1. Yigong Zhang.

a) SNR. we have no signal but only noise. ~~So SNR is the~~

b) Resolution. The water drop on the mirror distorts the image behind it. (blurs)

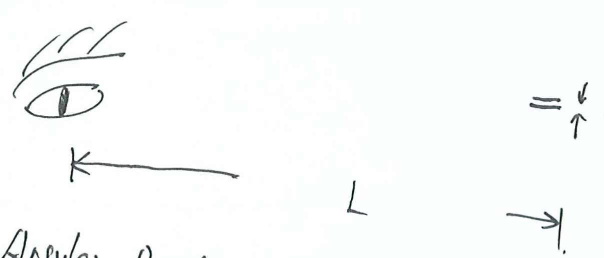
c) Resolution. The first image we got from Hubble telescope is blurred because it's not able to focus the light in our eye. So the resolution is the challenge.

d) Contrast. It's so hard to disting Waldo from ~~the~~ the others. Because other people have the similar pattern as Waldo. Waldo is not able to stand out.

e) Resolution. Our eye's sampling rate is not high enough to ~~sample~~ sample/resolve the Led light is turning on/off

f) SNR. The noise, which comes from the reflection of the sun from window, is too high comparing to the signal (friend).

2. My eye can seperate them at ~ 2m away.



Angular Resolution $= \frac{w}{L} = \frac{0.5 \times 10^{-3}}{2} = 2.5 \times 10^{-4}$ rad

3.

a)

$$\frac{df(x)}{dx} + a \cdot f(x) = u(x)$$

Check Linear.

$$h(x) = C_1 \cdot f_1(x) + C_2 \cdot f_2(x)$$

$$\mathcal{L}\{h(x)\} = \frac{dh(x)}{dx} + a\,h(x) = \frac{d\left[C_1 f_1(x) + C_2 f_2(x)\right]}{dx} + a \cdot \left[C_1 f_1(x) + C_2 f_2(x)\right]$$

$$= C_1 \cdot \left[\frac{df_1(x)}{dx} + a \cdot f_1(x)\right] + C_2 \cdot \left[\frac{df_2(x)}{dx} + a f_2(x)\right] \quad ①$$

$$\mathcal{L}[C_1 f_1(x)] + \mathcal{L}[C_2 f_2(x)] = \frac{dC_1 f_1(x)}{dx} + a \cdot C_1 f_1(x) + \frac{dC_2 f_2(x)}{dx} + a \cdot C_2 f_2(x)$$

$$= C_1\left[\frac{df_1(x)}{dx} + a f_1(x)\right] + C_2\left[\frac{df_2(x)}{dx} + a f_2(x)\right] \quad ②$$

① ≡ ② so proof!

Check shift-Invariant:

$$\frac{df(x-a)}{d(x-a)} + a f(x-a) = \frac{df(x-a)}{dx} + a \cdot f(x-a) \quad ③$$

$$u(x-a) = \frac{df(x-a)}{dx} + a \cdot f(x-a) \quad ④$$

③ ≡ ④ So it's shift invariant

So it's LTI!

b). $\frac{df(x)}{dx} = -b \cdot U(x) \cdot f(x)$

$-\frac{1}{b}\frac{1}{f(x)} \cdot \frac{df(x)}{dx} = U(x)$

$h(x) = a_1 f_1(x) + a_2 \cdot f_2(x)$

$\mathcal{L}\{h(x)\} = -\frac{1}{b}\frac{1}{h(x)} \cdot \frac{d\,h(x)}{dx}$

$= -\frac{1}{b} \cdot \frac{1}{a_1 f_1(x) + a_2 f_2(x)} \cdot \frac{d[a_1 f_1(x) + a_2 f_2(x)]}{dx}$ ⑤

$\mathcal{L}\{a_1 f_1(x)\} + \mathcal{L}\{a_2 f_2(x)\}$

$= -\frac{1}{b} \cdot \left[\frac{1}{a_1 f_1(x)} \cdot \frac{d\,a_1 f_1(x)}{dx} + \frac{1}{a_2 f_2(x)} \cdot \frac{d\,a_2 f_2(x)}{dx}\right]$ ⑥

⑤ ≠ ⑥ so, it's not linear.

check. shift invariant:

$f(x-a)$.

$\mathcal{L}\{f(x-a)\} = -\frac{1}{b}\frac{1}{f(x-a)} \cdot \frac{df(x-a)}{dx}$ ⑦

$U(x-a) = -\frac{1}{b}\frac{1}{f(x-a)} \cdot \frac{df(x-a)}{dx}$ ⑧

(7) $\equiv$ (8) So it's shift-invariant.

So this system is non-linear shift invariant system.

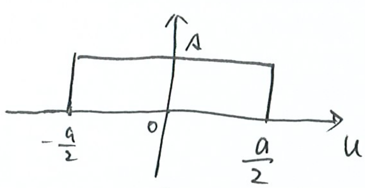
4.

$$f(x) * h(x) = \int_{-\infty}^{\infty} f(u)\cdot h(x-u)\cdot du$$

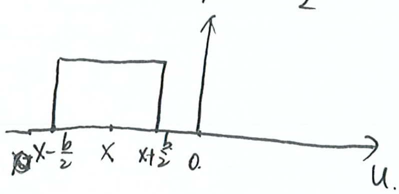
$$f(x) \Leftrightarrow A\cdot \mathrm{rect}(x/a)$$

$$h(x) \Leftrightarrow B\cdot \mathrm{rect}(x/b)$$

So $f(u)$:



$h(x-u)$



$\shortparallel$

$B\cdot \mathrm{rect}\left(\frac{x-u}{b}\right)$

Region I :

$$\left\{\begin{array}{l} x+\frac{b}{2} < -\frac{a}{2} \\ \text{or} \\ x-\frac{b}{2} > \frac{a}{2} \end{array}\right. \Rightarrow \quad x < -\frac{a+b}{2} \quad \text{or} \quad x > \frac{a+b}{2}$$

$$\int_{-\infty}^{\infty} f(u)\cdot h(x-u)\, du = 0$$

Region II :

$$x+\frac{b}{2} > -\frac{a}{2} \quad \& \quad x-\frac{b}{2} < -\frac{a}{2}$$

$$\Rightarrow \quad -\frac{a+b}{2} < x < -\frac{a-b}{2}.$$

$\cancel{\text{or}}$ $\cancel{x+\frac{b}{2} > \frac{a}{2} \ \& \ x-\frac{b}{2} < \frac{a}{2}}$

$\cancel{\frac{a-b}{2} < x < \frac{a+b}{2}}$

$$\int_{-\infty}^{\infty} f(u)\cdot h(x-u)\,du = \left(x+\frac{b}{2}+\frac{a}{2}\right)\times A\cdot B = \left(x+\frac{a+b}{2}\right)\cdot AB$$

Region $\underline{\text{III}}$:

$$x+\frac{b}{2} > \frac{a}{2} \quad \& \quad x-\frac{b}{2} < \frac{a}{2}$$

$$\frac{a-b}{2} < x < \frac{a+b}{2}$$

$$\int_{-\infty}^{\infty} f(u)\cdot h(x-u)\,du = \left[\frac{a}{2}-\left(x-\frac{b}{2}\right)\right]\times A\cdot B = \left(-x+\frac{a+b}{2}\right)\cdot AB$$

Region IV:

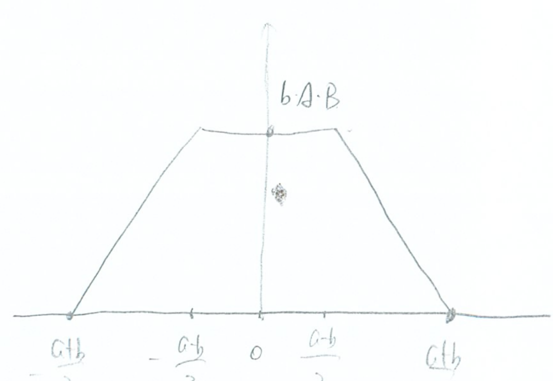
$$x-\frac{b}{2} > -\frac{a}{2} \quad \& \quad x+\frac{b}{2} < \frac{a}{2}$$

$$\frac{b-a}{2} < x < \frac{a-b}{2}$$

$$\int_{-\infty}^{\infty} f(u)\,h(x-u)\,du = b\cdot A\cdot B$$

So

$$f(x)*h(x) = \begin{cases} 0 & x < -\frac{a+b}{2} \\ \left(x+\frac{a+b}{2}\right)AB & -\frac{a+b}{2} < x < -\frac{a-b}{2} \\ b\cdot A\cdot B & -\frac{a-b}{2} < x < \frac{a-b}{2} \\ \left(-x+\frac{a+b}{2}\right)AB & \frac{a-b}{2} < x < \frac{a+b}{2} \\ 0 & x > \frac{a+b}{2} \end{cases}$$



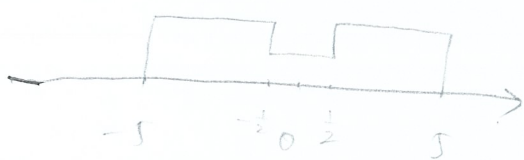
No. The asymmetry doesn't bother me. When ~~x~~ the peak reaches its amp, the range of x that we are integrating is b, not a. So we'll only get b in the product.

5.

$$f(x) = \mathrm{rect}(x/10) - 0.5\,\mathrm{rect}(x)$$

$$h(x) = \mathrm{rect}(x)$$

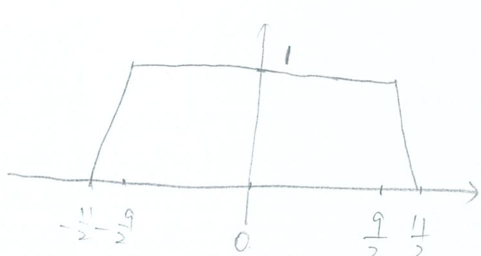
$f(x)$:



$$f(x) * h(x) = \underbrace{\mathrm{rect}(x/10) * \mathrm{rect}(x)}_{①} - \underbrace{0.5 \cdot \mathrm{rect}^2(x)}_{②}$$

①: Just like problem 4 with $\begin{cases} A = 1 \\ a = 10 \\ B = 1 \\ b = 1 \end{cases}$

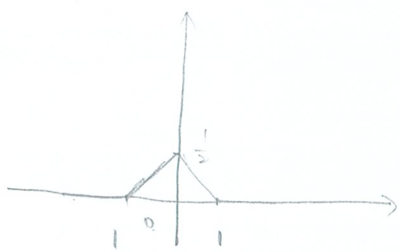
①:



③ Also like 4:

$$\begin{cases} A = 0.5 \\ a = 1 \\ B = 1 \\ b = 1 \end{cases}$$

②:



So ① - ②

Comment: On the original image, the contrast is always $\frac{1}{2}$ in the depression. But in the blurred image, the max contrast we can get is $\frac{1}{2}$. If we average the contrast in the depression, we get a constrast $= \frac{1}{4}$, which drops to the half of the original contrast.

6.

$$f(x/b) * h(x/b) \xrightarrow[w(x)=h(x/b)]{v(x)=f(x/b)} v(x) * w(x) = \int_{-\infty}^{\infty} v(u)\cdot w(x-u)\cdot du$$

$$= \int_{-\infty}^{\infty} f\left(\frac{u}{b}\right)\cdot h\left(\frac{x-u}{b}\right)\cdot du$$

$$\overset{\alpha = \frac{u}{b}}{=\!=} \int_{-\infty}^{\infty} f(\alpha)\cdot h\left(\frac{x}{b}-\alpha\right)\cdot d(\alpha\cdot b)$$

The absolute value comes from the flip of the integrating range when $b<0$.

$$= |b|\cdot \int_{-\infty}^{\infty} f(\alpha)\cdot h\left(\frac{x}{b}-\alpha\right) d\alpha$$

$$= |b|\cdot g\left(\frac{x}{b}\right) \qquad \text{if } f(x) * h(x) = g(x)$$

7.

$$f(x) * g(x) = \cancel{\iint} \int_{-\infty}^{\infty} f(u) \cdot g(x-u) \cdot du$$

Area under a convol.

$$\Rightarrow \int_{-\infty}^{\infty} f(x) * g(x) \cdot dx = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} du\, f(u) \cdot g(x-u) = \int_{-\infty}^{\infty} du\, f(u) \cdot \int_{-\infty}^{\infty} dx\, g(x-u)$$

$$\overset{v=x-u}{=\!=} \underbrace{\int_{-\infty}^{\infty} du \cdot f(u)}_{\text{Area under } f(x)} \cdot \underbrace{\int_{-\infty}^{\infty} dv \cdot g(v)}_{\text{Area under } g(x)}$$

So. Proof!

8.

~~1d Green's function~~

$\cancel{U_t = kU_{xx} + f}$

$$\phi(x,t) = \frac{1}{\sqrt{4\pi k t}} \exp\left(-\frac{x^2}{4kt}\right).$$

a) Input: $\delta(x)$

Output $u(x,t) = \phi(x,t) = \frac{1}{\sqrt{4\pi k \cdot t}} \exp\left(-\frac{x^2}{4kt}\right)$

b) Green's function:

$$\phi(x,t) = \frac{1}{\sqrt{4\pi k t}} \exp\left(-\frac{x^2}{4kt}\right).$$

X_0 is the FWHM

$\phi(x_0,t) = \frac{1}{2} \cdot \phi(0,t)$ because $\phi(0,t)$ gives the maximum at time t.

$\Rightarrow \exp\left(-\frac{x_0^2}{4kt}\right) = \frac{1}{2}$

$-\frac{x_0^2}{4kt} = \ln\left(\frac{1}{2}\right)$

$x_0^2 = 4 \cdot \ln 2 \cdot kt$

$X_0 = \sqrt{2 \cdot \ln 2 \cdot k \cdot t}$

So $\text{FWHM} = 2 \cdot X_0 = 2 \cdot \sqrt{2 \cdot \ln 2 \cdot k \cdot t}$

d) FWHM tells us how the heat diffuses. Since heat diffuses with time, FWHM changes with time.