Incentive Compatibility

Helen Yi, Steven Huang

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1 Incentive Compatibility in LMSR

Chen and Pennock[1] stated that "incentive compatibility means that every agent's best strategy is to honestly report all of their information as soon as they have it". We are going to mathematically show that it is in the agent's best interest to make their belief to be as close to the true distribution as possible, and truthfully report this belief in the prediction market.

We will first look at LMSR. We assume that the original market belief is $\bar{r}$, the agent's reported belief is $\bar{r}'$, and the true probability distribution of the event outcomes is $\bar{p}$. The expected profit that the agent can get (here we assume that the agent only conducts one transaction in the market) is

$$E[\pi_{LMSR}] = \sum_{i=1}^{N} \bar{p_i} \log{(\frac{\bar{r_i}'}{\bar{r_i}})} \tag{1}$$

where N is the number of outcomes of the event. Two constraints is this case are

$$\sum_{i=1}^{N} \bar{r_i}' = 1 \tag{2}$$

$$\sum_{i=1}^{N} \bar{p_i} = 1 \tag{3}$$

Assuming that the agents in the market would like to gain as much profit as possible, we can use Lagrange multiplier to optimize the expected profit under the two contraints provided above. The dual formulation in this case is

$$\min_{\alpha} \max_{\bar{r}'} \quad \sum_{i=1}^{N} \bar{p_i} \log{(\frac{\bar{r_i}'}{\bar{r_i}})} + \alpha(\sum_{i=1}^{N} \bar{r_i}' - 1) \tag{4}$$

Let $f = \sum_{i=1}^{N} \bar{p_i} \log{(\frac{\bar{r_i}'}{\bar{r_i}})} + \alpha(\sum_{i=1}^{N} \bar{r_i}' - 1)$. If we take the gradient of f with respect to $\bar{r}'$ and set the gradient to zero,

$$\frac{\partial f}{\partial \bar{r_i'}} = \frac{\bar{p_i}}{\bar{r_i}'} + \alpha = 0$$

we can see that

$$\bar{p_i} = -\alpha \bar{r_i}' \tag{5}$$

Using the constraints Eq.2 and Eq.3, we get

$$\sum_{i=1}^{N} \bar{p}_i = -\alpha \sum_{i=1}^{N} \bar{r}_i{}' \implies \alpha = -1 \tag{6}$$

If we plug this result back in Eq.5, we can get

$$\bar{p}_i = \bar{r}_i{}' \tag{7}$$

This result shows that the profit of the agent would be maximized if the agent reports a probability distribution that is the same as the true distribution. Agents trading in LMSR market would therefore be incentivised to use their best judgement to come up with the true distribution of the event outcomes, and reveal this belief in order to maximize expected profit.

2 Incentive Compatibility in Cost Function Based Market (CFBM)

Since we have proved the equivalence between LMSR and a CFBM based on cost function $C(\bar{q}) = \log \sum_{j=1}^{m} e^{\bar{q}_j}$, we expect the latter is also incentive compatible. For clarity, we are going to prove that CFBM is also incentive compatible in its own setting. We again assume that the original market belief, indicated by the market price for each security, is $\bar{r}$, the agent's reported belief is $\bar{r}'$ and the true probability distribution of the event is $\bar{p}$. Here, the agent's belief is revealed via the number of securities he or she buys(sells) that corresponds to each outcome of the event, denoted by vector $\bar{c}$. If CFBM is incentive compatible, an agent will carefully choose a $\bar{c}$ such that the new market price for each security will be updated from $\bar{r}$ to $\bar{r}'$

The expected profit in CFBM is

$$\sum_{i=1}^{N} \bar{p}_i(\bar{c}_i - (C(\bar{q} + \bar{c}) - C(\bar{q}))) \tag{8}$$

where $\bar{q}$ is the original number of outstanding shares (number of each security that has been sold) when the agent enters the market. Let $F = \sum_{i=1}^{N} \bar{p}_i(\bar{c}_i - (C(\bar{q} + \bar{c}) - C(\bar{q})))$. If we plug in the LMSR corresponding cost function.

$$F = \sum_{i=1}^{N} \bar{p}_i \left[\bar{c}_i - \log \sum_{j=1}^{N} e^{\bar{q}_j + \bar{c}_j} + \log \sum_{j=1}^{N} e^{\bar{q}_j} \right] \tag{9}$$

In order to maximize this expected profit, we just need to take the gradient of F with respect to $\bar{c}$, and set the gradient to $\bar{0}$ to solve for the optimal $\bar{c}$. We will start with taking the partial derivative with respect to $\bar{c}_i$ first.

$$\frac{\partial F}{\partial \bar{c}_i} = \bar{p}_i - \frac{\bar{p}_i}{\sum_{j=1}^{N} e^{\bar{q}_j + \bar{c}_j}} e^{\bar{q}_i + \bar{c}_i} + \sum_{j \neq i}^{N} (-\bar{p}_j \frac{e^{\bar{q}_i + \bar{c}_i}}{\sum_{t=1}^{N} e^{\bar{q}_t + \bar{c}_t}}) = 0 \tag{10}$$

$$\bar{p}_i = \frac{\bar{p}_i}{\sum_{j=1}^{N} e^{\bar{q}_j + \bar{c}_j}} e^{\bar{q}_i + \bar{c}_i} - \sum_{j \neq i}^{N} (-\bar{p}_j \frac{e^{\bar{q}_i + \bar{c}_i}}{\sum_{t=1}^{N} e^{\bar{q}_t + \bar{c}_t}})$$

$$\bar{p_i} = (\bar{p_i} + \sum_{j \neq i}^{N} \bar{p_j})(\frac{e^{\bar{q_i}+\bar{c_i}}}{\sum_{t=1}^{N} e^{\bar{q_t}+\bar{c_t}}}) = \frac{e^{\bar{q_i}+\bar{c_i}}}{\sum_{t=1}^{N} e^{\bar{q_t}+\bar{c_t}}}$$

$$\bar{p_i} = \frac{e^{\bar{q_i}+\bar{c_i}}}{\sum_{t=1}^{N} e^{\bar{q_t}+\bar{c_t}}}$$

Notice the right side of the equation above the first order derivative of our selected cost function, which denotes the market price of each security. This result shows that the profit of the agent would be maximized if the agent can carefully select a $\bar{c}$ such that the new market prices of the securities to be the same of the probabilities of the outcomes. Agents trading in CFBM would therefore be incentivised to use their best judgement to come up with the true distribution of the event outcomes, and revealing it by buying(selling) shares $\bar{c}$ to push the market price to the agent's belief.

3 Budget Constraints of Agents in LMSR and CFBM

Suppose we are only considering agents in LMSR since LMSR and CFBM have been proven to be equivalent.

3.1 Budget Matters Toy Example

Consider the scenario where the market is trying to predict the weather of tomorrow, and the weather can be one of the following

1. rainy
2. sunny

A risk neutral agent Bob enters the market with budget b. Suppose the market's initial belief of whether the weather tomorrow is going to be sunny or rainy is $\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix}$ (where the first entry is the probability of tomorrow being sunny, and the second entry is the probability of tomorrow being rainy). Bob has the belief that the probability of tomorrow being sunny is $\frac{1}{3}$, and the probability of tomorrow being rainy is $\frac{2}{3}$. Suppose Bob proceeds to push market belief from $\begin{bmatrix}\frac{1}{2}\frac{1}{2}\end{bmatrix}$ to $\begin{bmatrix}\frac{1}{3}\frac{2}{3}\end{bmatrix}$

The amount of money Bob stands to gain if tomorrow is rainy is

$$\pi_{rainy} = -\log\frac{1}{2} + \log\frac{1}{3} = \log\frac{2}{3} < 0 \tag{11}$$

The amount of money Bob stands to gain if tomorrow is sunny is

$$\pi_{sunny} = -\log\frac{1}{2} + \log\frac{2}{3} = \log\frac{4}{3} > 0 \tag{12}$$

Assume Bob's budget b is less than $-\log\frac{2}{3} = \log\frac{3}{2}$, then Bob would not be able to pay the market. Therefore we know that Bob's ability to push the market's belief is limited by his budget.

3.2 Incentive Compatibility with Budget Constraint in LMSR

Suppose there is an event with only two outcomes. There are three types of probabilities, the **true** probability distribution of the event, the probability of the market **before** a certain agent enters the market and the probability of the market **after** a certain agent enters the market.

1. **True Probability:** The true probability of outcome 1 occurring is p_1, consequently, the true probability of outcome 2 occurring is $1 - p_1$.

2. **Probability Before:** The market belief of outcome 1 occurring before a certain agent is denoted by r_1, consequently, the market maker's belief about the probability of outcome 2 occurring before a certain agent is denoted by $1 - r_1$.

3. **Probability After:** The market belief of outcome 1 occurring after a certain agent is denoted by r'_1, consequently, the market maker's belief about the probability of outcome 2 occurring after a certain agent is denoted by $1 - r'_1$.

In order to maximize expected profit, the objective function that needs to be maximized here is

$$p_1 \log \frac{r'_1}{r_1} + (1 - p_1) \log \left(\frac{1 - r'_1}{1 - r_1}\right) \tag{13}$$

Since r_1 is a known value, we can simplify our objective function to

$$\mathcal{O} = p_1 \log r'_1 + (1 - p_1) \log (1 - r'_1) \tag{14}$$

Now, we are going to consider budget limits in this case. Specifically, we need to add the following constraints.

$$b + \log \frac{r'_1}{r_1} \geq 0$$
$$b + \log \frac{1 - r'_1}{1 - r_1} \geq 0$$

With further simplification, we can get

$$\frac{r_1}{e^b} \leq r'_1 \leq 1 - \frac{1}{e^b} + \frac{r_1}{e^b} \tag{15}$$

The shape of the objective function is shown below

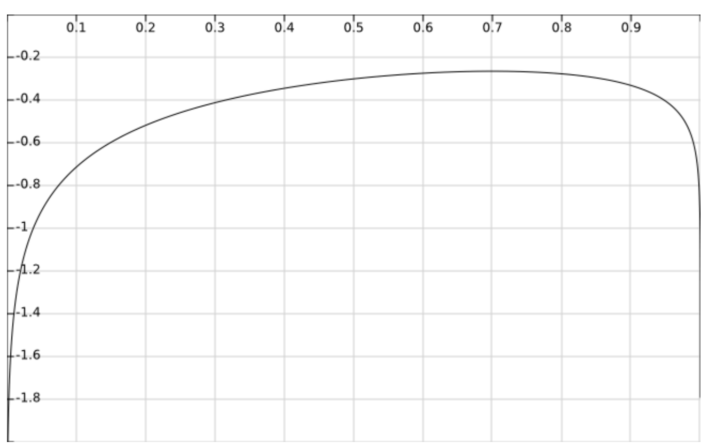

Figure 1: $\mathcal{O}$ with $p_1 = 0.7$

We see a convex shape of the objective function. This result can be backed up by taking the derivative of $\mathcal{O}$ with respect to r_1'. When $r_1' = p_1$, the derivative is equal to 0, when $r_1' < p_1$, the derivative is greater than 0, and when $r_1' > p_1$, the derivative is less than 0.
Therefore we can get the following conclusion

$$r_1' = \begin{cases} 1 - \frac{1}{e^b} + \frac{r_1}{e^b} & 1 - \frac{1}{e^b} + \frac{r_1}{e^b} < p_1 \\ p_1 & \frac{r_1}{e^b} \leq p_1 \leq 1 - \frac{1}{e^b} + \frac{r_1}{e^b} \\ \frac{r_1}{e^b} & p_1 < \frac{r_1}{e^b} \end{cases} \tag{16}$$

3.3 Back to Toy Example

Suppose Bob has $\log \frac{4}{3}$ on his hand. How should he spend the money to maximize profit?
In this case we can see that Bob still needs to have a good idea of what p_1 should be in order to make the best decision. Suppose Bob has made the perfect prediction which is exactly the same as the true prediction (the true distribution is $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}$). With the values given in section 3.1, we can compute the following values

$$1 - \frac{1}{e^b} + \frac{r_1}{e^b} = 1 - \frac{1}{\frac{4}{3}} + \frac{\frac{1}{2}}{\frac{4}{3}} = \frac{5}{8}$$
$$\frac{r_1}{e^b} = \frac{\frac{1}{2}}{\frac{4}{3}} = \frac{3}{8}$$

With Eq.16, we can see that it is in Bob's best interest to push the market's belief to $\begin{bmatrix} \frac{3}{8} & \frac{5}{8} \end{bmatrix}$. The amount of money Bob stands to gain if tomorrow is rainy is

$$\pi_{rainy} = -\log \frac{1}{2} + \log \frac{3}{8} = \log \frac{3}{4} < 0 \tag{17}$$

The amount of money Bob stands to gain if tomorrow is sunny is

$$\pi_{sunny} = -\log \frac{1}{2} + \log \frac{5}{8} = \log \frac{5}{4} > 0 \tag{18}$$

We can see that the amount of money Bob stands to lose is indeed smaller (compare Eq.11 and Eq.17), but the money he can potentially gain as profit is also lesser (compare Eq.12 and Eq.18).

References

[1] Yiling Chen, David M. Pennock, *Designing Markets For Prediction*. AI Magazine 31(4):42-52.