- **Article:** self-organized criticality
- **Authors:**Per Bak, Chao Tang and Kurt Wiesenfeld

1 Main idea

2 The sand pile model

Consider a one dimension model sand pile model. If we denote z_n as the difference of high between the point n and $n+1$. That is $z_n = h(n) - h(n+1)$.

If we add one sand at n. The result could be described as

$$z_n \to z_n + 1 \qquad z_{n-1} \to z_{n-1} - 1 \tag{2.0.1}$$

Let z_c be the threshold of z and if z_n exceeds the threshold, one unit of sand tumbles to a lower level:

$$z_n \to z_n - 2 \qquad z_{n\pm1} \to z_{n\pm1} + 1 \quad for \quad z_n > z_c \tag{2.0.2}$$

This model can be easily generated to higher dimension.

3 Argue that the system can involve into a critical state

We can approach "critical state" in two directions. One of them is that we can star from a far from equilibrium state. That's we can star from an initial state that all values of z at each point take an random value which is bigger than the threshold. At the beginning, a signal starts from any point will be able to propagate through the whole system. Also, it is not hard to imagine that as the system involves, some more-than-minimum stable points will show up. So when the distribution of those points are able to make some noise unable to propagate through the whole system, the system will stop involving. We could say that the system arrives at the critical state.

From my point of view, besides the critical state, there are another two domains that system can lie in. One is the "far from equilibrium phase" at which the noise origins from any points in the system can propagate through the whole system. Another is the "stable phase" at which the noise can not propagate through the whole system. Maybe we could say that the "critical state" is the boundary between this two phases. So noise originates from some points can only propagate in length that is much small than the size of the system while noise at some points can cause an avalanche that is comparable to the size of the whole system. In a word, by adding a unit of sand, the influence can happen at all length scales, from a shift of a single unit to and avalanche which can change the landscape of the whole system. The lack of length scale is equivalent to the lack of scale. Also, at the critical state, we may expect a power-law correction function.

4 Discussion of the sandpile model in different dimensions and different boundary conditions

4.1 1D

This is a trivial case. It is not hard to imagine that if we add sands randomly, at each point, the high difference will finally reach the threshold z_c. Not matter where you add another unit if sand, it will rumble along the chain and leave the system at its boundary.

4.2 2D with closed boundary

In closed boundary condition, we assume $z(0, y) = z(x, 0) = z(N + 1, y) = z(x, N + 1) = 0$. (Actually, I have to admit that I'm not yet been able to interpret what dose it mean by "colsed boundary").

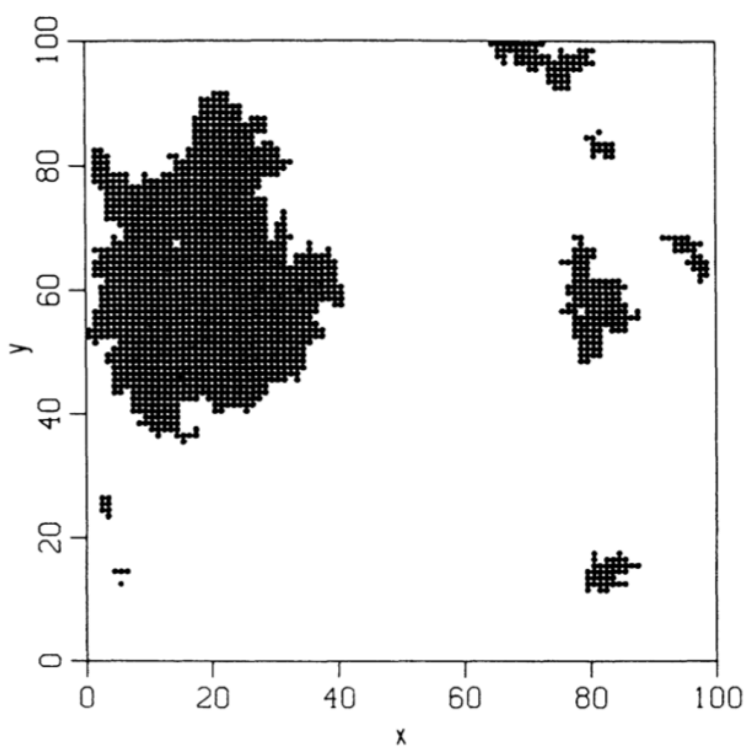

Figure 1: Local pertabations: the dark sites are domains affected by perturbing a single interior site.

By adding a unit of sand at different points of system when it has already reached the critical state, we find that as show in Fig 1, the domains of different sizes exists. We can further get the distribution of the size of the domains as showed in Fig 2. The log-log plot follows a straight line with slope -1.0. i.e.,

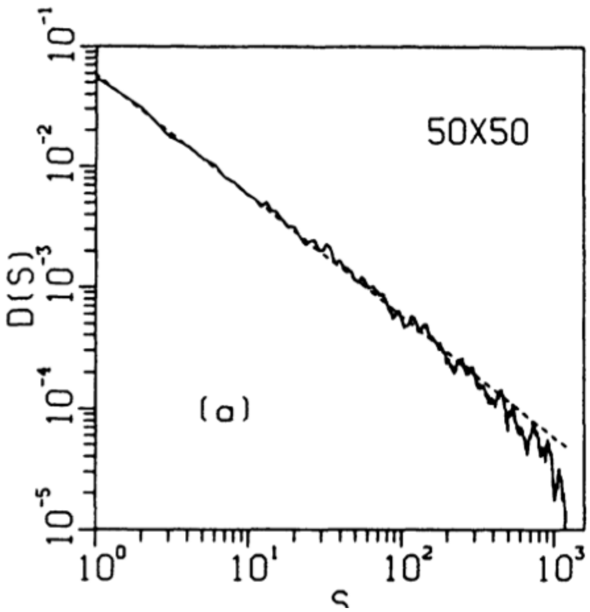

Figure 2:

$$D(s) \approx s^{-\tau} \quad \tau \approx 1.0 \tag{4.2.1}$$

When S is big, the deviation from a straight line is caused by a finite-size effect. Also, we could also find that the distribution of life time weighted by the average response s/T follows the similar rule:

$$D(T) \approx s^{-\tau} \quad \tau \approx 0.43 \tag{4.2.2}$$

Now we can pay attention of the energy dissipation process. We define $f(x,t)$ representing dissipation of energy at site x at time t. The total sliding at the t could be calculated by $F(t) = \int f(x,t)dx$. Also, the total cluster size $s = \int F(t)dt$. The power spectrum $S(f)$ of $F(t)$ is defined by:

$$S(f) = \int < F(t_0+t)F(t_0) > exp(2\pi i f t)dt \tag{4.2.3}$$

4.3 2D with open boundary