Formulation of the problem:

$$H(X) = -\int_{-\infty}^{\infty} p(x) \ln p(x)\,dx$$ ← Maximize this

Constraints: $\int p(x)\,dx = 1$, $\int p(x)\,x\,dx = \mu$,

$$\int p(x)(x-\mu)^2 dx = \sigma^2$$

$$L(p, \alpha_1, \alpha_2, \alpha_3) = \int p(x) \ln p(x)\,dx + \alpha_1\left(1 - \int p(x)\,dx\right)$$
$$+ \alpha_2\left(\mu - \int p(x)\,x\,dx\right) + \alpha_3\left(\sigma^2 - \int p(x)(x-\mu)^2 dx\right)$$

$$\alpha_1, \alpha_2, \alpha_3 \geq 0$$

$$\underline{\underline{\min_{p} \max_{\alpha_1,\alpha_2,\alpha_3} L(p, \alpha_1, \alpha_2, \alpha_3)}}$$ ← objective function

Why can we use dual formulation here?

$$\frac{\partial L}{\partial p} = p \cdot \frac{1}{p} + \ln p - \alpha_1 - \alpha_2 x - \alpha_3 (x-\mu)^2 = 0$$
$$= 1 + \ln p - \alpha_1 - \alpha_2 x - \alpha_3 (x^2 - 2\mu x + \mu^2) = 0$$

$$\ln p = -1 + \alpha_1 + \alpha_2 x + \alpha_3 (x-\mu)^2$$

$$p = \exp(-1+\alpha_1+\alpha_2 x+\alpha_3(x-\mu)^2)$$

$$= \exp(-1+\alpha_1+\alpha_2 x+\alpha_3(x^2-2\mu x+\mu^2))$$

$$= \exp[\alpha_3 x^2 + (\alpha_2 - 2\mu\alpha_3)x + (\alpha_3\mu^2+\alpha_1-1)]$$

$$\int p\,dx = 1$$

$$\boxed{\int p x\,dx = \mu}$$

$$\boxed{\mu\int p\,dx = \mu}$$

$$\alpha_3(x-\mu+k)^2$$

$$= \alpha_3(x^2+\mu^2+k^2-2\mu x-2k\mu+2kx)$$

$$= \alpha_3 x^2 + \alpha_3\mu^2 + \alpha_3 k^2 - 2\mu\alpha_3 x - 2k\alpha_3\mu + 2k\alpha_3 x$$

$$= \alpha_3 x^2 + \alpha_3\mu^2 + (2k\alpha_3 - 2\mu\alpha_3)x + \alpha_3 k^2 - 2k\alpha_3\mu$$

$$\alpha_2 = 2k\alpha_3 \Longrightarrow k = \frac{\alpha_2}{2\alpha_3}$$

$$p = \exp\left[\alpha_3\left(x-\mu+\frac{\alpha_2}{2\alpha_3}\right)^2 - \frac{\alpha_2^2}{4\alpha_3} + \alpha_2\mu + \alpha_1 - 1\right]$$

$$y = x-\mu+k \Longrightarrow x = y+\mu-k = y+\mu-\frac{\alpha_2}{2\alpha_3}$$

$$p = \exp\left[\alpha_3 y^2 - \frac{\alpha_2^2}{4\alpha_3} + \alpha_2\mu + \alpha_1 - 1\right]$$

$$\int p x\,dx = \mu$$

$$\int p\, x\, dx = \mu$$

$$\int_{-\infty}^{\infty} \exp\left[\alpha_3 y^2 - \frac{\alpha_2^2}{4\alpha_3} + \alpha_2 \mu + \alpha_1 - 1\right] (y + \mu - k)\, dy = \mu$$

odd function

$\int p\, \mu\, dy = \mu$

$$\Rightarrow k = 0 \Rightarrow \alpha_2 = 0$$

$$P = \exp\left[\alpha_3 x^2 - 2\mu\alpha_3 x + \alpha_3 \mu^2 + \alpha_1 - 1\right]$$

$$= \exp\left[\alpha_3 (x-\mu)^2 + (\alpha_1 - 1)\right] \quad \text{———} \quad (*)$$

$$\int p(x)\, dx = 1$$

$$\int \exp\left[\alpha_3 (x-\mu)^2 + (\alpha_1 - 1)\right] dx = 1$$

$$\int \exp(\alpha_3 (x-\mu)^2) \cdot \exp(\alpha_1 - 1)\, dx = 1$$

$$\int \exp(\alpha_3 (x-\mu)^2)\, dx = \exp(1 - \alpha_1)$$

According to Gaussian integral

$$\int_{-\infty}^{\infty} \exp(-a(x+b)^2)\, dx = \sqrt{\frac{\pi}{a}} \qquad a > 0$$

$$\sqrt{\frac{\pi}{-\alpha_3}} = \exp(1 - \alpha_1) \quad \text{———} \quad ①$$

$$\int p(x)(x-\mu)^2 dx = \sigma^2$$

$$\int \exp[\alpha_3(x-\mu)^2 + (\alpha_1-1)](x-\mu)^2 dx = \sigma^2$$

$$z = x-\mu, \quad dz = dx$$

$$\int \exp[\alpha_3 z^2 + (\alpha_1-1)] z^2 dz = \sigma^2$$

$$\int \exp[\alpha_3 z^2] z^2 dz = \sigma^2 \cdot \exp(1-\alpha_1)$$

$$\frac{1}{2}\sqrt{\frac{\pi}{(-\alpha_3)^3}} = \sigma^2 \exp(1-\alpha_1) \quad ②$$

With ①② $$\frac{1}{2}\sqrt{\frac{\pi}{(-\alpha_3)^3}} = \sigma^2 \sqrt{\frac{\pi}{-\alpha_3}}$$

$$\sqrt{\frac{1}{(-\alpha_3)^3}} = 2\sigma^2 \sqrt{\frac{1}{-\alpha_3}}$$

$$1 = 2\sigma^2 \sqrt{\frac{1}{-\alpha_3}} \cdot \sqrt{(-\alpha_3)^3}$$

$$1 = 2\sigma^2 \sqrt{(-\alpha_3)^2} = 2\sigma^2 |-\alpha_3| = -2\sigma^2\alpha_3$$

$$\Rightarrow \alpha_3 = \frac{1}{-2\sigma^2}$$

Hence, $$\sqrt{\frac{\pi}{\frac{1}{2\sigma^2}}} = \exp(1-\alpha_1) \Rightarrow \sqrt{2\sigma^2\pi} = \exp(1-\alpha_1)$$

$$1-\alpha_1 = \ln(\sqrt{2\sigma^2\pi})$$

$$\alpha_1 = 1 - \ln\left(\sqrt{2\sigma^2\pi}\right)$$

Plug all of these back into (*)

$$p = \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2 - \ln\left(\sqrt{2\sigma^2\pi}\right)\right]$$

$$= \frac{\exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)}{\sqrt{2\sigma^2\pi}}$$

This is Gaussian distribution!