Report

Yihan Shi, Medy Mu, Edna Zhang

10/31/2022

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##  [1]  9.462029  9.462029 25.000000 25.000000 25.000000 12.500000 25.000000
##  [8] 25.000000 25.000000  9.462029 25.000000 25.000000 25.000000  9.462029
## [15]  9.462029 12.500000 25.000000 12.500000  9.462029  9.462029 12.500000
## [22] 25.000000 25.000000  9.462029 25.000000  9.462029  9.462029  9.462029
## [29]  9.462029  9.462029  9.462029 25.000000  9.462029 25.000000 25.000000
## [36] 12.500000 25.000000  9.462029 12.500000  9.462029 25.000000 12.500000
## [43] 12.500000 25.000000  9.462029 12.500000 12.500000  9.462029 12.500000
## [50] 12.500000 25.000000 25.000000 25.000000  9.462029  9.462029  9.462029
## [57] 25.000000  9.462029 12.500000 25.000000 12.500000 12.500000 25.000000
## [64] 25.000000  9.462029  9.462029 12.500000  9.462029  9.462029  9.462029
## [71] 12.500000 12.500000  9.462029  9.462029 25.000000 12.500000  9.462029
## [78] 12.500000  9.462029  9.462029  9.462029  9.462029 12.500000  9.462029
## [85] 12.500000 25.000000 12.500000 25.000000 25.000000
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```
##      St  Re    Fr R_moment_1 R_moment_2 R_moment_3 R_moment_4  Fr_logit
## 1  0.10 224 0.052 0.00215700 1.3035e-01 1.4374e+01 1.5865e+03  9.462029
## 2  3.00 224 0.052 0.00379030 4.7042e-01 6.9940e+01 1.0404e+04  9.462029
## 3  0.70 224   Inf 0.00290540 4.3499e-02 8.2200e-01 1.5551e+01 25.000000
## 4  0.05  90   Inf 0.06352800 9.0653e-02 4.6746e-01 3.2696e+00 25.000000
## 5  0.70 398   Inf 0.00036945 6.2242e-03 1.2649e-01 2.5714e+00 25.000000
## 6  2.00  90 0.300 0.14780000 2.0068e+00 3.6249e+01 6.7167e+02 12.500000
## 7  0.20  90   Inf 0.08127300 3.2450e-01 3.0363e+00 3.2976e+01 25.000000
## 8  3.00 224   Inf 0.00574730 1.1966e-01 2.7480e+00 6.3159e+01 25.000000
## 9  0.90 224   Inf 0.00302150 4.5244e-02 8.4530e-01 1.5809e+01 25.000000
## 10 0.60 398 0.052 0.00031431 4.4672e-03 8.2060e-02 1.5077e+00  9.462029
## 11 0.90  90   Inf 0.09102700 5.9539e-01 7.2454e+00 9.5166e+01 25.000000
## 12 0.30 398   Inf 0.00036022 6.2830e-03 1.3546e-01 2.9211e+00 25.000000
## 13 2.00 224   Inf 0.00447250 8.0804e-02 1.6668e+00 3.4408e+01 25.000000
## 14 1.00 224 0.052 0.00312380 3.6478e-01 5.3322e+01 7.7958e+03  9.462029
## 15 0.50  90 0.052 0.12670000 6.8596e+02 5.4300e+06 4.2900e+10  9.462029
## 16 0.60 224 0.300 0.00257400 3.6621e-02 6.7102e-01 1.2309e+01 12.500000
## 17 0.10  90   Inf 0.07722700 2.2120e-01 1.8833e+00 2.0190e+01 25.000000
## 18 1.00  90 0.300 0.11236000 1.1261e+00 1.7335e+01 2.8261e+02 12.500000
## 19 0.70 224 0.052 0.00285610 3.1273e-01 4.4529e+01 6.3423e+03  9.462029
## 20 0.20  90 0.052 0.11760000 5.1774e+02 3.8100e+06 2.8000e+10  9.462029
## 21 0.10  90 0.300 0.06125200 6.9867e-02 2.4338e-01 1.1379e+00 12.500000
## 22 0.50 398   Inf 0.00036800 6.3559e-03 1.3341e-01 2.8013e+00 25.000000
## 23 0.20 224   Inf 0.00269160 3.9016e-02 7.6384e-01 1.4978e+01 25.000000
## 24 1.50 398 0.052 0.00038321 5.9338e-03 1.1156e-01 2.1004e+00  9.462029
## 25 0.90 398   Inf 0.00038344 6.4432e-03 1.2925e-01 2.5935e+00 25.000000
## 26 0.50 224 0.052 0.00274240 3.0355e-01 4.3911e+01 6.3530e+03  9.462029
## 27 0.10 398 0.052 0.00027479 3.2549e-03 5.8006e-02 1.0344e+00  9.462029
## 28 0.40 224 0.052 0.00268090 2.8897e-01 4.1585e+01 5.9861e+03  9.462029
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## 29 0.30  90 0.052 0.12261000 6.2727e+02 4.9100e+06 3.8500e+10  9.462029
## 30 0.05 224 0.052 0.00173740 1.6633e-03 2.0228e-02 3.6438e-01  9.462029
## 31 1.50 224 0.052 0.00341630 4.0300e-01 5.8417e+01 8.4710e+03  9.462029
## 32 0.80  90   Inf 0.09107400 6.1825e-01 7.4973e+00 9.7048e+01 25.000000
## 33 1.50  90 0.052 0.15181000 9.9690e+02 8.5500e+06 7.3300e+10  9.462029
## 34 0.05 398   Inf 0.00022202 1.0055e-03 1.0857e-02 1.1782e-01 25.000000
## 35 0.80 224   Inf 0.00298090 4.4580e-02 8.3764e-01 1.5759e+01 25.000000
## 36 0.90 224 0.300 0.00280490 4.1143e-02 7.5132e-01 1.3729e+01 12.500000
## 37 0.40 224   Inf 0.00292630 4.6261e-02 9.2914e-01 1.8681e+01 25.000000
## 38 0.80 398 0.052 0.00033341 4.9036e-03 9.1143e-02 1.6948e+00  9.462029
## 39 1.50 224 0.300 0.00341050 5.4101e-02 1.0222e+00 1.9340e+01 12.500000
## 40 0.20 224 0.052 0.00257870 2.6830e-01 3.9080e+01 5.6959e+03  9.462029
## 41 0.30 224   Inf 0.00283770 4.3589e-02 8.6962e-01 1.7373e+01 25.000000
## 42 0.30 224 0.300 0.00250630 3.5881e-02 6.8596e-01 1.3132e+01 12.500000
## 43 2.00 224 0.300 0.00381230 6.1927e-02 1.1844e+00 2.2705e+01 12.500000
## 44 1.00  90   Inf 0.09691800 6.7696e-01 8.2384e+00 1.0602e+02 25.000000
## 45 0.80 224 0.052 0.00295750 3.3361e-01 4.8161e+01 6.9539e+03  9.462029
## 46 1.00 224 0.300 0.00289530 4.2300e-02 7.6755e-01 1.3941e+01 12.500000
## 47 0.70  90 0.300 0.09471100 6.9751e-01 9.1793e+00 1.3187e+02 12.500000
## 48 0.30 224 0.052 0.00256750 2.6547e-01 3.7665e+01 5.3451e+03  9.462029
## 49 0.40 224 0.300 0.00262070 3.9502e-02 7.6851e-01 1.4966e+01 12.500000
## 50 0.10 224 0.300 0.00221530 2.4475e-02 4.2167e-01 7.2842e+00 12.500000
## 51 3.00  90   Inf 0.17234000 2.2386e+00 4.0454e+01 7.6198e+02 25.000000
## 52 1.00 224   Inf 0.00309680 4.6454e-02 8.6381e-01 1.6077e+01 25.000000
## 53 2.00 398   Inf 0.00053647 1.0022e-02 2.1023e-01 4.4109e+00 25.000000
## 54 0.80  90 0.052 0.13793000 8.2524e+02 6.8000e+06 5.6100e+10  9.462029
## 55 0.40 398 0.052 0.00029691 4.1375e-03 7.6124e-02 1.4014e+00  9.462029
## 56 0.50 398 0.052 0.00030716 4.3494e-03 8.0143e-02 1.4770e+00  9.462029
## 57 0.70  90   Inf 0.09217600 5.6482e-01 6.7191e+00 8.8723e+01 25.000000
## 58 2.00  90 0.052 0.15433000 1.0269e+03 8.8700e+06 7.6700e+10  9.462029
## 59 0.90  90 0.300 0.10962000 1.0319e+00 1.5797e+01 2.6136e+02 12.500000
## 60 0.30  90   Inf 0.07694500 3.2652e-01 3.4052e+00 4.1042e+01 25.000000
## 61 0.50 224 0.300 0.00250710 3.5152e-02 6.4378e-01 1.1801e+01 12.500000
## 62 0.50  90 0.300 0.08477300 4.9728e-01 6.0317e+00 8.3287e+01 12.500000
## 63 0.80 398   Inf 0.00037399 6.2457e-03 1.2542e-01 2.5193e+00 25.000000
## 64 0.20 398   Inf 0.00033521 5.4505e-03 1.1408e-01 2.3884e+00 25.000000
## 65 0.70  90 0.052 0.13173000 7.3694e+02 5.8700e+06 4.6700e+10  9.462029
## 66 2.00 398 0.052 0.00039644 6.1040e-03 1.1209e-01 2.0593e+00  9.462029
## 67 0.70 224 0.300 0.00260870 3.6438e-02 6.5445e-01 1.1765e+01 12.500000
## 68 0.60 224 0.052 0.00279390 3.0594e-01 4.3745e+01 6.2554e+03  9.462029
## 69 0.30 398 0.052 0.00030066 4.3488e-03 8.3446e-02 1.6023e+00  9.462029
## 70 0.90 224 0.052 0.00305410 3.5419e-01 5.1795e+01 7.5758e+03  9.462029
## 71 0.80 224 0.300 0.00268160 3.7714e-02 6.7549e-01 1.2112e+01 12.500000
## 72 0.20 224 0.300 0.00246950 3.4818e-02 6.7088e-01 1.2939e+01 12.500000
## 73 3.00 398 0.052 0.00040188 5.4492e-03 9.1871e-02 1.5565e+00  9.462029
## 74 0.90  90 0.052 0.14184000 8.7019e+02 7.2500e+06 6.0400e+10  9.462029
## 75 0.40 398   Inf 0.00036977 6.4986e-03 1.3933e-01 2.9880e+00 25.000000
## 76 0.20  90 0.300 0.07798500 2.5598e-01 2.0965e+00 2.0849e+01 12.500000
## 77 0.60  90 0.052 0.12946000 7.1816e+02 5.7200e+06 4.5600e+10  9.462029
## 78 1.50  90 0.300 0.13678000 1.8254e+00 3.2833e+01 6.0903e+02 12.500000
## 79 0.10  90 0.052 0.10464000 1.6015e+02 6.9900e+05 3.0700e+09  9.462029
## 80 3.00  90 0.052 0.15538000 1.0443e+03 9.1400e+06 8.0000e+10  9.462029
## 81 2.00 224 0.052 0.00363470 4.4512e-01 6.5387e+01 9.6105e+03  9.462029
## 82 0.05  90 0.052 0.08786800 5.3449e-01 2.2205e+01 1.5679e+03  9.462029
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## 83 0.40  90 0.300 0.08095700 3.9996e-01 4.3303e+00 5.3618e+01 12.500000
## 84 0.90 398 0.052 0.00034145 5.0555e-03 9.4083e-02 1.7522e+00  9.462029
## 85 3.00  90 0.300 0.16433000 2.3317e+00 4.4516e+01 8.8779e+02 12.500000
## 86 0.05 224   Inf 0.00153380 2.5653e-04 3.0407e-04 5.4466e-04 25.000000
## 87 0.05 224 0.300 0.00135380 1.0303e-04 5.1400e-05 4.1600e-05 12.500000
## 88 0.60 224   Inf 0.00291710 4.4317e-02 8.5282e-01 1.6431e+01 25.000000
## 89 1.50 224   Inf 0.00370310 6.0910e-02 1.1829e+00 2.2990e+01 25.000000
##    C_moment_1   C_moment_2 C_moment_3  C_moment_4     St_log R_moment_3_log
## 1           0 1.303453e-01  305.42800  93371.6556 -2.3025851     2.66542102
## 2           0 4.704056e-01  216.76225  47012.2515  1.0986123     4.24763773
## 3           0 4.349056e-02   90.58974   8216.7778 -0.3566749    -0.19601488
## 4           0 8.661719e-02   17.67980    420.2523 -2.9957323    -0.76044150
## 5           0 6.224064e-03  257.58554  66372.7845 -0.3566749    -2.06759203
## 6           0 1.984955e+00   12.64607    165.0999  0.6931472     3.59041179
## 7           0 3.178947e-01   16.50481    316.6692 -1.6094379     1.11063967
## 8           0 1.196270e-01   66.36620   4409.0254  1.0986123     1.01087337
## 9           0 4.523487e-02   87.81933   7721.0616 -0.1053605    -0.16806369
## 10          0 4.467101e-03  274.83405  75549.8545 -0.5108256    -2.50030459
## 11          0 5.871041e-01   15.74800    268.5223 -0.1053605     1.98036678
## 12          0 6.282870e-03  271.98952  73994.8041 -1.2039728    -1.99907889
## 13          0 8.078400e-02   72.54591   5267.8372  0.6931472     0.51090562
## 14          0 3.647702e-01  242.01901  58584.7754  0.0000000     3.97634900
## 15          0 6.859439e+02  302.23589  91170.0466 -0.6931472    15.50744969
## 16          0 3.661437e-02   95.73594   9176.4722 -0.5108256    -0.39895634
## 17          0 2.152360e-01   18.35624    423.4300 -2.3025851     0.63302556
## 18          0 1.113475e+00   14.43312    221.7275  0.0000000     2.85272758
## 19          0 3.127218e-01  254.61236  64847.8397 -0.3566749     3.79614066
## 20          0 5.177262e+02  323.41040 104455.1834 -1.6094379    15.15313975
## 21          0 6.611519e-02   13.58821    247.0250 -2.3025851    -1.41313127
## 22          0 6.355765e-03  263.27756  69341.5542 -0.6931472    -2.01432819
## 23          0 3.900876e-02   99.10147   9837.6456 -1.6094379    -0.26939694
## 24          0 5.933653e-03  244.06118  59651.6332  0.4054651    -2.19319272
## 25          0 6.443053e-03  249.90106  62469.7333 -0.1053605    -2.04600677
## 26          0 3.035425e-01  262.55447  68945.6663 -0.6931472     3.78216486
## 27          0 3.254824e-03  312.36435  97635.2387 -2.3025851    -2.84720883
## 28          0 2.889628e-01  267.70070  71684.9030 -0.9162907     3.72773953
## 29          0 6.272550e+02  312.53231  97846.5118 -1.2039728    15.40678450
## 30          0 1.660281e-03  298.87833 132136.7312 -2.9957323    -3.90068750
## 31          0 4.029883e-01  228.33361  52156.5453  0.4054651     4.06760694
## 32          0 6.099555e-01   15.38687    253.5907 -0.2231436     2.01454296
## 33          0 9.968770e+02  271.63187  73754.7680  0.4054651    15.96144184
## 34          0 1.005451e-03  340.51941 116536.4836 -2.9957323    -4.52294525
## 35          0 4.457111e-02   88.97550   7927.6854 -0.2231436    -0.17716687
## 36          0 4.113513e-02   90.01305   8108.6103 -0.1053605    -0.28592362
## 37          0 4.625244e-02   93.36614   8727.2623 -0.9162907    -0.07349585
## 38          0 4.903489e-03  265.42521  70481.7900 -0.2231436    -2.39532558
## 39          0 5.408937e-02   81.21436   6605.7096  0.4054651     0.02195717
## 40          0 2.682934e-01  281.20085  79124.6502 -1.6094379     3.66561083
## 41          0 4.358095e-02   95.54307   9141.8631 -1.2039728    -0.13969894
## 42          0 3.587472e-02  100.91256  10198.2682 -1.2039728    -0.37693596
## 43          0 6.191247e-02   76.83720   5918.6110  0.6931472     0.16923632
## 44          0 6.675669e-01   14.74677    230.8204  0.0000000     2.10880615
## 45          0 3.336013e-01  249.93512  62479.4951 -0.2231436     3.87454956
## 46          0 4.229162e-02   88.20990   7789.4788  0.0000000    -0.26455166
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## 47          0 6.885398e-01   15.72240    270.8991 -0.3566749     2.21695095
## 48          0 2.654634e-01  275.36415  75842.9250 -1.2039728     3.62873128
## 49          0 3.949513e-02   97.87204   9589.2537 -0.9162907    -0.26330170
## 50          0 2.447009e-02  110.11634  12158.7198 -2.3025851    -0.86353226
## 51          0 2.208899e+00   11.97304    150.5336  1.0986123     3.70016553
## 52          0 4.644441e-02   86.25833   7448.1608  0.0000000    -0.14640244
## 53          0 1.002171e-02  209.53109  43913.5897  0.6931472    -1.55955311
## 54          0 8.252210e+02  286.83472  82374.5966 -0.2231436    15.73243317
## 55          0 4.137412e-03  286.02727  81860.9139 -0.9162907    -2.57539169
## 56          0 4.349306e-03  279.39201  78074.9444 -0.6931472    -2.52394274
## 57          0 5.563236e-01   15.82009    278.7572 -0.3566749     1.90495422
## 58          0 1.026876e+03  269.54000  72732.4468  0.6931472    15.99818535
## 59          0 1.019883e+00   15.01038    244.6804 -0.1053605     2.75982005
## 60          0 3.205995e-01   18.34832    389.2185 -1.2039728     1.22530368
## 61          0 3.514571e-02   97.66761   9548.5285 -0.6931472    -0.44039823
## 62          0 4.900935e-01   17.21504    338.3256 -0.6931472     1.79702889
## 63          0 6.245560e-03  254.08875  64580.9984 -0.2231436    -2.07608717
## 64          0 5.450388e-03  283.49641  80394.1536 -1.6094379    -2.17085532
## 65          0 7.369226e+02  293.41594  85989.2793 -0.3566749    15.58536519
## 66          0 6.103843e-03  235.03564  55268.2162  0.6931472    -2.18845316
## 67          0 3.643119e-02   94.07562   8859.1692 -0.3566749    -0.42396009
## 68          0 3.059322e-01  258.50276  66829.8954 -0.5108256     3.77837732
## 69          0 4.348710e-03  290.96751  84721.9253 -1.2039728    -2.48355556
## 70          0 3.541807e-01  245.71029  60386.8688 -0.1053605     3.94729362
## 71          0 3.770681e-02   92.21343   8513.6635 -0.2231436    -0.39231693
## 72          0 3.481190e-02  103.24909  10671.4338 -1.6094379    -0.39916500
## 73          0 5.449038e-03  228.38500  52416.5178  1.0986123    -2.38736986
## 74          0 8.701699e+02  282.42930  79762.5936 -0.1053605    15.79651203
## 75          0 6.498463e-03  265.95378  70750.4657 -0.9162907    -1.97091006
## 76          0 2.498983e-01   16.31043    323.5310 -1.6094379     0.74026929
## 77          0 7.181432e+02  297.20645  88412.6643 -0.5108256    15.55947936
## 78          0 1.806691e+00   13.21391    181.1416  0.4054651     3.49143411
## 79          0 1.601391e+02  344.90546 119702.2981 -2.3025851    13.45740602
## 80          0 1.044276e+03  270.83214  73354.8229  1.0986123    16.02817094
## 81          0 4.451068e-01  220.17238  48503.6684  0.6931472     4.18032346
## 82          0 5.267692e-01   57.71417   5622.3422 -2.9957323     3.10031749
## 83          0 3.934060e-01   17.15980    337.4809 -0.9162907     1.46563682
## 84          0 5.055383e-03  261.73157  68555.7093 -0.1053605    -2.36357791
## 85          0 2.304696e+00   12.39718    161.7028  1.0986123     3.79584868
## 86          0 2.541775e-04   74.74621   8401.6432 -2.9957323    -8.09825262
## 87          0 1.011972e-04   50.08439   4035.0817 -2.9957323    -9.87587239
## 88          0 4.430849e-02   91.39647   8364.2508 -0.5108256    -0.15920677
## 89          0 6.089629e-02   78.67076   6194.7862  0.4054651     0.16796905
```
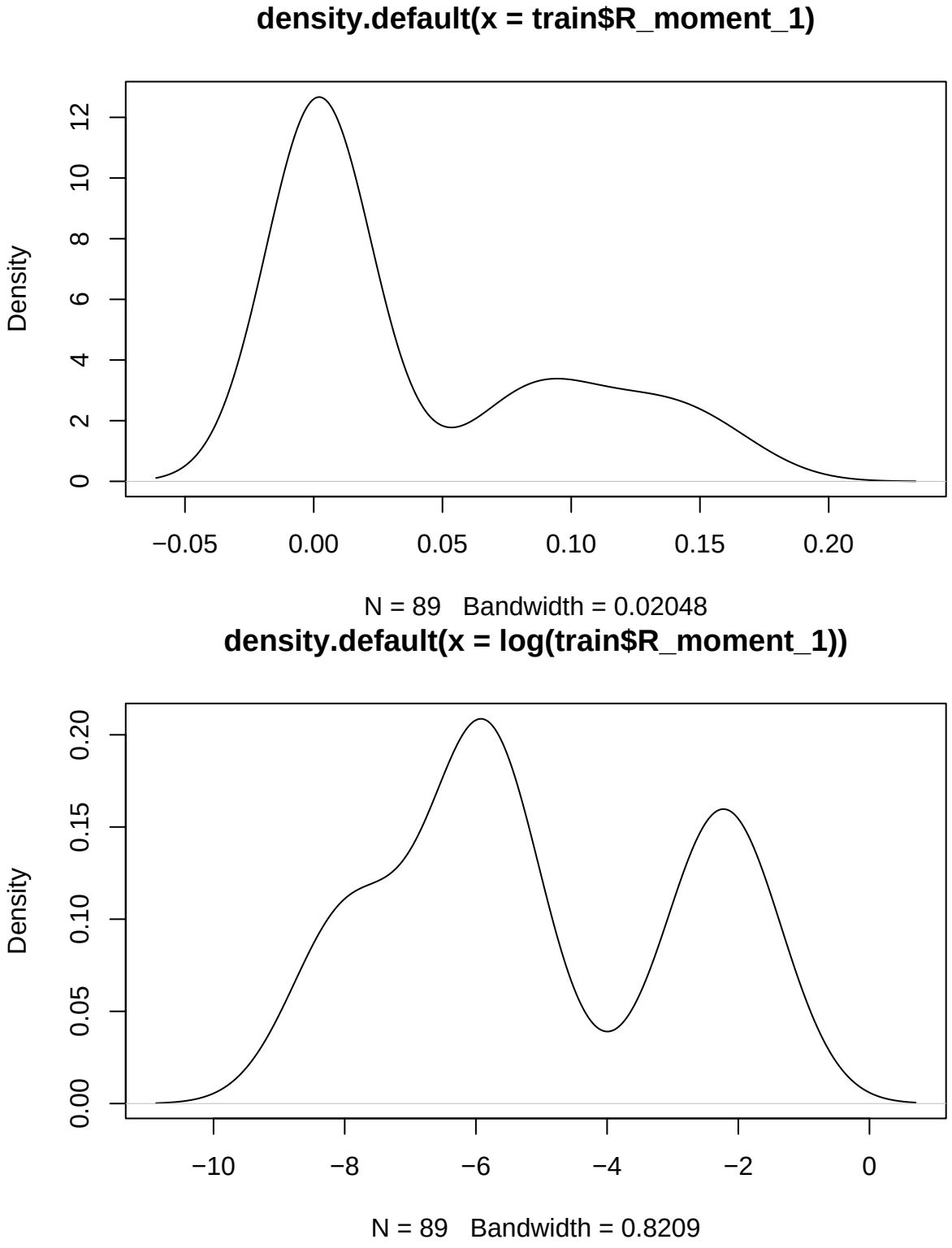

Introduction

"We are experiencing some turbulence, please fasten your seat belt." Many of us might have heard this radio on the plane and felt bumpy. When we mix paint in water, we can also observe turbulence as the color dissipates. Turbulence is so common and easily observed in daily life, yet its causes and effects are hard to predict. In fluid dynamics, turbulence is "characterized by chaotic changes in pressure and flow velocity". With some knowledge and observation in parameters such as fluid density, flow speed, and the property of particles that cluster inside turbulent flows, we can gain insights into the distribution and clustering of particles in idealized turbulence. In this case study, we will investigate 3 observed features that might contribute to

particle distribution in turbulence: Reynolds number (Re), which takes flow speed, viscosity, and density into account; Gravitational acceleration (Fr); Stokes number (St) that quantifies particle characteristics like size, relaxation time, and particle diameter. We hope to use these 3 features to explain changes in particle distribution as well as extrapolate beyond the scope of the known observations.

Model

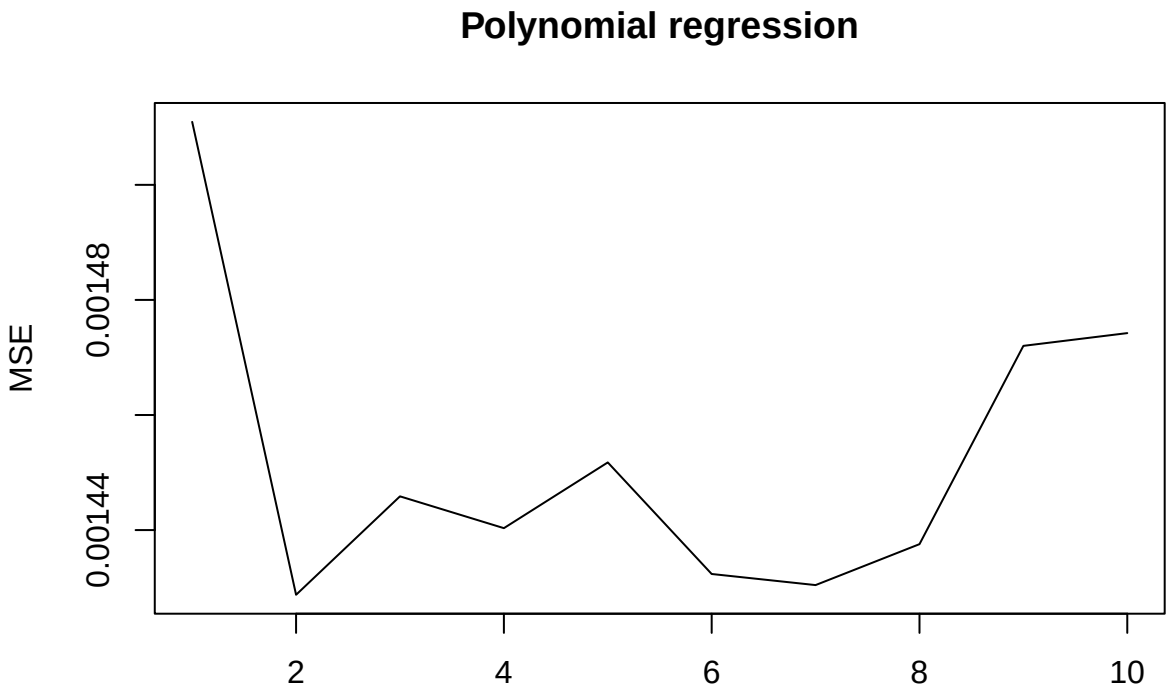
Raw Moment 1

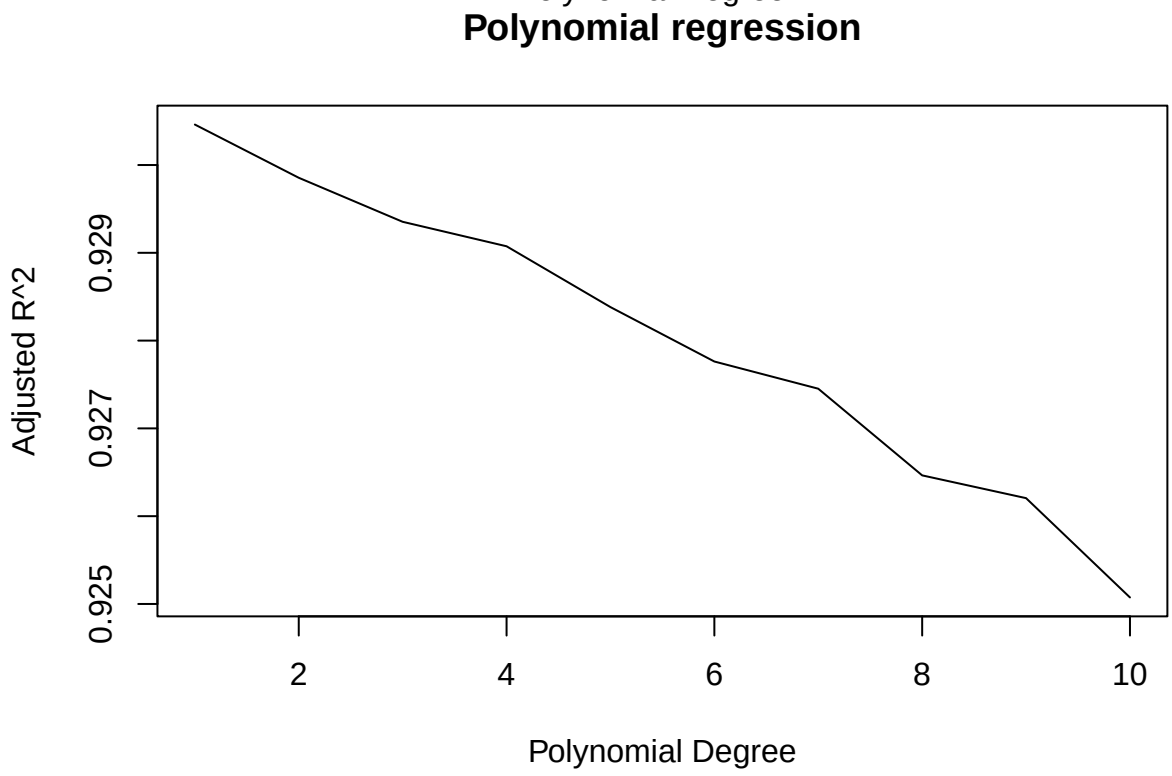


Since R_moment_1 is skewed, we do a log transformation to make it more normal.

Linear regression

Polynomial regression



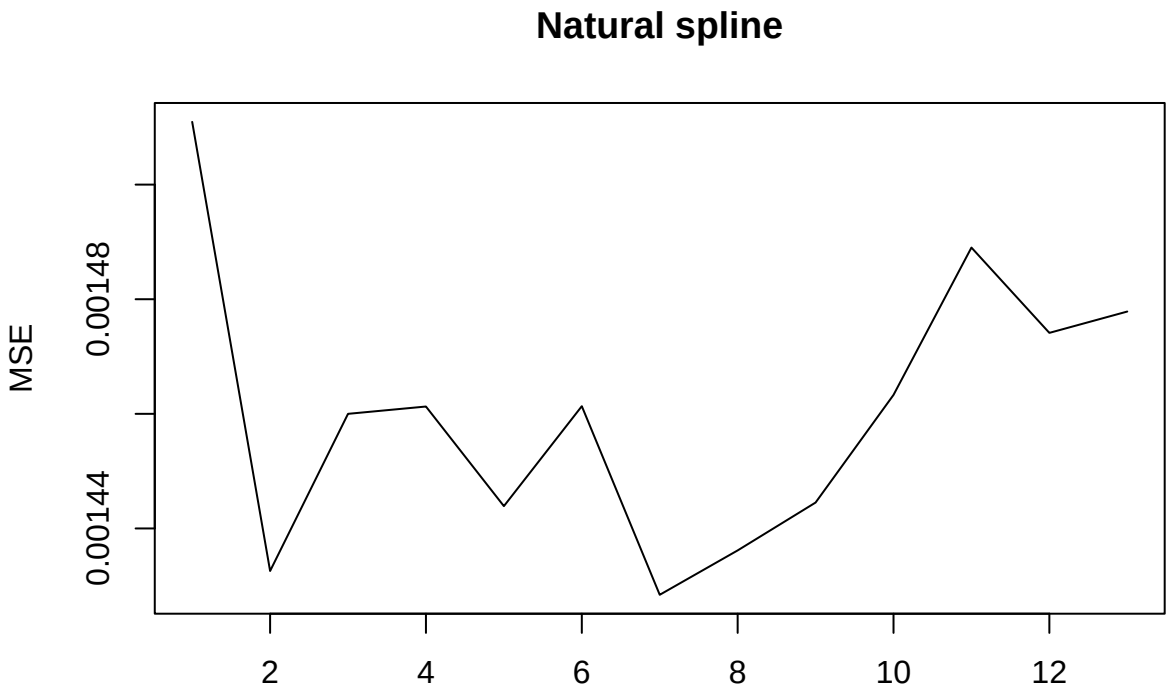


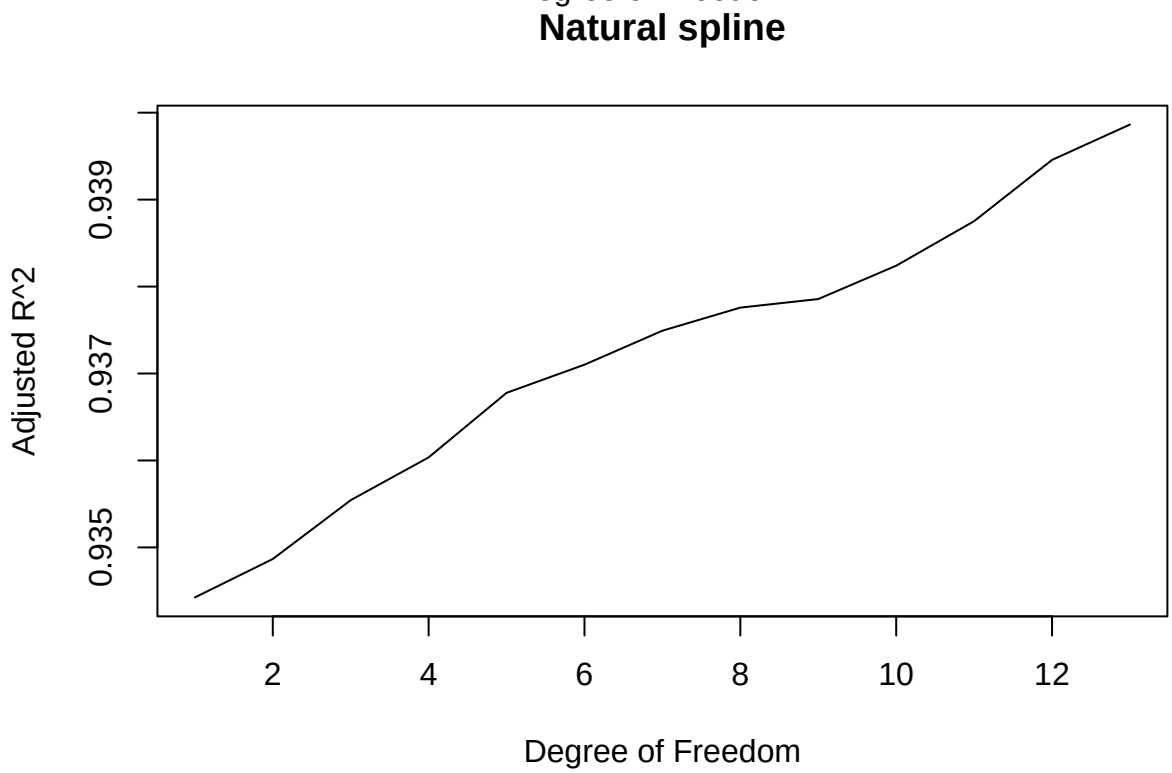
Using cross-validation with 5 folds, we see the adjusted R^2 decreases as polynomial degree increases. The MSE indicates that linear regression is not sufficient. The minimum MSE is achieved with polynomial order 2 on St. We decided that the optimal polynomial order for St is 2 for better interpretation.

We tried removing data of both high leverage and residual. However, this didn't change the MSE and adjusted R^2 greatly. Since these observations take up 7% of the full training data, we decided that we don't want to exclude them.

```
##
## Call:
## lm(formula = log(R_moment_1) ~ poly(St, 2) + Re + Fr_logit +
##     Re * Fr_logit, data = train)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.0934 -0.5734  0.2895  0.5002  0.7878
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)  -4.935e-01  3.326e-01  -1.483 0.141734
## poly(St, 2)1  2.030e+00  5.919e-01   3.430 0.000944 ***
## poly(St, 2)2 -2.693e-01  5.920e-01  -0.455 0.650303
## Re           -2.137e-02  1.348e-03 -15.845  < 2e-16 ***
## Fr_logit     -3.101e-02  1.989e-02  -1.559 0.122902
## Re:Fr_logit   1.488e-04  7.711e-05   1.929 0.057108 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.5903 on 83 degrees of freedom
## Multiple R-squared:  0.9344, Adjusted R-squared:  0.9305
## F-statistic: 236.5 on 5 and 83 DF,  p-value: < 2.2e-16
```

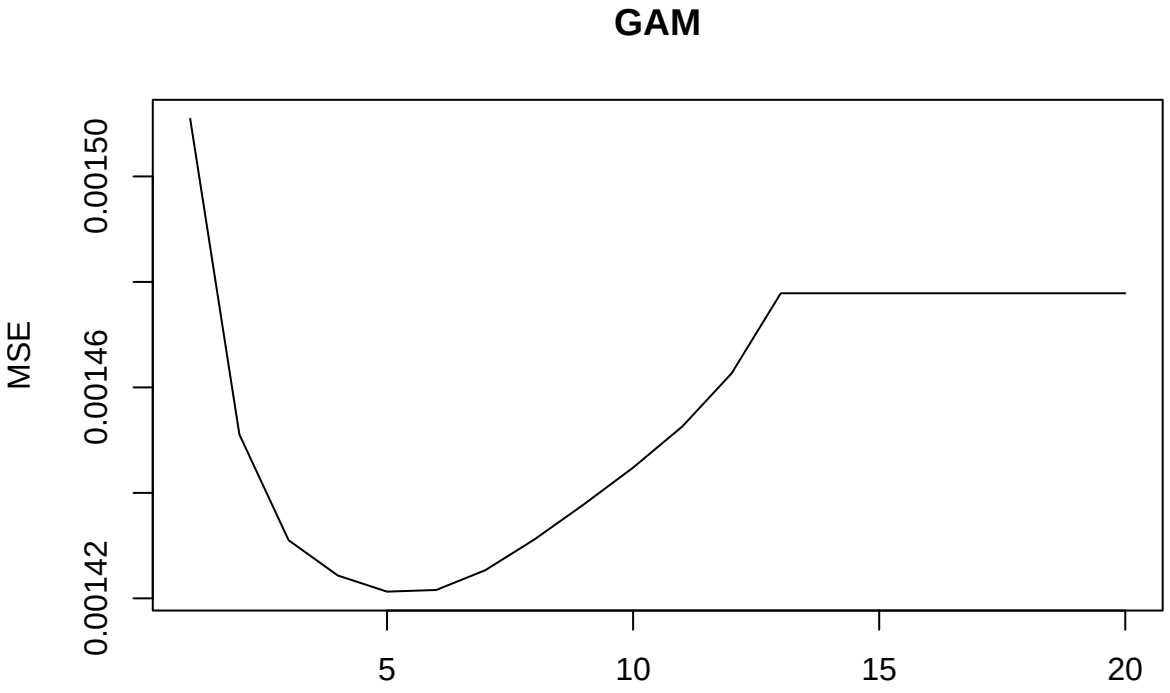
Natural spline

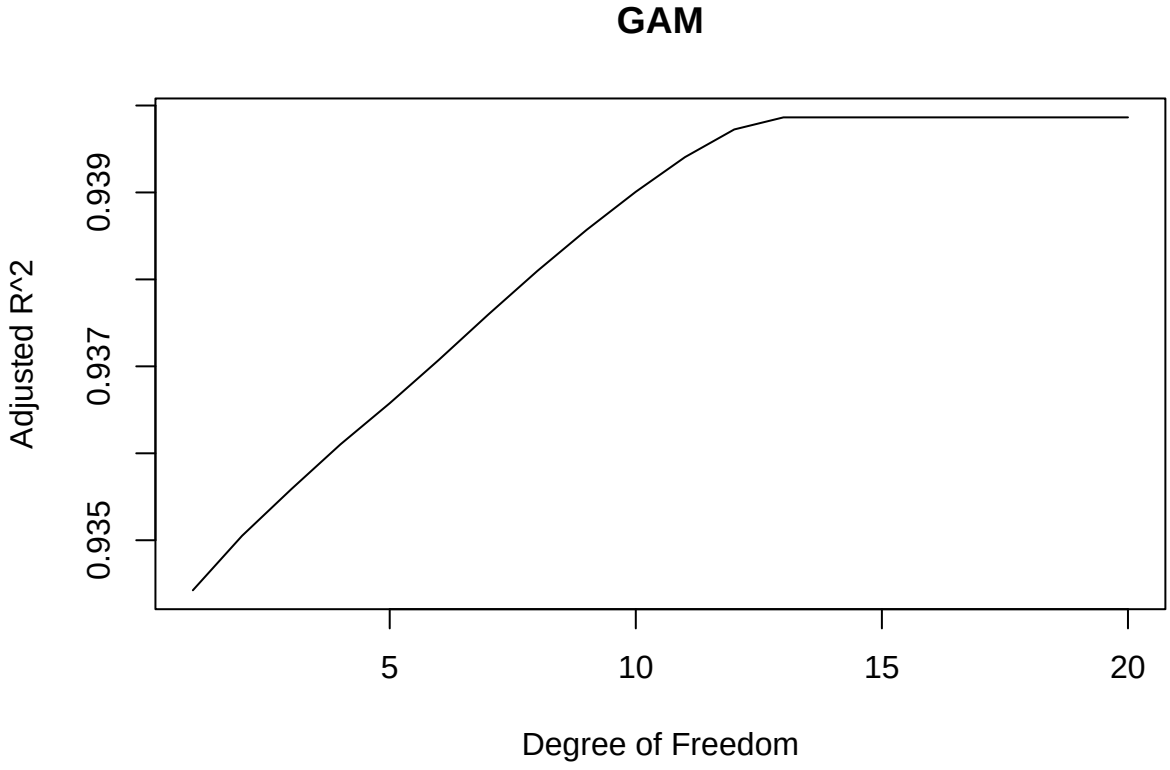




Using cross-validation with 5 folds, we see the adjusted R^2 increases as the degree of freedom increases, and the lowest MSE is achieved at degree of 7. The optimal polynomial degrees of freedom for St is 7, which achieves the lowest MSE and a decent adjusted R^2.

GAM





Using cross-validation with 5 folds, we see the adjusted R^2 increases as the degree of freedom increases to an extent (around 13). The optimal degrees of freedom for GAM model to achieve the lowest MSE is around 5, but MSE stops increasing after lifting degrees of freedom to 13. For more prediction power, we conclude that a degree of freedom around 13 is optimal.

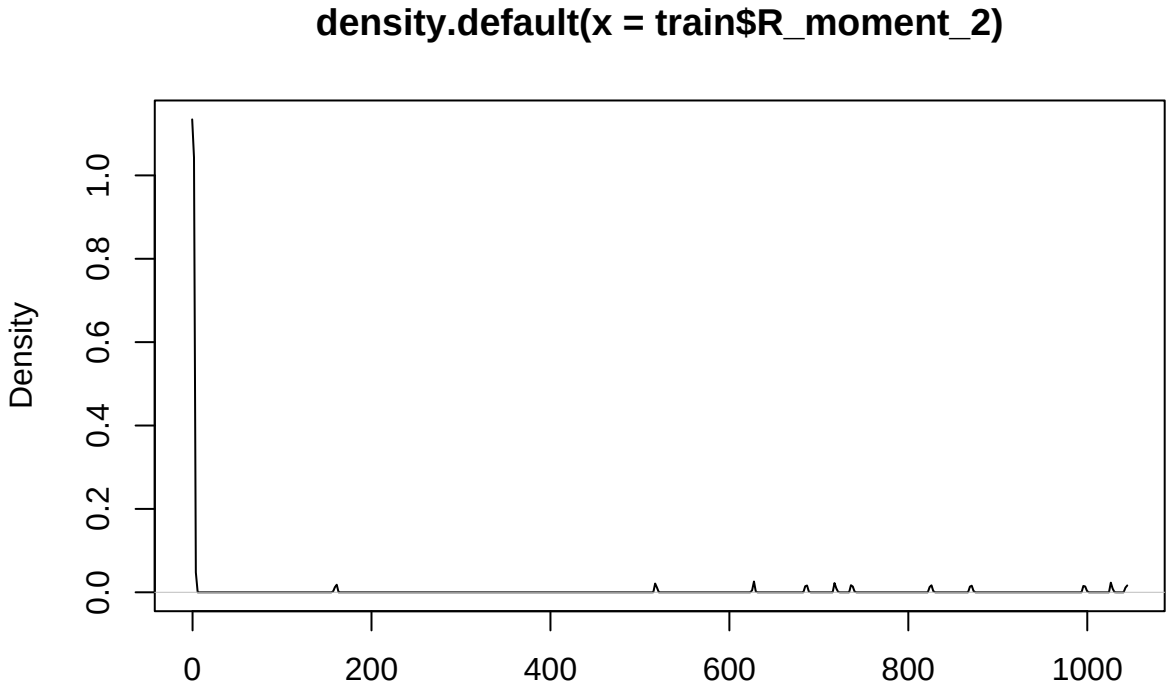
Table 1: Model MSE and Adjusted R^2

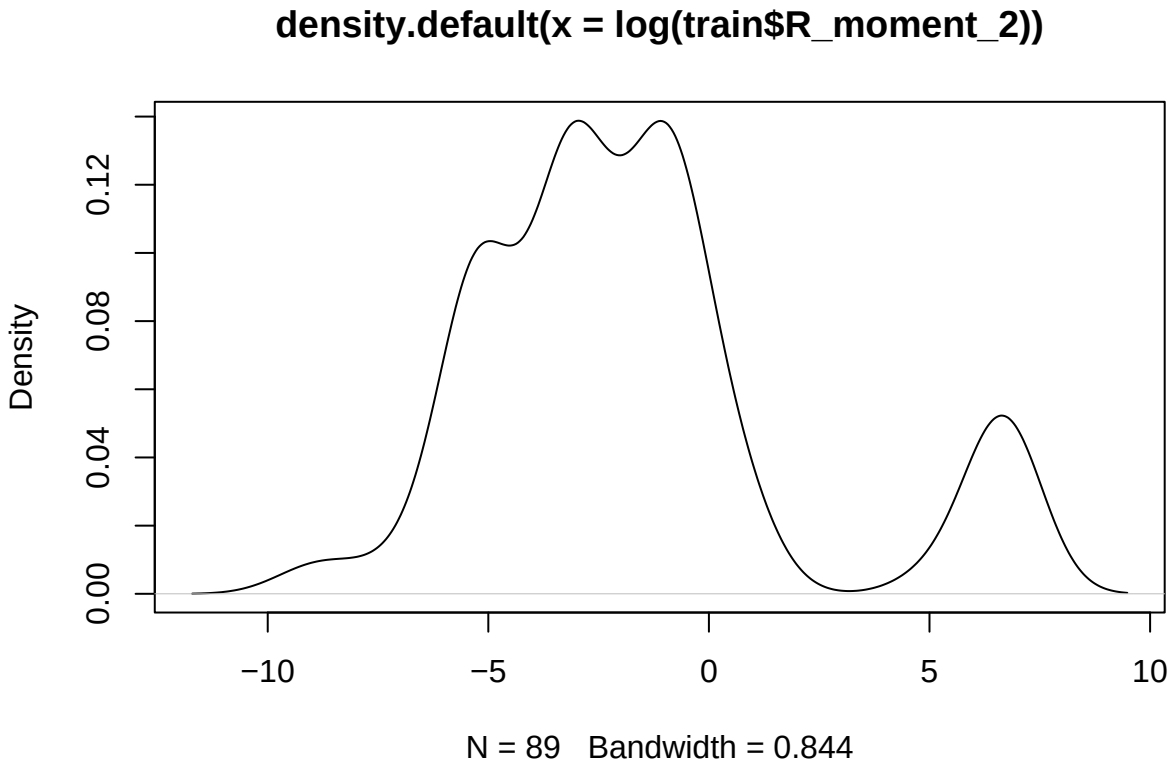
models	formula	mse	adj.R
Linear regresion	R_moment_1 ~ Fr + Re + St + Fr * Re	0.0015109	0.9304611
Polynomial regression	R_moment_1 ~ Fr + Re + poly(St, 2) + Fr * Re	0.0008570	0.9365328
Natural spline	R_moment_1 ~ ns(St, df = 13) + Fr + Re + Fr * Re	0.0008711	0.9436787
Generalized additive model	R_moment_1 ~ s(St, 13) + Re + Fr + Fr * Re	0.0008381	0.9436787

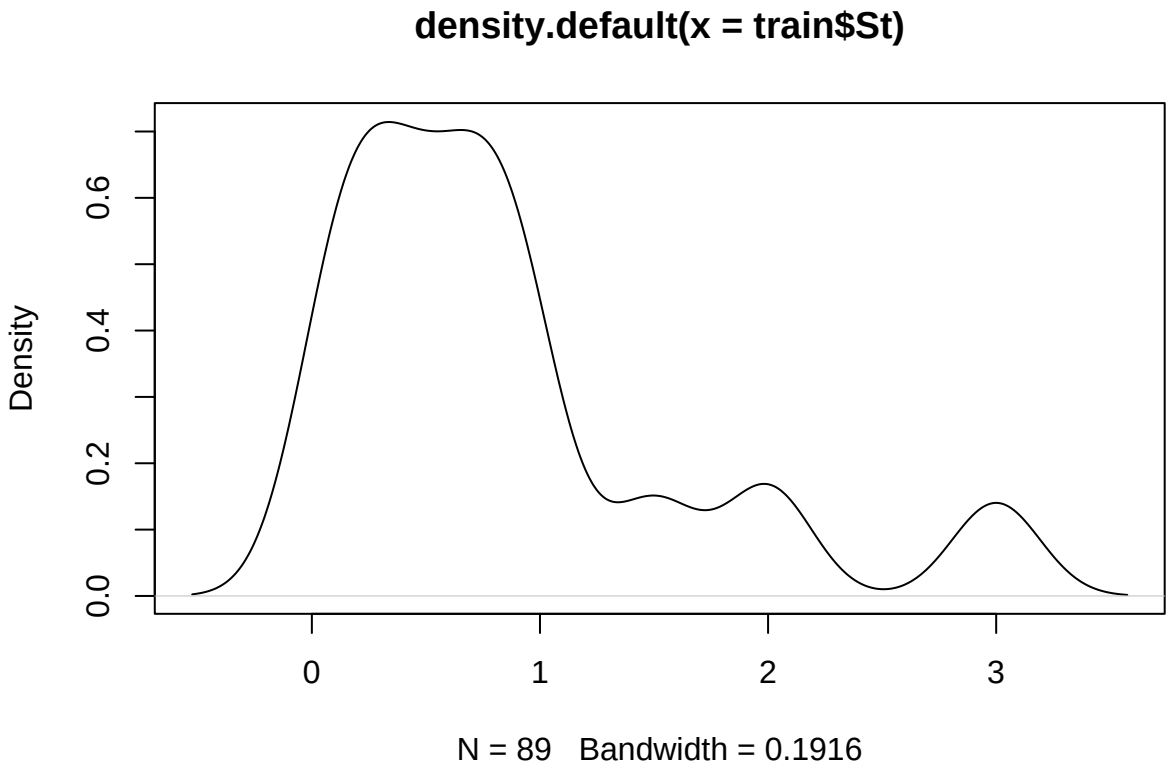
* Fr logit-transformed

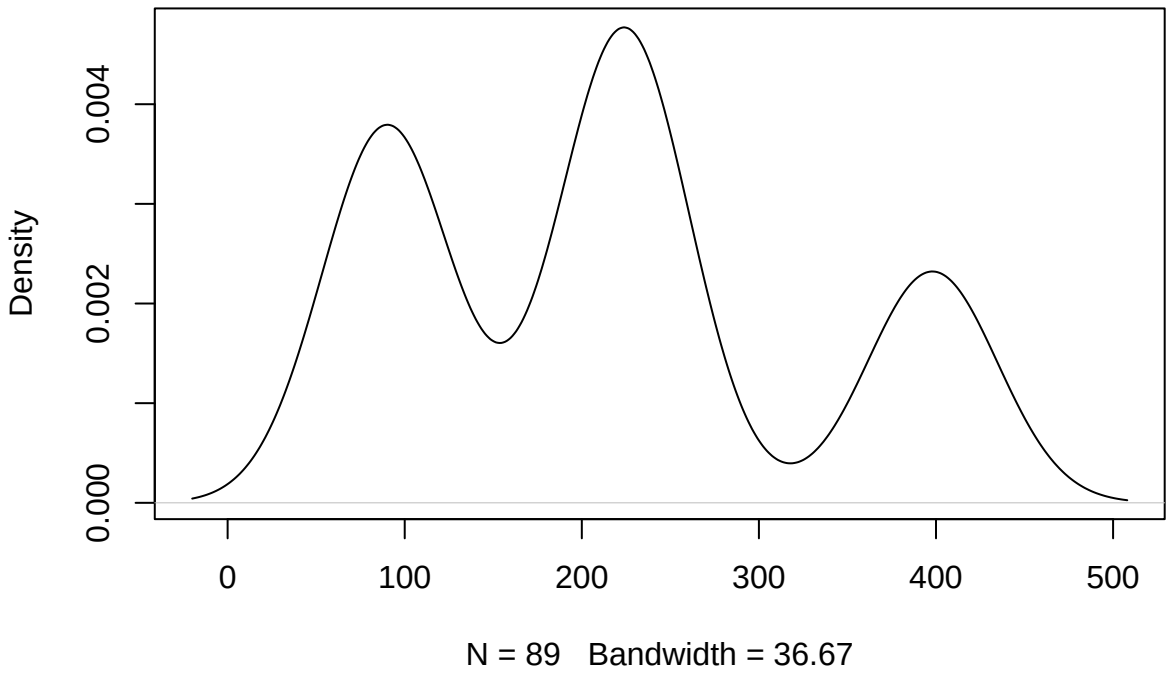
For Raw_moment_1 (mean), GAM has the best results since it presents the highest adjusted R^2 and the lowest MSE. The second best model is natural spline, with only slight increase in MSE. Since both are hard to interpret, we use it for prediction and polynomial regression for inference.

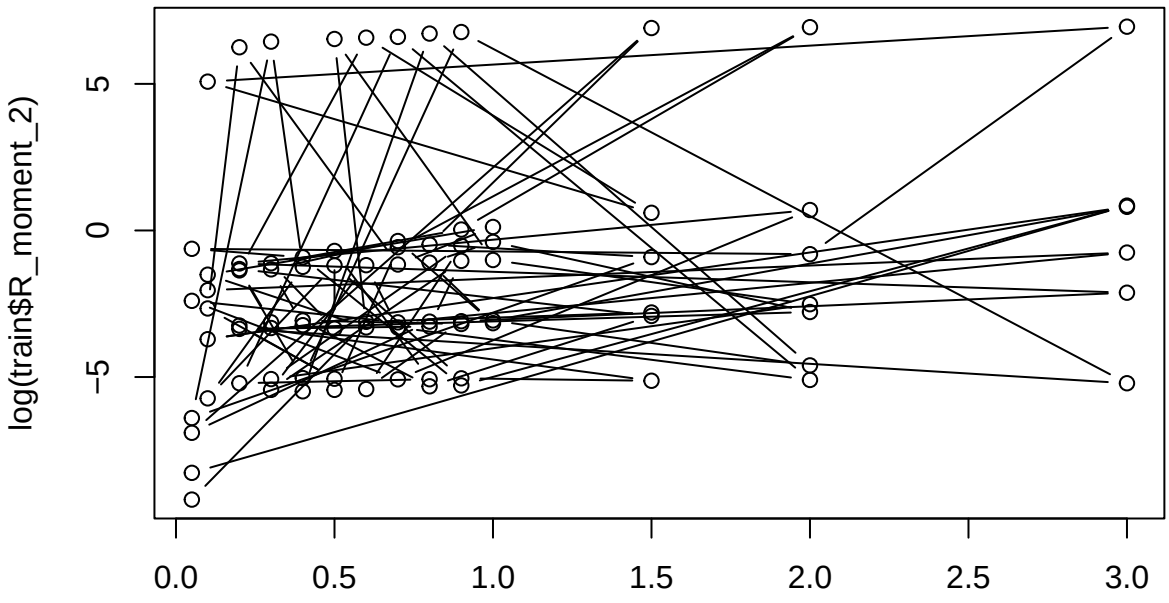
Raw Moment 2

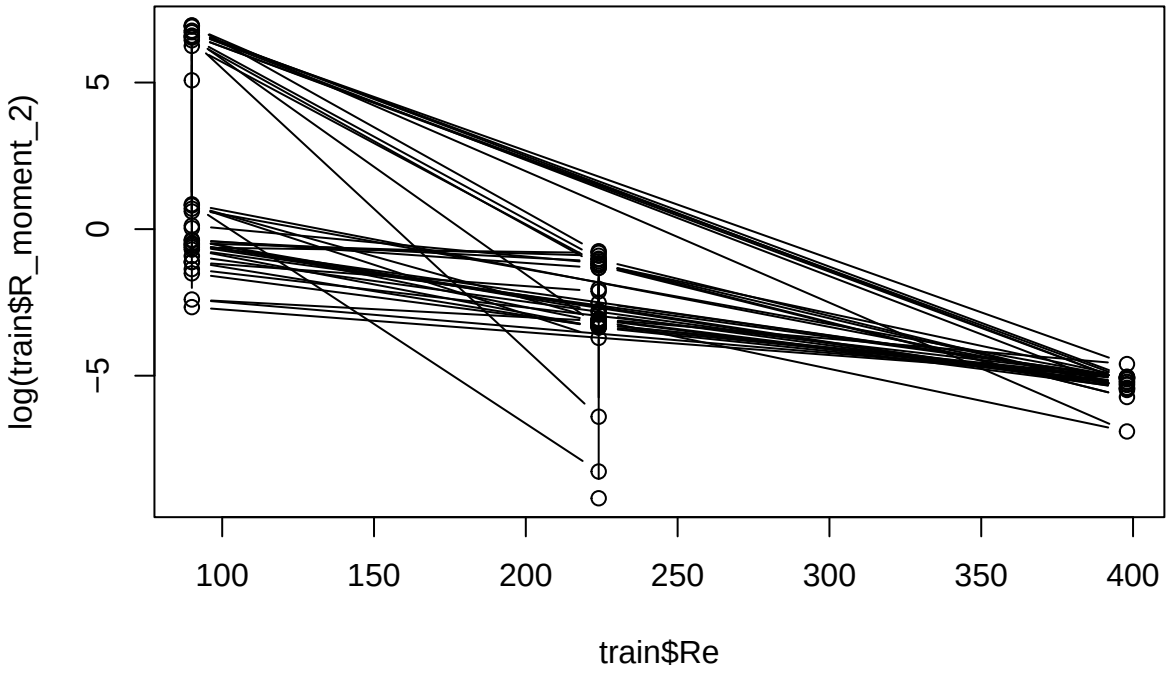

density.default(x = train$R_moment_2)
Density
0.0
0.2
0.4
0.6
0.8
1.0
0
200
400
600
800
1000
N = 89 Bandwidth = 0.1396


density.default(x = log(train$R_moment_2))
Density
0.00
0.04
0.08
0.12
−10
−5
0
5
10
N = 89 Bandwidth = 0.844


density.default(x = train$St)
Density
0.0
0.2
0.4
0.6
0
1
2
3
N = 89 Bandwidth = 0.1916


density.default(x = train$Re)
Density
0.000
0.002
0.004
0
100
200
300
400
500
N = 89 Bandwidth = 36.67






```
##                  St          Re    Fr_logit
## St        1.00000000 -0.03169871 -0.04921517
## Re       -0.03169871  1.00000000  0.09619529
## Fr_logit -0.04921517  0.09619529  1.00000000
```

Least Square Regression

```
##
## Call:
## lm(formula = log(R_moment_2) ~ St + Re + Fr_logit, data = train)
##
## Residuals:
```

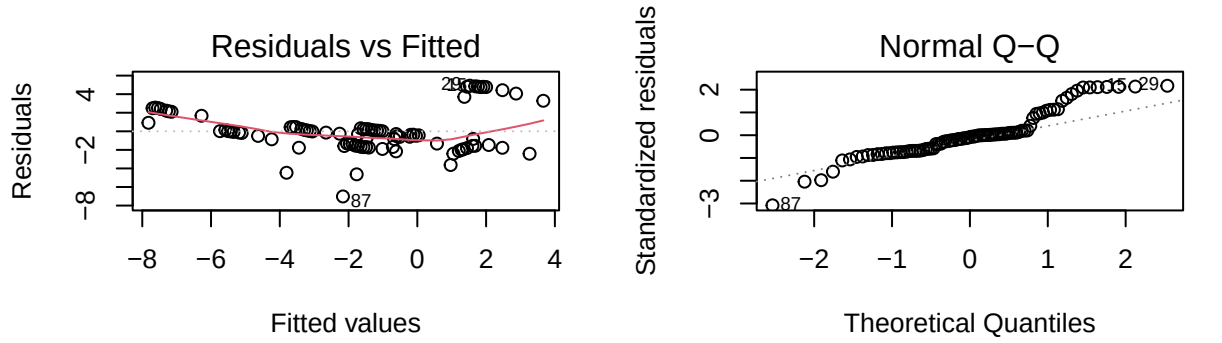
```
##     Min      1Q  Median      3Q     Max
## -7.0174 -1.5626 -0.1805  0.4415  4.9154
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  4.607479   0.793703   5.805 1.08e-07 ***
## St           0.791718   0.313510   2.525 0.013415 *
## Re          -0.023077   0.002188 -10.545  < 2e-16 ***
## Fr_logit    -0.131273   0.035707  -3.676 0.000413 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.308 on 85 degrees of freedom
## Multiple R-squared:  0.6265, Adjusted R-squared:  0.6133
## F-statistic: 47.52 on 3 and 85 DF,  p-value: < 2.2e-16
```

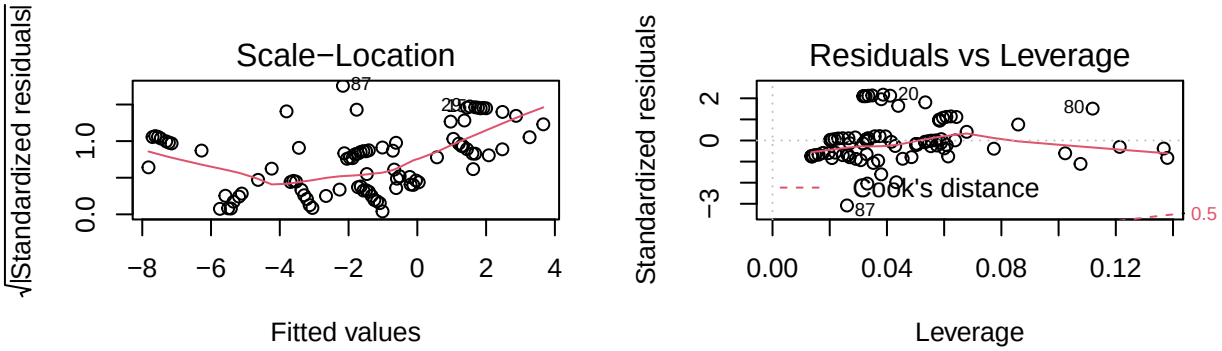
```
##
## Call:
## lm(formula = log(R_moment_2) ~ St + Re + Fr_logit + Re * Fr_logit,
##     data = train)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -6.8900 -1.4410  0.2603  1.1692  3.9378
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  8.9679045  1.1939132   7.511 5.71e-11 ***
## St           0.8518822  0.2826637   3.014  0.00341 **
## Re          -0.0427756  0.0047483  -9.009 5.70e-14 ***
## Fr_logit    -0.4146745  0.0699786  -5.926 6.63e-08 ***
## Re:Fr_logit  0.0012372  0.0002713   4.560 1.73e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.079 on 84 degrees of freedom
## Multiple R-squared:  0.7006, Adjusted R-squared:  0.6863
## F-statistic: 49.14 on 4 and 84 DF,  p-value: < 2.2e-16
```

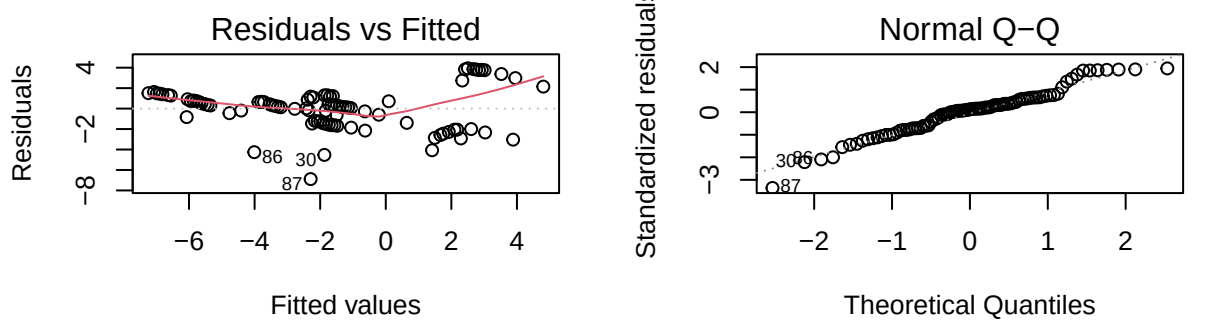
```
## [1] 70404.06
```

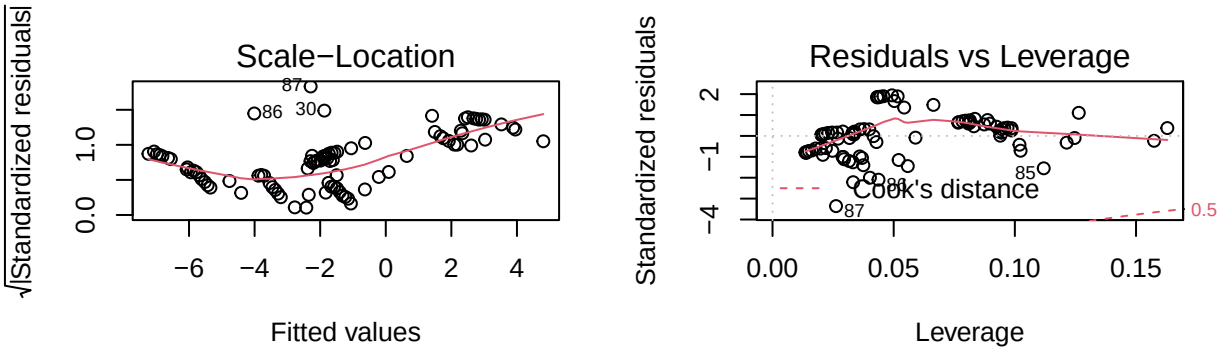
```
## [1] 70404.06
```

```
## [1] 0.7089149
```









Polynomial Model

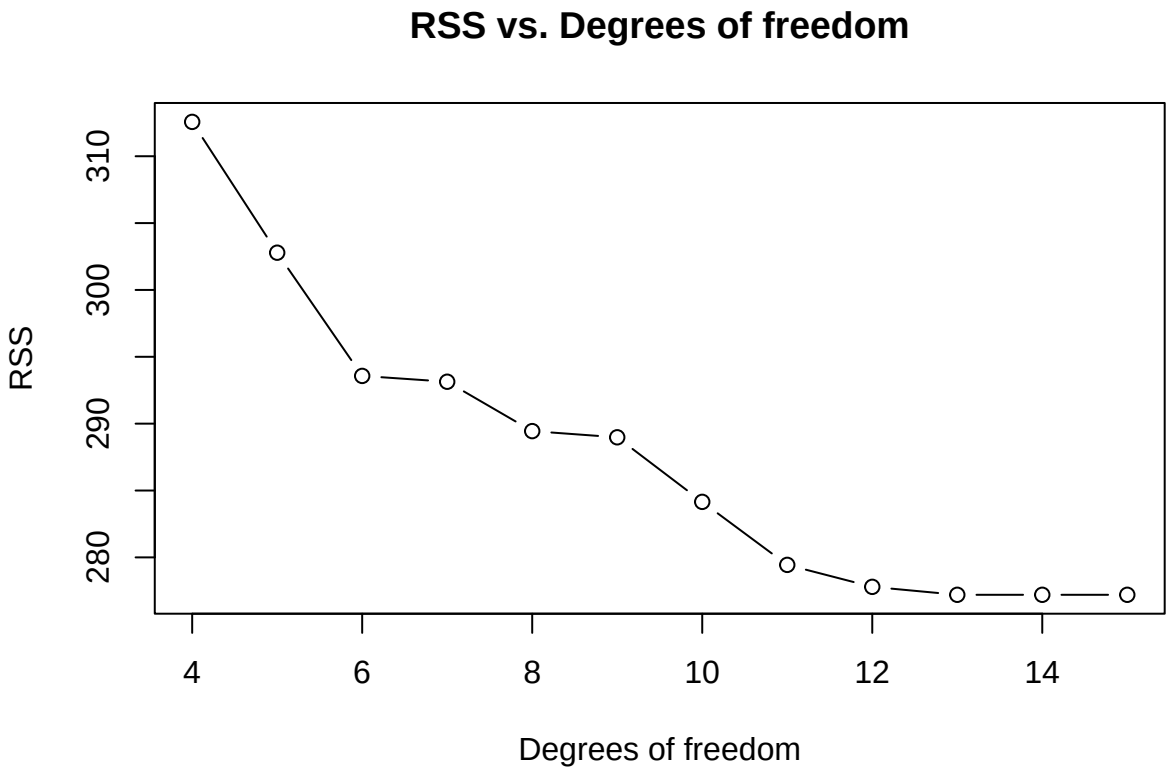
```
##  [1] 47248.71 45484.37 48325.80 46599.59 46116.85 47488.93 48226.22 50226.98
##  [9] 49930.20 50172.15
```

```
## [1] 45484.37
```

```
## [1] 0.8971457
```

```
##
## Call:
## lm(formula = log(R_moment_2) ~ poly(St, 5) + Re + poly(Fr_logit,
##     2) + Re * Fr_logit, data = train)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.3586 -0.7505  0.1445  0.7711  2.5673
##
## Coefficients: (1 not defined because of singularities)
##                       Estimate Std. Error t value Pr(>|t|)
## (Intercept)          4.164e+00  2.917e-01  14.277  < 2e-16 ***
## poly(St, 5)1         5.847e+00  1.233e+00   4.740 9.28e-06 ***
## poly(St, 5)2        -4.611e+00  1.236e+00  -3.729 0.000360 ***
## poly(St, 5)3         3.889e+00  1.232e+00   3.156 0.002264 **
## poly(St, 5)4        -4.315e+00  1.246e+00  -3.464 0.000863 ***
## poly(St, 5)5         3.597e+00  1.235e+00   2.914 0.004640 **
## Re                  -4.922e-02  2.868e-03 -17.163  < 2e-16 ***
## poly(Fr_logit, 2)1 -2.836e+01  2.720e+00 -10.426  < 2e-16 ***
## poly(Fr_logit, 2)2  1.420e+01  1.279e+00  11.101  < 2e-16 ***
## Fr_logit                   NA         NA      NA       NA
## Re:Fr_logit          1.383e-03  1.623e-04   8.526 8.21e-13 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.23 on 79 degrees of freedom
## Multiple R-squared:  0.9015, Adjusted R-squared:  0.8903
## F-statistic: 80.33 on 9 and 79 DF,  p-value: < 2.2e-16
```

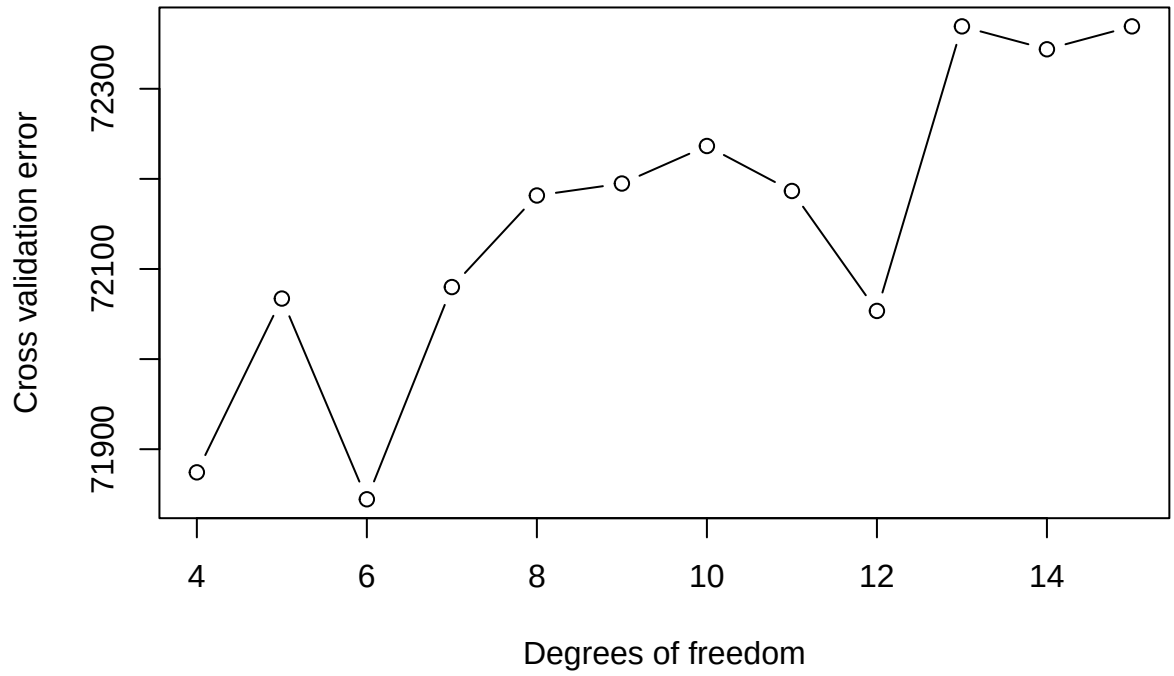
Natural spline




```
##  [1]       NA       NA       NA 71874.33 72067.20 71844.44 72079.94 72181.57
##  [9] 72194.86 72236.49 72186.55 72053.45 72369.24 72343.84 72369.24
```

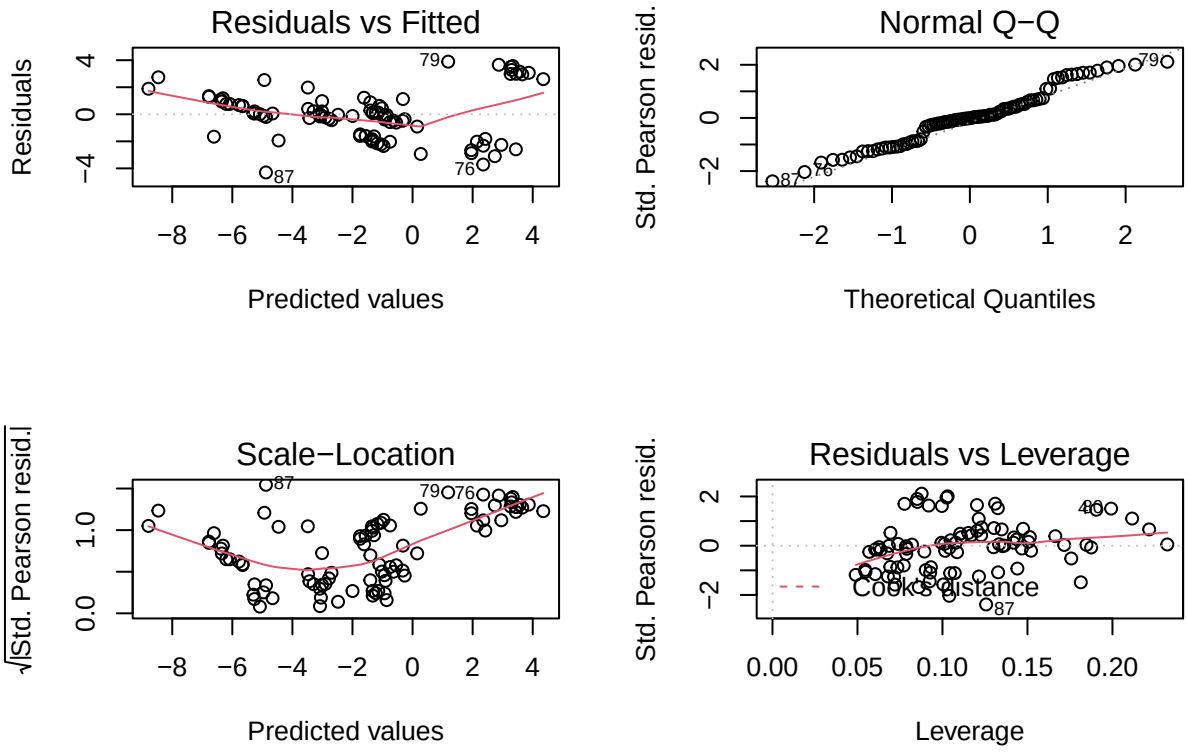
```
## [1] 71844.44
```

```
## [1] 0.7738726
```




```
## 16.66667% 33.33333%       50% 66.66667% 83.33333%
```

```
##       0.2       0.4       0.7       0.9       1.5
##
## Call:
## glm(formula = log(R_moment_2) ~ ns(St, df = 6) + Re + Fr_logit +
##     Re * Fr_logit, data = train)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -4.2999  -1.5999  -0.0138   0.9384   3.8899
##
## Coefficients:
##                   Estimate Std. Error t value Pr(>|t|)
## (Intercept)      6.4328374  1.2621064   5.097 2.31e-06 ***
## ns(St, df = 6)1  2.0668561  1.2378289   1.670 0.098928 .
## ns(St, df = 6)2  4.4491533  1.1955160   3.722 0.000369 ***
## ns(St, df = 6)3  2.8429188  1.2379387   2.296 0.024297 *
## ns(St, df = 6)4  2.4739182  1.2386729   1.997 0.049243 *
## ns(St, df = 6)5  8.6424790  1.8534022   4.663 1.25e-05 ***
## ns(St, df = 6)6  2.1680172  0.9283018   2.335 0.022057 *
## Re              -0.0427772  0.0044138  -9.692 4.37e-15 ***
## Fr_logit        -0.4095812  0.0656887  -6.235 2.08e-08 ***
## Re:Fr_logit      0.0012102  0.0002533   4.777 8.06e-06 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for gaussian family taken to be 3.716084)
##
##     Null deviance: 1212.42  on 88  degrees of freedom
## Residual deviance:  293.57  on 79  degrees of freedom
## AIC: 380.79
##
## Number of Fisher Scoring iterations: 2
```



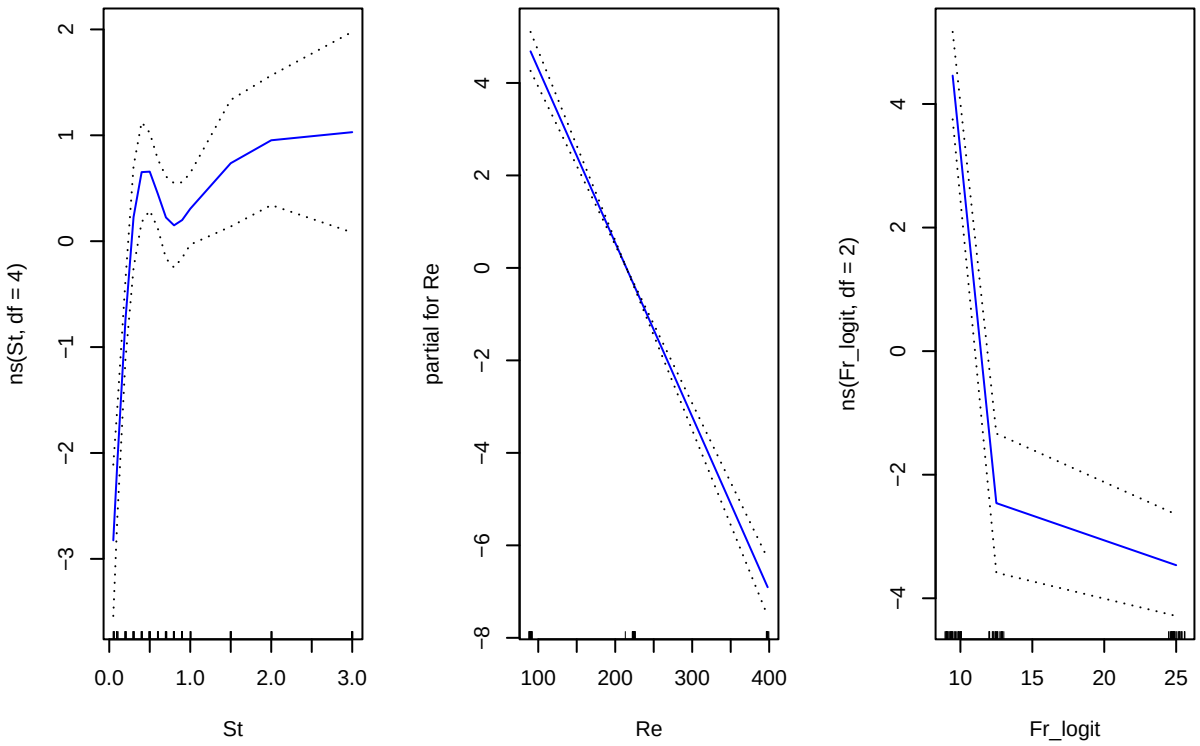
Generalized Additive Model

```
##  [1] 46427.17 45161.82 45284.31 44536.18 45287.51 45633.56 46105.45 45860.23
##  [9] 45544.69 45545.96 45984.79 45375.09 46566.39 46566.39 46566.39
```

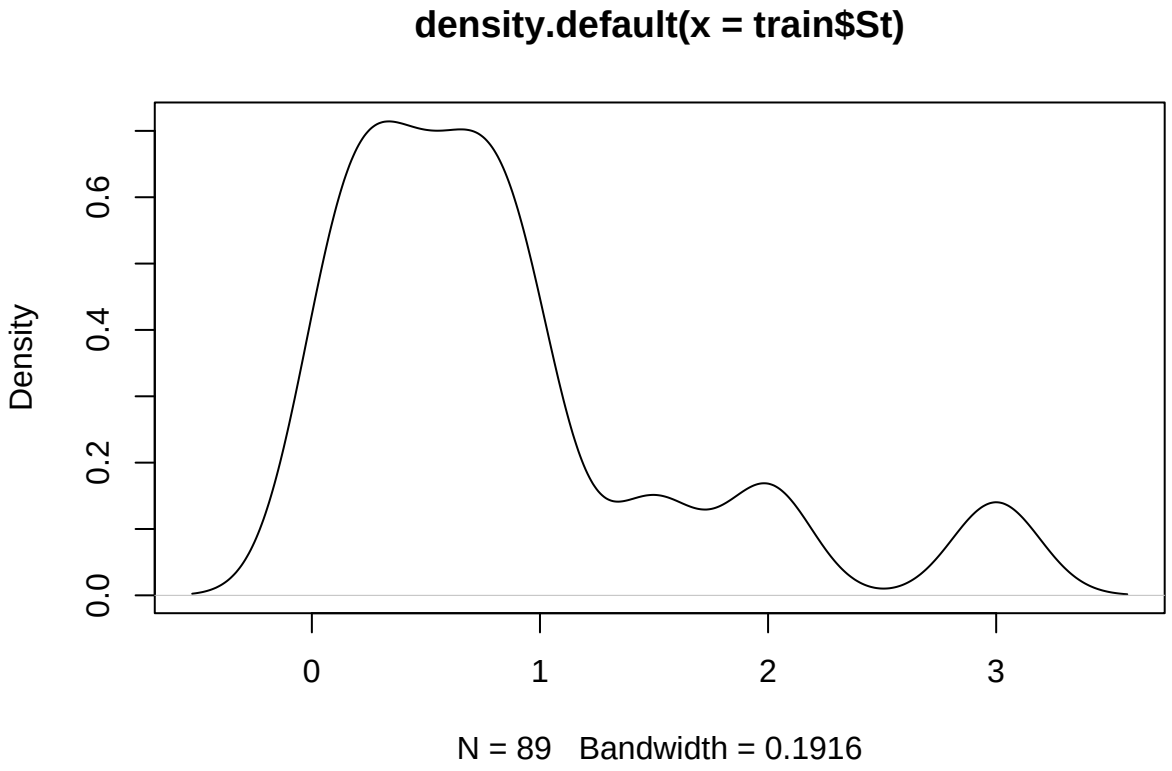
```
## [1] 44536.18
```

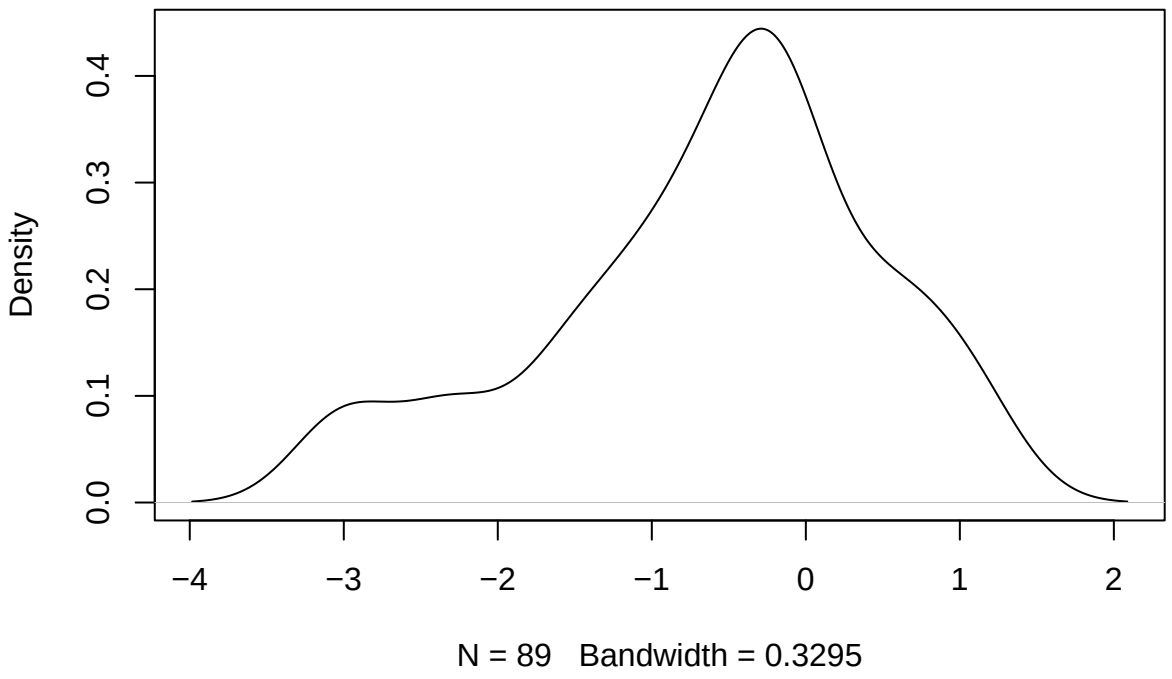
```
## [1] 0.9155727
```

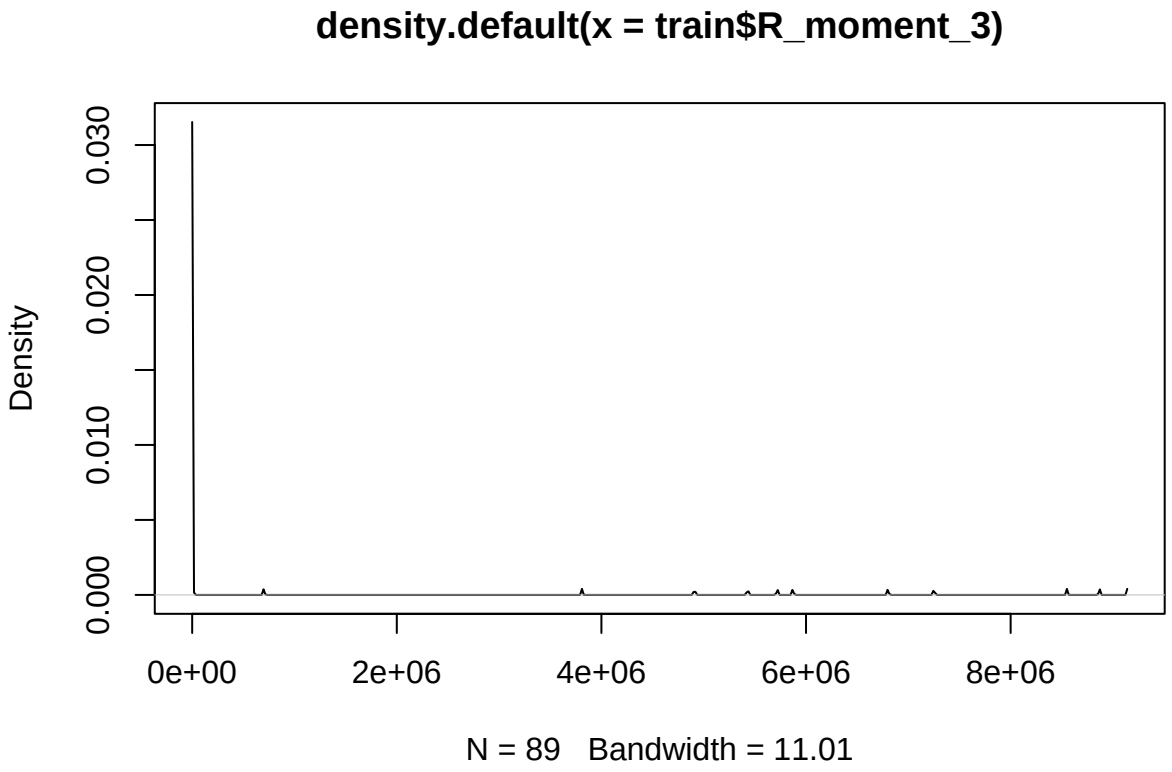
models	formula	mse	
Least square regression	log(R_moment_2) ~ Fr + Re + St + Fr * Re	70404.06	0.70
Polynomial regression	log(R_moment_2) ~ poly(Fr,2) + Re + poly(St, 5) + Fr * Re	45484.37	0.89
Natural spline	log(R_moment_2) ~ ns(St, df = 6) + Fr + Re + Fr * Re	71844.44	0.77
Generalized additive model	log(R_moment_2) ~ ns(St, 4) + Re + ns(Fr_logit,2) + Re:ns(Fr_logit,2)	44536.18	0.91

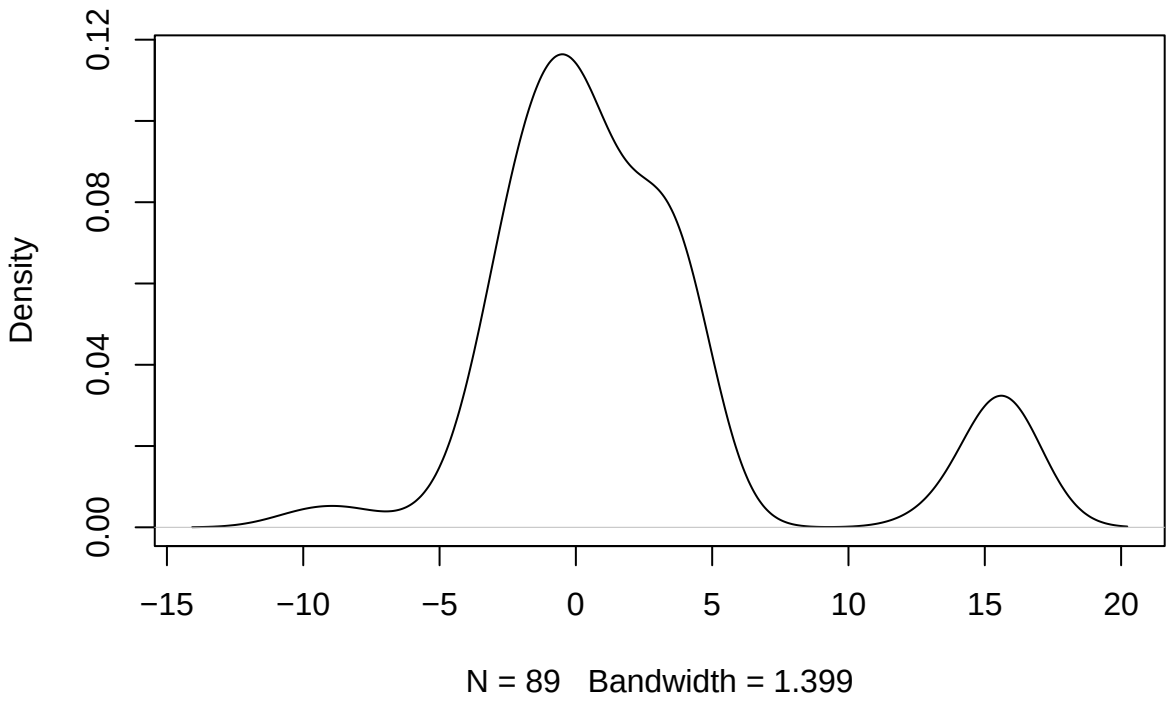


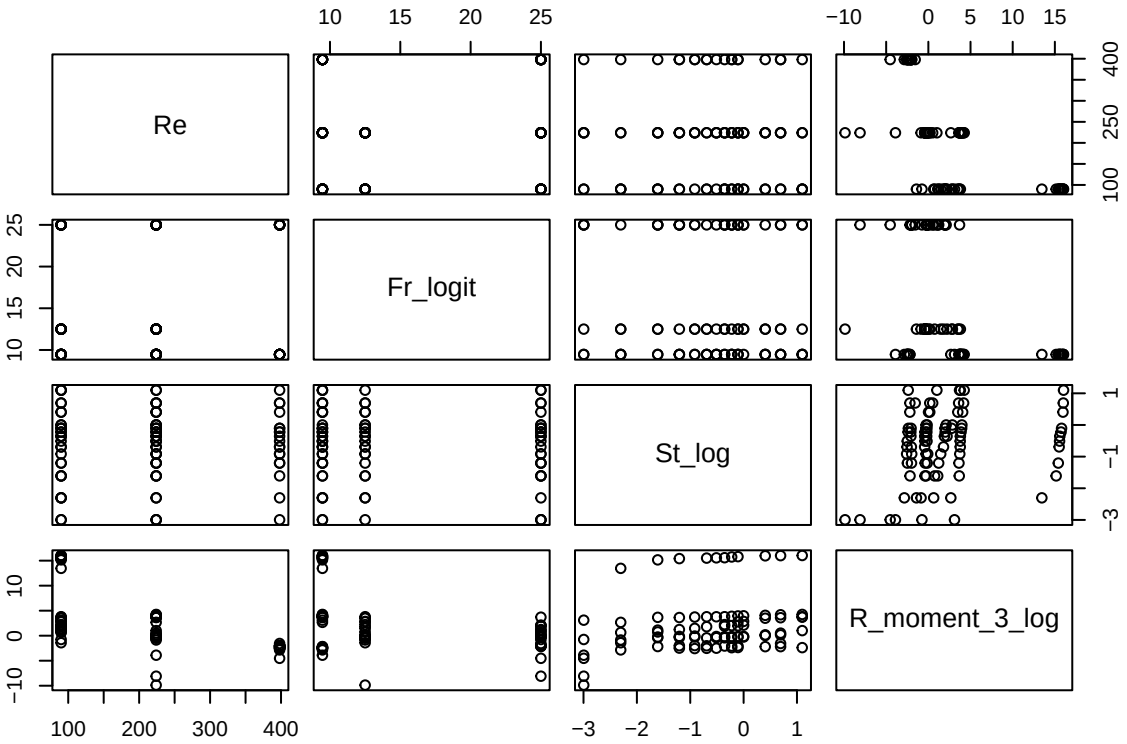
Raw Moment 3


density.default(x = train$St)
Density
0.0
0.2
0.4
0.6
0
1
2
3
N = 89 Bandwidth = 0.1916


density.default(x = log(train$St))
Density
0.0
0.1
0.2
0.3
0.4
−4
−3
−2
−1
0
1
2
N = 89 Bandwidth = 0.3295


density.default(x = train$R_moment_3)
Density
0.000
0.010
0.020
0.030
0e+00
2e+06
4e+06
6e+06
8e+06
N = 89 Bandwidth = 11.01

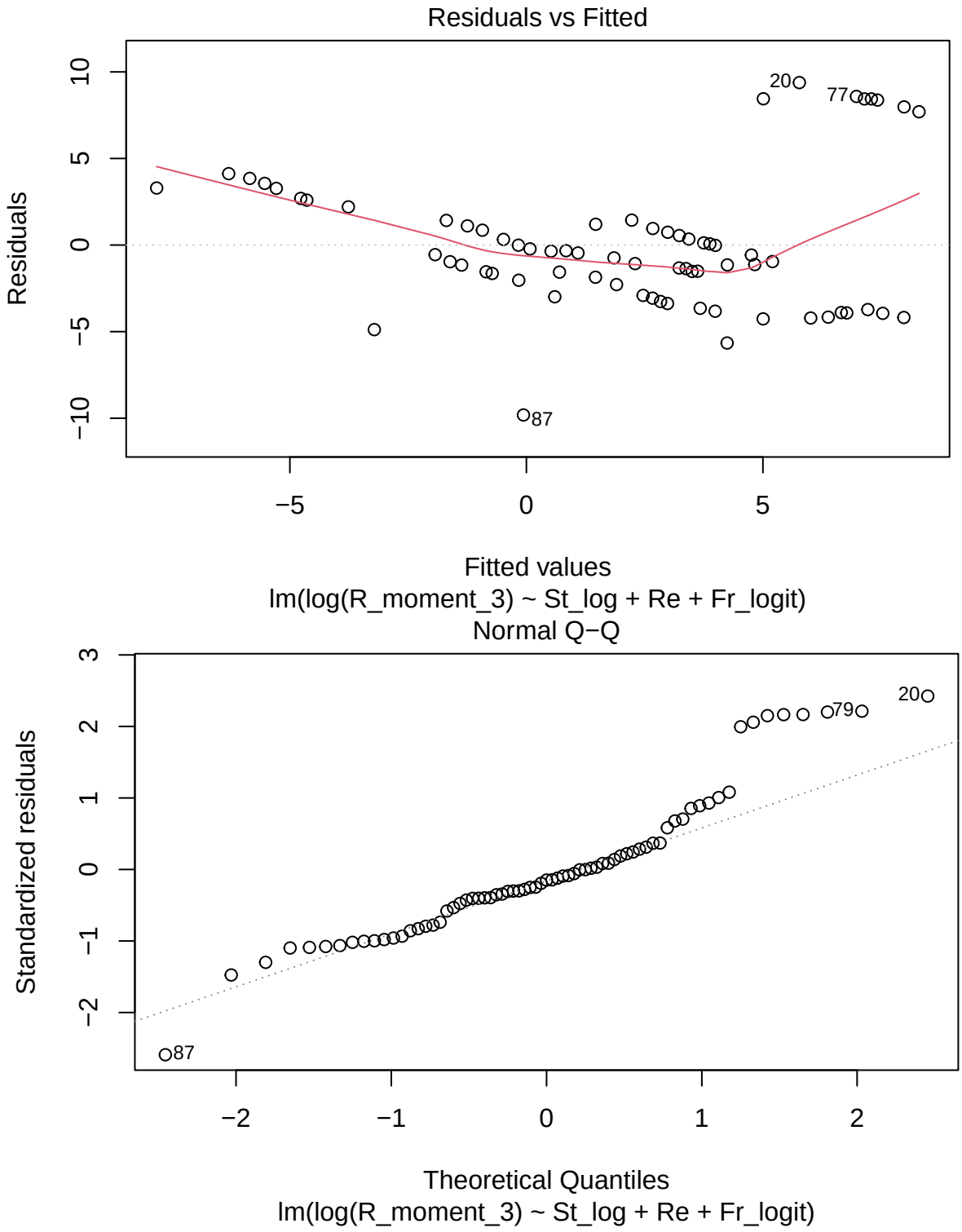

density.default(x = log(train$R_moment_3))
Density
0.00
0.04
0.08
0.12
−15
−10
−5
0
5
10
15
20
N = 89 Bandwidth = 1.399

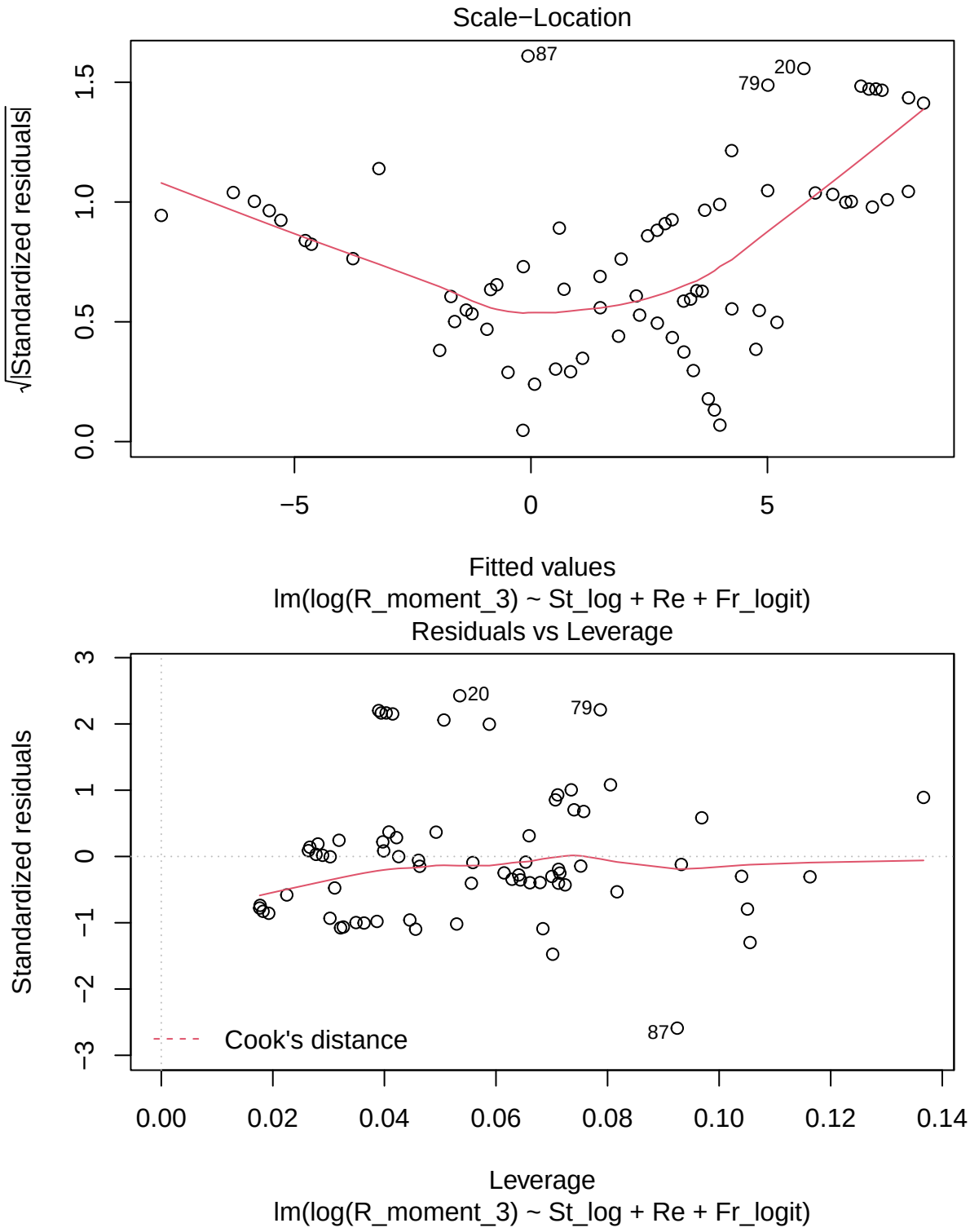



```
## NULL
```

Least Square Regression

```
##
## Call:
## lm(formula = log(R_moment_3) ~ St_log + Re + Fr_logit, data = trainData)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -9.814 -2.595 -0.576  1.311  9.383
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 12.30488    1.40914   8.732 1.16e-12 ***
## St_log       1.09853    0.45166   2.432 0.017687 *
## Re          -0.02645    0.00420  -6.297 2.68e-08 ***
## Fr_logit    -0.25218    0.06884  -3.663 0.000493 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.977 on 67 degrees of freedom
## Multiple R-squared:  0.5014, Adjusted R-squared:  0.4791
## F-statistic: 22.46 on 3 and 67 DF,  p-value: 3.544e-10
```


Residuals vs Fitted
Residuals
Fitted values
lm(log(R_moment_3) ~ St_log + Re + Fr_logit)
Normal Q−Q
Standardized residuals
Theoretical Quantiles
lm(log(R_moment_3) ~ St_log + Re + Fr_logit)
20
77
87
79



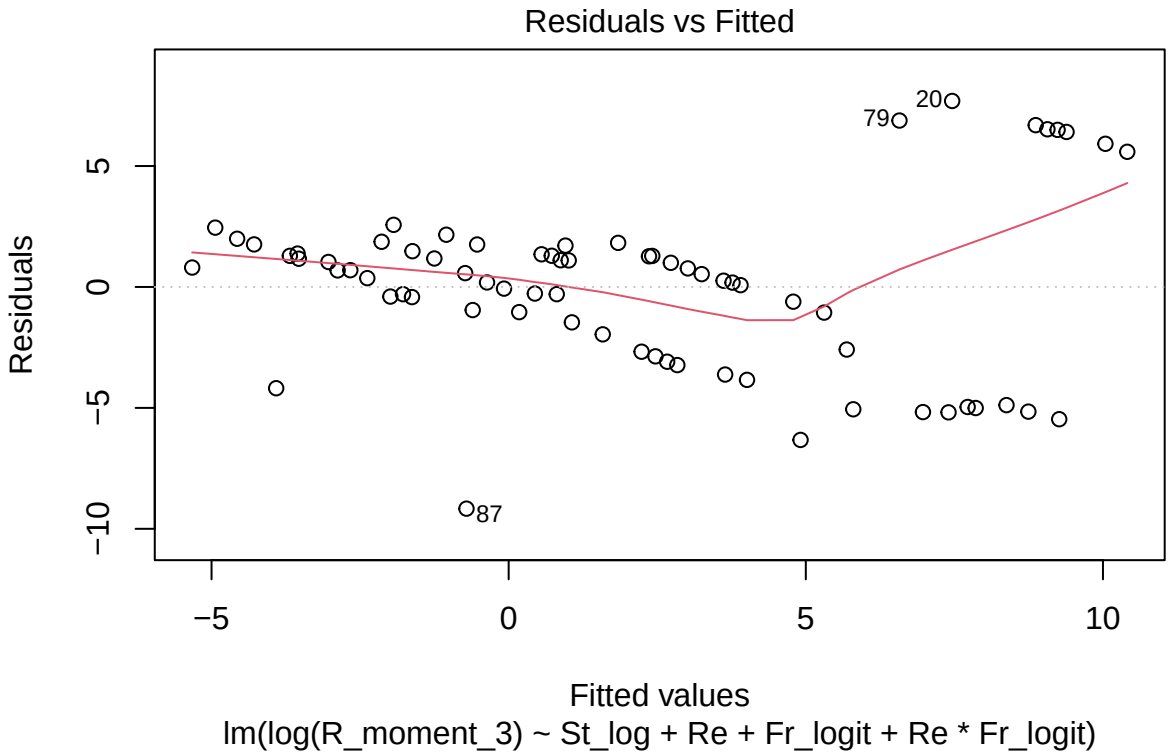

```
## [1] 5.216843e+12
```

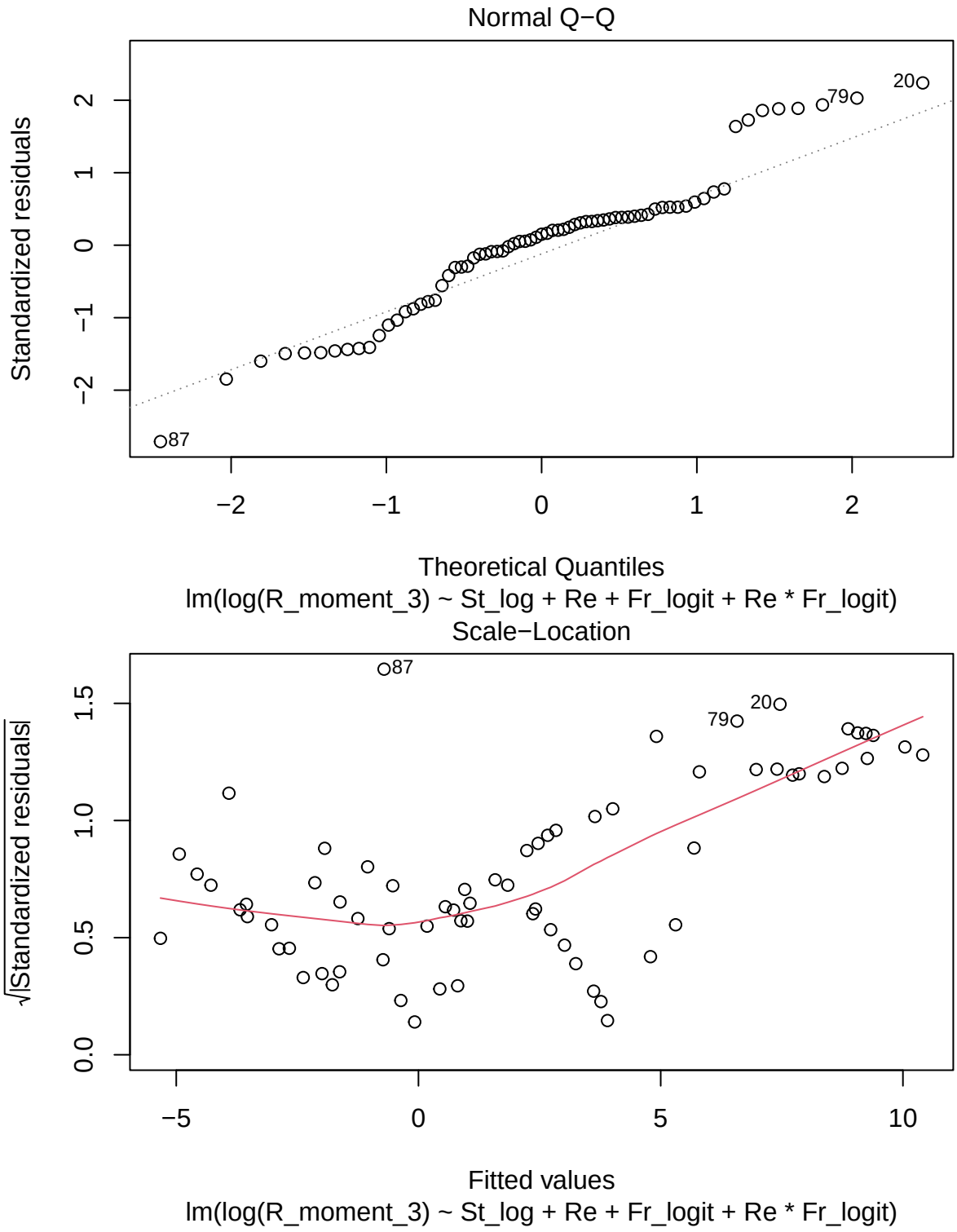
```
## [1] 0.5257978
```

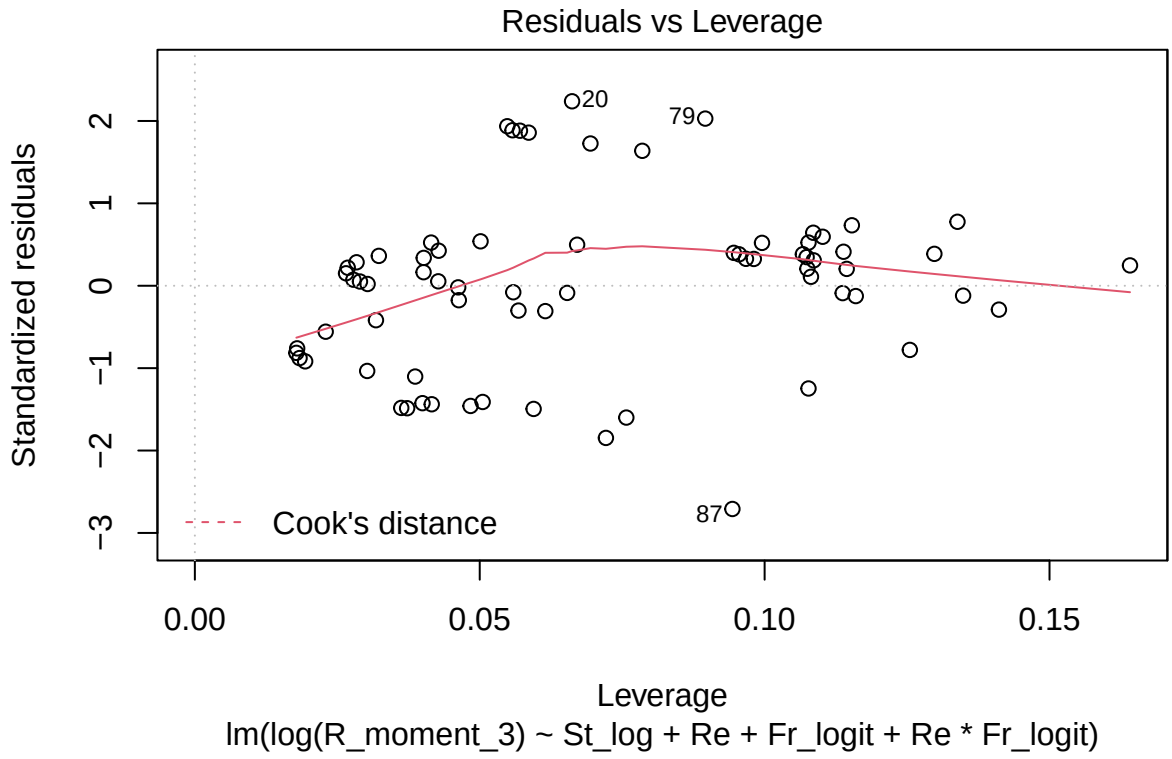
Least Square Regression with Interactions

```
##
```

```
## Call:
## lm(formula = log(R_moment_3) ~ St_log + Re + Fr_logit + Re *
##     Fr_logit, data = trainData)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -9.1657 -2.2740  0.5303  1.4301  7.6904
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) 20.3343402  2.2801849   8.918 6.09e-13 ***
## St_log       1.2797230  0.4060185   3.152  0.00244 **
## Re          -0.0625396  0.0093323  -6.701 5.48e-09 ***
## Fr_logit    -0.7437090  0.1316214  -5.650 3.69e-07 ***
## Re:Fr_logit  0.0021766  0.0005152   4.225 7.50e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.555 on 66 degrees of freedom
## Multiple R-squared:  0.6076, Adjusted R-squared:  0.5838
## F-statistic: 25.54 on 4 and 66 DF,  p-value: 8.275e-13
```




Normal Q−Q
Standardized residuals
Theoretical Quantiles
lm(log(R_moment_3) ~ St_log + Re + Fr_logit + Re * Fr_logit)
Scale−Location
√|Standardized residuals|
Fitted values
lm(log(R_moment_3) ~ St_log + Re + Fr_logit + Re * Fr_logit)




```
## [1] 5.188199e+12
```

```
## [1] 0.6362158
```

Polynomial Model

```
##  [1] 4.189000e+12 4.400576e+12 4.488424e+12 4.593392e+12 4.631683e+12
##  [6] 4.643822e+12 4.660484e+12 4.662366e+12 4.666616e+12 4.692031e+12
```

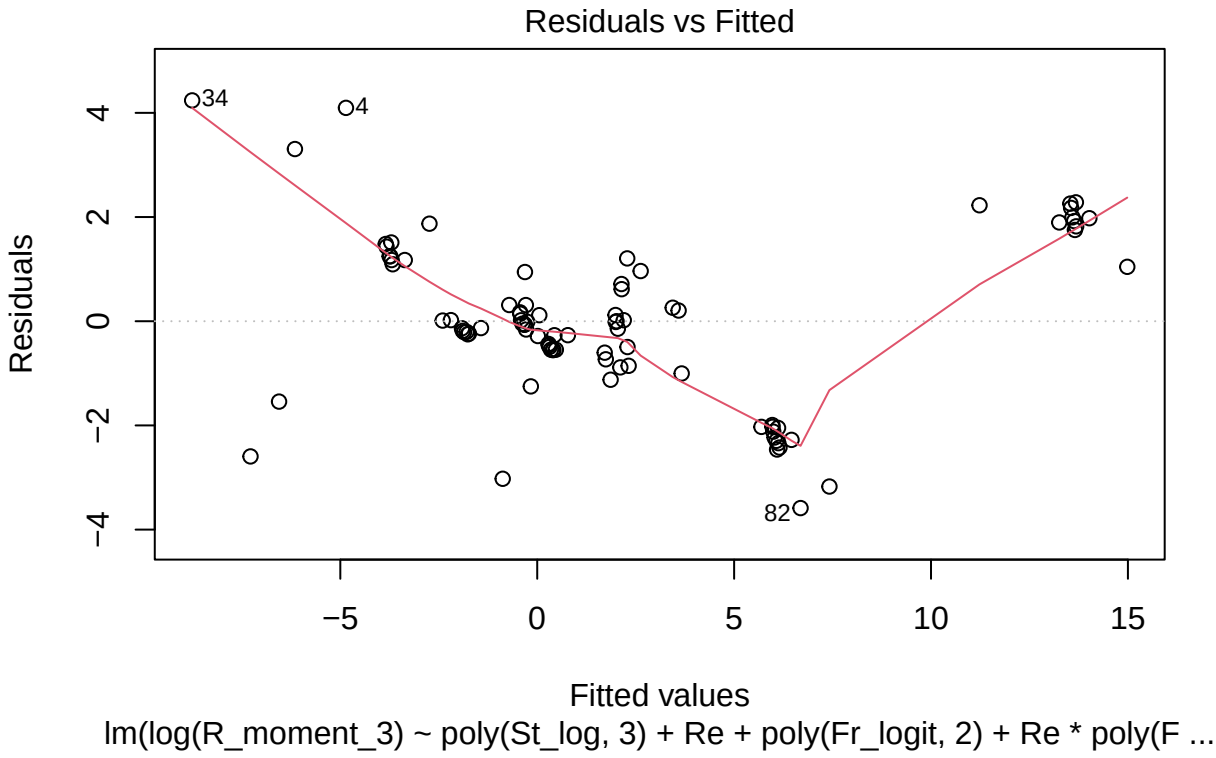
```
##  [1] 0.8538022 0.8775106 0.8895657 0.8915583 0.8909121 0.8897391 0.8881677
##  [8] 0.8864706 0.8862141 0.8846383
```

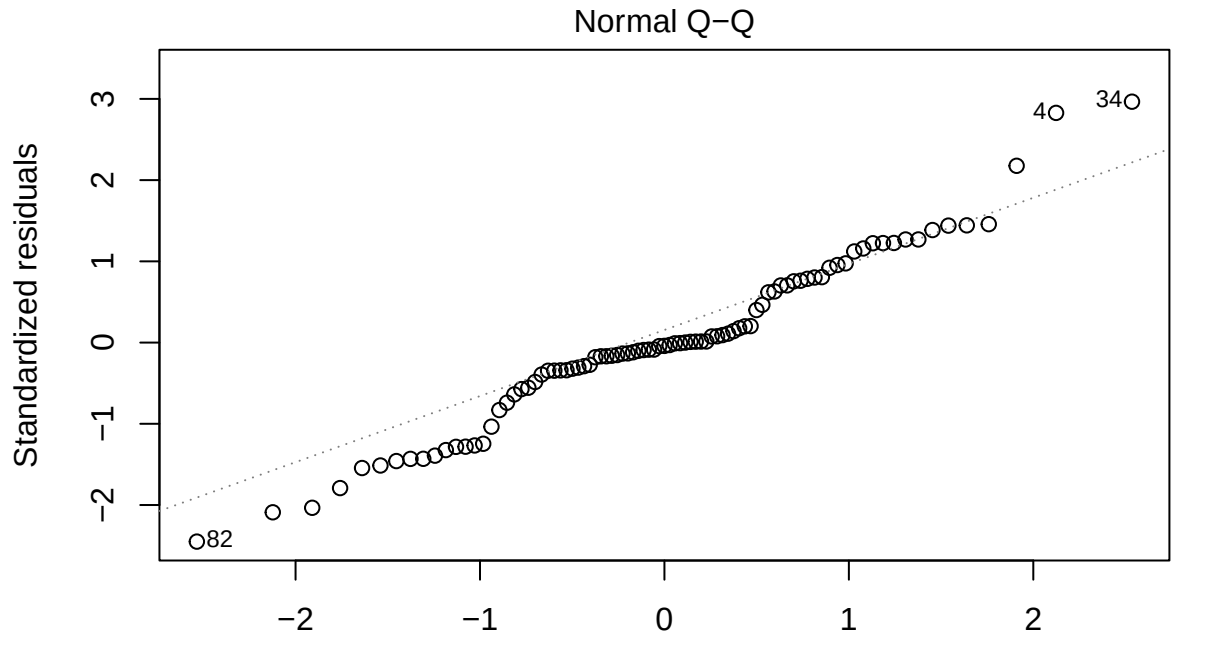
```
## [1] 4.189e+12
```

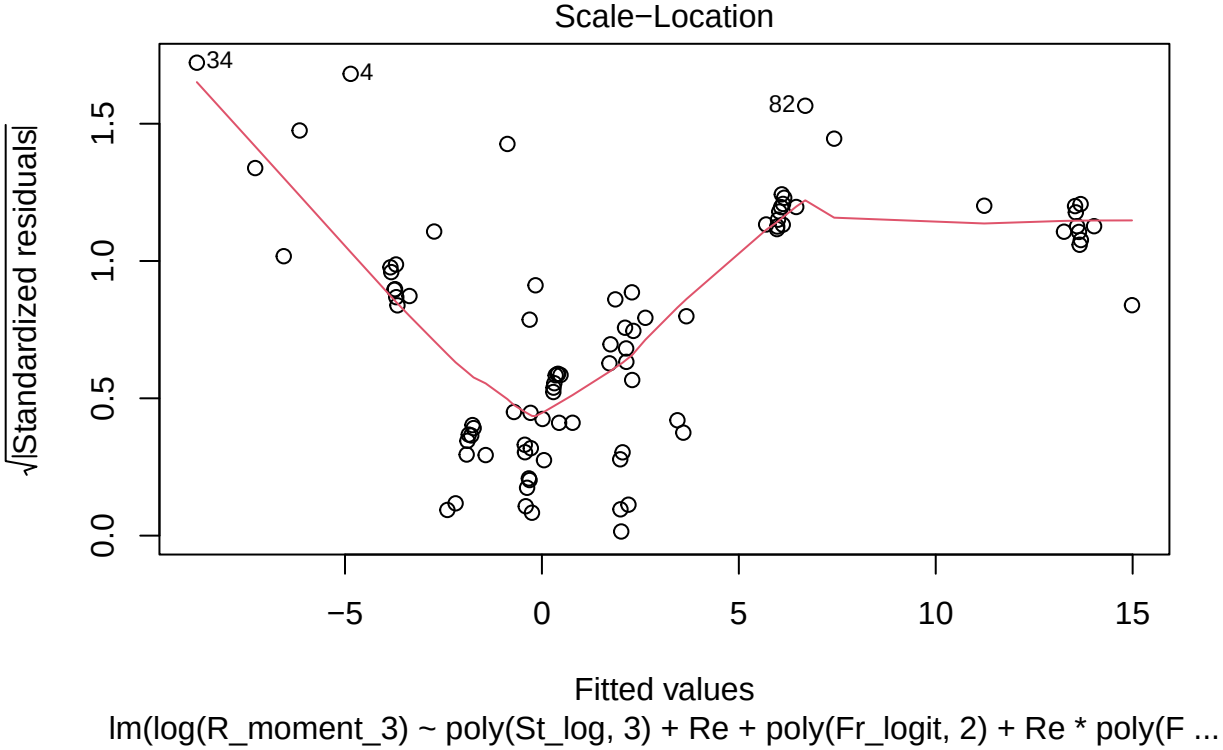
```
## [1] 0.8915583
```

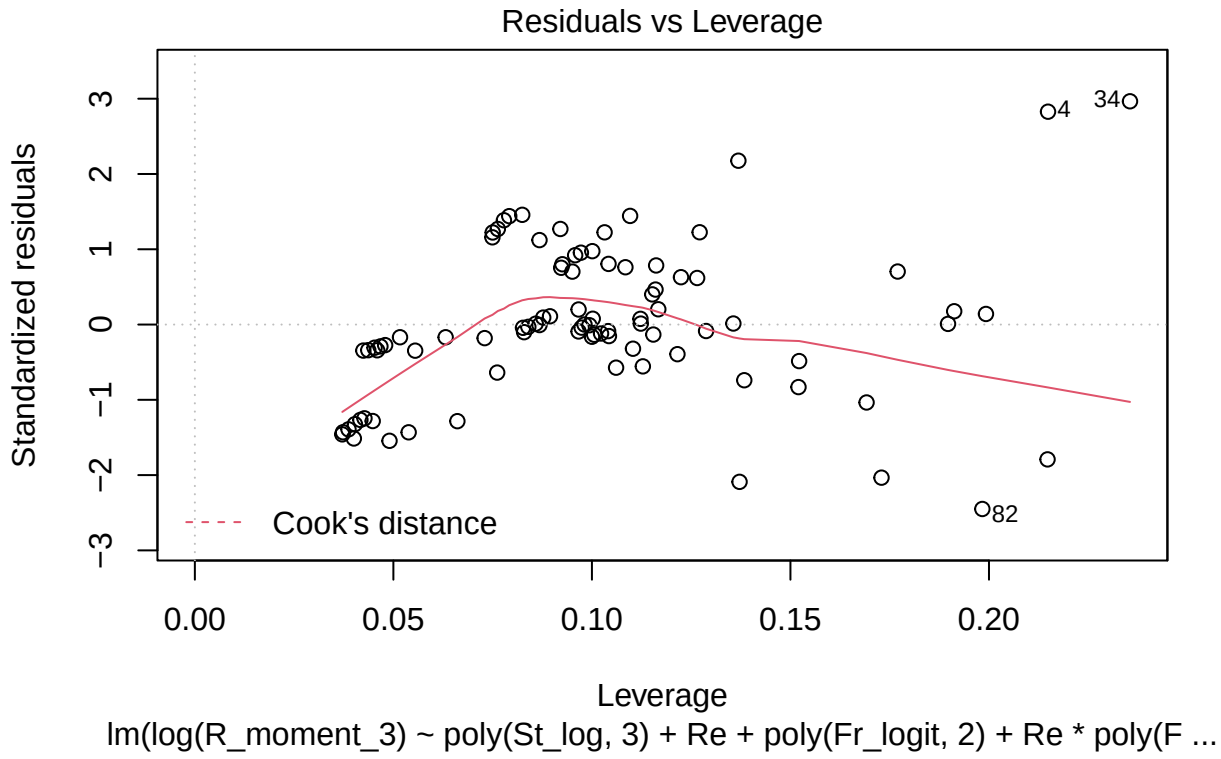
```
##
## Call:
## lm(formula = log(R_moment_3) ~ poly(St_log, 3) + Re + poly(Fr_logit,
##     2) + Re * poly(Fr_logit, 2), data = train)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.5865 -0.6032 -0.0636  1.0936  4.2391
##
## Coefficients:
##                           Estimate Std. Error t value Pr(>|t|)
## (Intercept)               9.136193   0.403721  22.630  < 2e-16 ***
## poly(St_log, 3)1         14.093749   1.647184   8.556 6.55e-13 ***
## poly(St_log, 3)2         -8.173098   1.643736  -4.972 3.70e-06 ***
## poly(St_log, 3)3          6.724747   1.644794   4.089 0.000103 ***
## Re                       -0.032069   0.001831 -17.517  < 2e-16 ***
## poly(Fr_logit, 2)1      -52.407049   3.625512 -14.455  < 2e-16 ***
```

```
## poly(Fr_logit, 2)2     46.785009   4.174440  11.207  < 2e-16 ***
## Re:poly(Fr_logit, 2)1   0.152829   0.014382  10.627  < 2e-16 ***
## Re:poly(Fr_logit, 2)2  -0.114500   0.022038  -5.196 1.52e-06 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.635 on 80 degrees of freedom
## Multiple R-squared:  0.9247, Adjusted R-squared:  0.9171
## F-statistic: 122.7 on 8 and 80 DF,  p-value: < 2.2e-16
```



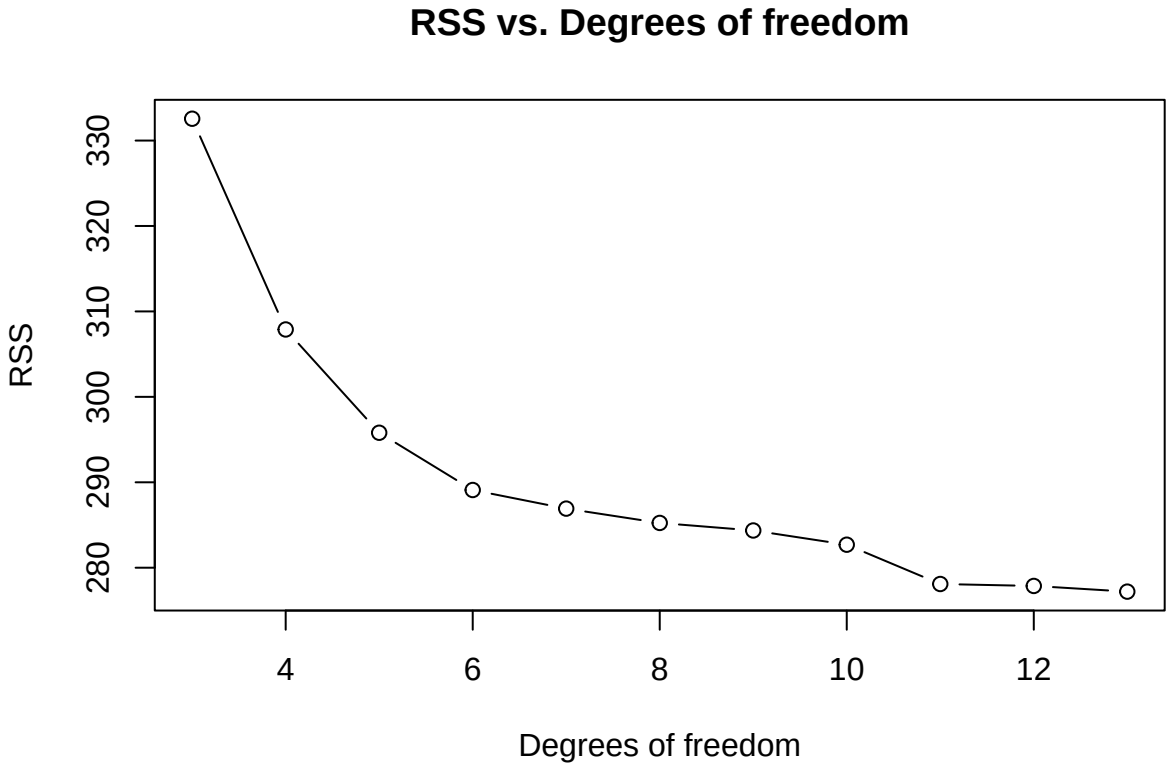

Normal Q−Q
Standardized residuals
Theoretical Quantiles
lm(log(R_moment_3) ~ poly(St_log, 3) + Re + poly(Fr_logit, 2) + Re * poly(F ...


Scale−Location
√|Standardized residuals|
Fitted values
lm(log(R_moment_3) ~ poly(St_log, 3) + Re + poly(Fr_logit, 2) + Re * poly(F ...




```
## [1] 4.488424e+12
## [1] 0.8895657
## [1] 14
```

Generalized additive model




```
##  [1]           NA           NA 4.465903e+12 4.578697e+12 4.602712e+12
##  [6] 4.690008e+12 4.709885e+12 4.726976e+12 4.696304e+12 4.720305e+12
## [11] 4.674294e+12 4.710038e+12 4.707467e+12
```

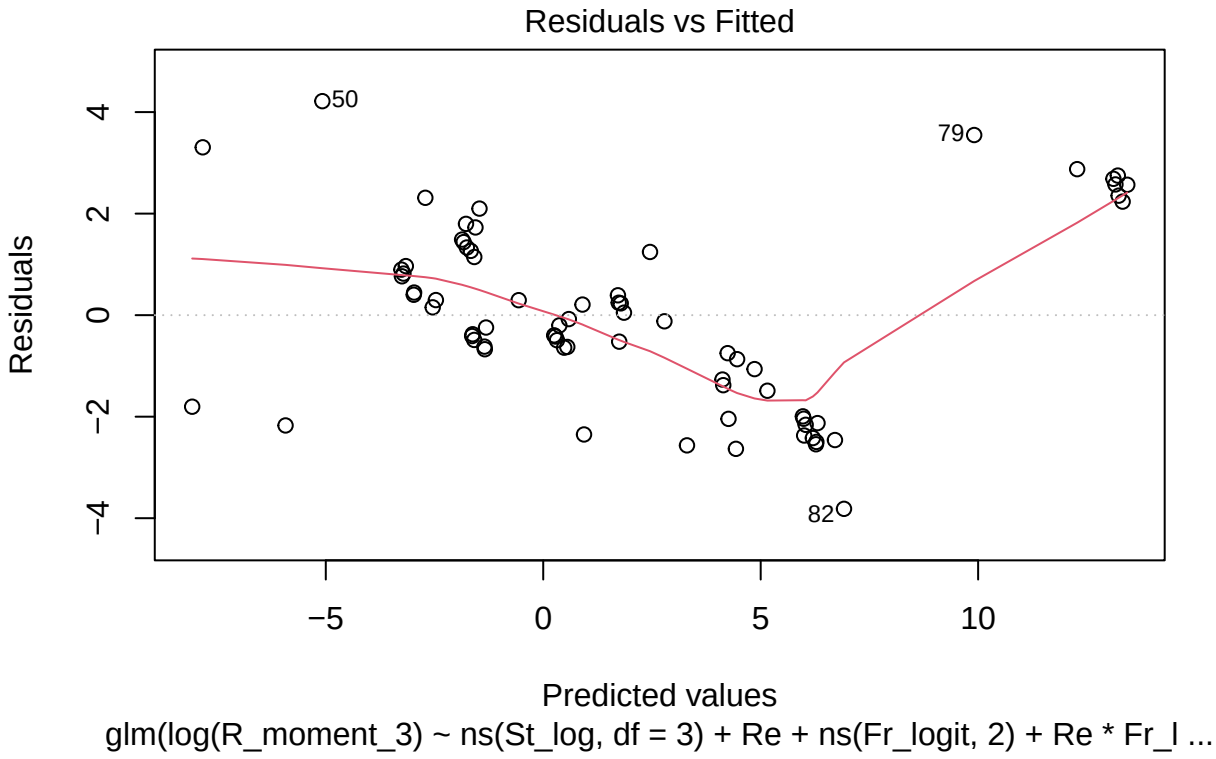
```
##  [1]        NA        NA 0.8946498 0.8983732 0.9007407 0.9034151 0.9045807
##  [8] 0.9048877 0.9050840 0.9058433 0.9084427 0.9088510 0.9091938
```

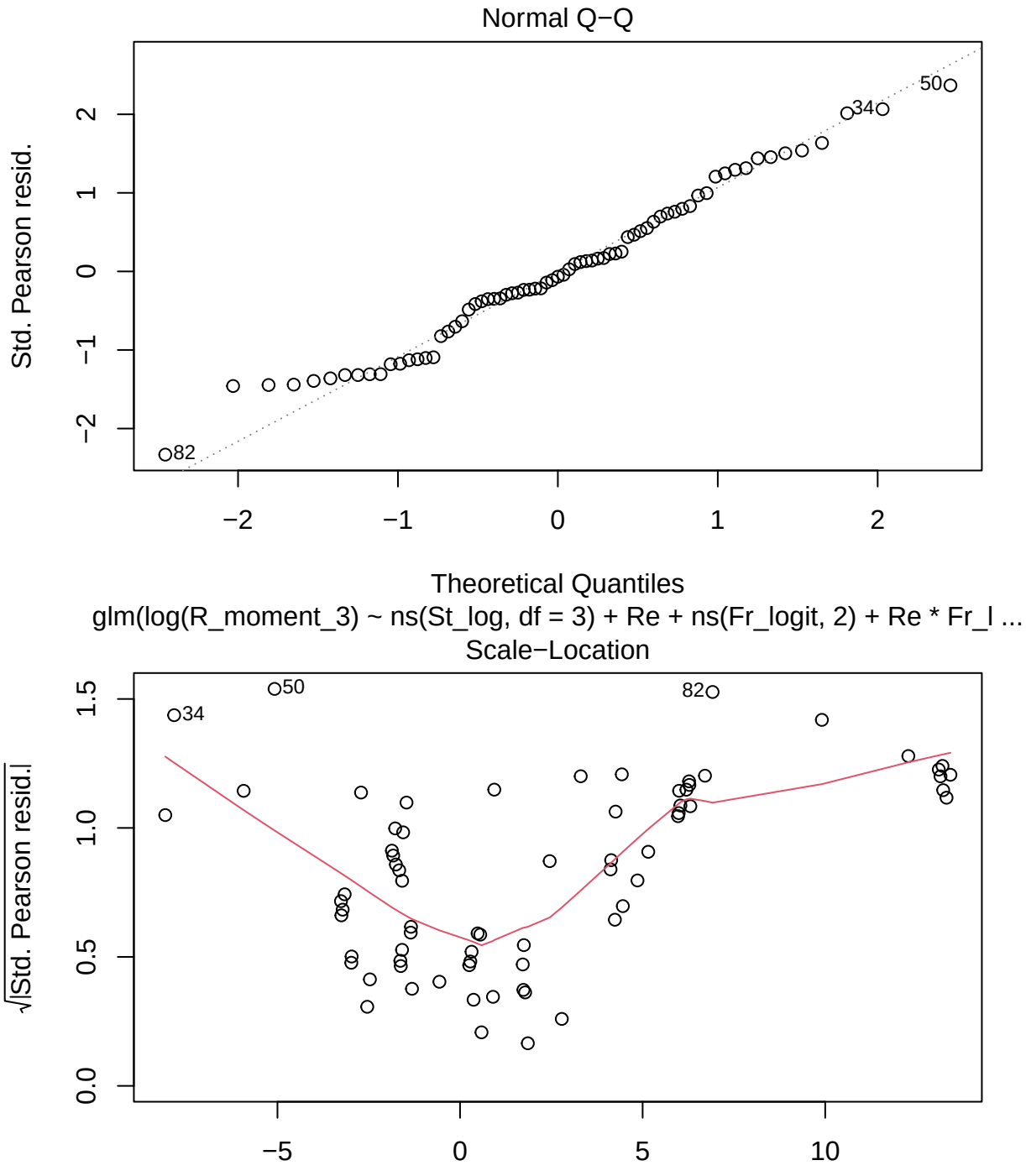
```
## [1] 4.465903e+12
```

```
## [1] 0.9091938
```

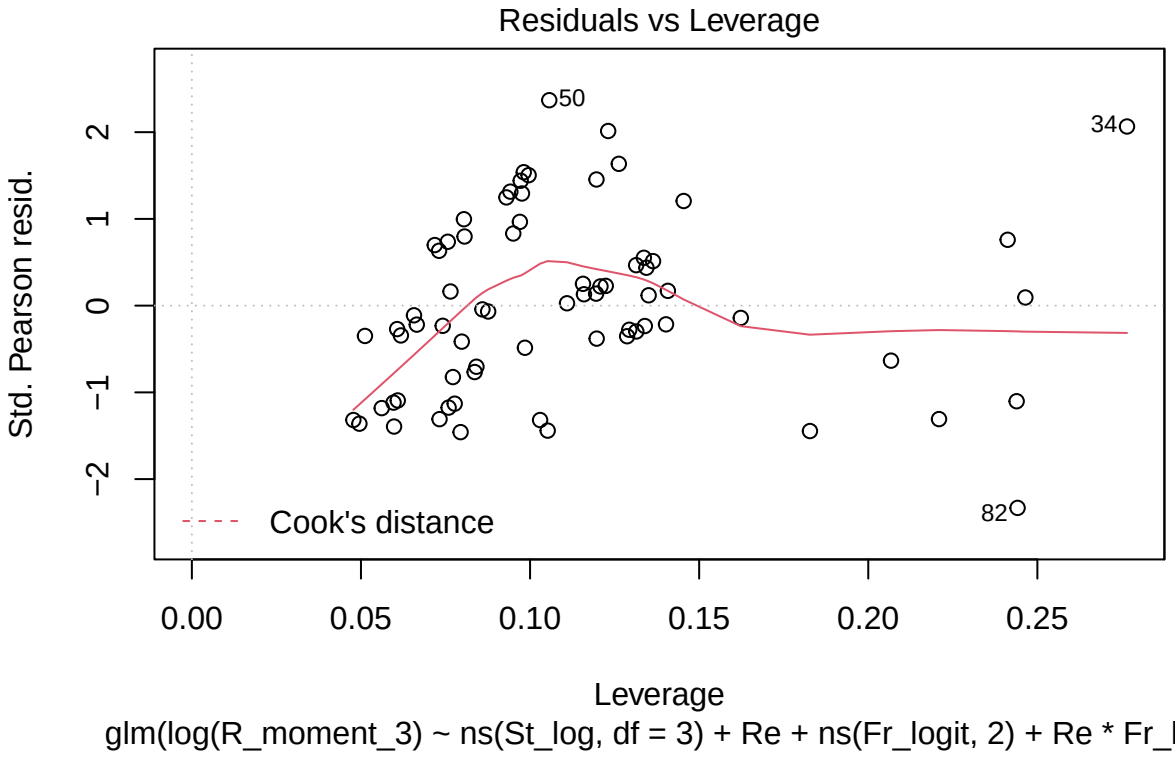
```
##
## Call:
## glm(formula = log(R_moment_3) ~ ns(St_log, df = 3) + Re + ns(Fr_logit,
##     2) + Re * Fr_logit, data = trainData)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -3.8156  -1.3240  -0.1211   1.2555   4.2154
##
## Coefficients: (1 not defined because of singularities)
##                       Estimate Std. Error t value Pr(>|t|)
## (Intercept)          1.170e+01  1.036e+00  11.289  < 2e-16 ***
## ns(St_log, df = 3)1  3.375e+00  8.485e-01   3.978 0.000182 ***
## ns(St_log, df = 3)2  1.298e+01  1.988e+00   6.529 1.31e-08 ***
## ns(St_log, df = 3)3  2.856e+00  8.243e-01   3.465 0.000961 ***
## Re                  -7.886e-02  5.110e-03 -15.431  < 2e-16 ***
## ns(Fr_logit, 2)1    -3.105e+01  1.899e+00 -16.353  < 2e-16 ***
## ns(Fr_logit, 2)2    -7.933e+00  9.618e-01  -8.248 1.31e-11 ***
## Fr_logit                    NA         NA      NA       NA
```

```
## Re:Fr_logit          2.717e-03  2.766e-04   9.823 2.50e-14 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for gaussian family taken to be 3.541814)
##
##     Null deviance: 2125.47  on 70  degrees of freedom
## Residual deviance:  223.13  on 63  degrees of freedom
## AIC: 300.79
##
## Number of Fisher Scoring iterations: 2
```




Normal Q−Q
Std. Pearson resid.
Theoretical Quantiles
glm(log(R_moment_3) ~ ns(St_log, df = 3) + Re + ns(Fr_logit, 2) + Re * Fr_l ...
Scale−Location
√|Std. Pearson resid.|
Predicted values
glm(log(R_moment_3) ~ ns(St_log, df = 3) + Re + ns(Fr_logit, 2) + Re * Fr_l ...

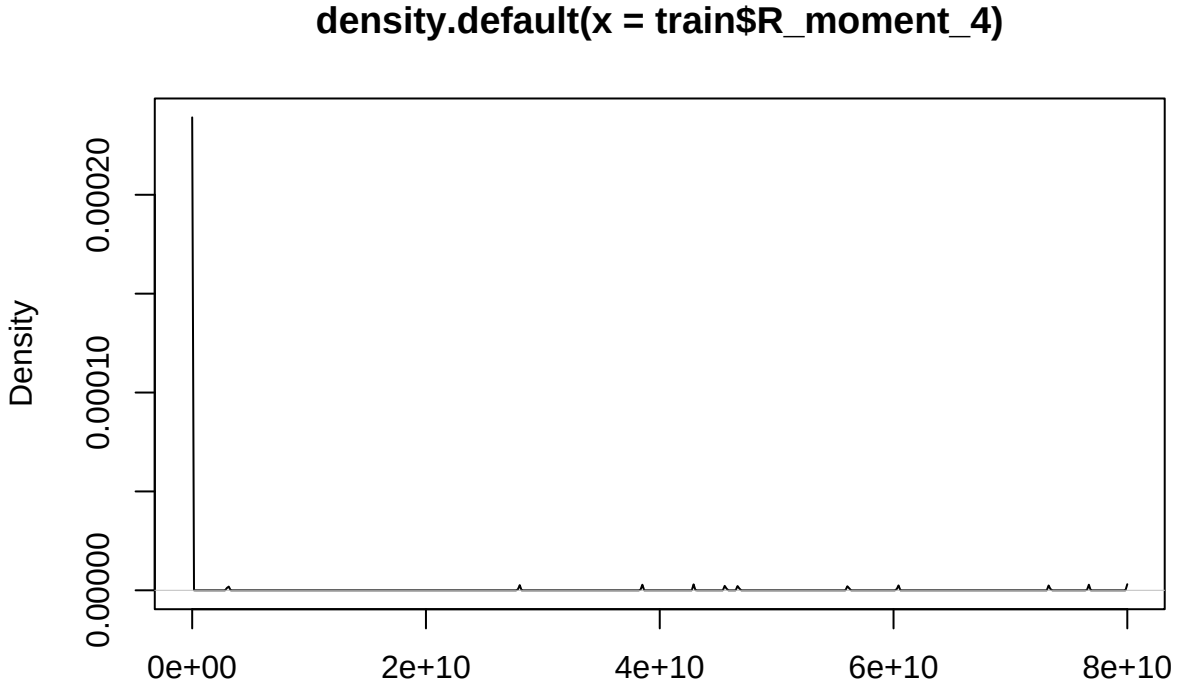
models	formula	mse
Least square regression	log(R_moment_3) ~ log(St)+Re+logit(Fr)	5.216843e+12
Interaction	log(R_moment_3) ~ log(St)+Re+logit(Fr)+Re*logit(Fr)	5.188199e+12
Polynomial regression	log(R_moment_3) ~ poly(log(St),3)+Re+poly(logit(Fr),2)+Re*logit(Fr)	4.488424e+12
Generalized additive model	log(R_moment_3) ~ ns(log(St),df=3)+Re+ns(logit(Fr),2)+Re*logit(Fr)	4.465903e+12

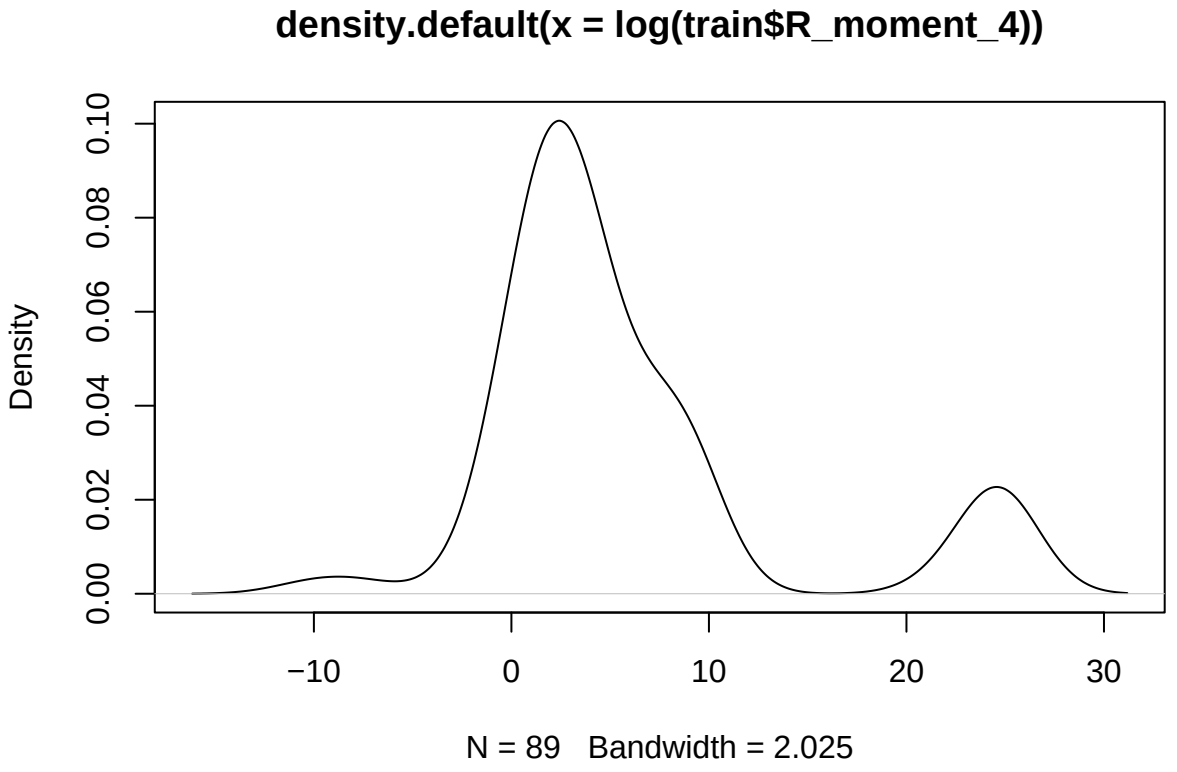


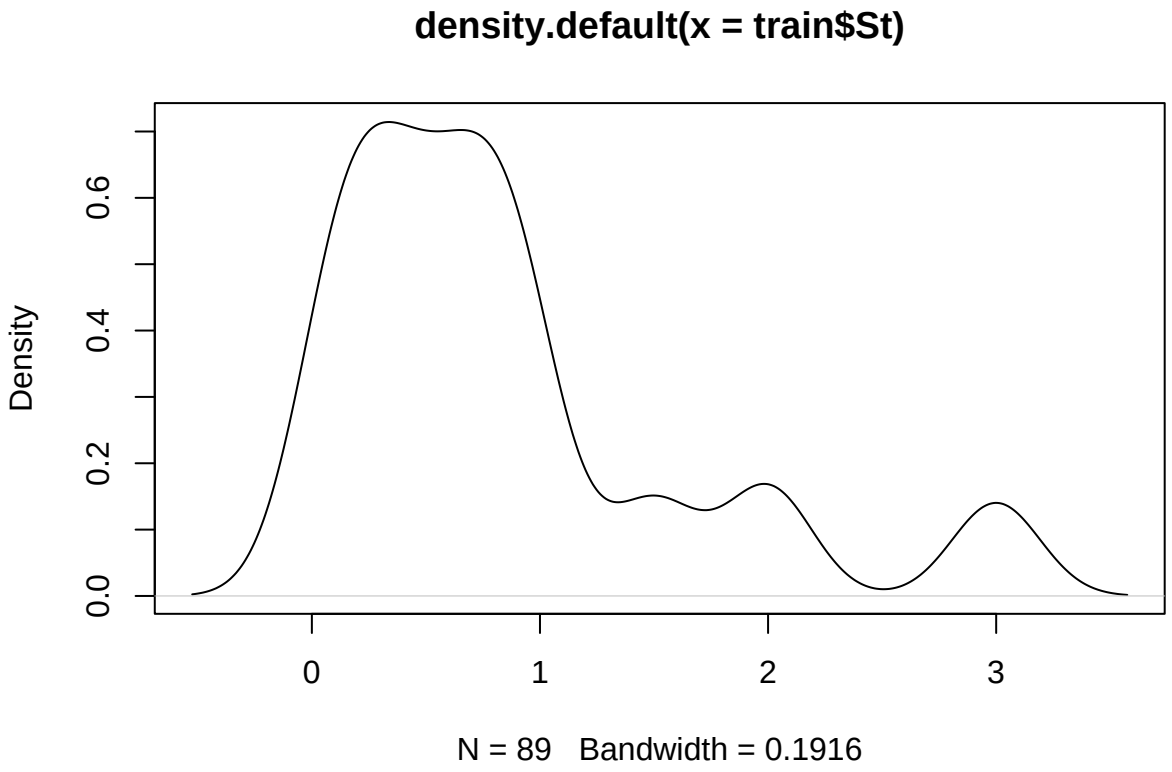

```
## [1] 4.465903e+12
```

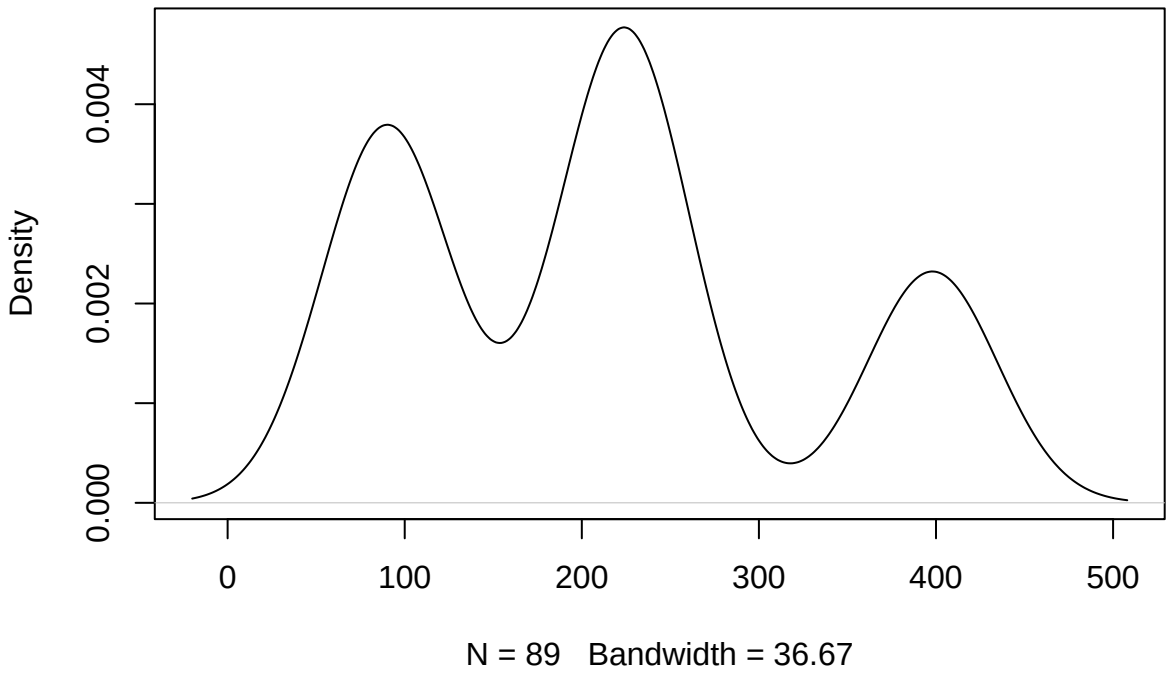
```
## [1] 0.8946498
```

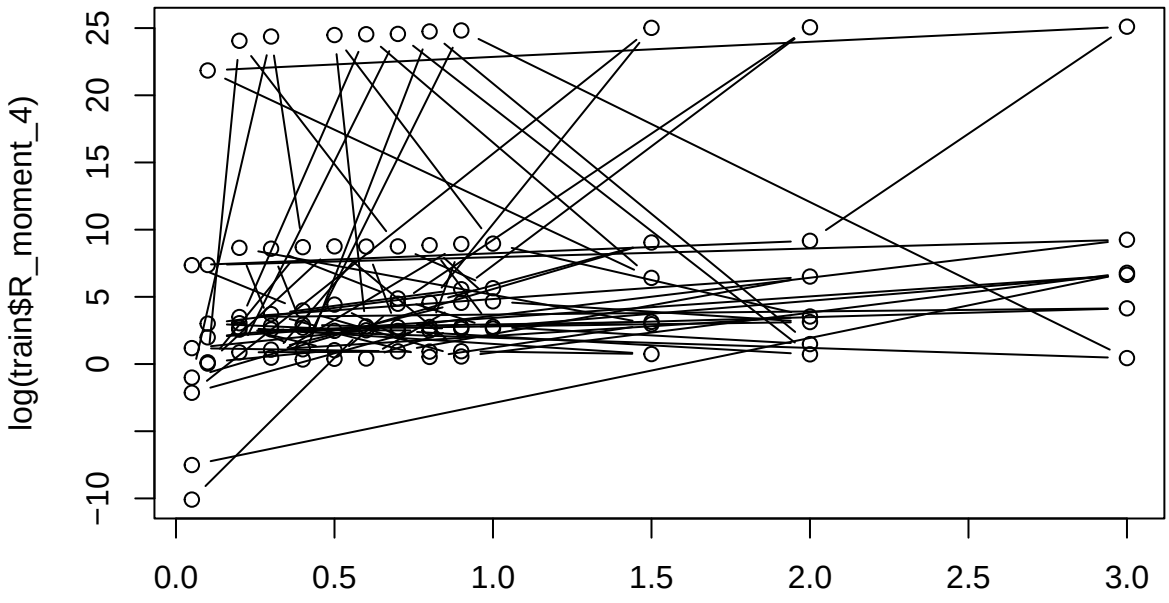
Raw Moment 4

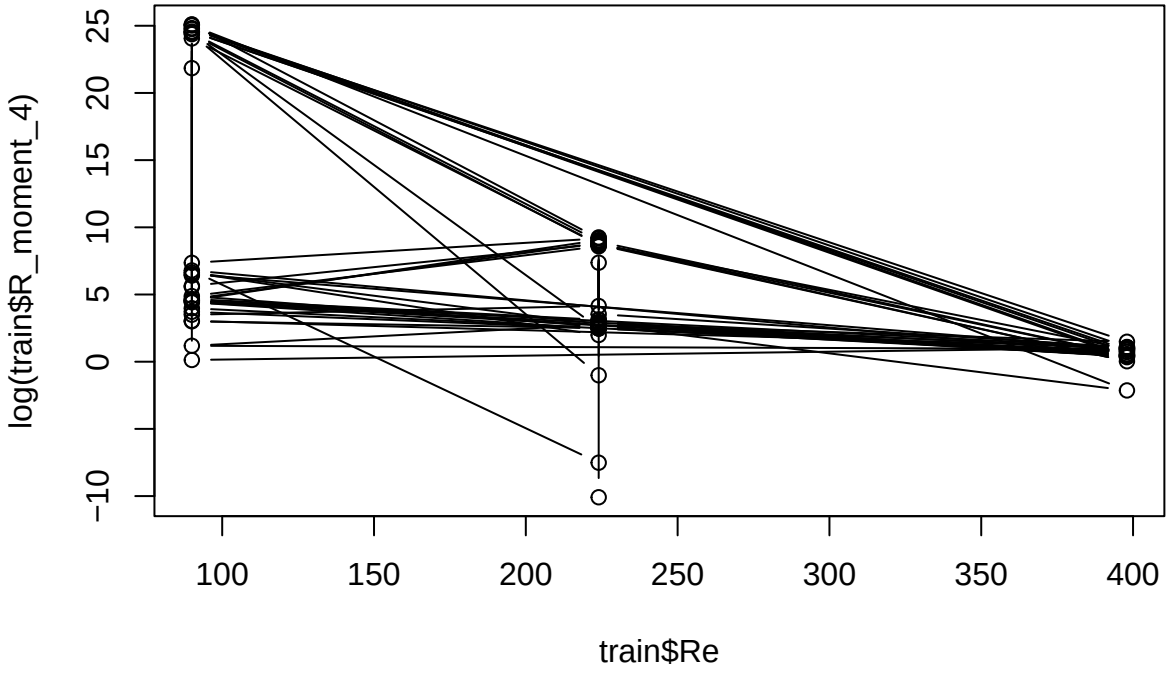

density.default(x = train$R_moment_4)
Density
0.00000
0.00010
0.00020
0e+00
2e+10
4e+10
6e+10
8e+10
N = 89 Bandwidth = 1462


density.default(x = log(train$R_moment_4))
Density
0.00
0.02
0.04
0.06
0.08
0.10
−10
0
10
20
30
N = 89 Bandwidth = 2.025


density.default(x = train$St)
Density
0.0
0.2
0.4
0.6
0
1
2
3
N = 89 Bandwidth = 0.1916


density.default(x = train$Re)
Density
0.000
0.002
0.004
0
100
200
300
400
500
N = 89 Bandwidth = 36.67





Least Square Regression

```
##
## Call:
## lm(formula = log(R_moment_4) ~ St + Re + Fr_logit, data = train)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -15.829  -3.955  -1.040   1.656  12.537
##
## Coefficients:
```

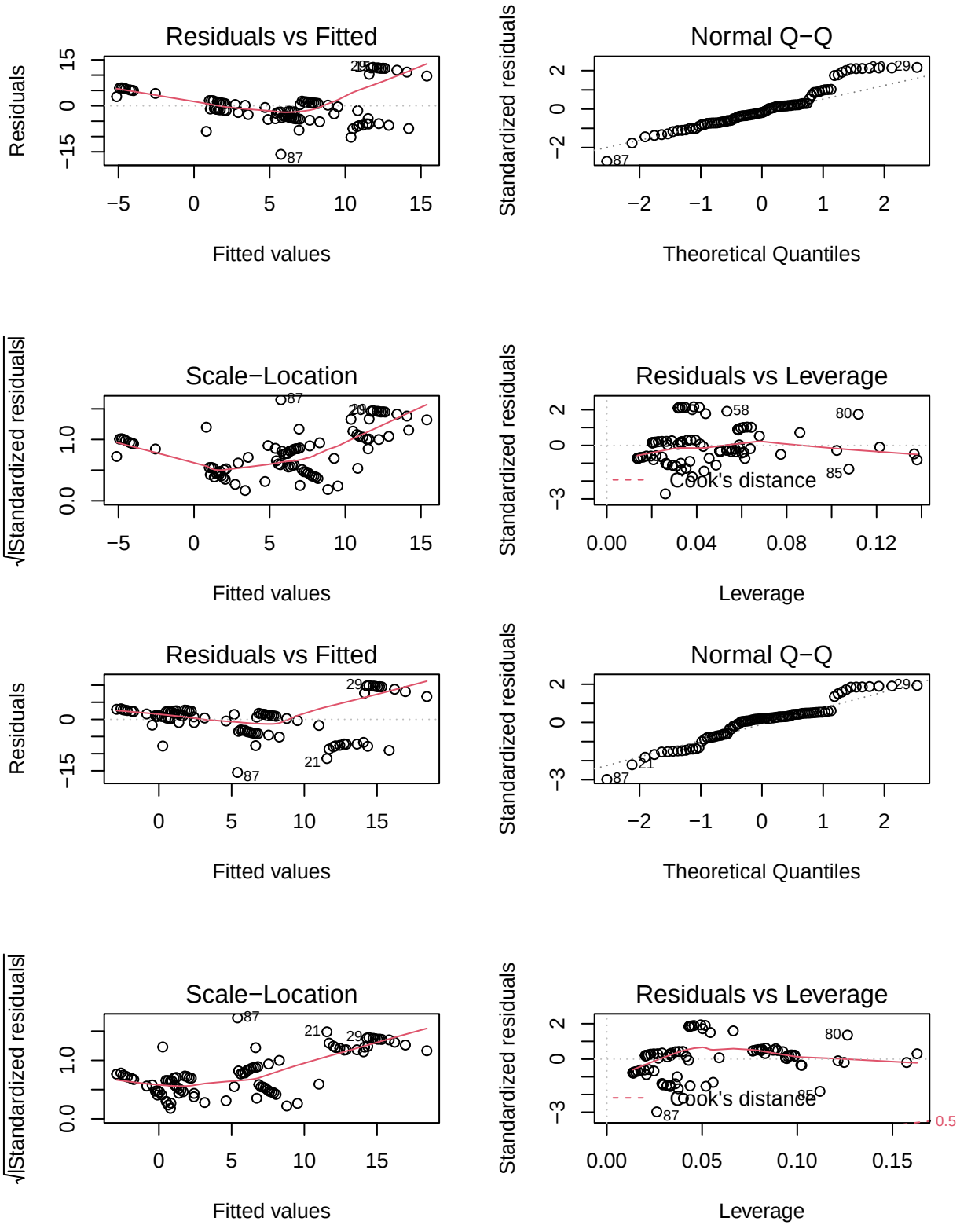
```
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 18.24334    2.03102   8.982 5.86e-14 ***
## St           1.31604    0.80225   1.640    0.105
## Re          -0.03409    0.00560  -6.088 3.19e-08 ***
## Fr_logit    -0.39447    0.09137  -4.317 4.26e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 5.907 on 85 degrees of freedom
## Multiple R-squared:  0.435,  Adjusted R-squared:  0.4151
## F-statistic: 21.81 on 3 and 85 DF,  p-value: 1.439e-10
```

```
##
## Call:
## lm(formula = log(R_moment_4) ~ St + Re + Fr_logit + Re * Fr_logit,
##     data = train)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -15.490  -3.490   1.041   2.405   9.932
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) 29.8585137  3.0233425   9.876 1.03e-15 ***
## St           1.4763025  0.7157884   2.062   0.0423 *
## Re          -0.0865652  0.0120242  -7.199 2.37e-10 ***
## Fr_logit    -1.1493810  0.1772067  -6.486 5.77e-09 ***
## Re:Fr_logit  0.0032957  0.0006871   4.797 6.92e-06 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 5.264 on 84 degrees of freedom
## Multiple R-squared:  0.5565, Adjusted R-squared:  0.5354
## F-statistic: 26.35 on 4 and 84 DF,  p-value: 3.608e-14
```

```
## [1] 3.646495e+20
```

```
## [1] 3.646495e+20
```

```
## [1] 0.5660468
```



Polynomial Model

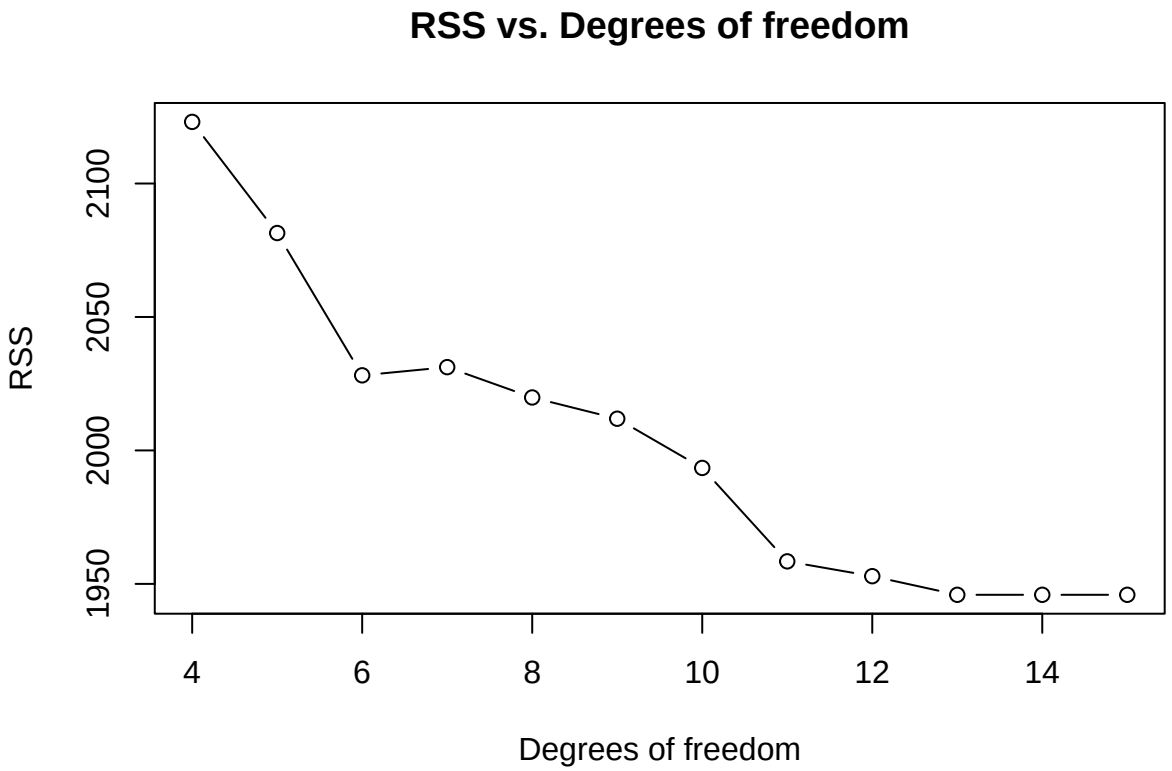
```
##  [1] 3.445536e+20 3.396746e+20 3.476248e+20 3.465095e+20 3.487530e+20
##  [6] 3.498513e+20 3.508899e+20 3.544240e+20 3.536081e+20 3.538072e+20
```

```
## [1] 3.396746e+20
```

```
## [1] 0.8534053
```

```
##
## Call:
## lm(formula = log(R_moment_4) ~ poly(St, 2) + Re + poly(Fr_logit,
##     2) + Re * Fr_logit, data = train)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -10.4985  -1.6887   0.2832   2.0563   5.4626
##
## Coefficients: (1 not defined because of singularities)
##                       Estimate Std. Error t value Pr(>|t|)
## (Intercept)          1.538e+01  7.601e-01  20.236  < 2e-16 ***
## poly(St, 2)1         9.719e+00  3.238e+00   3.002  0.00356 **
## poly(St, 2)2        -9.279e+00  3.246e+00  -2.859  0.00539 **
## Re                  -1.036e-01  7.518e-03 -13.779  < 2e-16 ***
## poly(Fr_logit, 2)1 -8.063e+01  7.095e+00 -11.364  < 2e-16 ***
## poly(Fr_logit, 2)2  3.932e+01  3.354e+00  11.721  < 2e-16 ***
## Fr_logit                   NA         NA      NA       NA
## Re:Fr_logit          3.767e-03  4.238e-04   8.888  1.2e-13 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.228 on 82 degrees of freedom
## Multiple R-squared:  0.8372, Adjusted R-squared:  0.8253
## F-statistic:  70.3 on 6 and 82 DF,  p-value: < 2.2e-16
```

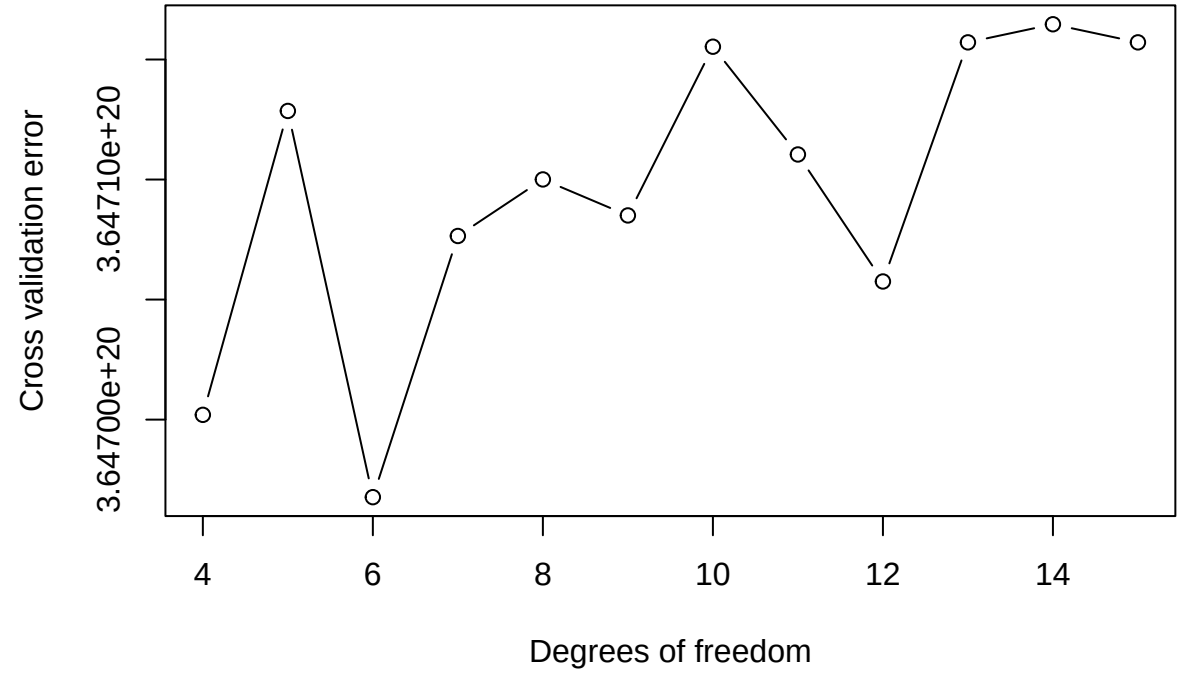
Natural spline




```
##  [1]           NA           NA           NA 3.647002e+20 3.647129e+20
##  [6] 3.646968e+20 3.647076e+20 3.647100e+20 3.647085e+20 3.647155e+20
## [11] 3.647110e+20 3.647058e+20 3.647157e+20 3.647165e+20 3.647157e+20
```

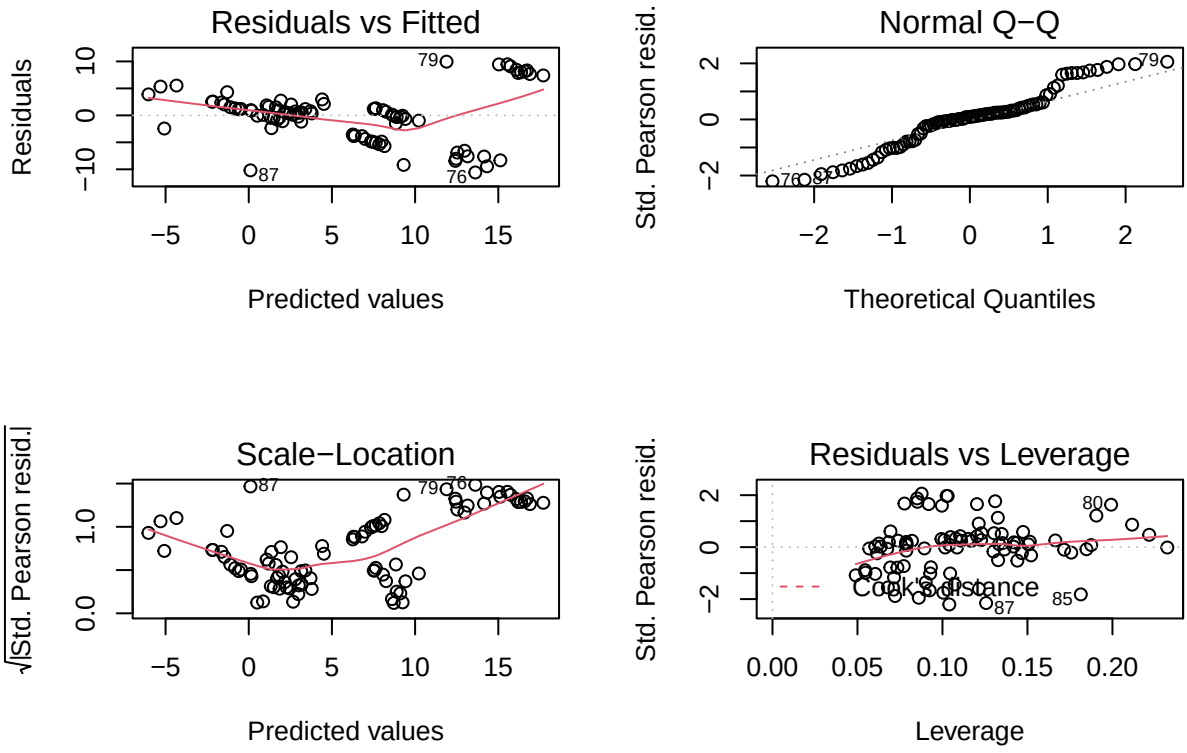
```
## [1] 3.646968e+20
```

```
## [1] 0.65005
```



```
## 16.66667% 33.33333%       50% 66.66667% 83.33333%
##       0.2       0.4       0.7       0.9       1.5
```

```
##
## Call:
## glm(formula = log(R_moment_4) ~ ns(St, df = 6) + Re + Fr_logit +
##     Re * Fr_logit, data = train)
##
## Deviance Residuals:
##      Min        1Q    Median        3Q       Max
## -10.5812   -2.4393    0.4298    2.0073    9.9542
##
## Coefficients:
##                   Estimate Std. Error t value Pr(>|t|)
## (Intercept)     24.6583738  3.3173272   7.433 1.10e-10 ***
## ns(St, df = 6)1  3.5983874  3.2535161   1.106 0.272083
## ns(St, df = 6)2  9.4555987  3.1423008   3.009 0.003515 **
## ns(St, df = 6)3  5.0013034  3.2538047   1.537 0.128274
## ns(St, df = 6)4  4.4495341  3.2557345   1.367 0.175604
## ns(St, df = 6)5 17.3831868  4.8714924   3.568 0.000614 ***
## ns(St, df = 6)6  3.7106203  2.4399533   1.521 0.132309
## Re              -0.0864862  0.0116013  -7.455 9.97e-11 ***
## Fr_logit        -1.1403722  0.1726567  -6.605 4.21e-09 ***
## Re:Fr_logit      0.0032407  0.0006658   4.867 5.69e-06 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for gaussian family taken to be 25.67262)
##
##     Null deviance: 5248.7  on 88  degrees of freedom
## Residual deviance: 2028.1  on 79  degrees of freedom
## AIC: 552.81
##
## Number of Fisher Scoring iterations: 2
```



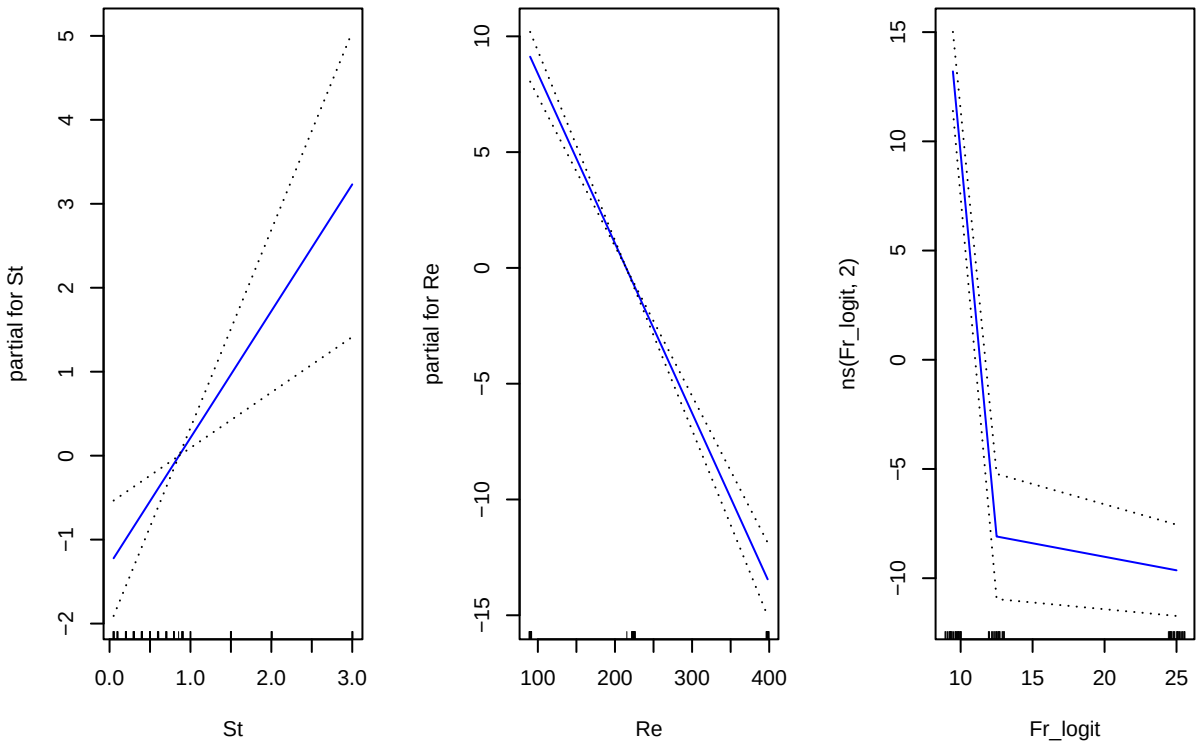
Generalized Additive Model

```
##  [1] 3.206759e+20 3.303028e+20 3.319019e+20 3.339522e+20 3.371089e+20
##  [6] 3.384887e+20 3.393099e+20 3.388484e+20 3.379093e+20 3.379113e+20
## [11] 3.384665e+20 3.369819e+20 3.396346e+20 3.396346e+20 3.396346e+20
```

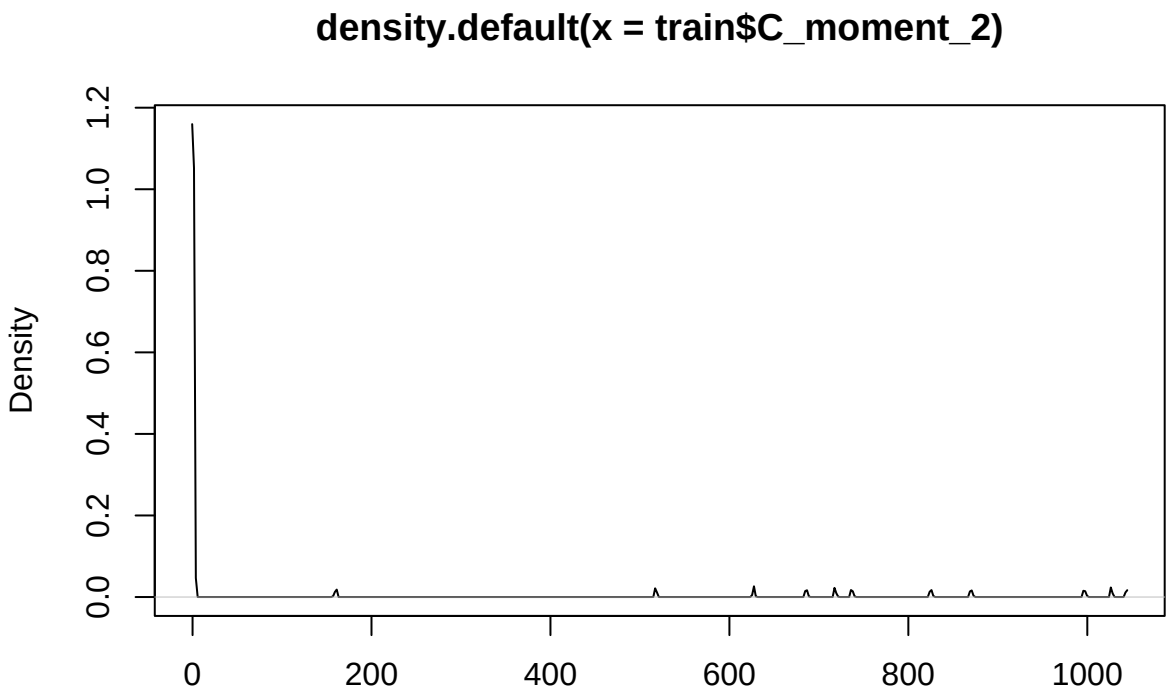
```
## [1] 3.206759e+20
```

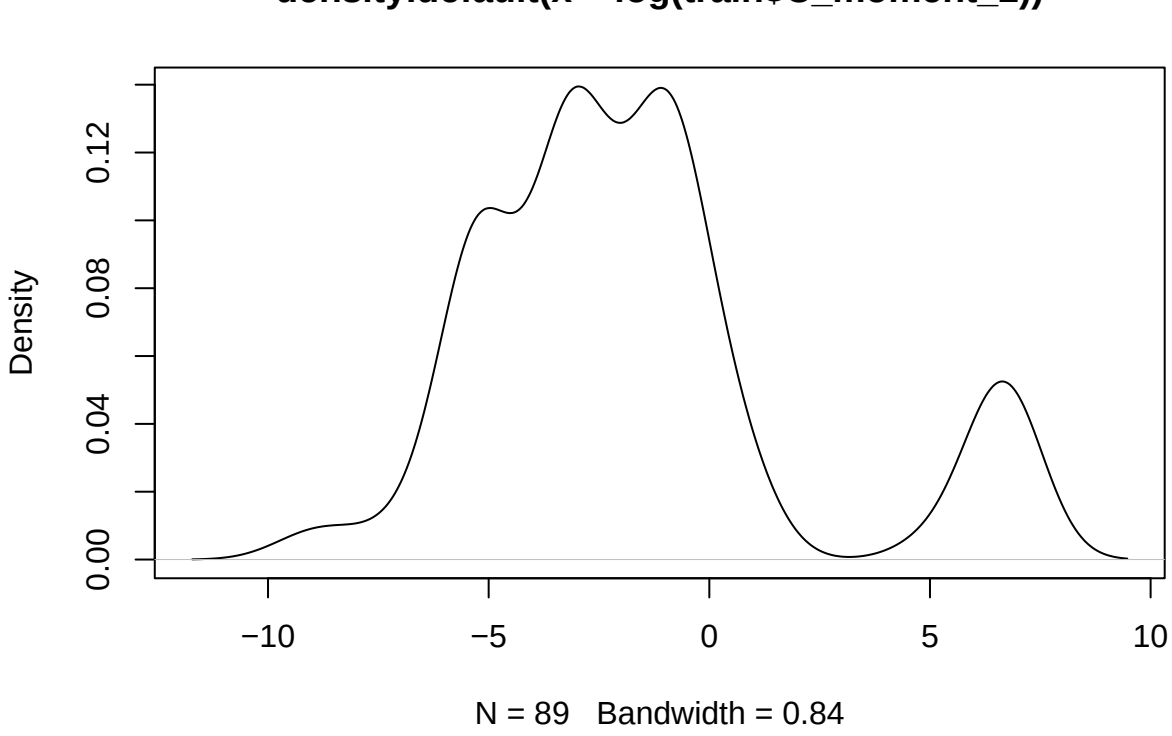
```
## [1] 0.884858
```

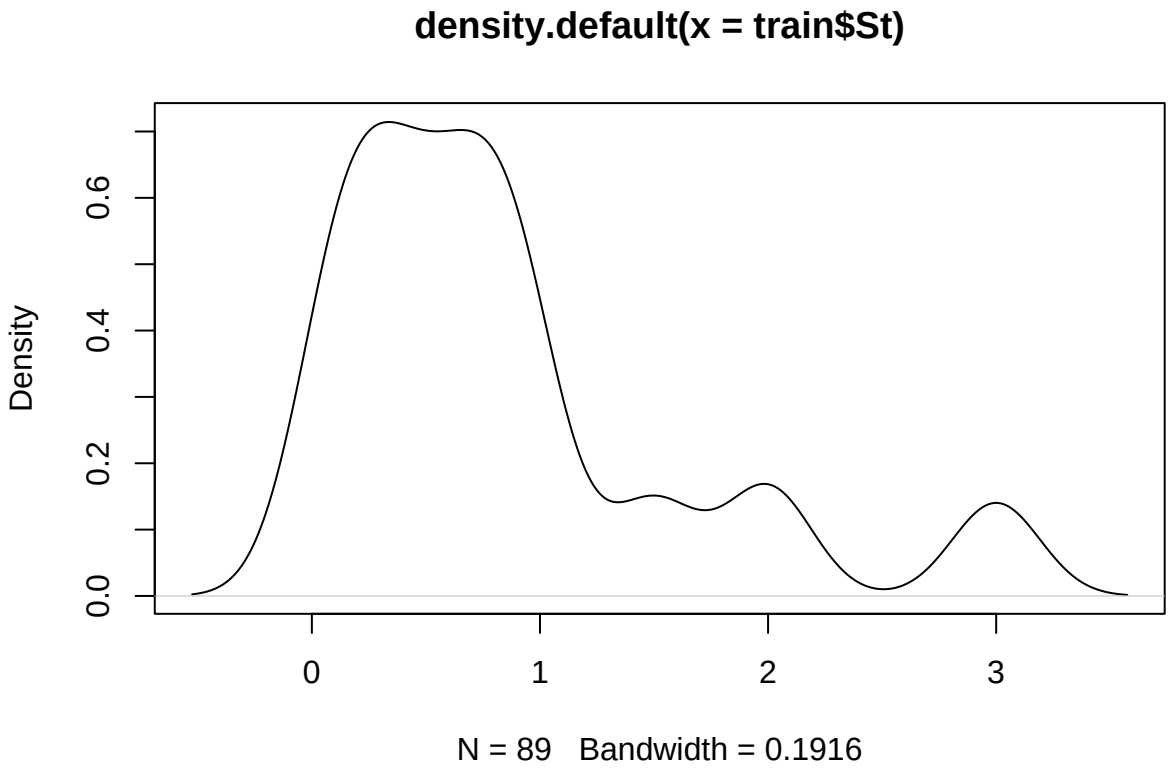
models	formula	mse	a
Least square regression	log(R_moment_4) ~ Fr + Re + St + Fr * Re	3.646495e+20	0.566
Polynomial regression	log(R_moment_4) ~ poly(Fr,2) + Re + poly(St, 2) + Fr * Re	3.396746e+20	0.853
Natural spline	log(R_moment_4) ~ ns(St, df = 6) + Fr + Re + Fr * Re	3.646968e+20	0.650
Generalized additive model	log(R_moment_4) ~ St + Re + ns(Fr_logit,2) + Re:ns(Fr_logit,2)	3.206759e+20	0.884

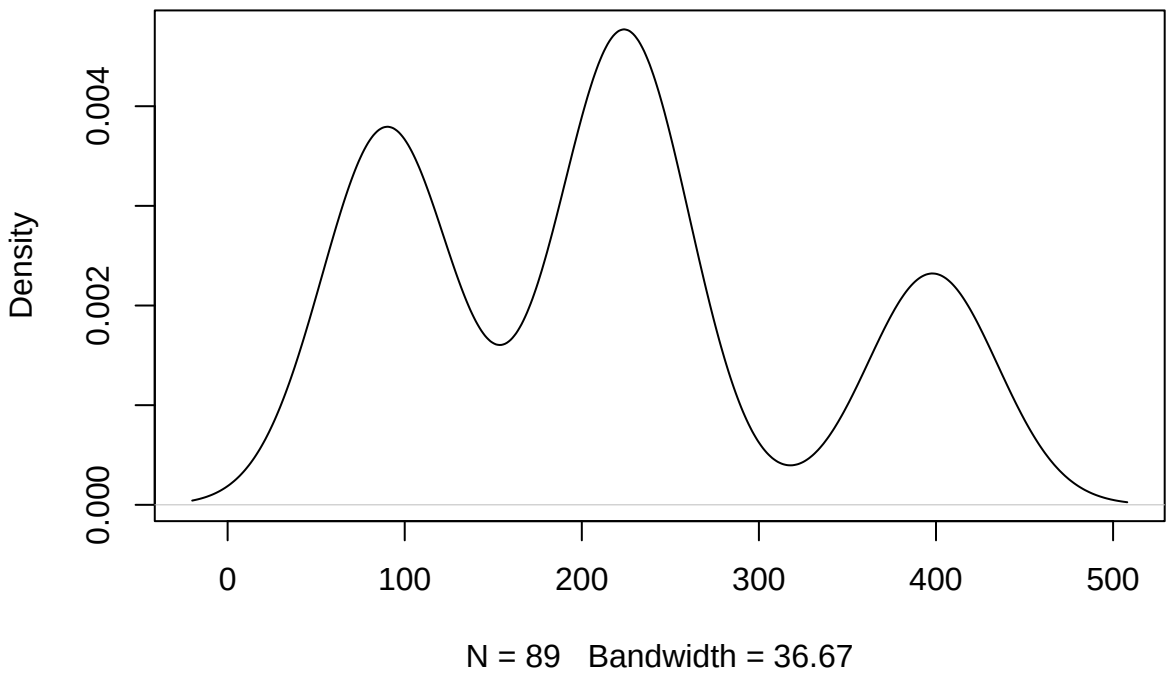


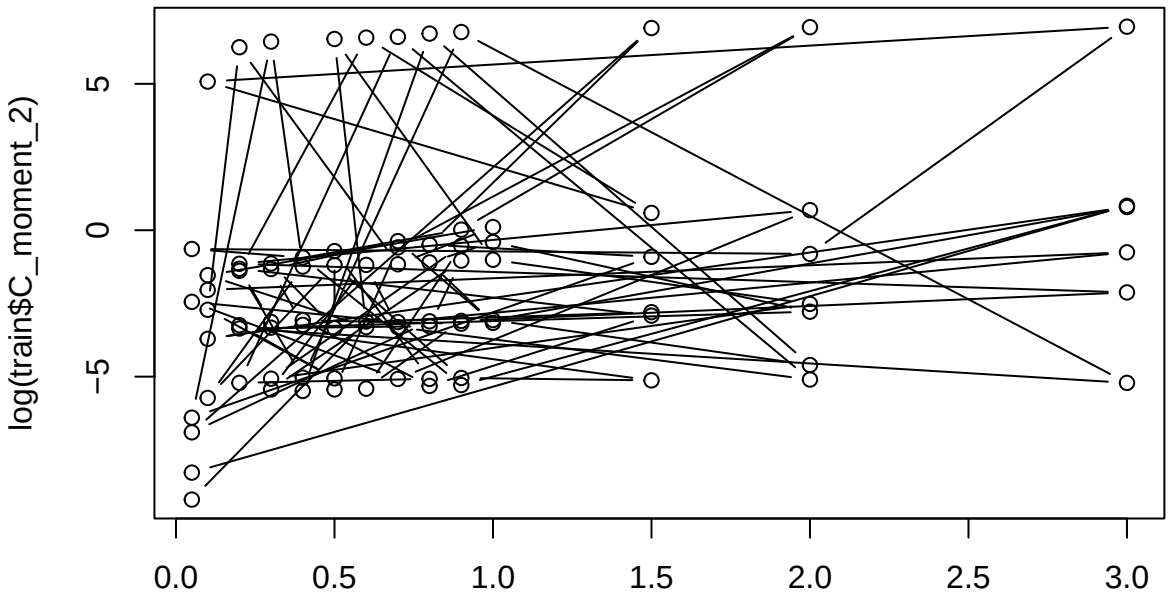
C_moment_2

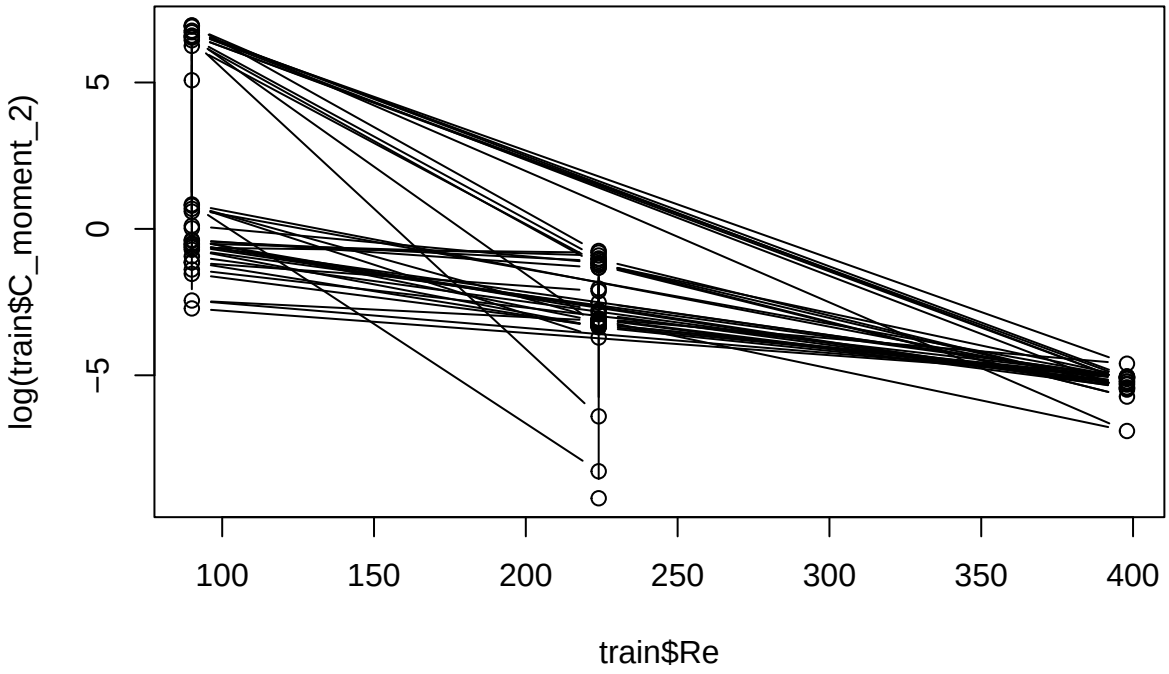

density.default(x = train$C_moment_2)
Density
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0
200
400
600
800
1000
N = 89 Bandwidth = 0.1375


density.default(x = log(train$C_moment_2))
Density
0.00
0.04
0.08
0.12
−10
−5
0
5
10
N = 89 Bandwidth = 0.84


density.default(x = train$St)
Density
0.0
0.2
0.4
0.6
0
1
2
3
N = 89 Bandwidth = 0.1916


density.default(x = train$Re)
Density
0.000
0.002
0.004
0
100
200
300
400
500
N = 89 Bandwidth = 36.67






```
##                   St          Re    Fr_logit
## St         1.00000000 -0.03169871 -0.04921517
## Re        -0.03169871  1.00000000  0.09619529
## Fr_logit -0.04921517  0.09619529  1.00000000
```

```
##
## Call:
## lm(formula = log(C_moment_2) ~ St + Re + Fr_logit, data = train)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -7.0303 -1.5651 -0.1860  0.4501  4.9247
```

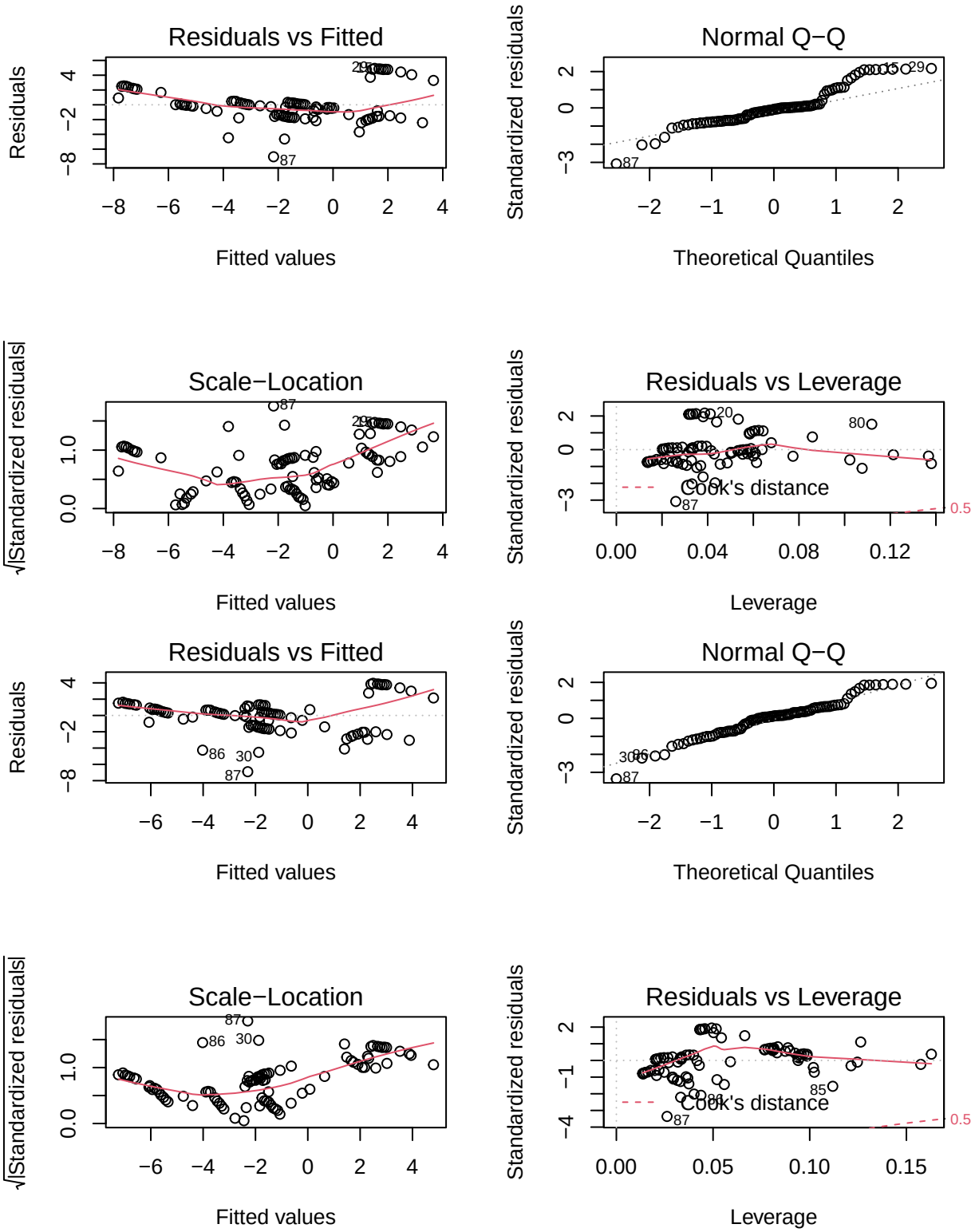
```
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  4.596698   0.795120   5.781  1.2e-07 ***
## St           0.793887   0.314070   2.528  0.01333 *
## Re          -0.023034   0.002192 -10.507  < 2e-16 ***
## Fr_logit    -0.131600   0.035771  -3.679  0.00041 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.312 on 85 degrees of freedom
## Multiple R-squared:  0.6251, Adjusted R-squared:  0.6119
## F-statistic: 47.25 on 3 and 85 DF,  p-value: < 2.2e-16
```

```
##
## Call:
## lm(formula = log(C_moment_2) ~ St + Re + Fr_logit + Re * Fr_logit,
##     data = train)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -6.9027 -1.4369  0.2679  1.1672  3.9448
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  8.9673082  1.1958823   7.498 6.06e-11 ***
## St           0.8541921  0.2831299   3.017  0.00338 **
## Re          -0.0427780  0.0047562  -8.994 6.09e-14 ***
## Fr_logit    -0.4156629  0.0700941  -5.930 6.51e-08 ***
## Re:Fr_logit  0.0012401  0.0002718   4.563 1.71e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.082 on 84 degrees of freedom
## Multiple R-squared:  0.6996, Adjusted R-squared:  0.6853
## F-statistic:  48.9 on 4 and 84 DF,  p-value: < 2.2e-16
```

```
## [1] 70420.79
```

```
## [1] 70420.79
```

```
## [1] 0.7079129
```




```
##  [1] 47279.28 45491.00 48355.20 46632.63 46157.62 47529.18 48264.68 50270.66
##  [9] 49970.83 50213.13
```

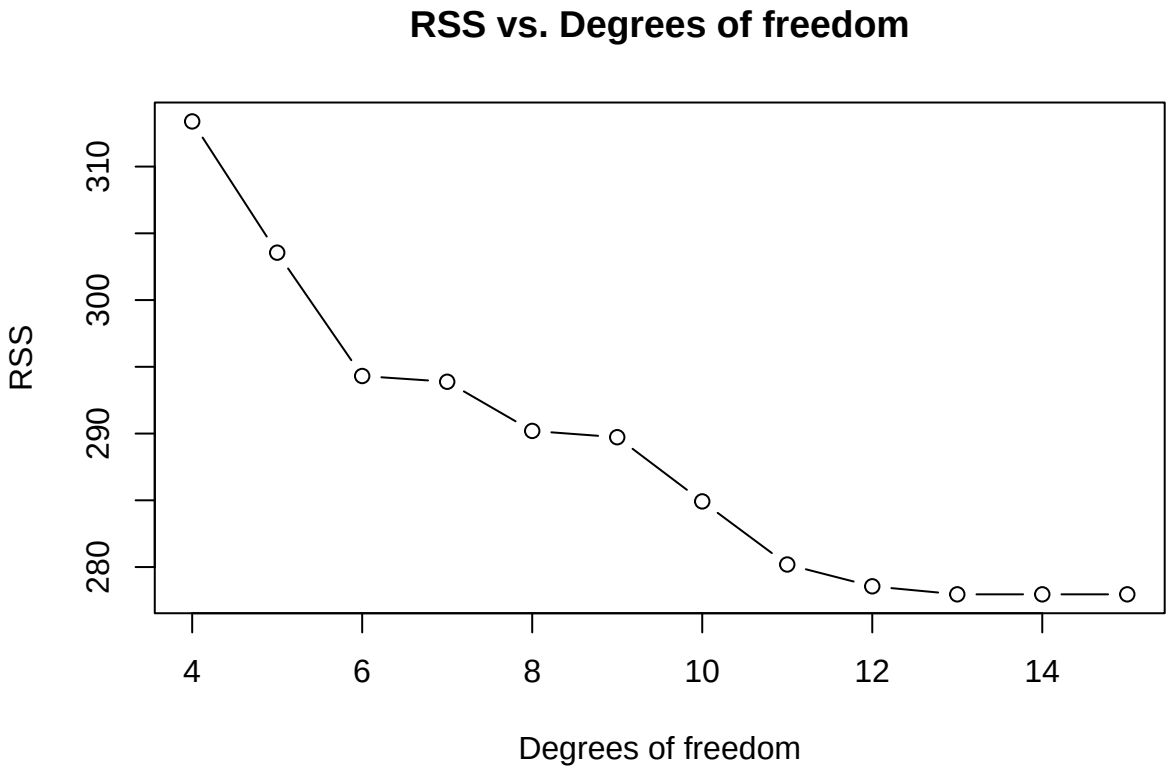
```
## [1] 45491
```

```
## [1] 0.8733799
```

```
##
## Call:
## lm(formula = log(C_moment_2) ~ poly(St, 2) + Re + poly(Fr_logit,
##     2) + Re * Fr_logit, data = train)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -5.1003 -0.6213  0.1176  0.8656  2.3951
##
## Coefficients: (1 not defined because of singularities)
##                       Estimate Std. Error t value Pr(>|t|)
## (Intercept)          4.018e+00  3.351e-01  11.988  < 2e-16 ***
## poly(St, 2)1         5.876e+00  1.428e+00   4.115 9.15e-05 ***
## poly(St, 2)2        -4.576e+00  1.431e+00  -3.198  0.00197 **
## Re                  -4.887e-02  3.315e-03 -14.742  < 2e-16 ***
## poly(Fr_logit, 2)1 -2.901e+01  3.129e+00  -9.274 2.05e-14 ***
## poly(Fr_logit, 2)2  1.415e+01  1.479e+00   9.564 5.44e-15 ***
## Fr_logit                   NA         NA      NA       NA
## Re:Fr_logit          1.403e-03  1.869e-04   7.508 6.54e-11 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.423 on 82 degrees of freedom
## Multiple R-squared:  0.863,  Adjusted R-squared:  0.853
## F-statistic: 86.08 on 6 and 82 DF,  p-value: < 2.2e-16
```

Non linear modeling

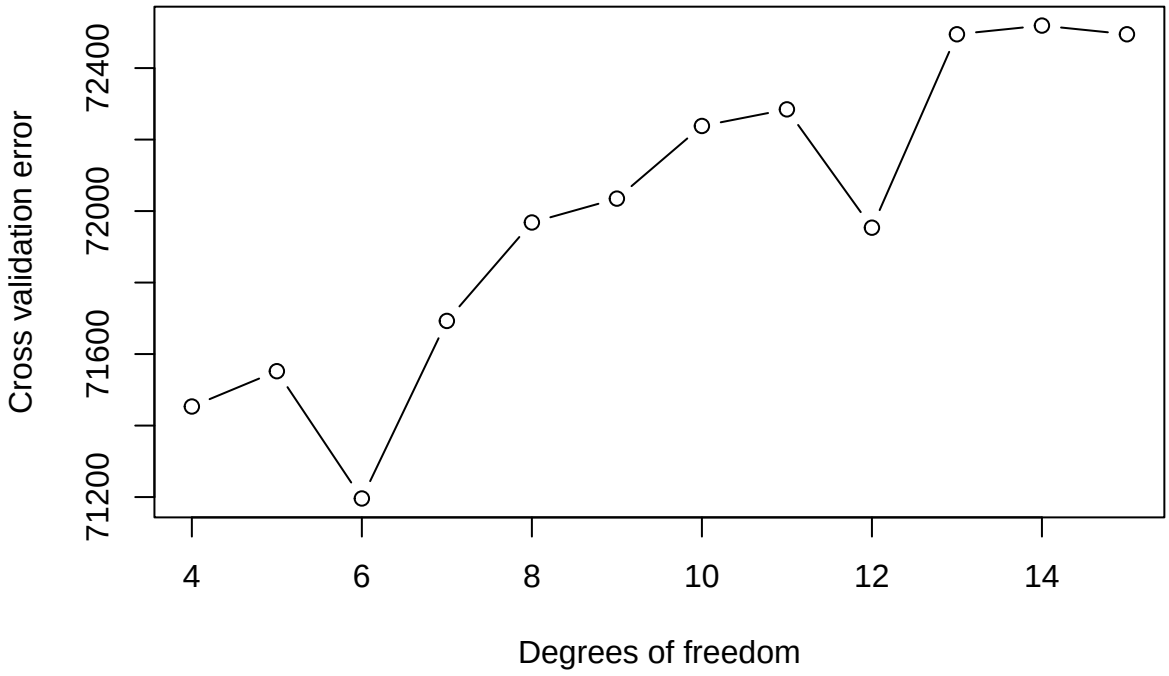
Natural spline



```
##  [1]       NA       NA       NA 71453.28 71552.09 71196.05 71692.72 71968.11
##  [9] 72034.87 72238.01 72284.53 71953.52 72494.62 72518.88 72494.62
```

```
## [1] 71196.05
```

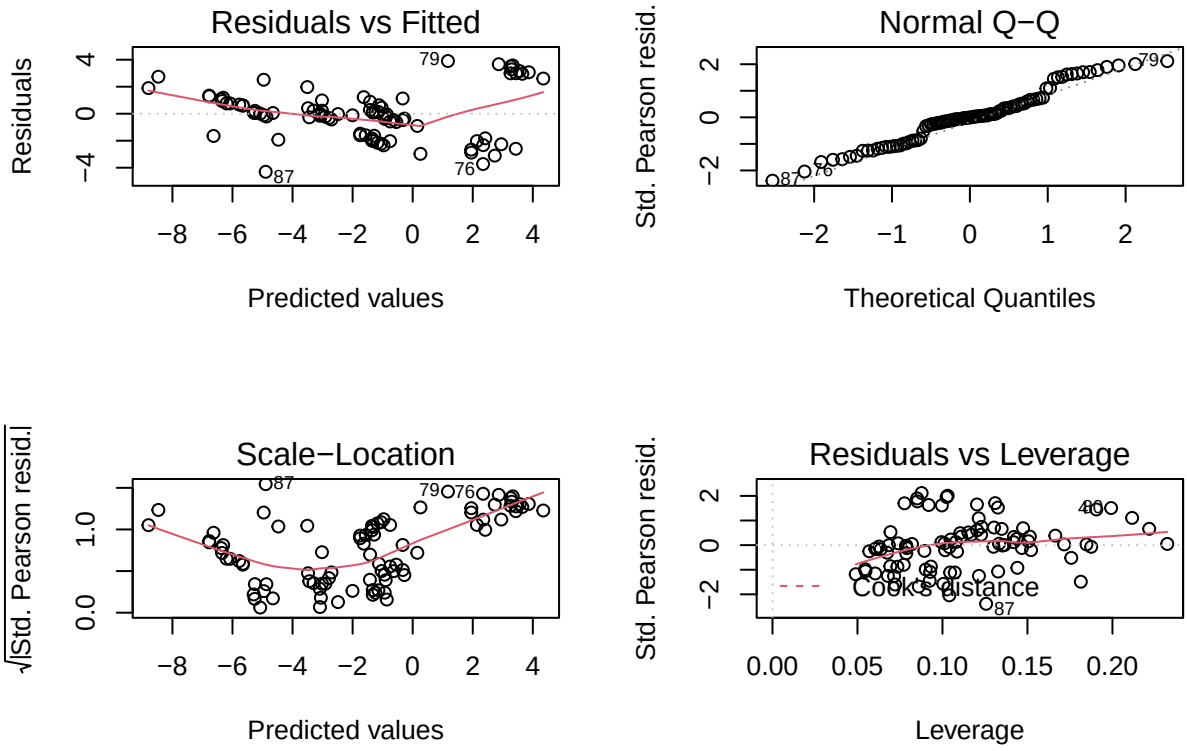
```
## [1] 0.7688989
```




```
## 16.66667% 33.33333%       50% 66.66667% 83.33333%
##       0.2       0.4       0.7       0.9       1.5
```

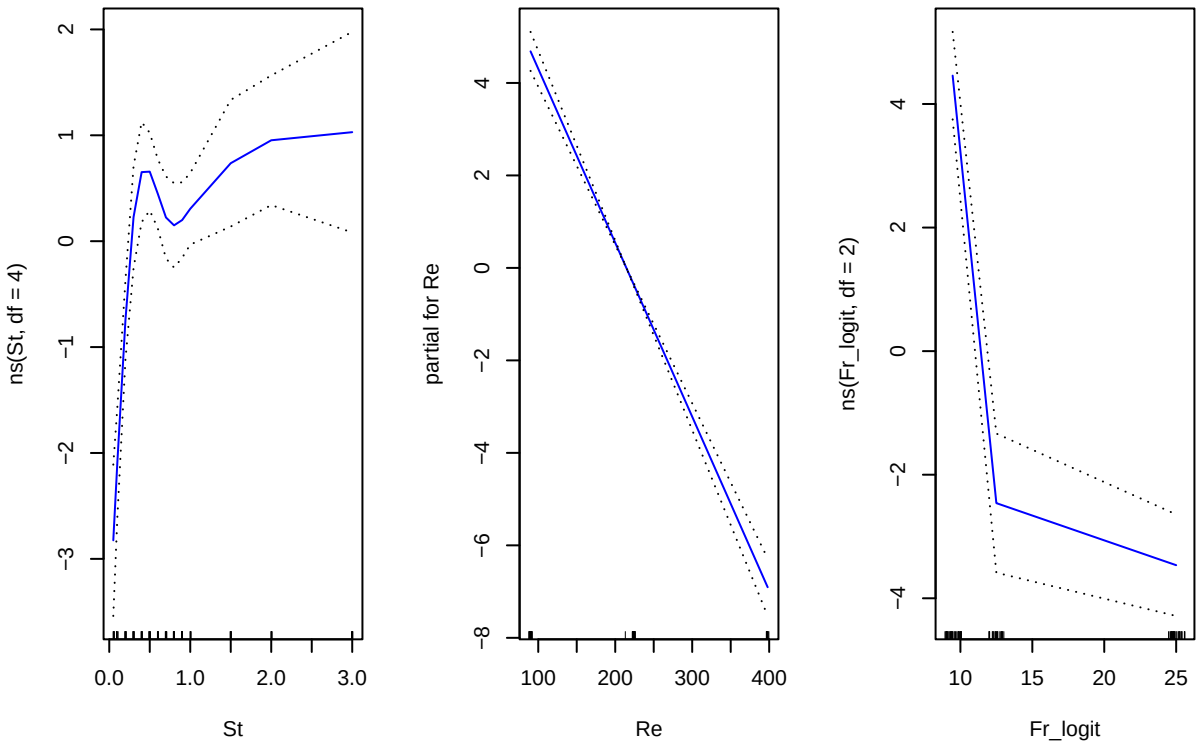
```
##
## Call:
## glm(formula = log(C_moment_2) ~ ns(St, df = 6) + Re + Fr_logit +
##     Re * Fr_logit, data = train)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -4.3039  -1.5973  -0.0084   0.9353   3.9021
##
## Coefficients:
##                   Estimate Std. Error t value Pr(>|t|)
## (Intercept)      6.4238096  1.2636985   5.083 2.43e-06 ***
## ns(St, df = 6)1  2.0764937  1.2393904   1.675 0.097807 .
## ns(St, df = 6)2  4.4622417  1.1970242   3.728 0.000362 ***
## ns(St, df = 6)3  2.8551956  1.2395004   2.304 0.023880 *
## ns(St, df = 6)4  2.4815789  1.2402355   2.001 0.048839 *
## ns(St, df = 6)5  8.6658024  1.8557403   4.670 1.22e-05 ***
## ns(St, df = 6)6  2.1742183  0.9294728   2.339 0.021853 *
## Re              -0.0427802  0.0044194  -9.680 4.60e-15 ***
## Fr_logit        -0.4105495  0.0657716  -6.242 2.02e-08 ***
## Re:Fr_logit      0.0012130  0.0002536   4.782 7.90e-06 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
```

```
## (Dispersion parameter for gaussian family taken to be 3.725466)
##
##     Null deviance: 1212.39  on 88  degrees of freedom
## Residual deviance:  294.31  on 79  degrees of freedom
## AIC: 381.02
##
## Number of Fisher Scoring iterations: 2
```

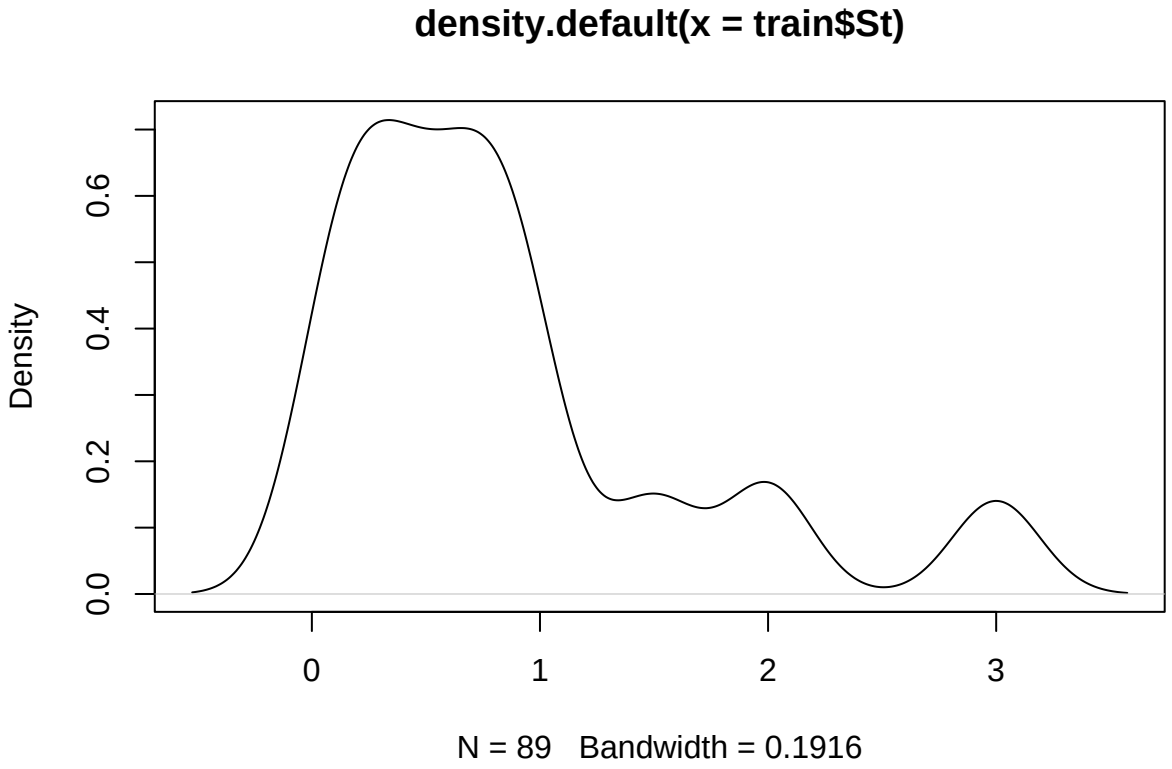


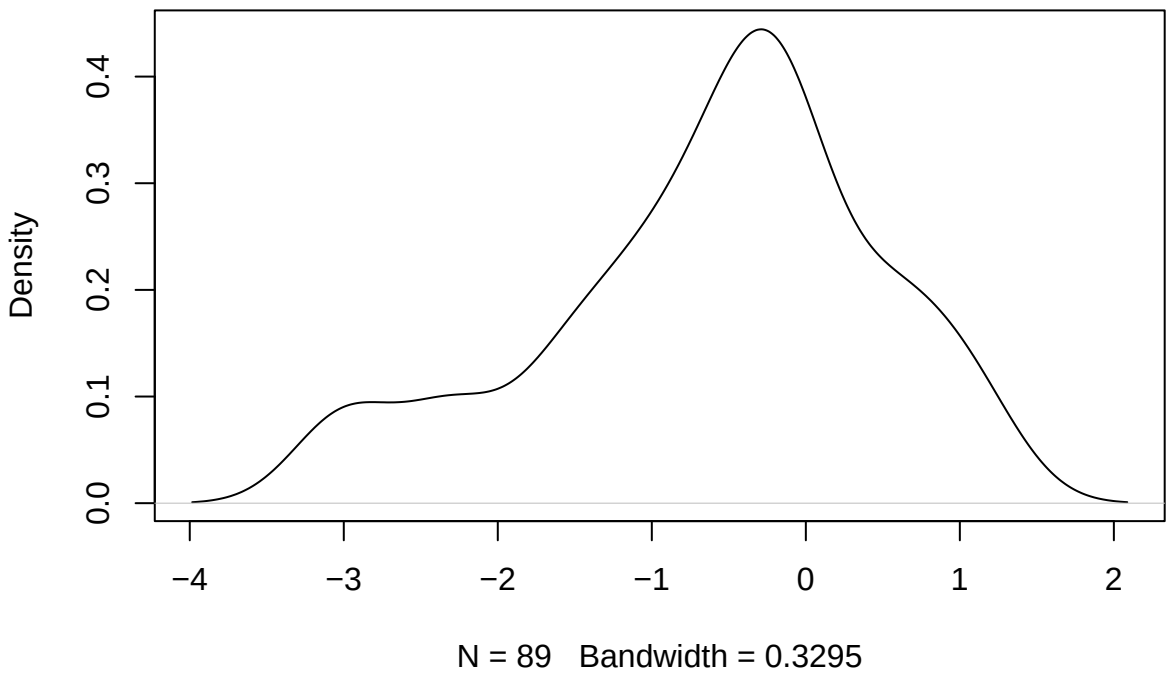

```
##  [1] 46403.34 45122.30 45251.36 44496.82 45253.49 45599.14 46072.96 45827.87
##  [9] 45509.21 45511.36 45949.90 45339.74 46531.93 46531.93 46531.93

## [1] 44496.82

## [1] 0.9156065
```

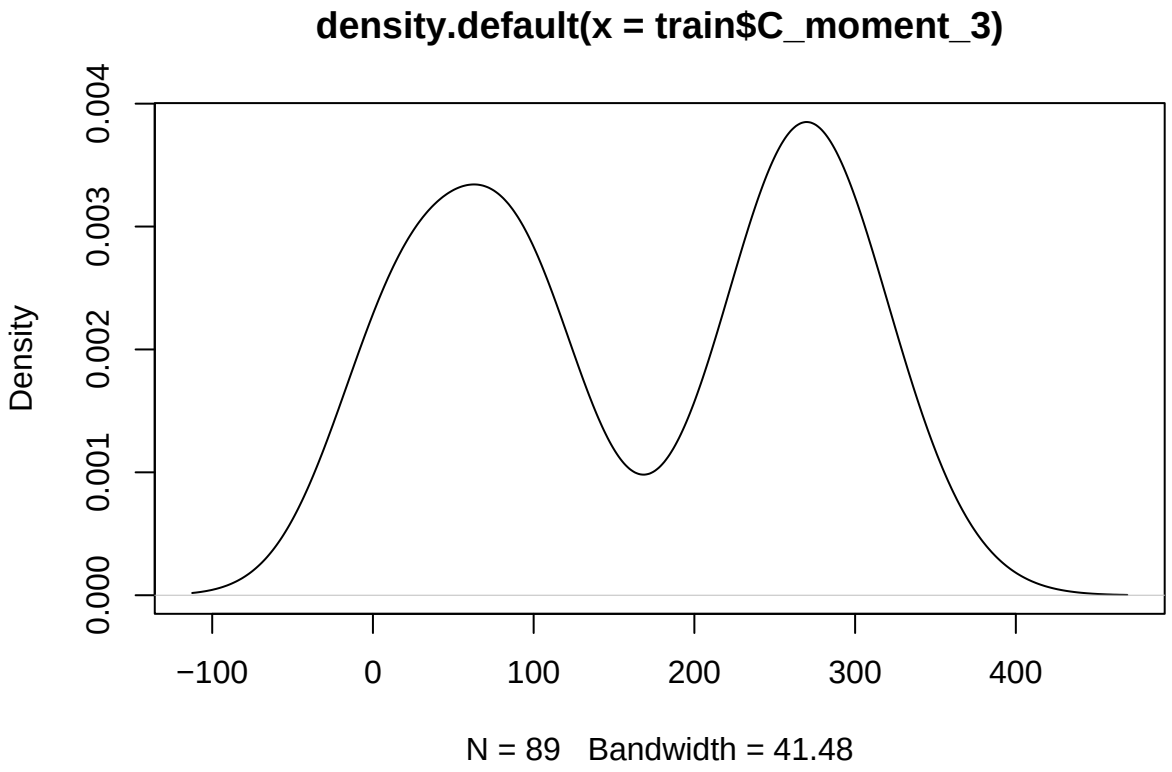
models	formula	mse	
Least square regression	log(C_moment_2) ~ Fr + Re + St + Fr * Re	70420.79	0.70
Polynomial regression	log(C_moment_2) ~ poly(Fr,2) + Re + poly(St, 2) + Fr * Re	45491.00	0.87
Natural spline	log(C_moment_2) ~ ns(St, df = 6) + Fr + Re + Fr * Re	71196.05	0.76
Generalized additive model	log(C_moment_2) ~ ns(St, 4) + Re + ns(Fr_logit,2) + Re:ns(Fr_logit,2)	44496.82	0.91

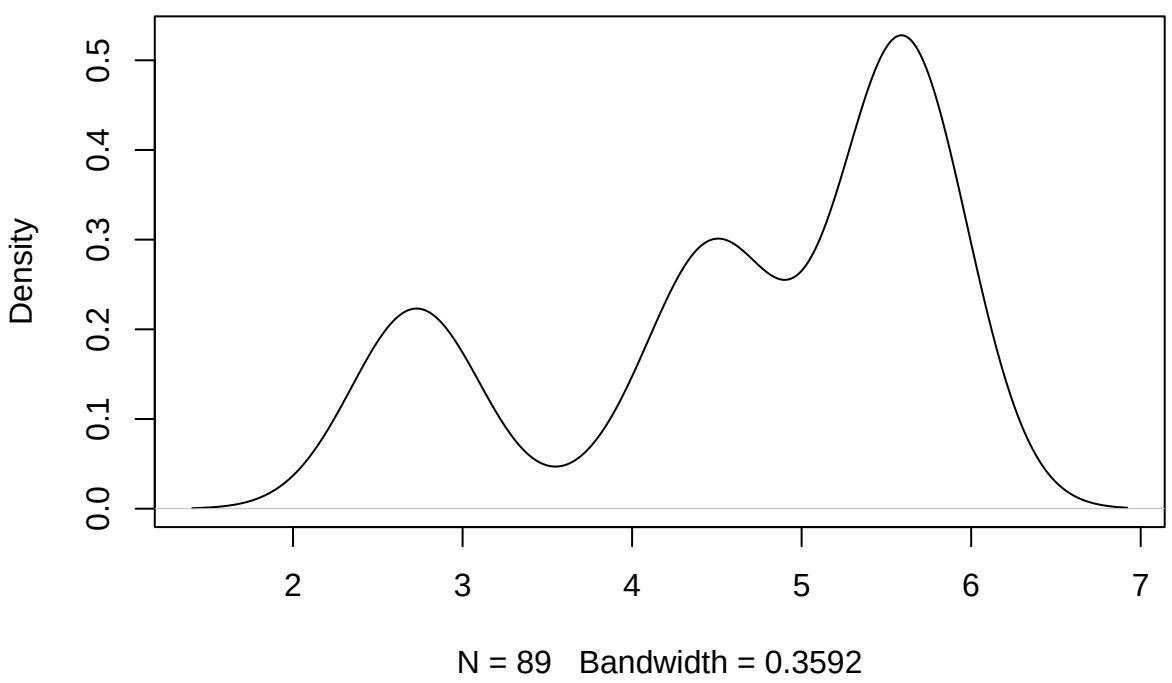


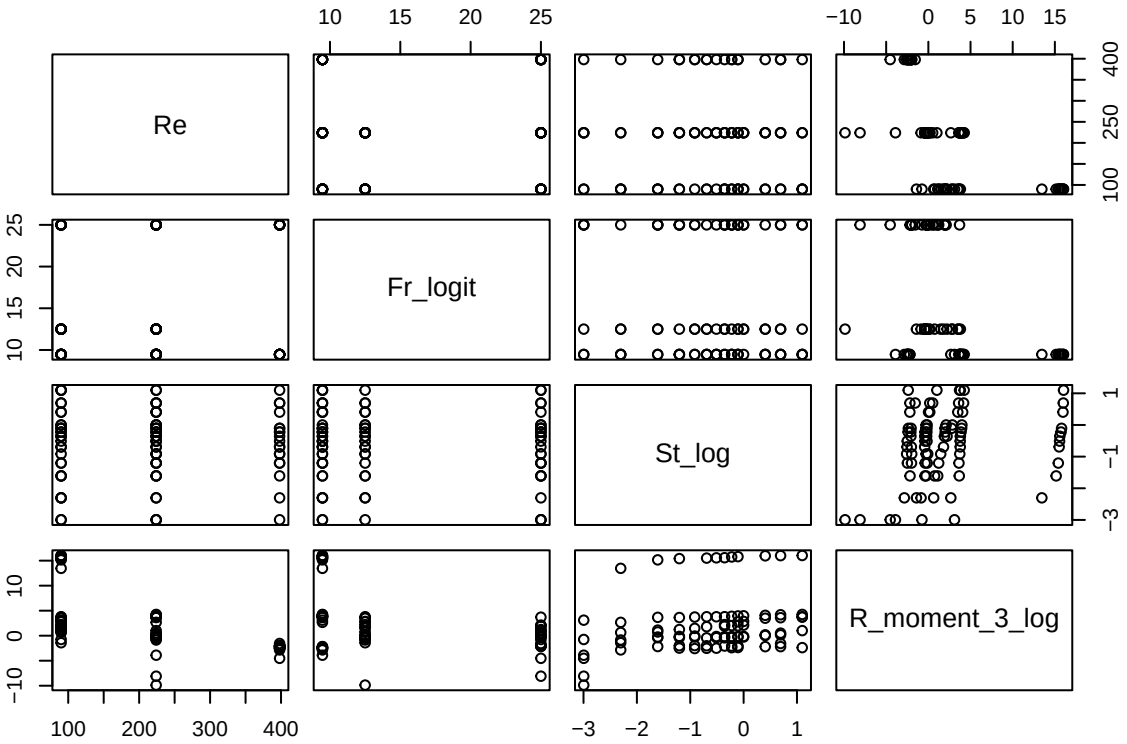
C_moment_3


density.default(x = train$St)
Density
0.0
0.2
0.4
0.6
0
1
2
3
N = 89 Bandwidth = 0.1916


density.default(x = log(train$St))
Density
0.0
0.1
0.2
0.3
0.4
−4
−3
−2
−1
0
1
2
N = 89 Bandwidth = 0.3295


density.default(x = train$C_moment_3)
Density
0.000
0.001
0.002
0.003
0.004
−100
0
100
200
300
400
N = 89 Bandwidth = 41.48

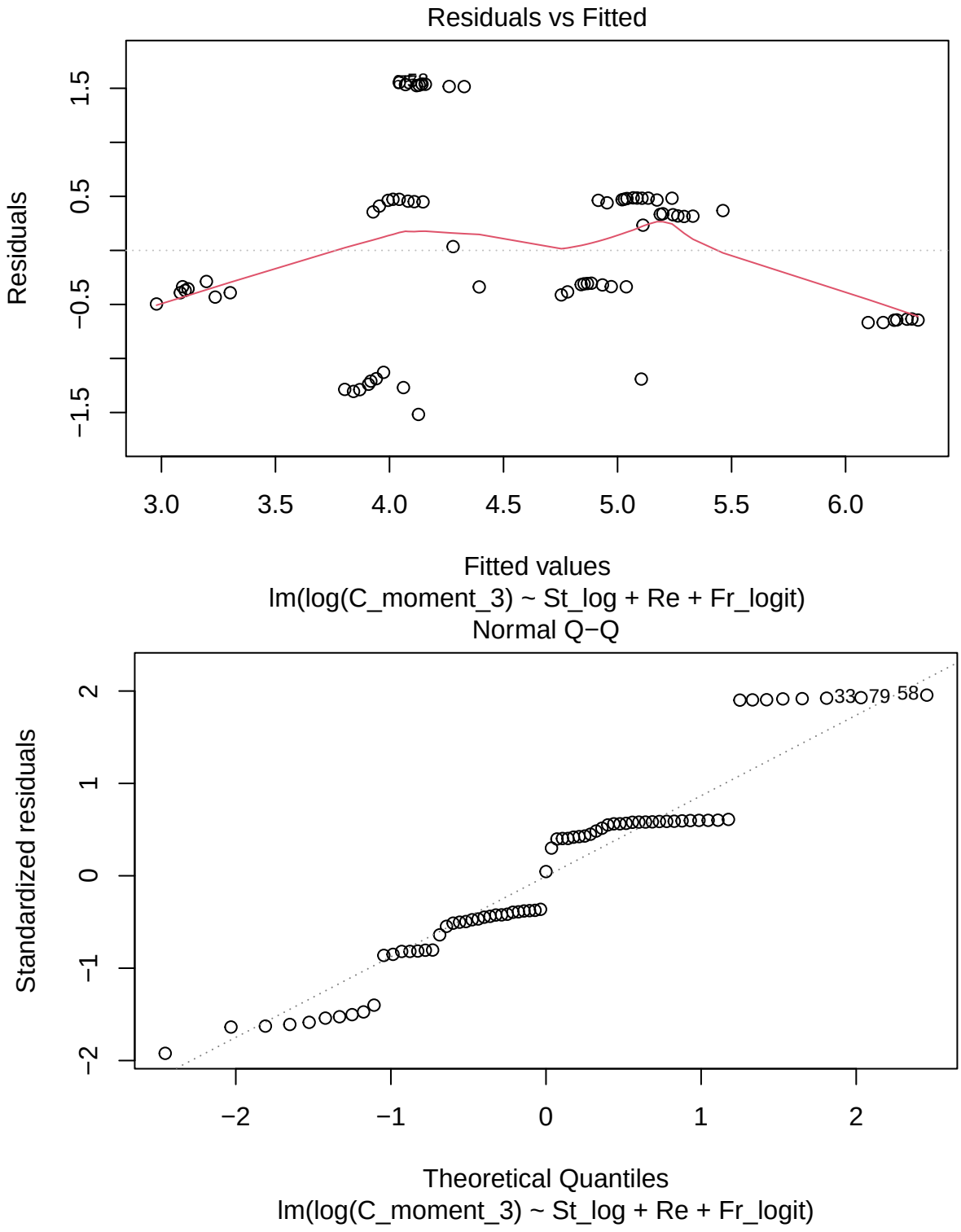

density.default(x = log(train$C_moment_3))
Density
0.0
0.1
0.2
0.3
0.4
0.5
2
3
4
5
6
7
N = 89 Bandwidth = 0.3592

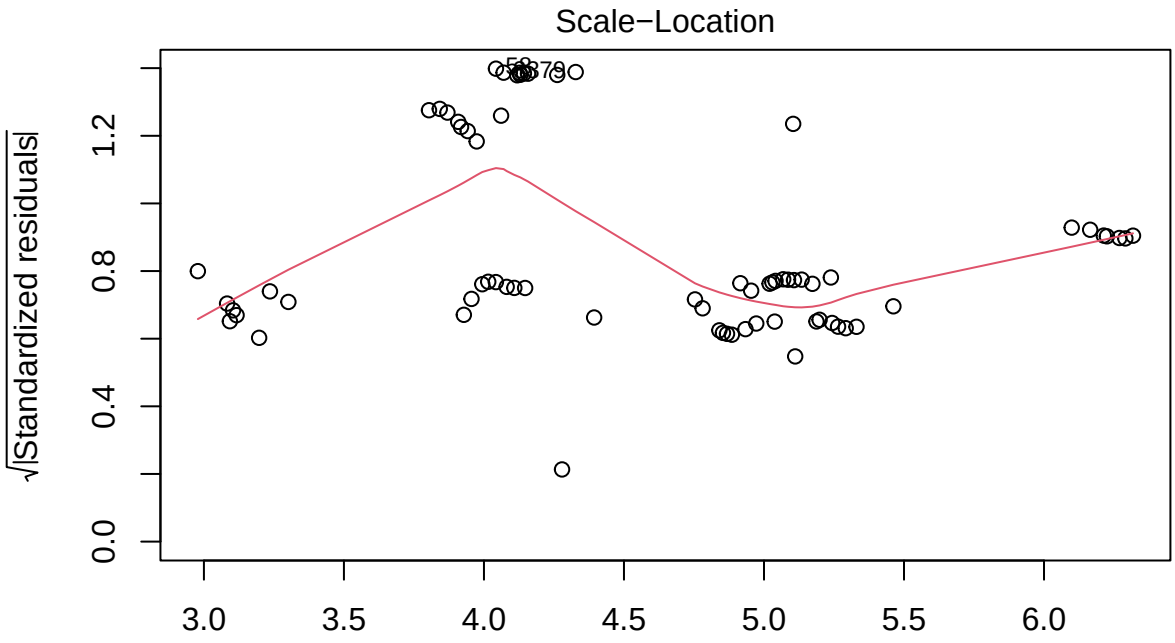


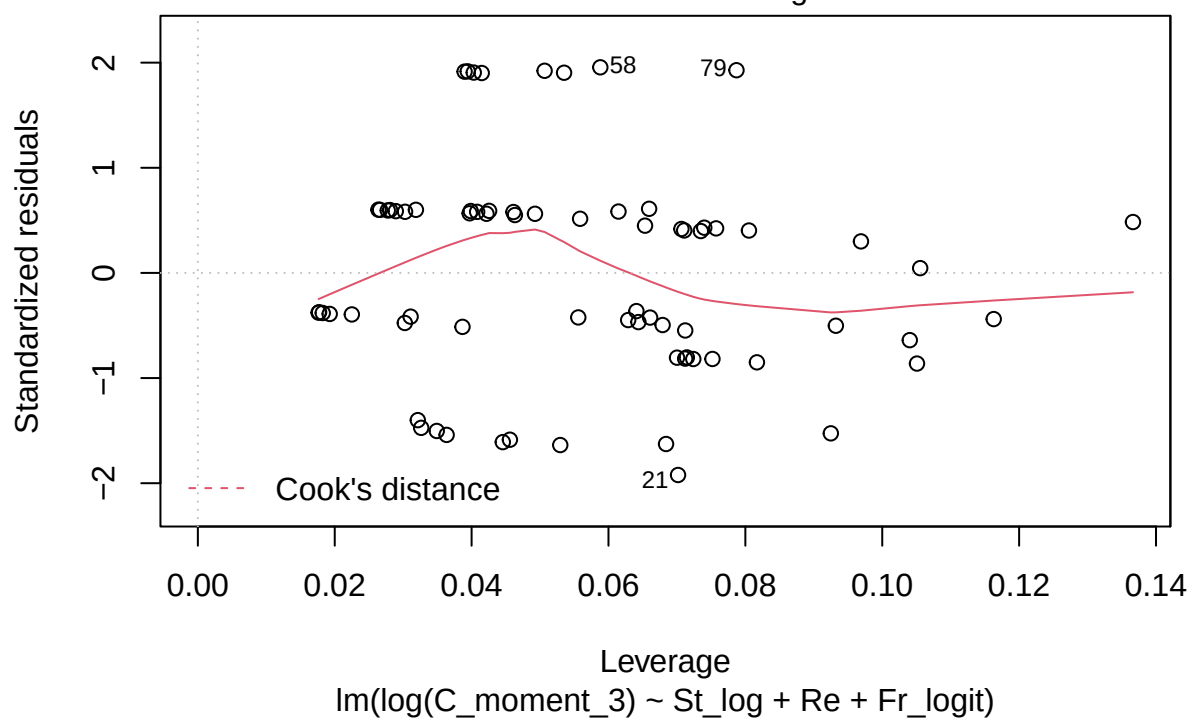

```
## NULL
```

Least Square Regression

```
##
## Call:
## lm(formula = log(C_moment_3) ~ St_log + Re + Fr_logit, data = trainData)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.51783 -0.46392  0.03527  0.46752  1.55384
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  4.1214824  0.2901992  14.202  < 2e-16 ***
## St_log      -0.0950549  0.0930159  -1.022     0.31
## Re           0.0068005  0.0008649   7.863 4.26e-11 ***
## Fr_logit    -0.0660294  0.0141768  -4.658 1.57e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.819 on 67 degrees of freedom
## Multiple R-squared:  0.5325, Adjusted R-squared:  0.5116
## F-statistic: 25.44 on 3 and 67 DF,  p-value: 4.229e-11
```


Residuals vs Fitted
Residuals
1.5
0.5
−0.5
−1.5
3.0
3.5
4.0
4.5
5.0
5.5
6.0
Fitted values
lm(log(C_moment_3) ~ St_log + Re + Fr_logit)
Normal Q−Q
Standardized residuals
2
1
0
−1
−2
33
79
58
−2
−1
0
1
2
Theoretical Quantiles
lm(log(C_moment_3) ~ St_log + Re + Fr_logit)





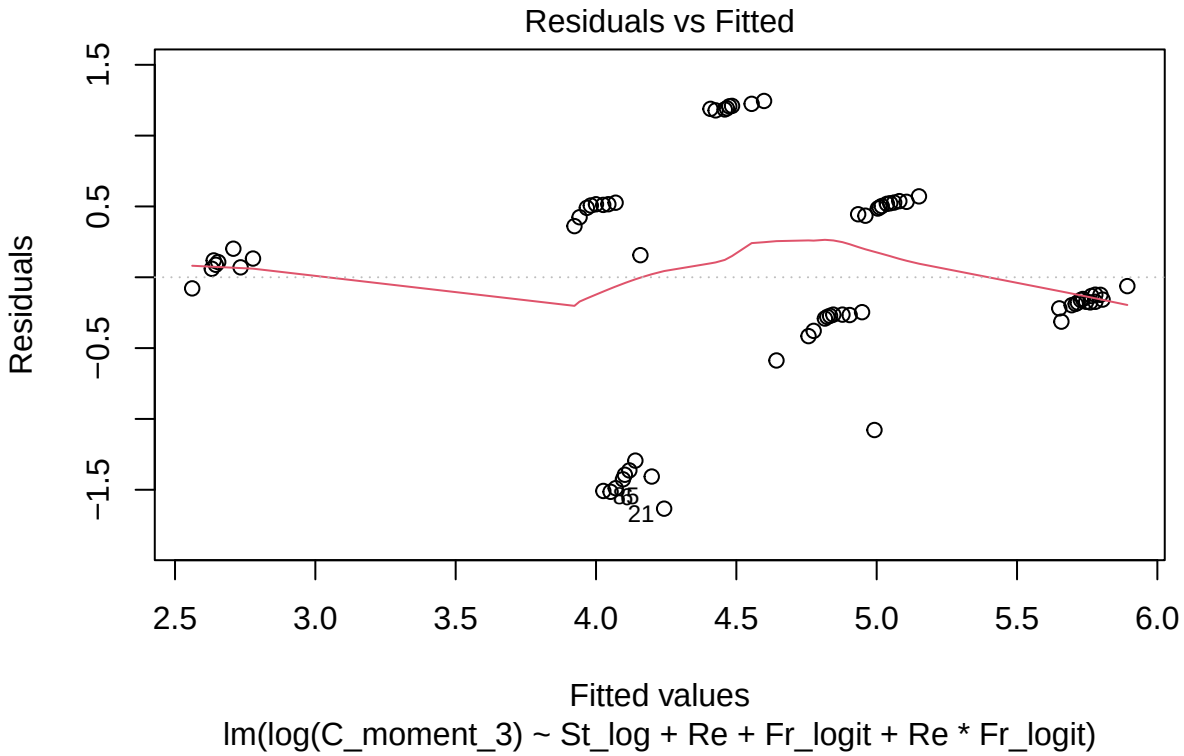

```
## [1] 15634.66
```

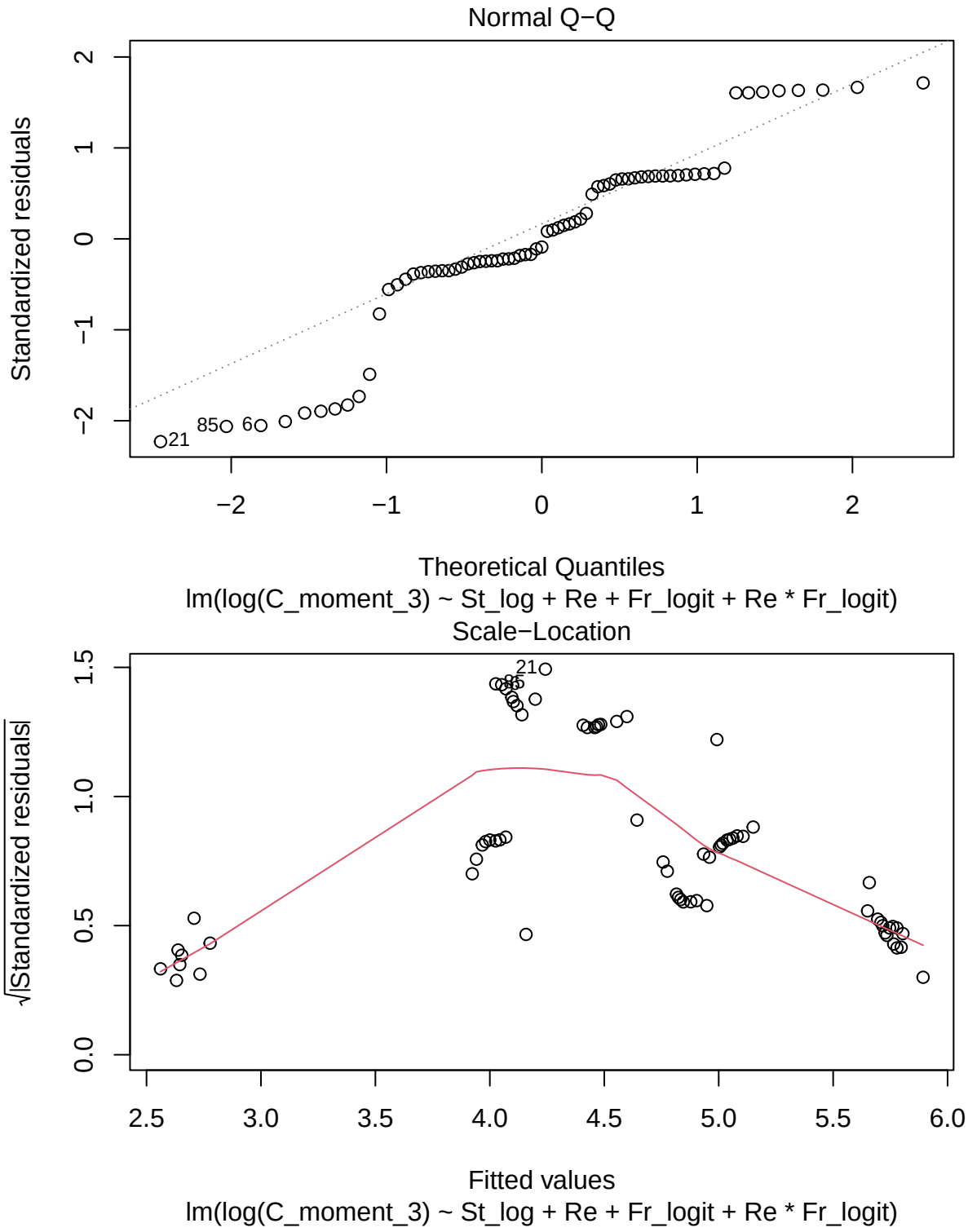
```
## [1] 0.5115609
```

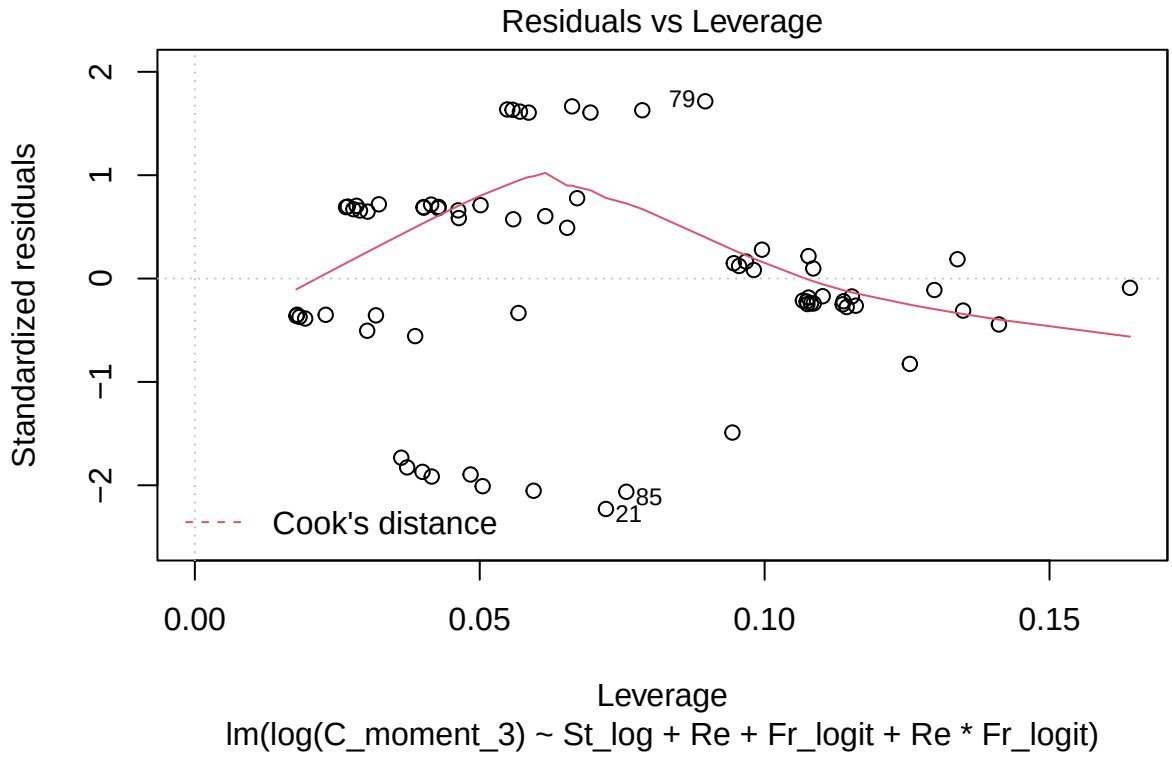
Least Square Regression with Interactions

```
##
```

```
## Call:
## lm(formula = log(C_moment_3) ~ St_log + Re + Fr_logit + Re *
##     Fr_logit, data = trainData)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.63349 -0.26521 -0.06257  0.50888  1.24457
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  5.5109322  0.4878894  11.295  < 2e-16 ***
## St_log      -0.0636999  0.0868755  -0.733  0.46601
## Re           0.0005547  0.0019968   0.278  0.78203
## Fr_logit    -0.1510857  0.0281629  -5.365 1.12e-06 ***
## Re:Fr_logit  0.0003766  0.0001102   3.417  0.00109 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.7607 on 66 degrees of freedom
## Multiple R-squared:  0.6028, Adjusted R-squared:  0.5787
## F-statistic: 25.04 on 4 and 66 DF,  p-value: 1.228e-12
```




Normal Q−Q
Standardized residuals
Theoretical Quantiles
lm(log(C_moment_3) ~ St_log + Re + Fr_logit + Re * Fr_logit)
Scale−Location
√|Standardized residuals|
Fitted values
lm(log(C_moment_3) ~ St_log + Re + Fr_logit + Re * Fr_logit)




```
## [1] 7639.658
```

```
## [1] 0.5790335
```

Polynomial Model

```
##  [1] 3966.051 3739.825 4014.669 4258.430 4747.412 5144.345 5378.789 5466.231
##  [9] 5987.388 5928.257
```

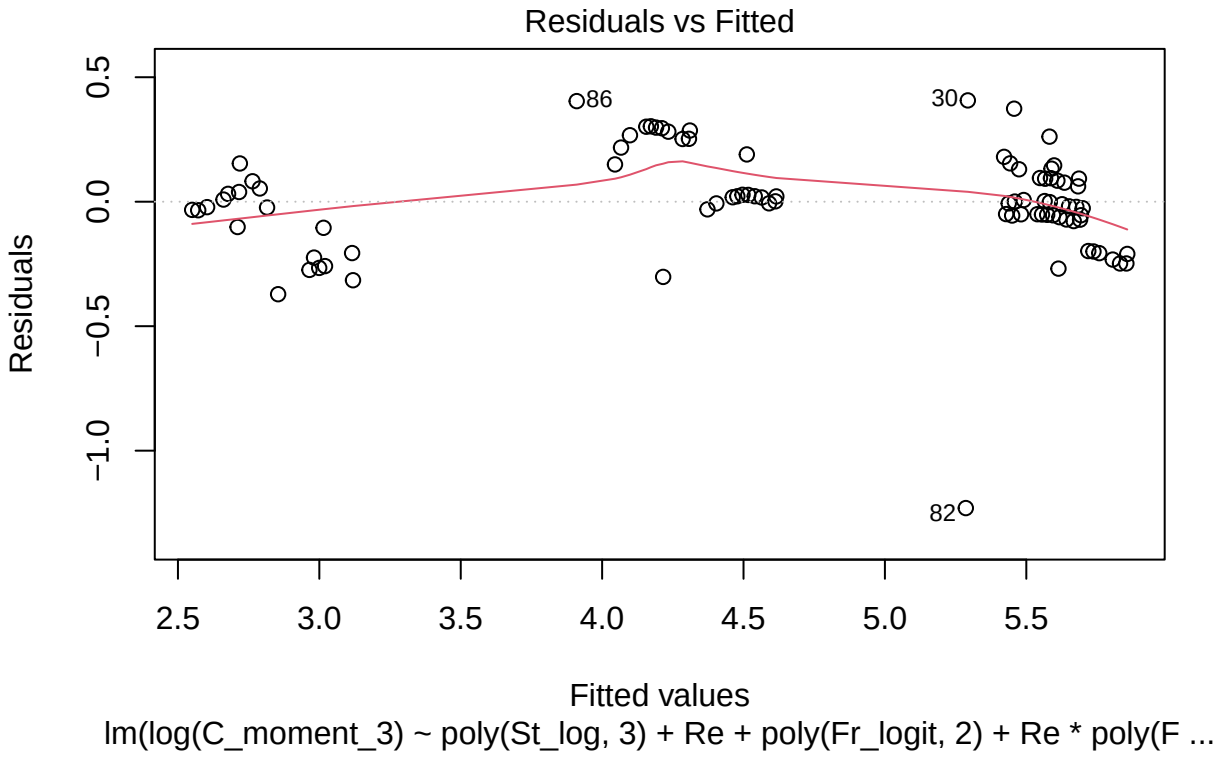
```
##  [1] 0.8388636 0.8452883 0.8439009 0.8419529 0.8405346 0.8398282 0.8375259
##  [8] 0.8354358 0.8348120 0.8321407
```

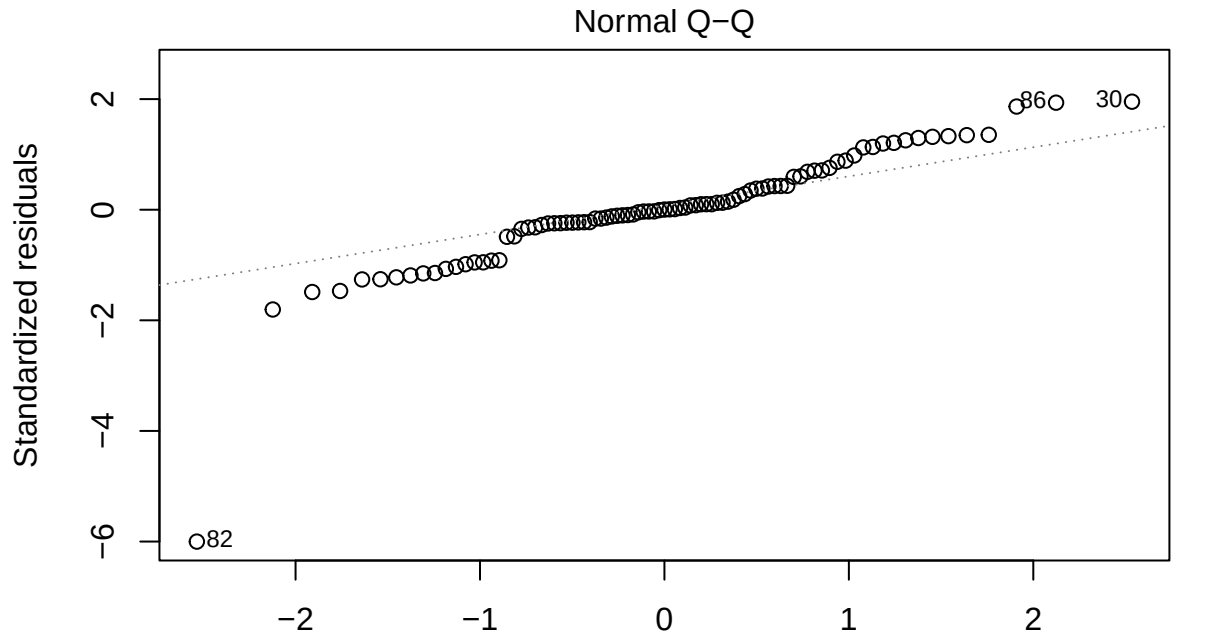
```
## [1] 3739.825
```

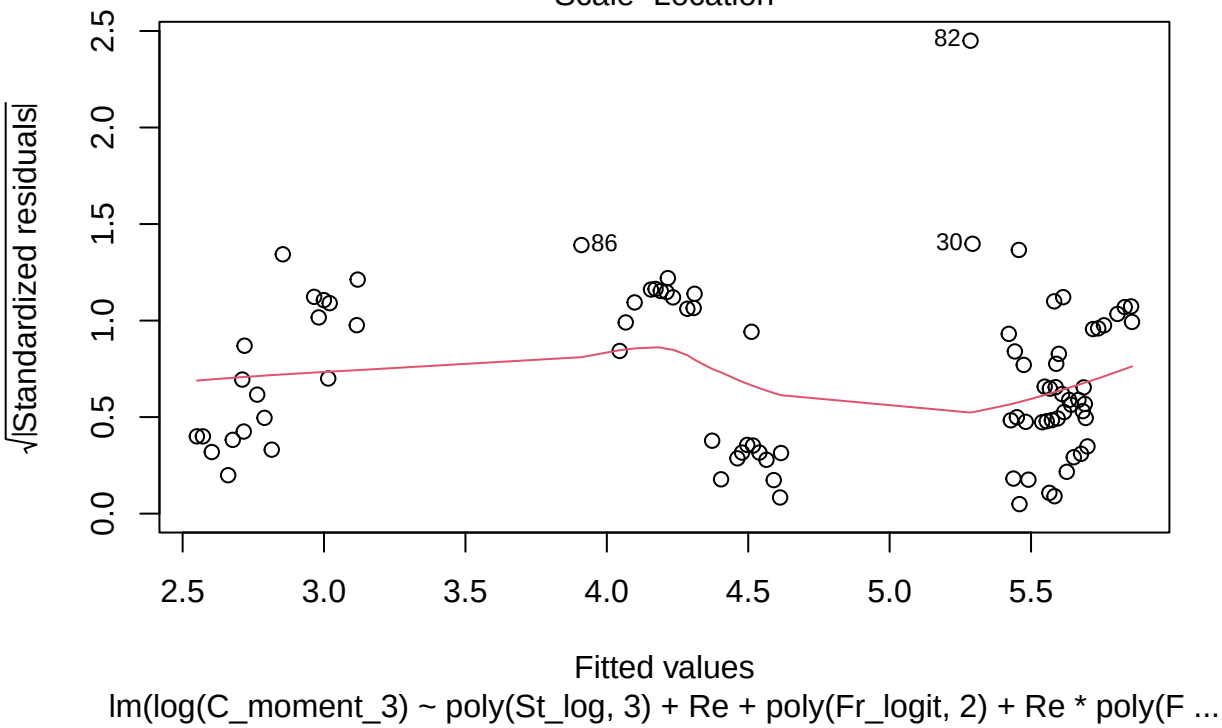
```
## [1] 0.8452883
```

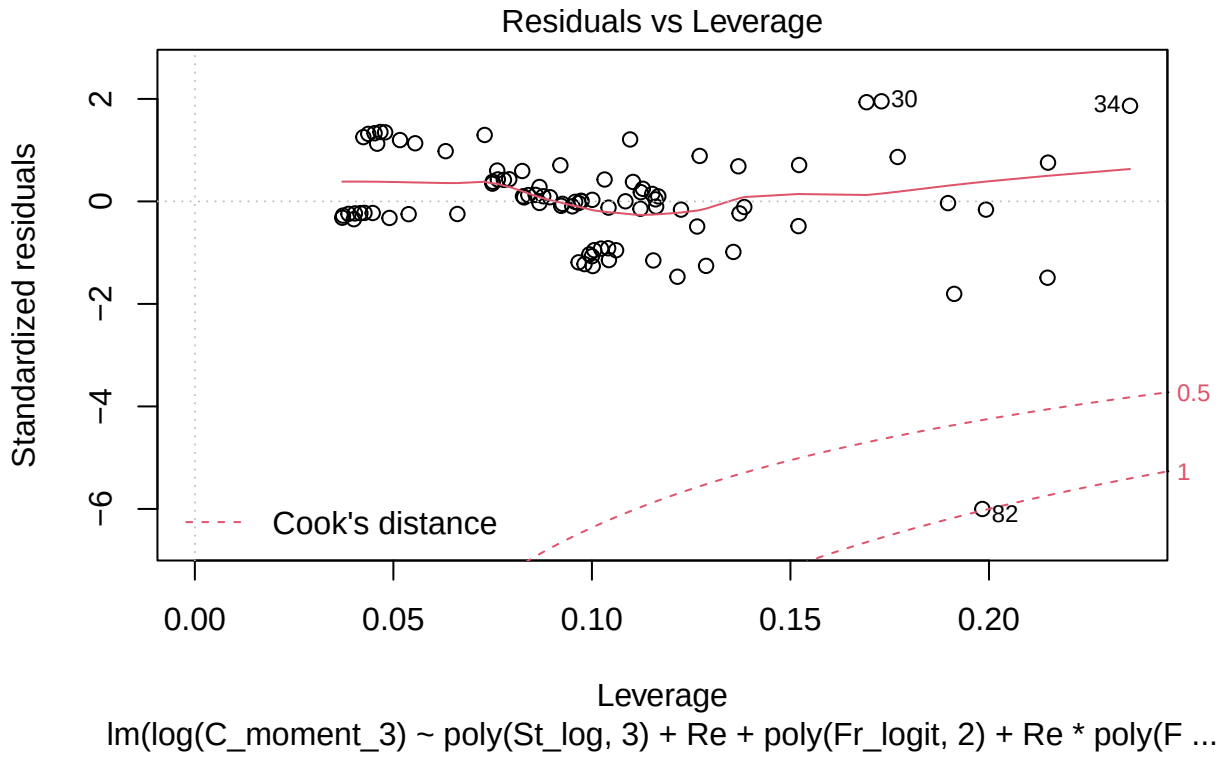
```
##
## Call:
## lm(formula = log(C_moment_3) ~ poly(St_log, 3) + Re + poly(Fr_logit,
##     2) + Re * poly(Fr_logit, 2), data = train)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.23053 -0.06219  0.00052  0.09507  0.40684
##
## Coefficients:
##                          Estimate Std. Error t value Pr(>|t|)
## (Intercept)             3.361e+00  5.656e-02  59.433  < 2e-16 ***
## poly(St_log, 3)1       -9.635e-02  2.308e-01  -0.418 0.677394
## poly(St_log, 3)2       -9.162e-01  2.303e-01  -3.979 0.000151 ***
## poly(St_log, 3)3        4.362e-01  2.304e-01   1.893 0.061951 .
## Re                      6.491e-03  2.565e-04  25.308  < 2e-16 ***
## poly(Fr_logit, 2)1     -1.046e+01  5.079e-01 -20.594  < 2e-16 ***
```

```
## poly(Fr_logit, 2)2     1.365e+01  5.848e-01  23.338  < 2e-16 ***
## Re:poly(Fr_logit, 2)1  2.466e-02  2.015e-03  12.242  < 2e-16 ***
## Re:poly(Fr_logit, 2)2 -4.655e-02  3.087e-03 -15.076  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.229 on 80 degrees of freedom
## Multiple R-squared:  0.9627, Adjusted R-squared:  0.959
## F-statistic: 258.2 on 8 and 80 DF,  p-value: < 2.2e-16
```




Normal Q−Q
Standardized residuals
Theoretical Quantiles
lm(log(C_moment_3) ~ poly(St_log, 3) + Re + poly(Fr_logit, 2) + Re * poly(F ...
86
30
82


Scale−Location
√|Standardized residuals|
Fitted values
lm(log(C_moment_3) ~ poly(St_log, 3) + Re + poly(Fr_logit, 2) + Re * poly(F ...
82
86
30




```
## [1] 4014.669
```

```
## [1] 0.8439009
```

```
## [1] 14
```

Generalized additive model

```
##  [1]       NA       NA 4026.345 4416.845 4275.525 5020.343 5275.599 6087.578
##  [9] 5712.157 6284.989 6153.087 6428.869 6613.955
```

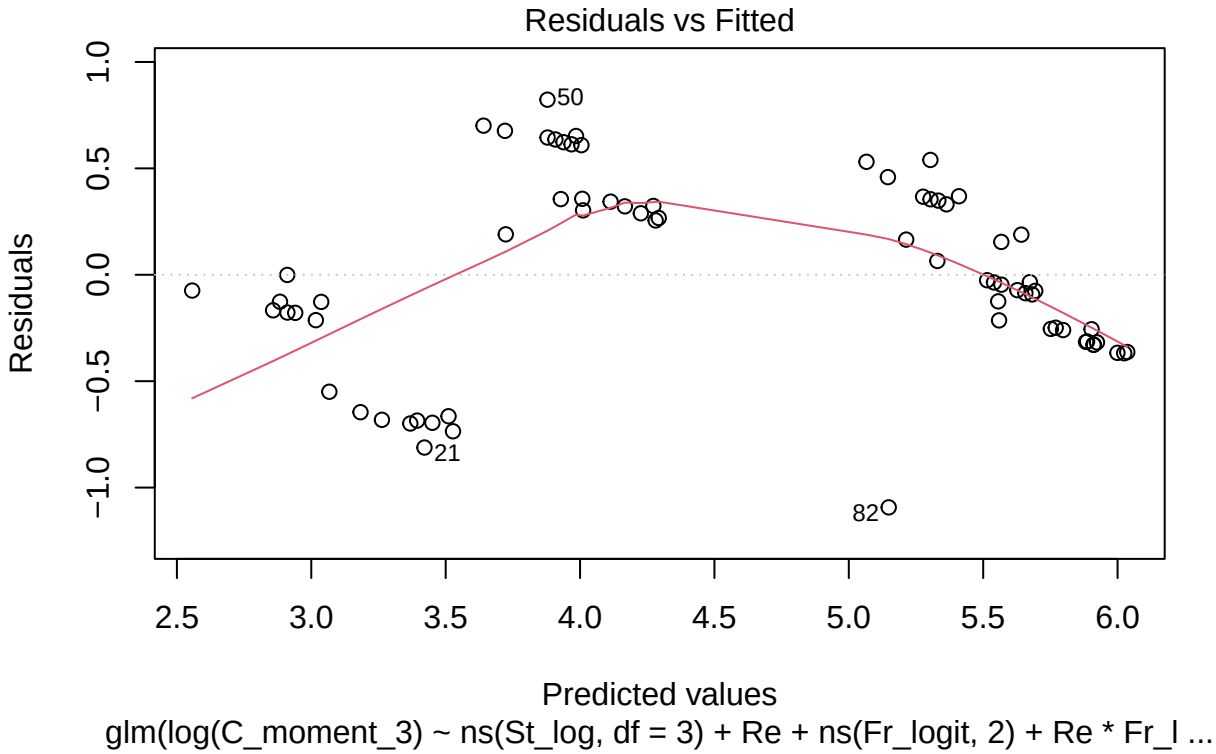
```
##  [1]        NA        NA 0.8590343 0.8598887 0.8605120 0.8620582 0.8633814
##  [8] 0.8641227 0.8646028 0.8653537 0.8666789 0.8667691 0.8678281
```

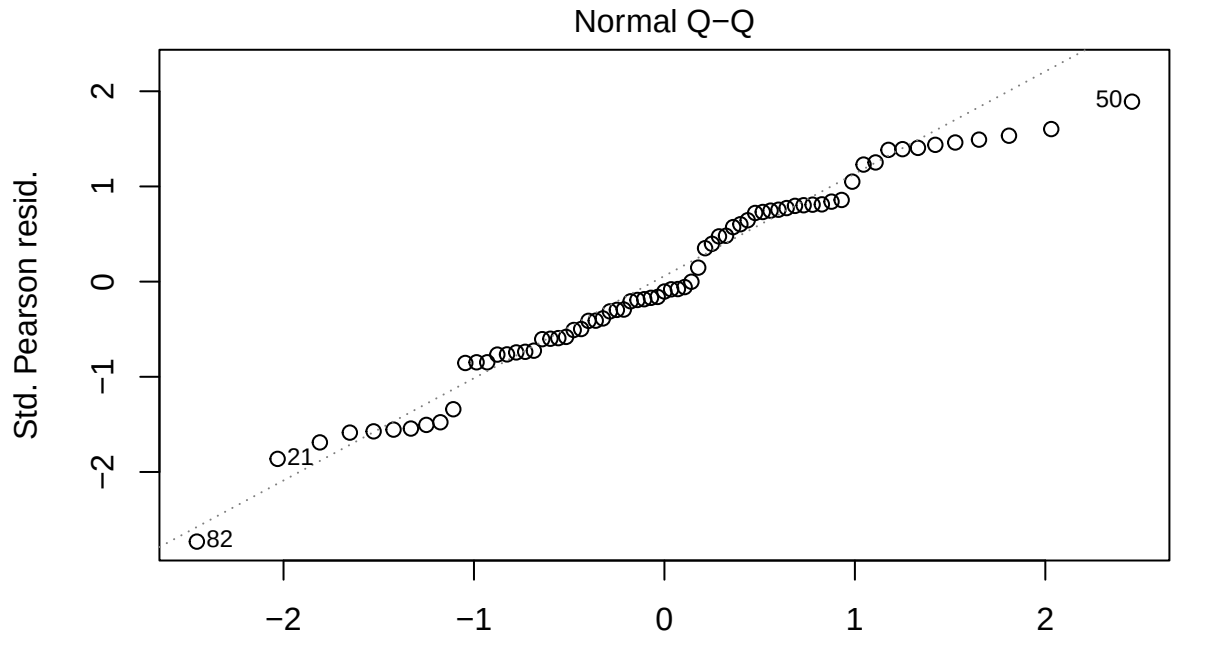
```
## [1] 4026.345
```

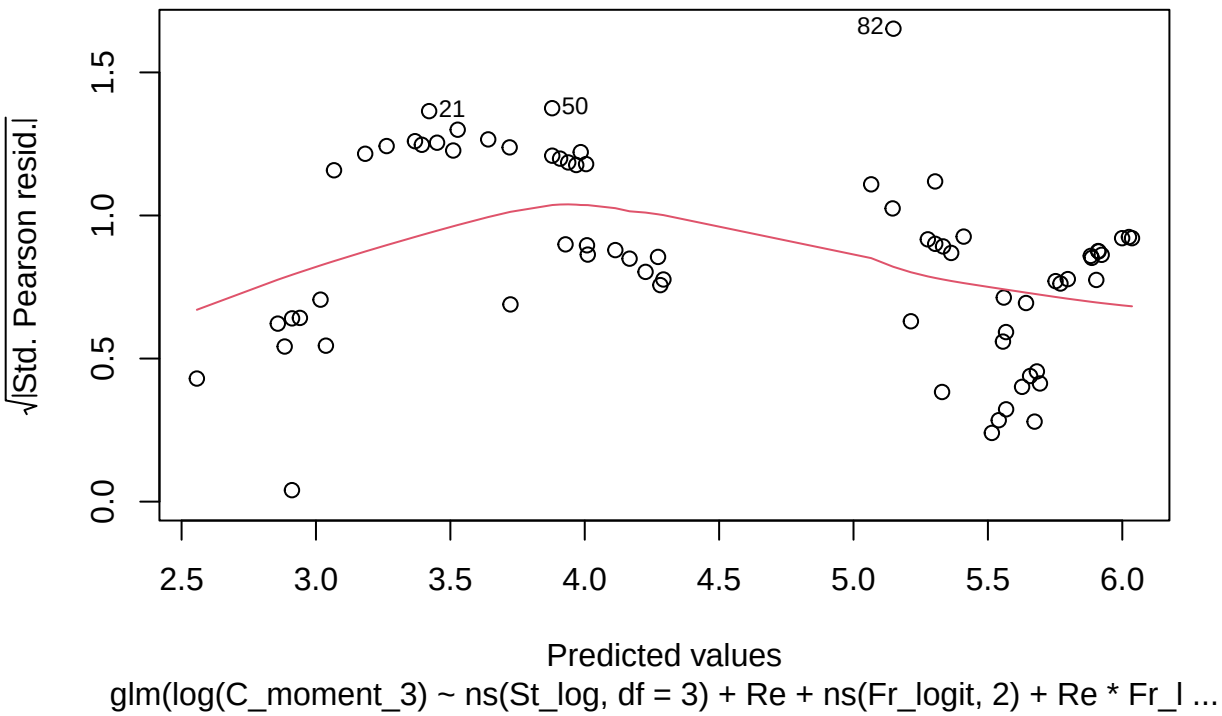
```
## [1] 0.8678281
```

```
##
## Call:
## glm(formula = log(C_moment_3) ~ ns(St_log, df = 3) + Re + ns(Fr_logit,
##     2) + Re * Fr_logit, data = trainData)
##
## Deviance Residuals:
##      Min        1Q    Median        3Q       Max
## -1.09297  -0.28652  -0.04652   0.34604   0.82281
##
## Coefficients: (1 not defined because of singularities)
##                        Estimate Std. Error t value Pr(>|t|)
## (Intercept)           4.971e+00  2.534e-01  19.615  < 2e-16 ***
## ns(St_log, df = 3)1 -1.767e-02  2.075e-01  -0.085   0.9324
## ns(St_log, df = 3)2  2.585e-01  4.862e-01   0.532   0.5968
## ns(St_log, df = 3)3 -4.127e-01  2.016e-01  -2.047   0.0448 *
```

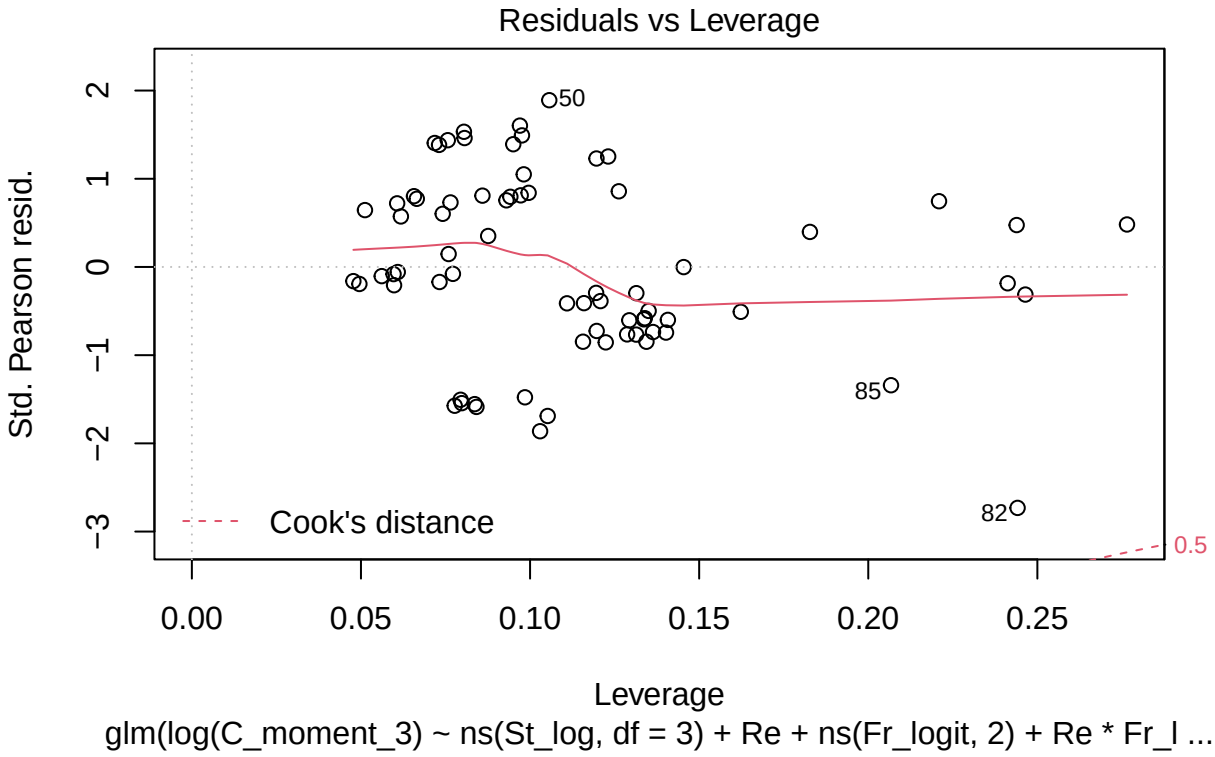
```
## Re                  -2.540e-03  1.250e-03  -2.032   0.0464 *
## ns(Fr_logit, 2)1    -6.391e+00  4.643e-01 -13.764  < 2e-16 ***
## ns(Fr_logit, 2)2    -1.557e+00  2.352e-01  -6.622 9.08e-09 ***
## Fr_logit                    NA         NA      NA       NA
## Re:Fr_logit          4.764e-04  6.763e-05   7.045 1.67e-09 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for gaussian family taken to be 0.211812)
##
##     Null deviance: 96.132  on 70  degrees of freedom
## Residual deviance: 13.344  on 63  degrees of freedom
## AIC: 100.81
##
## Number of Fisher Scoring iterations: 2
```




Normal Q−Q
Std. Pearson resid.
Theoretical Quantiles
glm(log(C_moment_3) ~ ns(St_log, df = 3) + Re + ns(Fr_logit, 2) + Re * Fr_l ...
50
21
82
−2
−1
0
1
2


Scale−Location
√|Std. Pearson resid.|
Predicted values
glm(log(C_moment_3) ~ ns(St_log, df = 3) + Re + ns(Fr_logit, 2) + Re * Fr_l ...
82
21
50
0.0
0.5
1.0
1.5
2.5
3.0
3.5
4.0
4.5
5.0
5.5
6.0

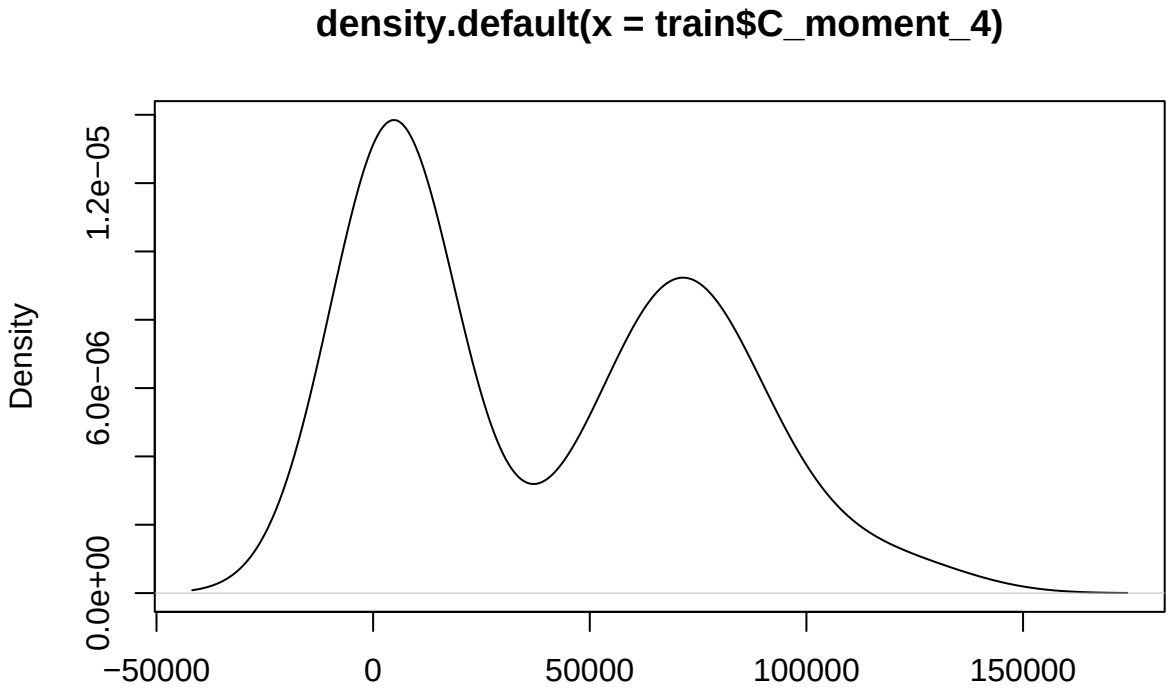
models	formula	mse	
Least square regression	log(C_moment_3) ~ log(St)+Re+logit(Fr)	15634.660	0.51
Interaction	log(C_moment_3) ~ log(St)+Re+logit(Fr)+Re*logit(Fr)	7639.658	0.57
Polynomial regression	log(C_moment_3) ~ poly(log(St),3)+Re+poly(logit(Fr),2)+Re*logit(Fr)	4014.669	0.84
Generalized additive model	log(C_moment_3) ~ ns(log(St),df=3)+Re+ns(logit(Fr),2)+Re*logit(Fr)	4026.345	0.85

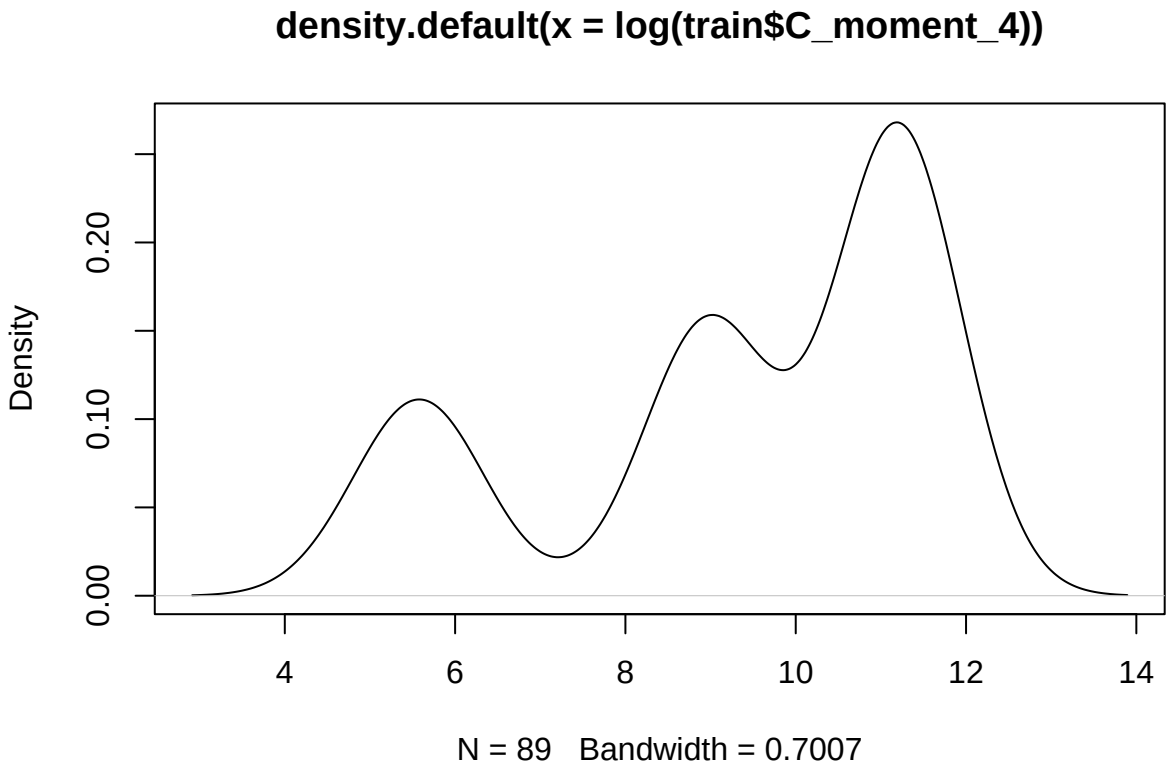


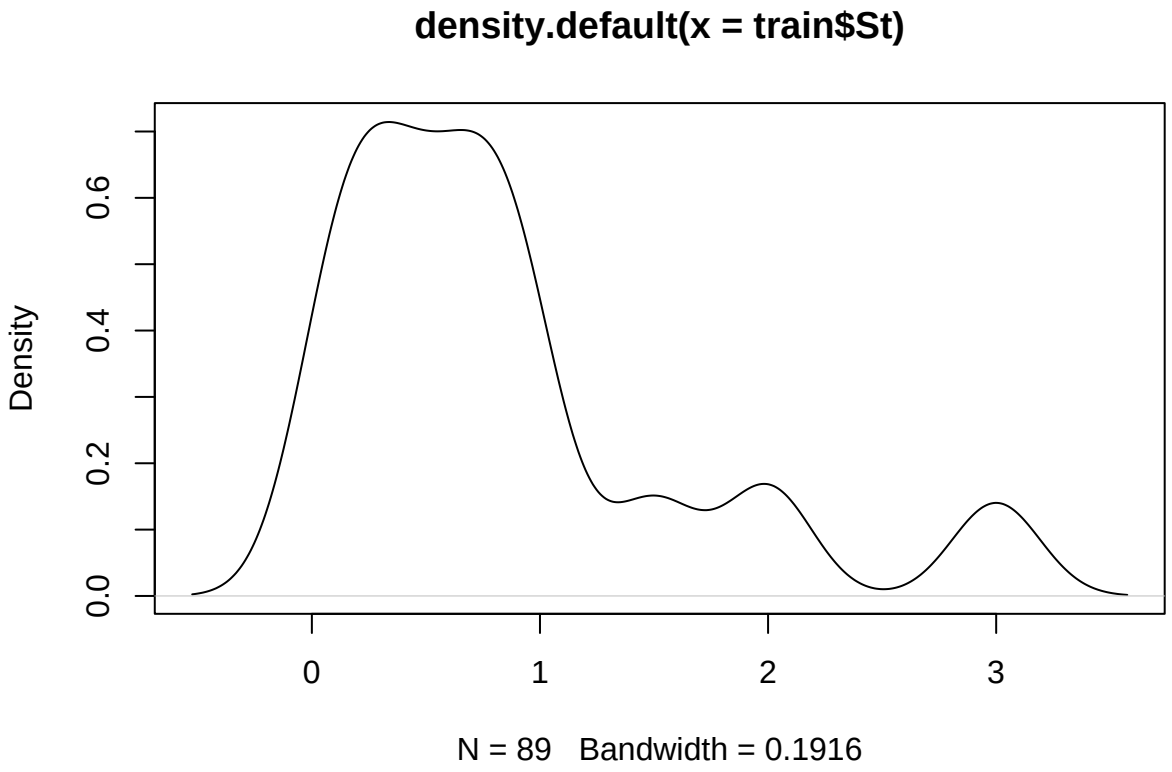

```
## [1] 4026.345
```

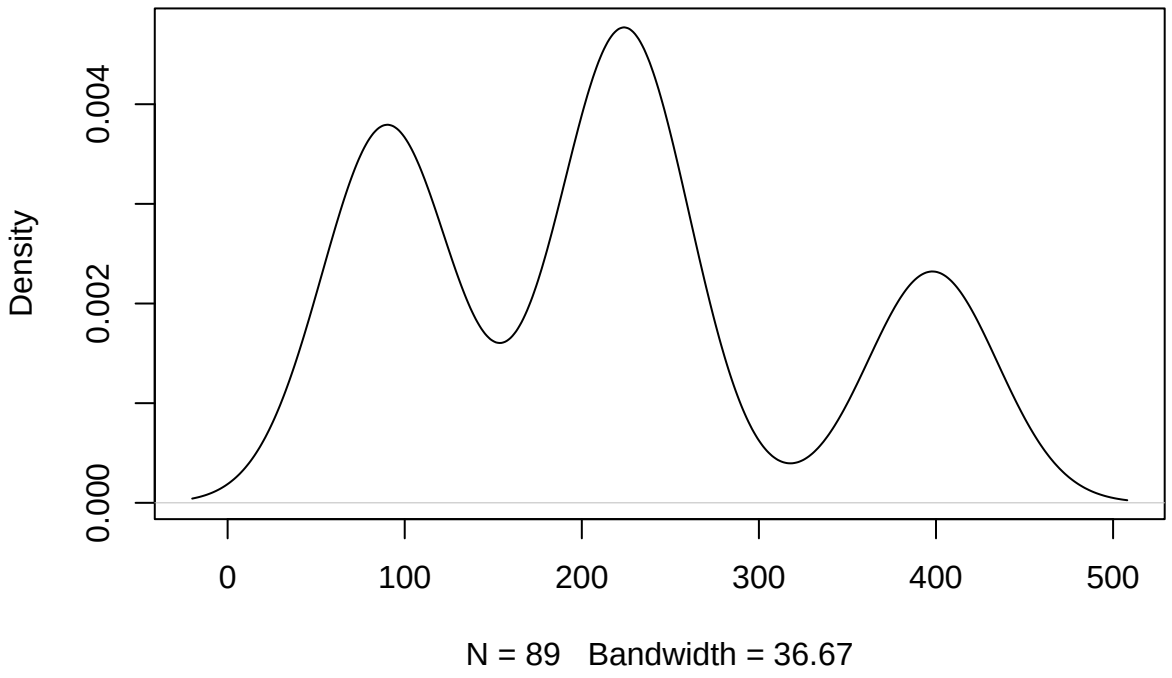
```
## [1] 0.8590343
```

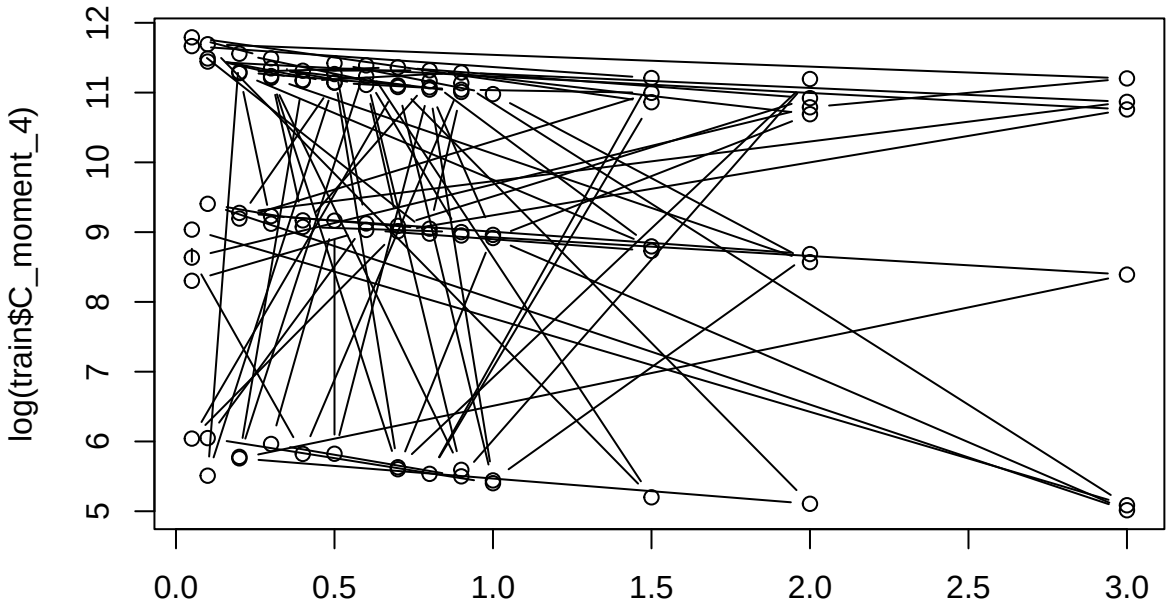
C_moment_4

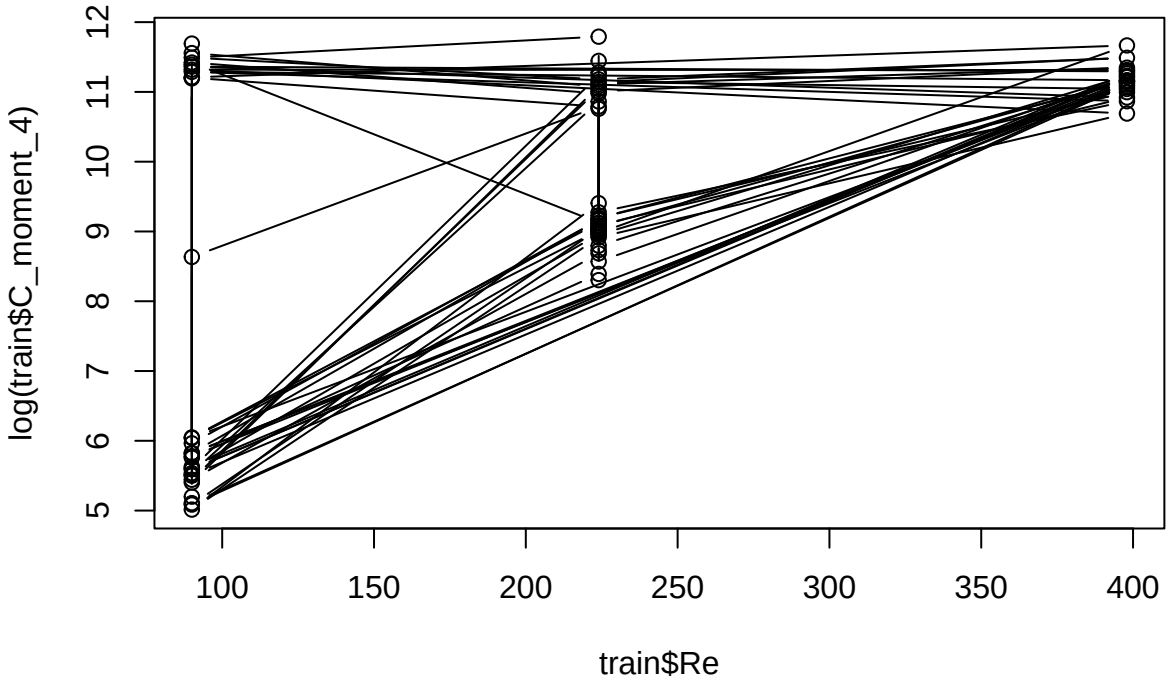

density.default(x = train$C_moment_4)
Density
0.0e+00
6.0e−06
1.2e−05
−50000
0
50000
100000
150000
N = 89 Bandwidth = 1.397e+04


density.default(x = log(train$C_moment_4))
Density
0.00
0.10
0.20
4
6
8
10
12
14
N = 89 Bandwidth = 0.7007


density.default(x = train$St)
Density
0.0
0.2
0.4
0.6
0
1
2
3
N = 89 Bandwidth = 0.1916


density.default(x = train$Re)
Density
0.000
0.002
0.004
0
100
200
300
400
500
N = 89 Bandwidth = 36.67






```
##
## Call:
## lm(formula = log(C_moment_4) ~ St + Re + Fr_logit, data = train)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -2.9367 -1.0139  0.6739  0.7860  3.1459
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  9.036759   0.544644  16.592  < 2e-16 ***
```

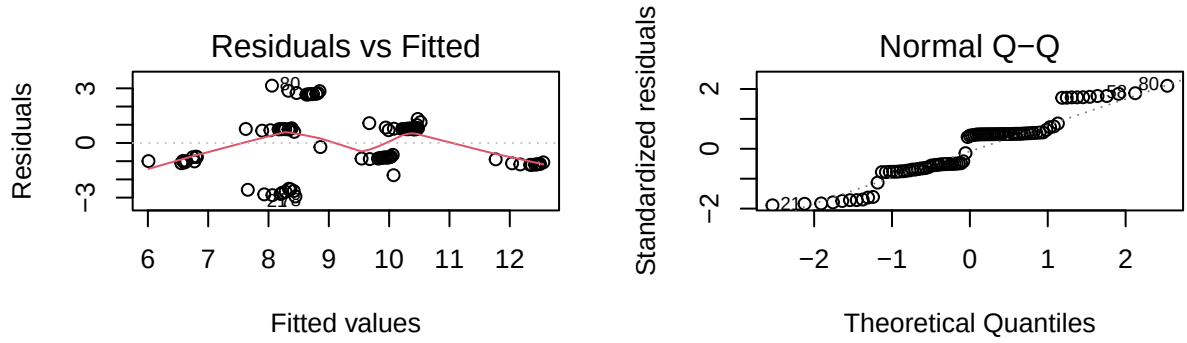
```
## St          -0.272165   0.215133  -1.265    0.209
## Re           0.012042   0.001502   8.019 5.19e-12 ***
## Fr_logit    -0.131772   0.024503  -5.378 6.52e-07 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.584 on 85 degrees of freedom
## Multiple R-squared:  0.507,  Adjusted R-squared:  0.4896
## F-statistic: 29.13 on 3 and 85 DF,  p-value: 4.737e-13
```

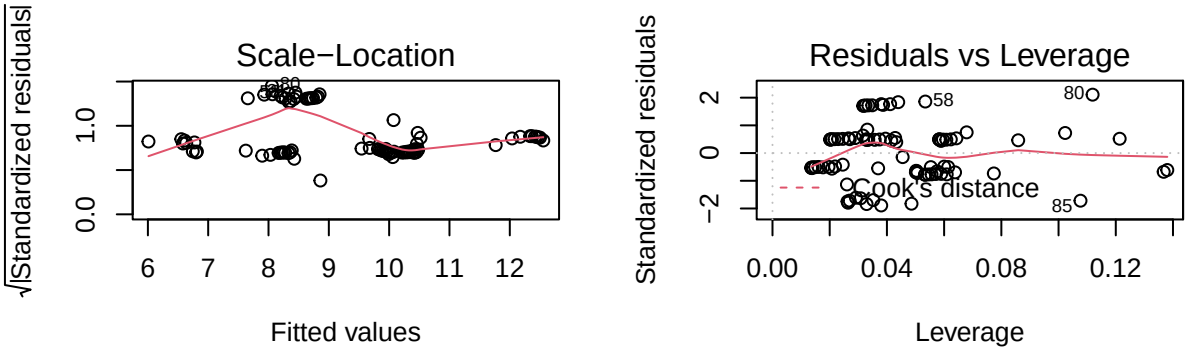
```
##
## Call:
## lm(formula = log(C_moment_4) ~ St + Re + Fr_logit + Re * Fr_logit,
##     data = train)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.2300 -0.6788  0.0201  0.8594  2.3907
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) 11.9250477  0.8261593  14.434  < 2e-16 ***
## St          -0.2323132  0.1955965  -1.188    0.238
## Re          -0.0010064  0.0032857  -0.306    0.760
## Fr_logit    -0.3194929  0.0484235  -6.598 3.52e-09 ***
## Re:Fr_logit  0.0008195  0.0001878   4.365 3.60e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.439 on 84 degrees of freedom
## Multiple R-squared:  0.5981, Adjusted R-squared:  0.579
## F-statistic: 31.25 on 4 and 84 DF,  p-value: 6.161e-16
```

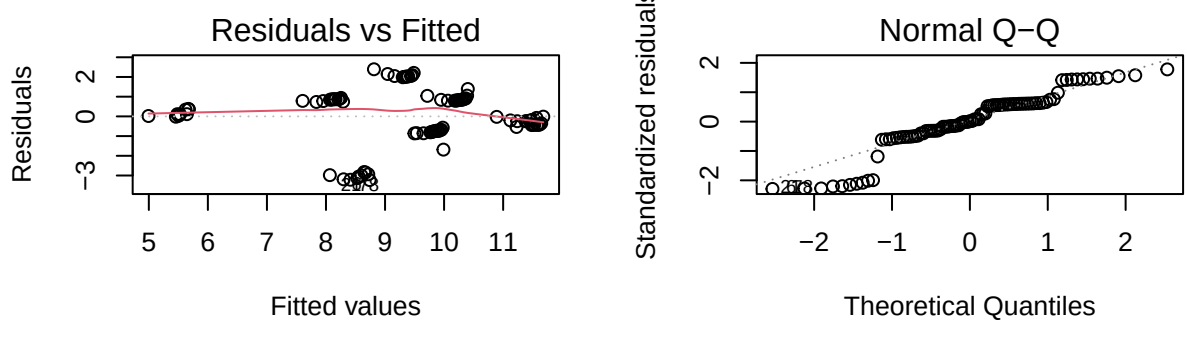
```
## [1] 1268309651
```

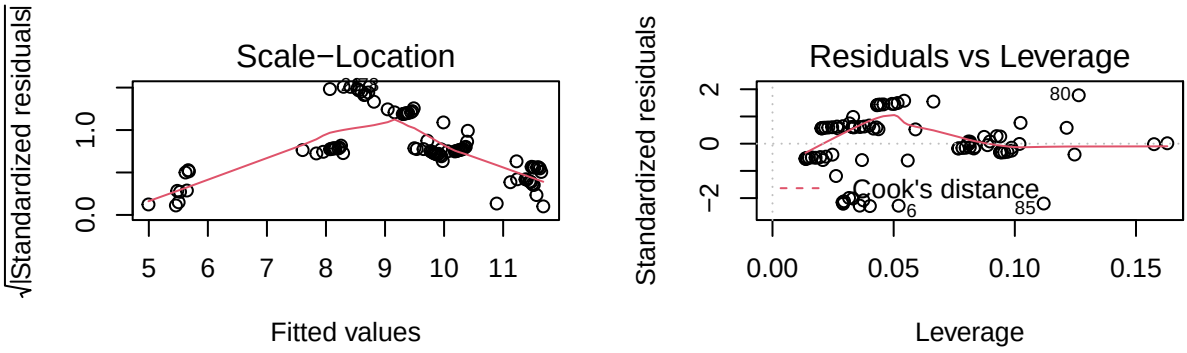
```
## [1] 1268309651
```

```
## [1] 0.5916249
```










```
##  [1]  892093168 1038182611 1065907001 1075330019 1213898931 1264705537
##  [7] 1408093004 1583710946 1743839229 1843753010
```

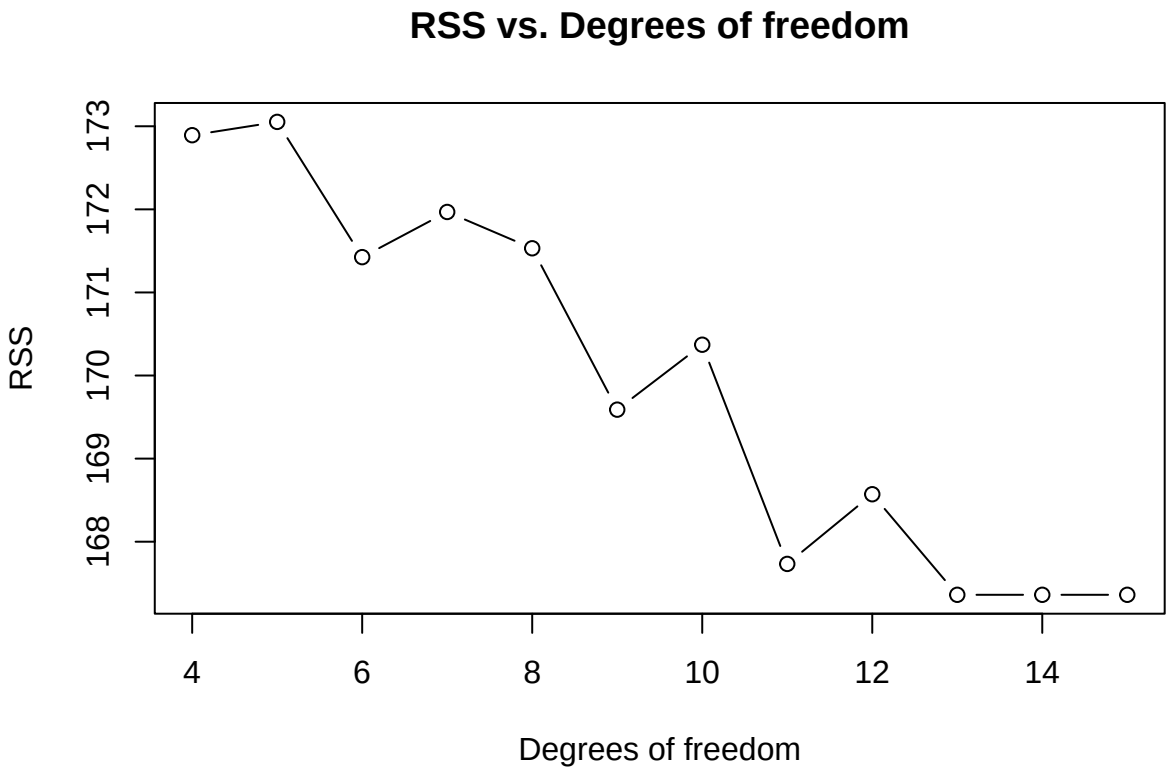
```
## [1] 892093168
```

```
## [1] 0.8544865
```

```
##
## Call:
## lm(formula = log(C_moment_4) ~ poly(St, 2) + Re + poly(Fr_logit,
##     2) + Re * Fr_logit, data = train)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -2.2781 -0.5129 -0.0980  0.6007  1.3765
##
## Coefficients: (1 not defined because of singularities)
##                       Estimate Std. Error t value Pr(>|t|)
## (Intercept)          7.325e+00  2.017e-01  36.311  < 2e-16 ***
## poly(St, 2)1        -2.036e+00  8.594e-01  -2.369  0.02020 *
## poly(St, 2)2        -1.382e-01  8.614e-01  -0.160  0.87295
## Re                  -5.864e-03  1.995e-03  -2.939  0.00428 **
## poly(Fr_logit, 2)1 -2.270e+01  1.883e+00 -12.053  < 2e-16 ***
## poly(Fr_logit, 2)2  1.106e+01  8.903e-01  12.421  < 2e-16 ***
## Fr_logit                   NA         NA      NA       NA
## Re:Fr_logit          9.652e-04  1.125e-04   8.580 4.93e-13 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.8567 on 82 degrees of freedom
## Multiple R-squared:  0.8609, Adjusted R-squared:  0.8507
## F-statistic: 84.55 on 6 and 82 DF,  p-value: < 2.2e-16
```

Non linear modeling

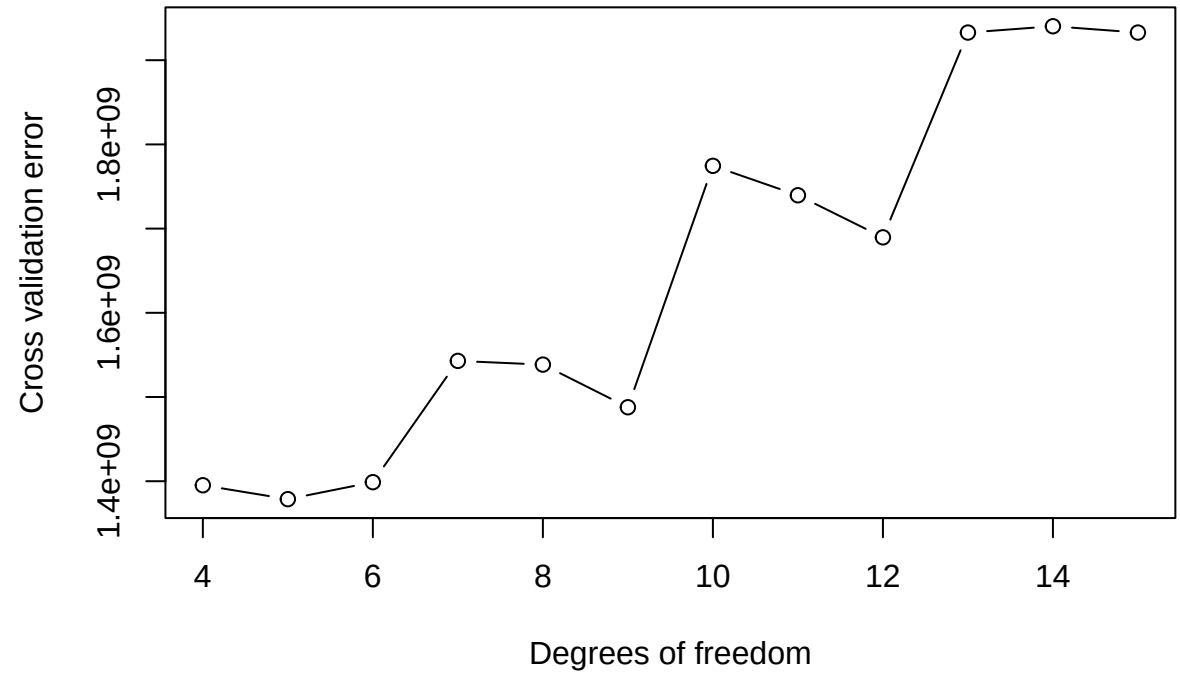
Natural spline




```
##  [1]          NA          NA          NA 1395231258 1378696765 1398866605
##  [7] 1542887933 1538492171 1487859135 1774559537 1739590296 1689615216
## [13] 1932851793 1940311523 1932851793
```

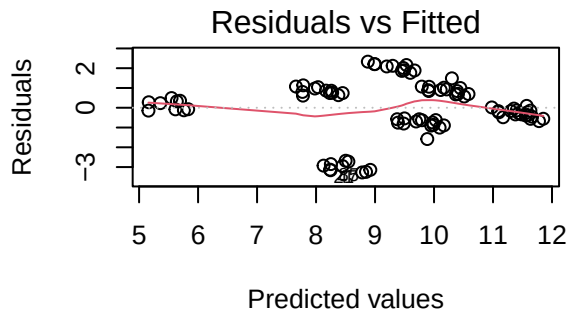
```
## [1] 1378696765
```

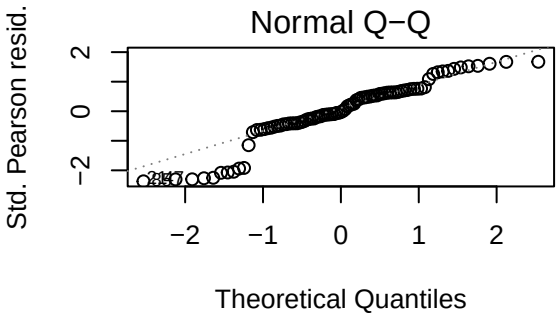
```
## [1] 0.6381364
```

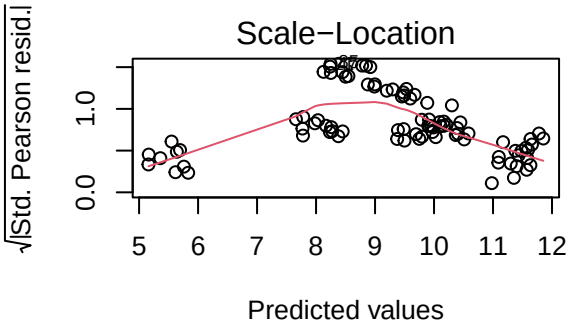


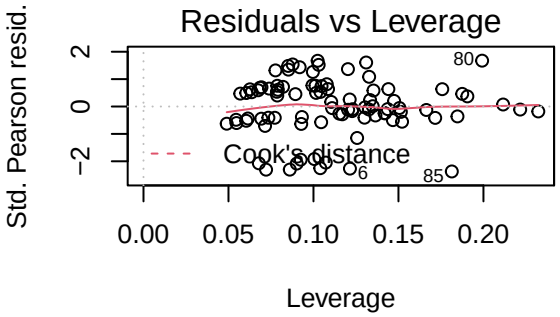
```
## 25% 50% 75%
## 0.3 0.7 1.0
```

```
##
## Call:
## glm(formula = log(C_moment_4) ~ ns(St, df = 6) + Re + Fr_logit +
##     Re * Fr_logit, data = train)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -3.2723  -0.5811  -0.0412   0.8865   2.3306
##
## Coefficients:
##                   Estimate Std. Error t value Pr(>|t|)
## (Intercept)     11.8085973  0.9644431  12.244  < 2e-16 ***
## ns(St, df = 6)1 -0.5521370  0.9458914  -0.584    0.561
## ns(St, df = 6)2  0.5370464  0.9135579   0.588    0.558
## ns(St, df = 6)3 -0.7074380  0.9459753  -0.748    0.457
## ns(St, df = 6)4 -0.5103850  0.9465363  -0.539    0.591
## ns(St, df = 6)5  0.0569556  1.4162840   0.040    0.968
## ns(St, df = 6)6 -0.6390810  0.7093651  -0.901    0.370
## Re              -0.0009227  0.0033728  -0.274    0.785
## Fr_logit        -0.3206860  0.0501963  -6.389 1.07e-08 ***
## Re:Fr_logit      0.0008187  0.0001936   4.229 6.25e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for gaussian family taken to be 2.169936)
##
##     Null deviance: 432.52  on 88  degrees of freedom
## Residual deviance: 171.42  on 79  degrees of freedom
## AIC: 332.91
##
## Number of Fisher Scoring iterations: 2
```





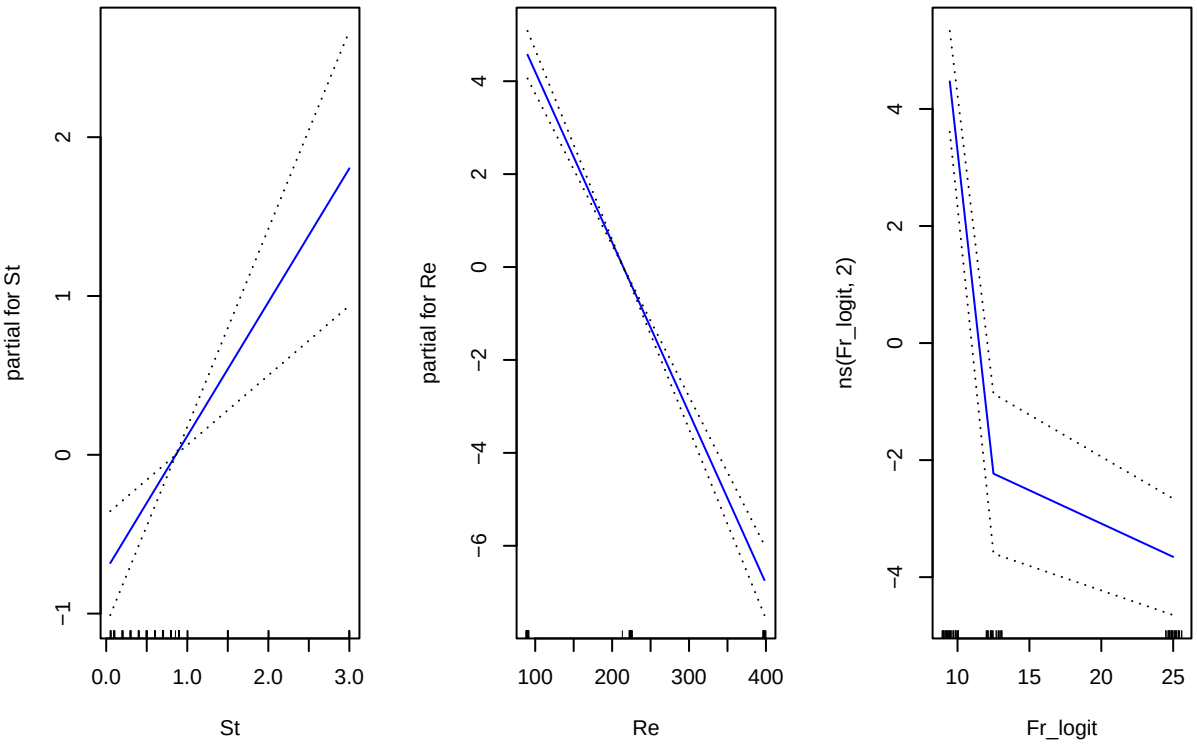





```
##  [1] 322033591 352599768 371416429 399773184 433743822 451606008 464517903
##  [8] 466304145 475248884 491615552 487871434 492795215 499203350 499203350
## [15] 499203350
```

```
## [1] 322033591
```

```
## [1] 0.9800513
```



models	formula	mse	adj
Least square regression	log(C_moment_4) ~ Fr + Re + St + Fr * Re	1268309651	0.59162
Polynomial regression	log(C_moment_4) ~ poly(Fr,2) + Re + poly(St, 2) + Fr * Re	892093168	0.85448
Natural spline	log(C_moment_4) ~ ns(St, df = 6) + Fr + Re + Fr * Re	1378696765	0.63813
Generalized additive model	log(C_moment_4) ~ St + Re + ns(Fr_logit,2) + Re:ns(Fr_logit,2)	322033591	0.98005