Blind Deconvolution for Multiple Images with Missing Values

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Introduction

- Multiple blurred images with missing values (noise)
- Tensor-based method
- Deconvolution (deblurring) & completion (denoising)

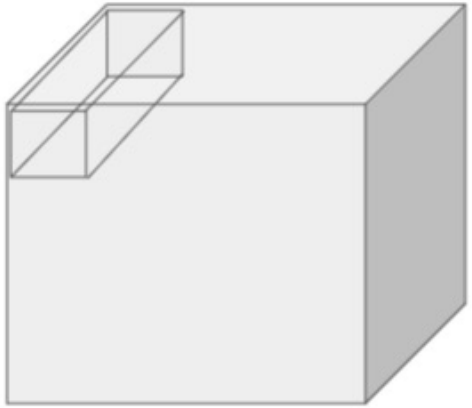
Transformed Tensor

- Let $\Phi \in \mathbb{C}^{n_3 \times n_3}$ be a unitary matrix with $\Phi\Phi^H = \Phi^H\Phi = I_{n_3}$. Denote $\hat{\mathcal{X}}_\Phi = \Phi[\mathcal{X}]$ as a tensor obtained via multiplying all tubes of the tensor $\mathcal{X} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ by the unitary matrix Φ along the third dimension, i.e.,

$$\text{vec}(\hat{\mathcal{X}}_\Phi(i,j,:)) = \Phi * \text{vec}(\mathcal{X}(i,j,:))$$

-

$$\mathcal{X} = \Phi^H[\hat{\mathcal{X}}_\Phi]$$



Kilmer, M. E., & Martin, C. D. (2011). Factorization strategies for third-order tensors. *Linear Algebra and its Applications*, *435*(3), 641-658.

Song, G., Ng, M. K., & Zhang, X. (2020). Robust tensor completion using transformed tensor singular value decomposition. *Numerical Linear Algebra with Applications*, *27*(3), e2299.

Tensor Product

- $$\bar{\mathcal{X}}_\Phi = blockdiag(\hat{\mathcal{X}}_\Phi) = \begin{pmatrix} \hat{\mathcal{X}}_{\Phi(1)} & & & \\ & \hat{\mathcal{X}}_{\Phi(2)} & & \\ & & \ddots & \\ & & & \hat{\mathcal{X}}_{\Phi(n_3)} \end{pmatrix}$$

- $$fold(blockdiag(\hat{\mathcal{X}}_\Phi)) = fold(\bar{\mathcal{X}}_\Phi) = \hat{\mathcal{X}}_\Phi$$

- (Φ-product) The Φ-product of two tensors $\mathcal{X} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ and $\mathcal{Y} \in \mathbb{C}^{n_2 \times n_4 \times n_3}$, denoted by $\mathcal{X} \diamond_\Phi \mathcal{Y}$ is an $n_1 \times n_4 \times n_3$ tensor given by

$$\mathcal{Z} = \mathcal{X} \diamond_\Phi \mathcal{Y} = \Phi^H[fold(blockdiag(\hat{\mathcal{X}}_\Phi) \cdot blockdiag(\hat{\mathcal{Y}}_\Phi))]$$

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Tensor Computation

- (conjugate transpose of tensor) The conjugate transpose of the tensor $\mathcal{X} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ is the tensor $\mathcal{X}^H \in \mathbb{C}^{n_2 \times n_1 \times n_3}$ obtained by
$$\mathcal{X}^H = \Phi^H[fold(blockdiag(\hat{\mathcal{X}}_\Phi)^H)]$$

- (identity tensor) The indentity tensor $\mathcal{I}_\Phi \in \mathbb{C}^{n \times n \times n_3}$ with respect to Φ, is defined as $\mathcal{I}_\Phi = \Phi[\mathcal{T}]$, with each frontal slice of $\mathcal{T} \in \mathbb{R}^{n \times n \times n_3}$ being the $n \times n$ identity matrix.

- (unitary tensor) A tensor $\mathcal{Q} \in \mathbb{C}^{n \times n \times n_3}$ is unitary with respect to Φ-product if it satisfies the following:
$$\mathcal{Q}^H \diamond_\Phi \mathcal{Q} = \mathcal{Q} \diamond_\Phi \mathcal{Q}^H = \mathcal{I}_\Phi$$
where, $\mathcal{I}_\Phi$ is the identity tensor.

Kilmer, M. E., & Martin, C. D. (2011). Factorization strategies for third-order tensors. *Linear Algebra and its Applications*, *435*(3), 641-658.

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Tensor SVD

- **Theorem (transformed tensor SVD)**
  Suppose that $\mathcal{X} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$, then $\mathcal{X}$ can be factorized as

  $$\mathcal{X} = \mathcal{U} \diamond_\Phi \mathcal{S} \diamond_\Phi \mathcal{V}^H$$

  where $\mathcal{U} \in \mathbb{C}^{n_1 \times n_1 \times n_3}$, $\mathcal{V} \in \mathbb{C}^{n_2 \times n_2 \times n_3}$ are unitary tensors with respect to Φ-product, and $\mathcal{S} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ is a diagonal tensor.

- **Definition (transformed tubal nuclear norm)**
  The transformed tubal nuclear norm (TTNN) of a tensor $\mathcal{X} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$, denoted as $||\mathcal{X}||_{TTNN}$, is the sum of nuclear norms of all frontal slices of $\hat{\mathcal{X}}_\Phi$, that is,

  $$||\mathcal{X}||_{TTNN} = \sum_{i=1}^{n_3} ||\hat{\mathcal{X}}_{\Phi(i)}||_* = \sum_{i=1}^{n_3} ||\hat{\mathcal{S}}_{\Phi(i)}||_* = \sum_{i=1}^{n_3} sum(diag(\hat{\mathcal{S}}_{\Phi(i)}))$$

  where sum operation is the summation of all elements of a vector, and diag extracts the diagonal elements of a matrix.

Song, G., Ng, M. K., & Zhang, X. (2020). Robust tensor completion using transformed tensor singular value decomposition. *Numerical Linear Algebra with Applications*, *27*(3), e2299.

Modeling

- we are given several blurred images $\mathcal{Y} \in \mathbb{R}^{n_v \times n_h \times n_b}$ that are only partially observed ($\mathcal{Y}(\Omega)$ is observed), and we aim to recover the corresponding single sharp image $X \in \mathbb{R}^{n_v \times n_h}$, blurred images $\mathcal{Y}$ and blur kernels $\mathcal{K} \in \mathbb{R}^{(2*k_v+1)\times(2*k_h+1)\times k_b}$.
- $\mathcal{Y}_t = \mathcal{K}_t \star X$ $\quad\quad$ $\mathcal{Y} = \mathcal{K} \star X$ $\quad\quad$ Periodic boundary condition
- Low TTNN
- $\mathcal{K}_t \geq 0$ and $\sum_{i,j} \mathcal{K}_t(i,j) = 1$

$$Tik(\mathcal{K}_t) = ||\mathcal{K}_t||_F^2 = \sum_{i,j} \mathcal{K}_t(i,j)^2 \leq \sum_{i,j} \mathcal{K}_t(i,j) = 1 = ||\delta||_F^2 \qquad Tik(\mathcal{K}) = ||\mathcal{K}||_F^2 = \sum_{t=1}^{T} ||\mathcal{K}_t||_F^2 \leq n_b$$

- Low total variation (TV)

Modeling

- $$\underset{\mathcal{Y},\mathcal{K},X}{\arg\min}\ \frac{1}{2}||\mathcal{Y}-\mathcal{K}\star X||_F^2+\alpha||\mathcal{Y}||_{TTNN}+\frac{\beta}{2}||\mathcal{K}||_F^2+\gamma(||D_1X||_1+||XD_2^T||_1)$$

$$s.t.\quad 0\leq X,\mathcal{Y}\leq 1,\ \ P_\Omega(\mathcal{Y}-\mathcal{M})=0,\ \ \mathcal{K}\geq 0,\ \ \sum_{i,j}\mathcal{K}_t(i,j)=1$$

$$D_1=\begin{pmatrix} 1 & & & -1 \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{pmatrix}_{n_v\times n_v} \qquad D_2=\begin{pmatrix} 1 & & & -1 \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{pmatrix}_{n_h\times n_h}$$

Modeling

- $$\arg\min_{\mathcal{Y},\mathcal{K},X} F(\mathcal{Y},\mathcal{K},X)$$

where

$$F(\mathcal{Y},\mathcal{K},X) = \frac{1}{2}||\mathcal{Y} - \mathcal{K} \star X||_F^2 + \alpha||\mathcal{Y}||_{TTNN} + \frac{\beta}{2}||\mathcal{K}||_F^2 + \delta_{\mathfrak{S}}(\mathcal{Y},\mathcal{K},X)$$

$$+\gamma(||D_1X||_1 + ||XD_2^T||_1)$$

$$\mathfrak{S} = \{(\mathcal{Y},\mathcal{K},X) \in \mathbb{R}^{n_v\times n_h\times n_b} \times \mathbb{R}^{(2*k_v+1)\times(2*k_h+1)\times k_b} \times \mathbb{R}^{n_v\times n_h} : 0 \le \mathcal{Y}, X \le 1,$$

$$P_\Omega(\mathcal{Y} - \mathcal{M}) = \mathbf{0},\ \ \mathcal{K} \ge 0,\ \sum_{i=-k_v}^{k_v}\sum_{j=-k_h}^{k_h} \mathcal{K}_t(i,j) = 1,\ \ for\ \ each\ \ t = 1,..,k_b\}$$

and $\delta_{\mathfrak{S}}$ is an indicator function defined on the whole space that

$$\delta_{\mathfrak{S}}(\mathcal{Y},\mathcal{K},X) = \begin{cases} 0, & (\mathcal{Y},\mathcal{K},X) \in \mathfrak{S} \\ +\infty, & otherwise \end{cases}$$

Proximal Alternating Method

- $$\mathcal{Y}^{i+1} = \arg\min_{\mathcal{Y}} F(\mathcal{Y}, \mathcal{K}^i, X^i) + \frac{\rho_1}{2}||\mathcal{Y} - \mathcal{Y}^i||_F^2$$
$$\mathcal{K}^{i+1} = \arg\min_{\mathcal{K}} F(\mathcal{Y}^{i+1}, \mathcal{K}, X^i) + \frac{\rho_2}{2}||\mathcal{K} - \mathcal{K}^i||_F^2$$
$$X^{i+1} = \arg\min_{X} F(\mathcal{Y}^{i+1}, \mathcal{K}^{i+1}, X) + \frac{\rho_3}{2}||X - X^i||_F^2$$

Convergence Analysis

- **Assumption**

  There exist constants $\epsilon_1 \in (0, \rho_1)$, $\epsilon_2 \in (0, \rho_2)$, $\epsilon_3 \in (0, \rho_3)$, *and* b_1, b_2, b_3, *such that for each iteration,*

$$F(\mathcal{Y}^{i+1}, \mathcal{K}^i, X^i) + \frac{\rho_1}{2}||\mathcal{Y}^{i+1} - \mathcal{Y}^i||_F^2 - F(\mathcal{Y}^i, \mathcal{K}^i, X^i) \le \frac{\epsilon_1}{2}||\mathcal{Y}^{i+1} - \mathcal{Y}^i||_F^2$$

$$F(\mathcal{Y}^{i+1}, \mathcal{K}^{i+1}, X^i) + \frac{\rho_2}{2}||\mathcal{K}^{i+1} - \mathcal{K}^i||_F^2 - F(\mathcal{Y}^{i+1}, \mathcal{K}^i, X^i) \le \frac{\epsilon_2}{2}||\mathcal{K}^{i+1} - \mathcal{K}^i||_F^2$$

$$F(\mathcal{Y}^{i+1}, \mathcal{K}^{i+1}, X^{i+1}) + \frac{\rho_3}{2}||X^{i+1} - X^i||_F^2 - F(\mathcal{Y}^{i+1}, \mathcal{K}^{i+1}, X^i) \le \frac{\epsilon_3}{2}||X^{i+1} - X^i||_F^2$$

and

$$\exists\ \xi_1^{i+1} \in \partial_{\mathcal{Y}} F(\mathcal{Y}^{i+1}, \mathcal{K}^i, X^i) + \rho_1(\mathcal{Y}^{i+1} - \mathcal{Y}^i)$$

$$\exists\ \xi_2^{i+1} \in \partial_{\mathcal{K}} F(\mathcal{Y}^{i+1}, \mathcal{K}^{i+1}, X^i) + \rho_2(\mathcal{K}^{i+1} - \mathcal{K}^i)$$

$$\exists\ \xi_3^{i+1} \in \partial_X F(\mathcal{Y}^{i+1}, \mathcal{K}^{i+1}, X^{i+1}) + \rho_3(X^{i+1} - X^i)$$

with

$$||\xi_1^{i+1}||_F \le b_1||\mathcal{Y}^{i+1} - \mathcal{Y}^i||_F$$

$$||\xi_2^{i+1}||_F \le b_2||\mathcal{K}^{i+1} - \mathcal{K}^i||_F$$

$$||\xi_1^{i+1}||_F \le b_3||X^{i+1} - X^i||_F$$

Convergence Analysis

- **Proposition**
  A real valued semi-algebraic function is a KL function.

- **Lemma**
  F *is semi-algebraic on* $\mathfrak{S}$. *Combined with the above proposition,* F *has KL property at each point* $\mathcal{V} = (\mathcal{Y}, \mathcal{K}, X) \in \mathfrak{S}$.

Bolte, J., Daniilidis, A., & Lewis, A. (2007). The Łojasiewicz inequality for nonsmooth subanalytic functions with applications to subgradient dynamical systems. *SIAM Journal on Optimization*, *17*(4), 1205-1223.

Convergence Analysis

Lemma

For a PLSC function $f : \mathbb{R}^n \to \mathbb{R}$, suppose that there is a sequence $\{x^k\}_{k\in\mathbb{N}}$ satisfying the following three conditions:

H1. *(sufficient decrease condition). For each $k \in \mathbb{N}$, there exists a fixed constant $\mathfrak{a} > 0$,*

$$f(x^{k+1}) + \mathfrak{a}||x^{k+1} - x^k||_2^2 \leq f(x^k)$$

H2. *(relative error condition). For each $k \in \mathbb{N}$, there exists a fixed constant $\mathfrak{b} > 0$, and $\exists w^{k+1} \in \partial f(x^{k+1})$, such that*

$$||w^{k+1}|| \leq \mathfrak{b}||x^{k+1} - x^k||_2$$

H3. *(continuity condition) There exists a subsequence $\{x^{k_j}\}_{j\in\mathbb{N}}$, and x^* such that*

$$x^{k_j} \to x^* \quad \text{and} \quad f(x^{k_j}) \to f(x^*) \quad \text{as} \quad j \to +\infty$$

And if f has the KL property at x^* in H3., then

- $x^k \to x^*$.
- $0 \in \partial f(x^*)$, *meaning that x^* is a critical point of f.*
- *The sequence $\{x^k\}_{k\in\mathbb{N}}$ has a finite length, i.e.,*

$$\sum_{k=0}^{+\infty} ||x^{k+1} - x^k||_2^2 < +\infty$$

Attouch, H., Bolte, J., & Svaiter, B. F. (2013). Convergence of descent methods for semi-algebraic and tame problems: proximal algorithms, forward–backward splitting, and regularized Gauss–Seidel methods. *Mathematical Programming*, *137*(1), 91-129.

Convergence Analysis

Theorem (global convergence of inexact iteration)

Let $(\mathcal{Y}^i, \mathcal{K}^i, X^i)$ *be the sequence generated by PAM Iteration with errors satisfying the Assumption. Then there exists* $(\mathcal{Y}^*, \mathcal{K}^*, X^*)$, *such that*

- $(\mathcal{Y}^i, \mathcal{K}^i, X^i) \to (\mathcal{Y}^*, \mathcal{K}^*, X^*)$.
- $0 \in \partial F(\mathcal{Y}^*, \mathcal{K}^*, X^*)$
- $\{\mathcal{Y}^i, \mathcal{K}^i, X^i\}_{i \in \mathbb{N}}$ *has a finite length, i.e.,*

$$\sum_{i=0}^{+\infty} \sqrt{||\mathcal{Y}^{i+1} - \mathcal{Y}^i||_F^2 + ||\mathcal{K}^{i+1} - \mathcal{K}^i||_F^2 + ||X^{i+1} - X^i||_F^2} < +\infty$$

Numerical Methods

$$\mathcal{Y}^{i+1} = \arg\min_{\mathcal{Y}} F(\mathcal{Y}, \mathcal{K}^i, X^i) + \frac{\rho_1}{2}||\mathcal{Y} - \mathcal{Y}^i||_F^2$$
$$\mathcal{K}^{i+1} = \arg\min_{\mathcal{K}} F(\mathcal{Y}^{i+1}, \mathcal{K}, X^i) + \frac{\rho_2}{2}||\mathcal{K} - \mathcal{K}^i||_F^2$$
$$X^{i+1} = \arg\min_{X} F(\mathcal{Y}^{i+1}, \mathcal{K}^{i+1}, X) + \frac{\rho_3}{2}||X - X^i||_F^2$$

Solving Y

$$\mathcal{Y}^{i+1} = \arg\min_{\mathcal{Y}} F(\mathcal{Y}, \mathcal{K}^i, X^i) + \frac{\rho_1}{2}||\mathcal{Y} - \mathcal{Y}^i||_F^2$$

$$= \arg\min_{0\le\mathcal{Y}\le1, P_\Omega(\mathcal{Y}-\mathcal{M})=0} \frac{1}{2}||\mathcal{Y} - \mathcal{K}^i \star X^i||_F^2 + \alpha||\mathcal{Y}||_{TTNN} + \frac{\rho_1}{2}||\mathcal{Y} - \mathcal{Y}^i||_F^2$$

is equivalent to

$$\arg\min_{0\le\mathcal{Y}\le1, P_\Omega(\mathcal{Y}-\mathcal{M})=0} \frac{1}{2}||\mathcal{Y} - \mathcal{K}^i \star X^i||_F^2 + \alpha||\mathcal{Q}_{\mathcal{Y}}||_{TTNN} + \frac{\rho_1}{2}||\mathcal{Y} - \mathcal{Y}^i||_F^2$$

$$s.t. \quad \mathcal{Q}_{\mathcal{Y}} = \mathcal{Y}$$

Solving Y

$$L_{\mathcal{Y}} = \frac{1}{2}||\mathcal{Y} - \mathcal{K}^i \star X^i||_F^2 + \alpha||\mathcal{Q}_{\mathcal{Y}}||_{TTNN} + \frac{\rho_1}{2}||\mathcal{Y} - \mathcal{Y}^i||_F^2$$
$$+\langle \mathcal{Z}_{\mathcal{Y}}, \mathcal{Y} - \mathcal{Q}_{\mathcal{Y}} \rangle + \frac{\mu_1}{2}||\mathcal{Y} - \mathcal{Q}_{\mathcal{Y}}||_F^2$$

$$\mathcal{Q}_{\mathcal{Y}}^{j+1} = \arg\min_{\mathcal{Q}_{\mathcal{Y}}} \ \alpha||\mathcal{Q}_{\mathcal{Y}}||_{TTNN} + \frac{\mu_1}{2}||\mathcal{Y}^{i,j} - \mathcal{Q}_{\mathcal{Y}} + \frac{\mathcal{Z}_{\mathcal{Y}}^j}{\mu_1}||_F^2$$

$$\mathcal{Y}^{i,j+1} = \arg\min_{0 \le \mathcal{Y} \le 1, P_\Omega(\mathcal{Y} - \mathcal{M}) = 0} \ \frac{1}{2}||\mathcal{Y} - \mathcal{K}^i \star X^i||_F^2 + \frac{\rho_1}{2}||\mathcal{Y} - \mathcal{Y}^i||_F^2 + \frac{\mu_1}{2}||\mathcal{Y} - \mathcal{Q}_{\mathcal{Y}}^{j+1} + \frac{\mathcal{Z}_{\mathcal{Y}}^j}{\mu_1}||_F^2$$

$$\mathcal{Z}_{\mathcal{Y}}^{j+1} = \mathcal{Z}_{\mathcal{Y}}^j + \mu_1(\mathcal{Y}^{i,j+1} - \mathcal{Q}_{\mathcal{Y}}^{j+1})$$

Solving Y

- $$\mathcal{Q}_{\mathcal{Y}}^{j+1} = \arg\min_{\mathcal{Q}_{\mathcal{Y}}} \ \alpha||\mathcal{Q}_{\mathcal{Y}}||_{TTNN} + \frac{\mu_1}{2}||\mathcal{Y}^{i,j} - \mathcal{Q}_{\mathcal{Y}} + \frac{\mathcal{Z}_{\mathcal{Y}}^{j}}{\mu_1}||_F^2$$

$$\begin{aligned}
&\alpha\|Q_{\mathcal{Y}}\|_{TTNN} + \frac{\mu_1}{2}\|\mathcal{Y}^{i,j} - Q_{\mathcal{Y}} + \frac{\mathcal{Z}_{\mathcal{Y}}^{j}}{\mu_1}\|_F^2 \\
=&\alpha\sum_{k=1}^{n_b}\|\hat{Q}_{\mathcal{Y}(k)}\|_* + \frac{\mu_1}{2}\sum_{k=1}^{n_b}\|\hat{Q}_{\mathcal{Y}(k)} - (\hat{\mathcal{Y}}_k^{i,j} + \frac{\hat{\mathcal{Z}}_{\mathcal{Y}(k)}^{j}}{\mu_1})\|_F^2 \\
=&\sum_{k=1}^{n_b}\left(\alpha\|\hat{Q}_{\mathcal{Y}(k)}\|_* + \frac{\mu_1}{2}\|\hat{Q}_{\mathcal{Y}(k)} - (\hat{\mathcal{Y}}_k^{i,j} + \frac{\hat{\mathcal{Z}}_{\mathcal{Y}(k)}^{j}}{\mu_1})\|_F^2\right)
\end{aligned}$$

$$\min\{\lambda\|\mathcal{X}\|_{TTNN} + \frac{1}{2}\|\mathcal{X} - \mathcal{Y}\|_F^2\}$$

is equivalent to

$$\min \lambda \sum_{i=1}^{n_3} \|\hat{\mathcal{X}}_{\Phi(i)}\|_* + \frac{1}{2}\|\hat{\mathcal{X}}_{\Phi} - \hat{\mathcal{Y}}_{\Phi}\|_F^2$$

Solving Y

- the singular value shrinkage operator

$$\min \lambda \|\hat{\mathcal{X}}_{\Phi(i)}\|_* + \frac{1}{2}\|\hat{\mathcal{X}}_{\Phi(i)} - \hat{\mathcal{Y}}_{\Phi(i)}\|_F^2$$

$$\hat{\mathcal{Y}}_{\Phi(i)} = U * S_0 * V^H$$

$$\hat{\mathcal{X}}_{\Phi(i)} = U * S * V^H$$

where

$$S = \max\{S_0 - \lambda, 0\}$$

Cai, J. F., Candès, E. J., & Shen, Z. (2010). A singular value thresholding algorithm for matrix completion. *SIAM Journal on optimization*, *20*(4), 1956-1982.

Solving Y

$$\mathcal{Q}_{\mathcal{Y}}^{j+1} = \arg\min_{\mathcal{Q}_{\mathcal{Y}}} \ \alpha||\mathcal{Q}_{\mathcal{Y}}||_{TTNN} + \frac{\mu_1}{2}||\mathcal{Y}^{i,j} - \mathcal{Q}_{\mathcal{Y}} + \frac{\mathcal{Z}_{\mathcal{Y}}^{j}}{\mu_1}||_F^2$$

$$Q_{\mathcal{Y}} = \mathcal{U} \diamond_{\Phi} \mathcal{S} \diamond_{\Phi} \mathcal{V}^H$$

where $\mathcal{Y}^{i,j} + \frac{\mathcal{Z}_{\mathcal{Y}j}}{\mu_1} = \mathcal{U} \diamond_{\Phi} \mathcal{S}_0 \diamond_{\Phi} \mathcal{V}^H$, $\mathcal{S} = \Phi^H[\hat{\mathcal{S}}]$ and $\hat{\mathcal{S}} = \max\{\Phi[\mathcal{S}_0] - \frac{\alpha}{\mu_1}, 0\}$

Solving K

$$\mathcal{K}^{i+1} = \arg\min_{\mathcal{K}} F(\mathcal{Y}^{i+1}, \mathcal{K}, X^i) + \frac{\rho_2}{2}||\mathcal{K} - \mathcal{K}^i||_F^2$$
$$= \arg\min_{\mathcal{K}\geq 0, \mathbf{1}^T\mathcal{K}_t(:)=1} \frac{1}{2}||\mathcal{Y}^{i+1} - \mathcal{K} \star X^i||_F^2 + \frac{\beta}{2}||\mathcal{K}||_F^2 + \frac{\rho_2}{2}||\mathcal{K} - \mathcal{K}^i||_F^2$$

is equivalent to

$$\arg\min_{\mathcal{K}\geq 0, \mathbf{1}^T\mathcal{K}_t(:)=1} \frac{1}{2}||\mathcal{Y}^{i+1} - \mathcal{Q}_{\mathcal{K}} \star X^i||_F^2 + \frac{\beta}{2}||\mathcal{K}||_F^2 + \frac{\rho_2}{2}||\mathcal{K} - \mathcal{K}^i||_F^2$$

$$s.t. \quad \mathcal{Q}_{\mathcal{K}} = \mathcal{K}$$

$$L_{\mathcal{K}} = \frac{1}{2}||\mathcal{Y}^{i+1} - \mathcal{Q}_{\mathcal{K}} \star X^i||_F^2 + \frac{\beta}{2}||\mathcal{K}||_F^2 + \frac{\rho_2}{2}||\mathcal{K} - \mathcal{K}^i||_F^2$$
$$+\langle \mathcal{Z}_{\mathcal{Y}}, \mathcal{K} - \mathcal{Q}_{\mathcal{K}}\rangle + \frac{\mu_2}{2}||\mathcal{K} - \mathcal{Q}_{\mathcal{K}}||_F^2$$

Solving K

$$\mathcal{Q}_{\mathcal{K}}^{j+1} = \arg\min_{\mathcal{Q}_{\mathcal{K}}} \ \frac{1}{2}||\mathcal{Y}^{i+1} - \mathcal{Q}_{\mathcal{K}} \star X^{i}||_F^2 + \frac{\mu_2}{2}||\mathcal{K}^{i,j} - \mathcal{Q}_{\mathcal{K}} + \frac{\mathcal{Z}_{\mathcal{K}}^{j}}{\mu_2}||_F^2$$

$$\mathcal{K}^{i,j+1} = \arg\min_{\mathcal{K}\geq 0, \mathbf{1}^T\mathcal{K}_t(:)=1} \ \frac{\beta}{2}||\mathcal{K}||_F^2 + \frac{\rho_2}{2}||\mathcal{K} - \mathcal{K}^{i}||_F^2 + \frac{\mu_2}{2}||\mathcal{K} - \mathcal{Q}_{\mathcal{K}}^{j+1} + \frac{\mathcal{Z}_{\mathcal{K}}^{j}}{\mu_2}||_F^2$$

$$\mathcal{Z}_{\mathcal{K}}^{j+1} = \mathcal{Z}_{\mathcal{K}}^{j} + \mu_2(\mathcal{K}^{i,j+1} - \mathcal{Q}_{\mathcal{K}}^{j+1})$$

Solving K

- $$\mathcal{Q}_{\mathcal{K}}^{j+1} = \arg\min_{\mathcal{Q}_{\mathcal{K}}} \ \frac{1}{2}||\mathcal{Y}^{i+1} - \mathcal{Q}_{\mathcal{K}} \star X^i||_F^2 + \frac{\mu_2}{2}||\mathcal{K}^{i,j} - \mathcal{Q}_{\mathcal{K}} + \frac{\mathcal{Z}_{\mathcal{K}}^j}{\mu_2}||_F^2$$

$$[\mathcal{Q}_{\mathcal{K}_t} \star X](:) = A\mathcal{Q}_{\mathcal{K}_t}(:)$$

A is circulant

- $$\begin{aligned}
\mathcal{K}^{i,j+1} &= \arg\min_{\mathcal{K}\geq 0, \mathbf{1}^T\mathcal{K}_t(:)=1} \ \frac{\beta}{2}||\mathcal{K}||_F^2 + \frac{\rho_2}{2}||\mathcal{K} - \mathcal{K}^i||_F^2 + \frac{\mu_2}{2}||\mathcal{K} - \mathcal{Q}_{\mathcal{K}}^{j+1} + \frac{\mathcal{Z}_{\mathcal{K}}^j}{\mu_2}||_F^2 \\
&= \arg\min_{\mathcal{K}\geq 0, \mathbf{1}^T\mathcal{K}_t(:)=1} \ ||\mathcal{K} - \frac{1}{\beta + \rho_2 + \mu_2}(\rho_2\mathcal{K}^i + \mu_2\mathcal{Q}_{\mathcal{K}}^{j+1} - \mathcal{Z}_{\mathcal{K}}^j)||_F^2 \\
&= \arg\min_{\mathcal{K}\geq 0, \mathbf{1}^T\mathcal{K}_t(:)=1} \ ||\mathcal{K} - \mathcal{W}||_F^2
\end{aligned}$$

Wang, W., & Carreira-Perpinán, M. A. (2013). Projection onto the probability simplex: An efficient algorithm with a simple proof, and an application. *arXiv preprint arXiv:1309.1541*.

Solving X

$$
\begin{aligned}
X^{i+1} &= \arg\min_{X} F(\mathcal{Y}^{i+1}, \mathcal{K}^{i+1}, X) + \frac{\rho_3}{2}||X - X^i||_F^2 \\
&= \arg\min_{0 \le X \le 1} \frac{1}{2}||\mathcal{Y}^{i+1} - \mathcal{K}^{i+1} \star X||_F^2 + \gamma(||D_1 X||_1 + ||X D_2^T||_1) \\
&\quad + \frac{\rho_3}{2}||X - X^i||_F^2
\end{aligned}
$$

is equivalent to

$$
\arg\min_{0 \le X \le 1} \frac{1}{2}||\mathcal{Y}^{i+1} - \mathcal{K}^{i+1} \star Q_X||_F^2 + \gamma(||Q_1||_1 + ||Q_2||_1) + \frac{\rho_3}{2}||X - X^i||_F^2
$$

$$
s.t. \quad Q_X = X, \;\; Q_1 = D_1 Q_X, \;\; Q_2 = Q_X D_2^T, \;\; -1 \le Q_1, Q_2 \le 1
$$

$$
\begin{aligned}
L_X = {} & \frac{1}{2}||\mathcal{Y}^{i+1} - \mathcal{K}^{i+1} \star Q_X||_F^2 + \gamma(||Q_1||_{l_1} + ||Q_2||_{l_1}) + \frac{\rho_3}{2}||X - X^i||_F^2 \\
& + \langle Z_X, X - Q_X \rangle + \langle Z_1, D_1 Q_X - Q_1 \rangle + \langle Z_2, Q_X D_2^T - Q_2 \rangle \\
& + \frac{\mu_3}{2}||X - Q_X||_F^2 + \frac{\mu_3}{2}||D_1 Q_X - Q_1||_F^2 + \frac{\mu_3}{2}||Q_X D_2^T - Q_2||_F^2
\end{aligned}
$$

Solving X

$$Q_1^{j+1} = \underset{-1\le Q_1\le 1}{\arg\min}\ \gamma\|Q_1\|_{l_1} + \frac{\mu_3}{2}\|D_1 Q_X^j - Q_1 + \frac{Z_1^j}{\mu_3}\|_F^2$$

$$Q_2^{j+1} = \underset{-1\le Q_2\le 1}{\arg\min}\ \gamma\|Q_2\|_{l_1} + \frac{\mu_3}{2}\|Q_X^j D_2^T - Q_2 + \frac{Z_2^j}{\mu_3}\|_F^2$$

$$\begin{aligned} Q_X^{j+1} = \underset{Q_X}{\arg\min}\ \frac{1}{2}\|\mathcal{Y}^{i+1} - \mathcal{K}^{i+1} \star Q_X\|_F^2 + \frac{\mu_3}{2}\Big[\|X^{i,j} - Q_X + \frac{Z_X^j}{\mu_3}\|_F^2 \\ + \|D_1 Q_X - Q_1^{j+1} + \frac{Z_1^j}{\mu_3}\|_F^2 + \|Q_X D_2^T - Q_2^{j+1} + \frac{Z_2^j}{\mu_3}\|_F^2\Big] \end{aligned}$$

$$X^{i,j+1} = \underset{0\le X\le 1}{\arg\min}\ \frac{\rho_3}{2}\|X - X^i\|_F^2 + \frac{\mu_3}{2}\|X - Q_X^{j+1} + \frac{Z_X^j}{\mu_3}\|_F^2$$

$$Z_1^{j+1} = Z_1^j + \mu_3(D_1 Q_X^{j+1} - Q_1^{j+1})$$

$$Z_2^{j+1} = Z_2^j + \mu_3(Q_X^{j+1} D_2^T - Q_2^{j+1})$$

$$Z_X^{j+1} = Z_X^j + \mu_3(X^{i,j+1} - Q_X^{j+1})$$

Solving X

- $$Q_X^{j+1} = \arg\min_{Q_X} \frac{1}{2}||\mathcal{Y}^{i+1} - \mathcal{K}^{i+1} \star Q_X||_F^2 + \frac{\mu_3}{2}\Big[||X^{i,j} - Q_X + \frac{Z_X^j}{\mu_3}||_F^2 + ||D_1 Q_X - Q_1^{j+1} + \frac{Z_1^j}{\mu_3}||_F^2 + ||Q_X D_2^T - Q_2^{j+1} + \frac{Z_2^j}{\mu_3}||_F^2\Big]$$

$$[\mathcal{K}^{i+1} \star Q_X](:) = AQ_X(:)$$

here, A is a circulant matrix.

$$D_1 = \begin{pmatrix} 1 & & & -1 \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{pmatrix}_{n_v \times n_v} \qquad D_2 = \begin{pmatrix} 1 & & & -1 \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{pmatrix}_{n_h \times n_h}$$

Numerical Experiments

- Φ: DCT, DFT matrix.
- The observation ratio is $\rho = \frac{|\Omega|}{n_v n_h n_b}$.
- Image size: 251×251.
- Insensitive parameters: $\rho_1 = \rho_3 = \rho_3 = 10^{-6}$.
- Gaussian kernels:

$$G(i,j) = \begin{cases} exp(-\frac{i^2+j^2}{2\sigma^2}), & |i| \leq k_v,\ |j| \leq k_h \\ 0, & otherwise. \end{cases}$$

- We adopt the following relative error as stopping criterion for each iteration step,

$$\max\{\frac{||\mathcal{Y}^{i+1}-\mathcal{Y}^i||_F^2}{||\mathcal{Y}^{i+1}||_F^2}, \frac{||\mathcal{K}^{i+1}-\mathcal{K}^i||_F^2}{||\mathcal{K}^{i+1}||_F^2}, \frac{||\mathcal{X}^{i+1}-\mathcal{X}^i||_F^2}{||\mathcal{X}^{i+1}||_F^2}\} < 5\times 10^{-7}$$

  The maximum number of PAM iteration for our model is set to 500.

- PSNR:

$$MSE = \frac{1}{n_v n_h n_b} \sum_{(i,j,t)} |\mathcal{Y}(i,j,t) - \hat{\mathcal{Y}}(i,j,t)|^2$$

$$PSNR = 10 \cdot \log_{10}(\frac{\max(\mathcal{Y})^2}{MSE})$$

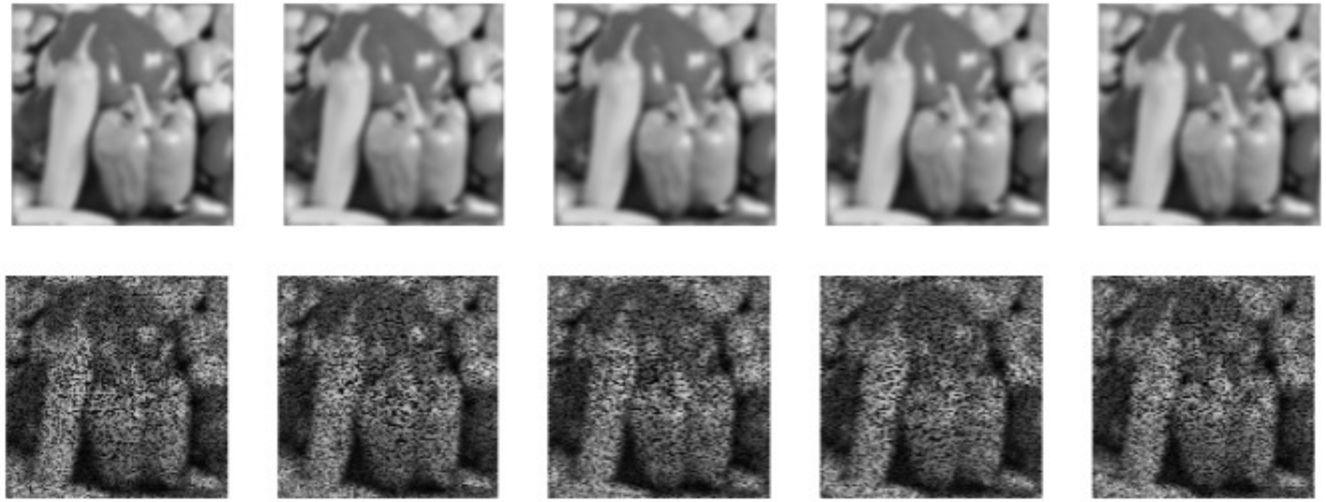
- It is a special case of our model that $n_b = k_b = 1$, thus we call it **one-dimension model**. Therefore, the model is

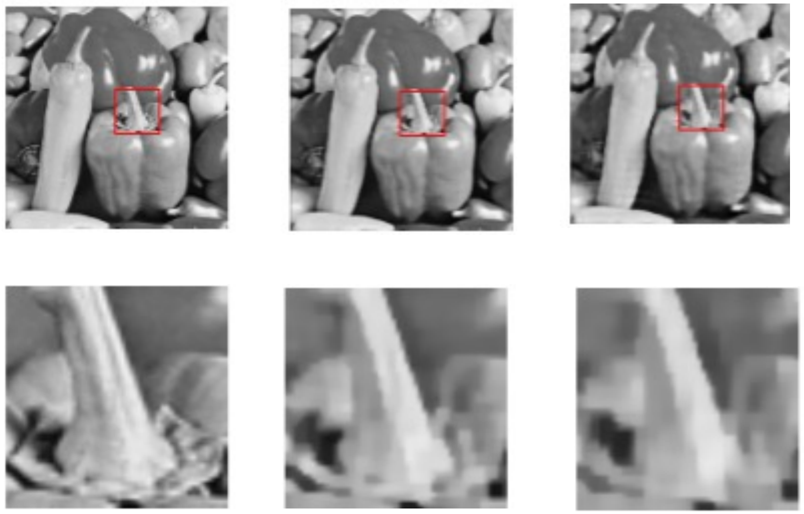
$$\arg\min_{Y,K,X} \frac{1}{2}||Y-K\star X||_F^2 + \alpha||Y||_* + \frac{\beta}{2}||K||_F^2 + \gamma(||D_1 X||_1 + ||X D_2^T||_1)$$

$$s.t.\quad 0 \leq X, Y \leq 1,\ P_\Omega(Y-M) = 0,\ K \geq 0,\ \sum_{i,j} K(i,j) = 1$$

Numerical Experiments

		PSNR of restored sharp images		
		DFT	DCT	one-dim
Pepper	$p = 0.6$	30.6261	**31.0788**	27.5730
	$p = 0.8$	31.0991	**31.3448**	28.5924
Dog	$p = 0.6$	34.2476	**34.4482**	31.5432
	$p = 0.8$	34.5492	**34.7374**	32.0961
Cameraman	$p = 0.6$	29.0374	**29.2544**	26.3951
	$p = 0.8$	29.3584	**29.4952**	27.0570
Barbara	$p = 0.6$	28.3122	**28.6385**	26.3631
	$p = 0.8$	29.0014	**29.0080**	27.3385

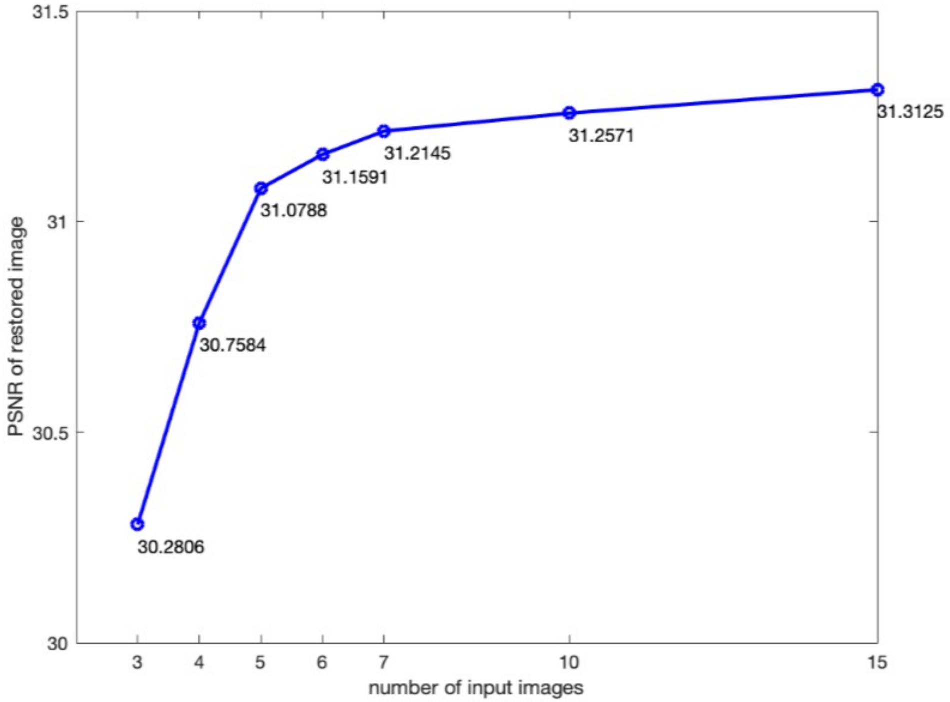




Numerical Experiments

- we fix the observation ratio $\rho = 0.6$ and the standard deviation of Gaussian kernels σ ranges from $[3, 5]$. The standard deviations are equally distributed in interval $[3, 5]$, i.e., if given $n_b = k_b$ blurred images, in matlab language, $\sigma \in \mathit{linspace}(3, 5, n_b)$, where

$$\sigma_1 = 3, \quad \sigma_{k_b} = 5, \quad \sigma_{i+1} - \sigma_i = \sigma_i - \sigma_{i-1}.$$



Numerical Experiments

In original mode, we set observation ratio $\rho = 0.6$ for each blurred image. For mode 1, the observation ratios are monotonically increasing, i.e.,

$$\rho = [\rho_1, ..., \rho_5] = [0.56, 0.58, 0.60, 0.62, 0.64],$$

where, ρ_i denotes the observation ratio of the i-th blurred image. For mode 2, the observation ratios are monotonically decreasing, i.e.,

$$\rho = [\rho_1, ..., \rho_5] = [0.64, 0.62, 0.60, 0.58, 0.56].$$

And for mode 3, the observation ratios are changing more complicated, for example,

$$\rho = [\rho_1, ..., \rho_5] = [0.60, 0.62, 0.58, 0.64, 0.56].$$

	original	mode 1	mode 2	mode 3
PSNR	31.0788	31.1078	31.1577	31.0544

Table: PSNR values of images in four observation modes

Thank you so much for listening!

Q & A