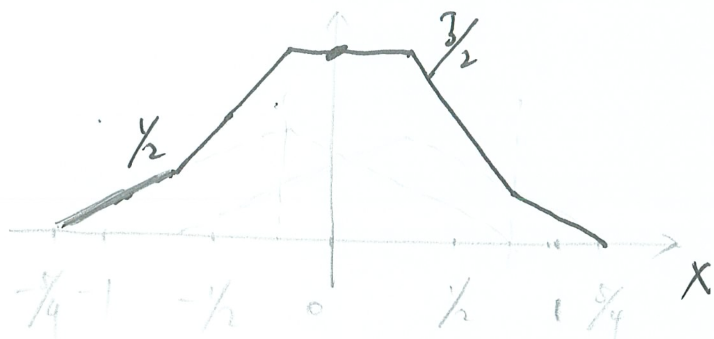
$a = 2$.



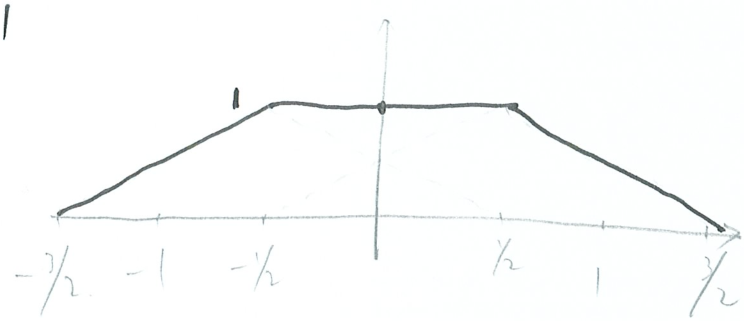
b) $\left[\delta(x-a/2)+\delta(x+a/2)\right] * \Delta(x)$

$= \Delta(x-a/2) + \Delta(x+a/2)$

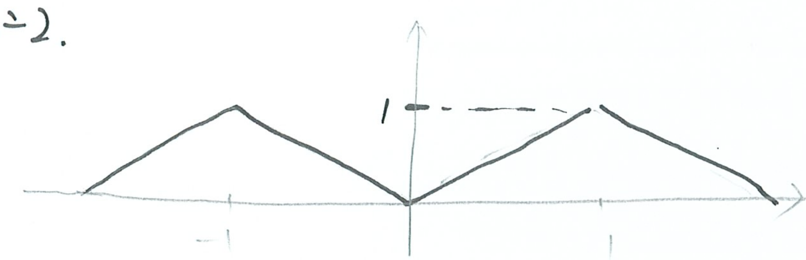
$a = 1/2$



$a = 1$



$a = 2.$



3.

$$\delta_a(x) \triangleq \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2} \quad \text{as} \quad a \to 0.$$

$$\delta(x/a) \triangleq \frac{1}{\sigma\sqrt{\pi}} \cdot e^{-\frac{(x/a)^2}{\sigma^2}} \quad \text{as} \quad \sigma \to 0.$$

$$= \frac{1}{\sigma\sqrt{\pi}} \cdot e^{-\frac{x^2}{(a\sigma)^2}}$$

$$\overset{t^2=(a\cdot\sigma)^2}{\Longrightarrow} \frac{|a|}{t\sqrt{\pi}} \cdot e^{-x^2/t^2}$$

the absolute sign comes from the arbitrary sign of a.

$$= |a| \cdot \delta(t) \qquad \text{because} \quad \frac{1}{t\cdot\sqrt{\pi}} \cdot e^{-x^2/t^2} = \delta(t)$$

$$\overset{x=t}{=} |a| \cdot \delta(x)$$

so $\delta(x/a) = |a| \cdot \delta(x)$

4. a)

For CT, to see the blurring, the ~~optimized resolution~~ phatom size should be ~ resolution = 200 μm.
And the material of the phantom needs to have high $\mu(x,y)$ to give contrast.
So a lead disk with diameter = 200 μm is a good candidate.

b) ~~For MRI~~ Same as (a), we wish to have an object which has comparable dimension as the resolution and high contrast to the background. So we would like to have a cubic of water in 1mm x 1mm x 1mm.

c). we would like to have a 1 mCi radioactive source in size of 2.5mm x 2.5mm x 2.5mm.

5.

It's hard to have a impulse $= \exp(i 2\pi k_n \cdot x)$. But we can benefit from the linearity of linear system to have a set of data look like the output of $\exp(i 2\pi k_n x)$

$$e^{i 2\pi k_n x} = \cos(2\pi k_n x) + i \sin(2\pi k_n x)$$

So we can pass $f(x) = \cos(2\pi k_n x)$ and $g(x) = \sin(2\pi k_n x)$ separately to the system. Let's say we get $F(x)$ and $G(x)$.

So $F(x) + i G(x)$ will give us the output look like we have an input $= e^{i 2\pi k_n x}$.

6.

a) $h_n = \exp(-\pi x^2 / W_n^2)$

$h_{tot} = h_1 * h_2 * h_3 * \cdots h_N$

$H_{tot}(k) = H_1 \cdot H_2 \cdot H_3 \cdots H_N$

$H_n(k) = \mathcal{F}\{h_n(x)\}(k) = \sqrt{\frac{\pi}{\pi/W_n^2}} \cdot e^{-\pi^2 k^2 / \frac{\pi}{W_n^2}}$

$= W_n \cdot e^{-\pi \cdot W_n^2 \cdot k^2}$

because $\mathcal{F}\{e^{-ax^2}\}(k) = \sqrt{\frac{\pi}{a}} \cdot e^{-\pi^2 k^2 / a}$

so $H_{tot}(k) = \prod_{i=1}^{n} W_i \cdot e^{-\pi \cdot k^2 \cdot \sum_{i=1}^{n} W_i^2}$

$h_{tot}(x) = \mathcal{F}^{-1}\{H_{tot}(k)\} = \prod_{i=1}^{n} W_i \cdot \sqrt{\sum_{i=1}^{n} W_i^2} \cdot e^{-\frac{\pi}{\sum_{i=1}^{n} W_i^2} \cdot x^2}$

~~b).~~

$\cancel{H_{tot}(k) = 1\times2\times3\times4 \cdot e^{-\pi \cdot k^2 (1+4+9+16)}}$

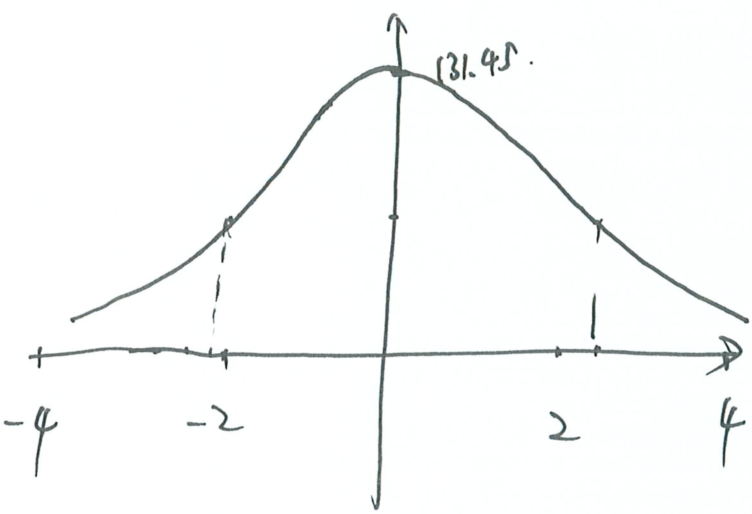
$\cancel{= 24 \cdot e^{-30\pi \cdot k^2}}$

so $h_{tot}(x) = \prod_{i=1}^{n} w_i \cdot \sqrt{\sum_{i=1}^{n} w_i^2} \cdot e^{\frac{-\pi}{\sum_{i=1}^{n} w_i^2}\cdot x^2}$

b). $$h_{tot}(x) = 1\times2\times3\times4\cdot\sqrt{1^2+2^2+3^2+4^2}\cdot e^{-\frac{\pi}{30}\cdot x^2}$$

$$= 24\cdot\sqrt{30}\cdot e^{-\frac{\pi}{30}\cdot x^2}.$$

$$=131.45\cdot e^{-0.105\cdot x^2}$$



c). $$h_{tot}(0)\Big|_{w_n=1 \text{ for } n=1,2,3,4} = 2\cdot e^{-\frac{\pi}{4}x^2} = 2$$

$$h_{tot}(0)\Big|_{w_n=4 \text{ for } n=1,2,3,4} = 4^4\cdot\sqrt{4^3}\cdot e^{-\frac{\pi}{4^3}\cdot x^2} = 2048.$$

$$\frac{h_{tot}(0)\big|_{w_n=1}}{h_{tot}(0)\big|_{w_n=4}} = \frac{2}{2048} = 9.77\times10^{-4}$$

7. a) No. Actually the ~~o~~ concepts of resolution and detection are mixed here. resolution limit defines how well we can resolve two objects when they are very close to each other. So resolution limit = 2.5×10^{-4} rad (from HW1) means we can only resolve two objects ~~to~~ at this limit, which doesn't necessarily mean we can only see things at this big.

b) No. It won't get bigger. Because we can't resolve the change in size as it's beyond our resolution limit. But ~~we~~ we can still see it because it's still not beyond our detection limit.

~~Actually, there's no diff~~

8 a)

rect function : base function.

$f(n \cdot \Delta x)$

$f'(n \cdot \Delta x)$ [derivative of $f(x)$ at $x = n \cdot \Delta x$]

b)

$$f(x) = \sum_{n=-\infty}^{\infty} \left[f(n \cdot \Delta x) + (x - n \cdot \Delta x) \cdot f'(n \cdot \Delta x) \right] \cdot \mathrm{rect}\left[\frac{(x - n \cdot \Delta x)}{\Delta x} \right]$$

c) Yes. Because we know that

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \mathrm{rect}\left[\frac{x - n \cdot \Delta x}{\Delta x} \right] = \delta(x)$$

$$\cancel{\lim_{\Delta x \to 0} \mathrm{rect}\left[\frac{x - n \cdot \Delta x}{\Delta x} \right] =}$$

d). $\mathcal{F}\left\{\frac{1}{\Delta x}\text{rect}\left[\frac{x-n\cdot\Delta x}{\Delta x}\right]\right\}(k)$

$$= \frac{1}{\Delta x}\cdot\mathcal{F}\left\{\text{rect}\left[\frac{x-n\cdot\Delta x}{\Delta x}\right]\right\} = \frac{1}{\Delta x}\cdot e^{-2\pi\cdot(n\cdot\Delta x)i\cdot k}\cdot\mathcal{F}\left\{\text{rect}\left[\frac{x}{\Delta x}\right]\right\}$$

$$= \cancel{\frac{1}{\Delta x}}\cdot e^{-2\pi(n\Delta x)\cdot i\cdot k}\,\cancel{\Delta x}\,\text{Sinc}(\Delta x\cdot k)$$

So

$$\text{So}\quad \mathcal{F}\left\{\lim_{\Delta x\to 0}\frac{1}{\Delta x}\text{rect}\left[\frac{x-n\cdot\Delta x}{\Delta x}\right]\right\}(k)$$

$$= \lim_{\Delta x\to 0}\cdot\mathcal{F}\left\{\frac{1}{\Delta x}\cdot\text{rect}\left[\frac{x-n\Delta x}{\Delta x}\right]\right\}(k)$$

$$= \lim_{\Delta x\to 0} e^{-2\pi(n\Delta x)\cdot i\cdot k}\cdot\text{Sinc}(\Delta x\cdot k)$$

$$= 1\cdot 1$$

$$= 1$$

we know that $\mathcal{F}\{\delta(x)\} = 1$

so the FT of the base matches with the FT of $\delta(x)$, which agrees with what I said in (c).