Week 3: Intro to Bayes

26/01/23

Question 1

Consider the happiness example from the lecture, with 118 out of 129 women indicating they are happy. We are interested in estimating θ, which is the (true) proportion of women who are happy. Calculate the MLE estimate $\hat{\theta}$ and 95% confidence interval.

Assume $Y|\theta \sim Bin(n, \theta)$.

$$L(\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y} l(\theta) = log\binom{n}{y} + ylog\theta + (n-y)log(1-\theta)\frac{\partial l}{\partial \theta} = \frac{y}{\theta} + \frac{n-y}{1-\theta}(-1)$$

setting the derivative to 0 gives

$$\begin{aligned}
\frac{y}{\theta} &= \frac{n-y}{1-\theta} \\
(n-y)\theta &= y(1-\theta) \\
n\theta - y\theta &= y - y\theta \\
n\theta &= y \\
\hat{\theta} &= \frac{y}{n}
\end{aligned}$$

to get the variance

$$\begin{aligned}
Var(\hat{\theta}) &= Var(\frac{y}{n}) \\
&= \frac{1}{n^2}Var(y) \\
&= \frac{1}{n^2}n\theta(1-\theta) \\
&= \frac{\theta(1-\theta)}{n}
\end{aligned}$$

$\hat{V}ar(\hat{\theta}) = \frac{\hat{\theta}(1-\hat{\theta})}{n}$

```
n=129
y=118
theta_hat=y/n
upper=theta_hat+1.96*sqrt(theta_hat*(1-theta_hat)/n)
lower=theta_hat-1.96*sqrt(theta_hat*(1-theta_hat)/n)
sprintf("MLE estimate of theta is %f",theta_hat)
```

```
[1] "MLE estimate of theta is 0.914729"
```

```
sprintf("confidence interval is (%f , %f)", lower, upper)
```

```
[1] "confidence interval is (0.866533 , 0.962924)"
```

Question 2

Assume a Beta(1,1) prior on θ. Calculate the posterior mean for $\hat{\theta}$ and 95% credible interval. The Beta (1, 1) distribution is the same as the Uniform (0, 1) distribution. Hence from the lecture we know that the posterior distribution is $p(\theta|y) \sim Beta(y+1, n-y+1)$

```
pos_mean=(y+1)/(y+1+n-y-1)
sprintf("posterior mean is %f",pos_mean)
```

```
[1] "posterior mean is 0.922481"
```

```
sprintf("credible interval is (%f,%f)", qbeta(0.25,y+1,n-y+1),qbeta(0.975,y+1,n-y+1))
```

```
[1] "credible interval is (0.892696,0.951389)"
```

Question 3

Now assume a Beta(10,10) prior on θ. What is the interpretation of this prior? Are we assuming we know more, less or the same amount of information as the prior used in Question 2?

The interpretation for a Beta(10,10) prior is: what is the most likely probability given α-1=10−1=9 success (happy), and β-1=10−1=9 of failures (not happy). The mean of Beta(10,10) and Beta(1,1) are both 0.5, meaning that using Beta(10,10) or Beta(1,1) prior gives the same expectation of the probability of being happy, hence we have the same amount of information as the prior used in Question 2.

Question 4

Create a graph in ggplot which illustrates

- The likelihood (easiest option is probably to use `geom_histogram` to plot the histogram of appropriate random variables)
- The priors and posteriors in question 2 and 3 (use `stat_function` to plot these distributions)

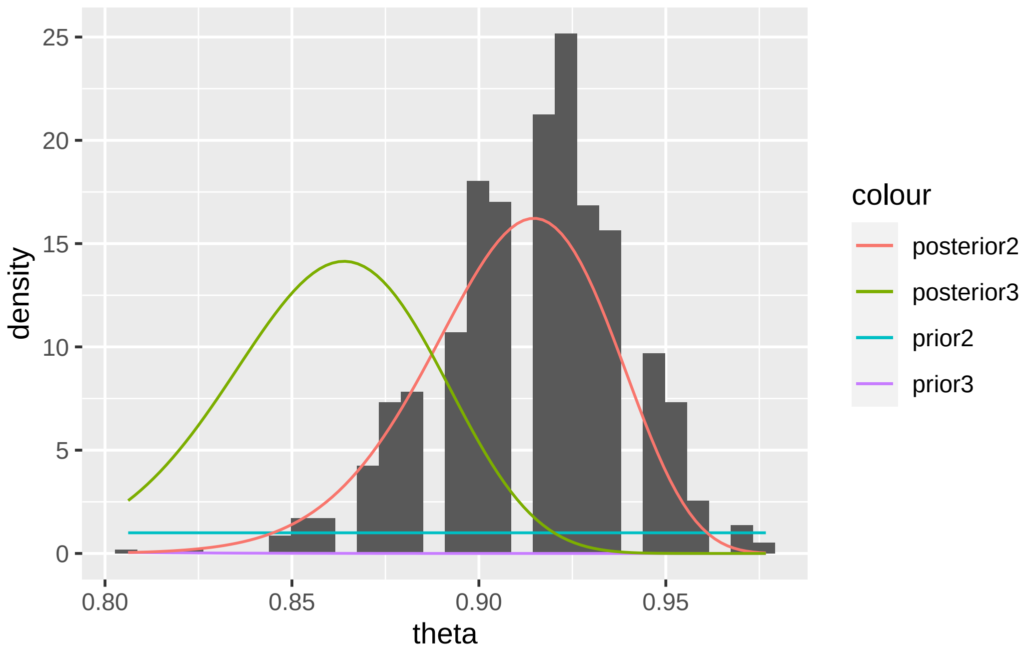
Comment on what you observe.

prior Q2: Beta(1,1). posterior Q2: Beta(y+1, n-y+1). prior Q3: Beta(10,10). posterior Q3: Beta(10+y, 10+n-y)

```r
library(ggplot2)

set.seed(1)
df=data.frame(num_happy=rbinom(n=1000,size=129,prob=118/129))
df$theta=df$num_happy/129
ggplot(data=df)+
  geom_histogram(aes(x = theta, y=..density..))+
  stat_function(aes(x = theta, color="prior2"),fun=dbeta,args=list(shape1=1,shape2=1))+
  stat_function(aes(x = theta, color="prior3"),fun=dbeta,args=list(shape1=10,shape2=10))+
  stat_function(aes(x = theta, color="posterior2"),fun=dbeta,
                args=list(shape1=118+1,shape2=129-118+1))+
  stat_function(aes(x = theta, color="posterior3"),fun=dbeta,
                args=list(shape1=118+10,shape2=129-118+10))
```

```
`stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```



We see that the posterior distribution looks like a combination of the likelihood and prior distribution. Posterior in Q2 is more closely aligned with the likelihood, suggesting that prior used in Q2 is weakly-informative.

Question 5

(No R code required) A study is performed to estimate the effect of a simple training program on basketball free-throw shooting. A random sample of 100 college students is recruited into the study. Each student first shoots 100 free-throws to establish a baseline success probability. Each student then takes 50 practice shots each day for a month. At the end of that time, each student takes 100 shots for a final measurement. Let θ be the average improvement in success probability. θ is measured as the final proportion of shots made minus the initial proportion of shots made.

Given two prior distributions for θ (explaining each in a sentence):

- A noninformative prior, and
- A subjective/informative prior based on your best knowledge.

A noninformative prior could be Uniform(0,1) since it assigns equal probability to all possible θ hence does not provide more information about θ other than the data itself.

θ should be in the range (-1,1), so an informative prior could be a tanh function $f(\theta) = tanh(\theta) = \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}$ which has the range (-1,1).