Bayesian Synthetic Control with a Soft Simplex Constraint

Yihong Xu[1]   Quan Zhou[2]

[1]Dept. of Economics, Texas A&M University
[2]Dept. of Statistics, Texas A&M University

2025 Southern Economic Association

Nov 22, 2025

Overview

- The Synthetic Control Method (SCM) has at least two limitations;

  - High-dimensional problem: the number of control units needs to be much smaller than the total observed time periods before the intervention;

  - Debate over the simplex constraint: all coefficients are restricted to be non-negative and sum-to-one.

- We propose a novel extension of SCM, using Bayesian variable selection (spike-and-slab prior) with a soft simplex constraint, BVS-SS;

  - Contribution **1**: BVS-SS solves the high dimension problem while still preserves the simplex constraint;

  - Contribution **2**: BVS-SS introduces a notion of "**soft**" simplex constraint, asking data's advice on whether the constraint is proper.

SCM Setup

- T_0 is # of pre-treatment periods; T_1 is # of post-treatment periods;

- $X_{1,t}$ is the outcome of the treated unit; $X_{0,t}$ is the $N \times 1$ outcome vector of the control units;

$$\hat{w} = \operatorname*{argmin}_{w \in \Delta^{N-1}} \frac{1}{T_0} \sum_{t=1}^{T_0} \left(X_{1,t} - X_{0,t}^\top w \right)^2, \tag{1}$$

$$\Delta^{N-1} = \left\{ u \in \mathbb{R}^N \colon u_i \geq 0 \text{ for each } i, \text{ and } \sum_{i=1}^{N} u_i = 1 \right\}$$

where Δ^{N-1} denotes the $(N-1)$-simplex.

- The average treatment on the treated:

$$\widehat{\mathrm{ATT}} = \frac{1}{T_1} \sum_{t=T_0+1}^{T} \left(X_{1,t} - X_{0,t}^\top \hat{w} \right)$$

Challenges

- How to solve the optimization (1) when X_0 is high-dimensional ($N \gg T_0$)?

  - Panel data approach: Elastic net (Doudchenko and Imbens, 2016), Lasso (Carvalho et al., 2018; Hollingsworth and Wing, 2020), forward selection (Shi and Huang, 2023), spike-and-slab (Kim et al., 2020) etc.;

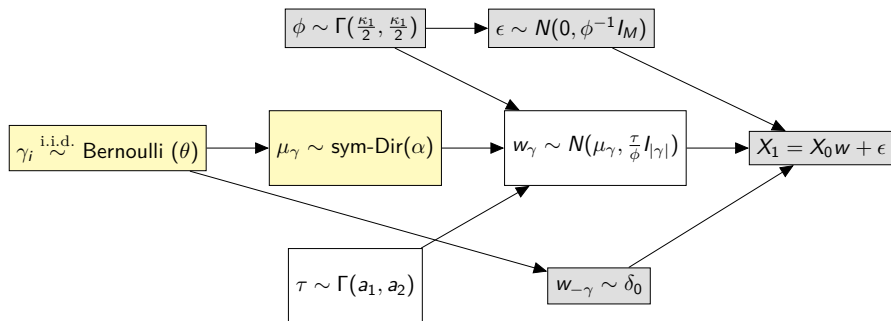 - Trade-off: dropping the simplex constraint;

Challenges

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  - Trade-off: dropping the simplex constraint;

- Is the simplex constraint necessary?

  - Pros: no hedging (e.g., $w_1 = 300\%$ and $w_2 = -250\%$); the counterfactual has economic interpretation;

  - Cons: too rigid (may have bad fitting); high-dimensional problem.

Summary

We propose a novel Bayesian SCM framework, based on Bayesian variable selection with a soft simplex constraint(BVS-SS). It

1. deals with the high-dimensional control group, while still keeping the simplex constraint;

Summary

Remarks:

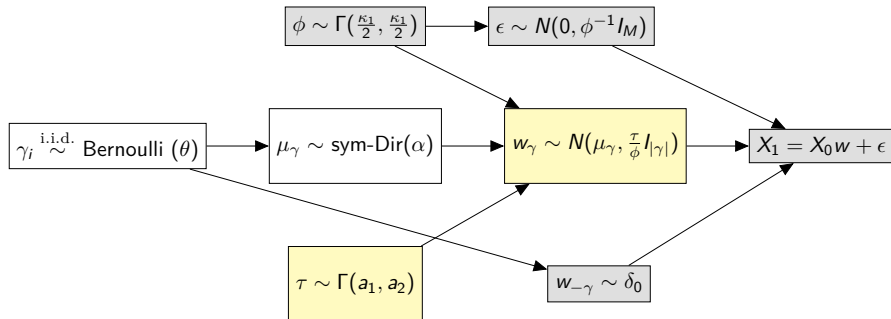- A small θ penalizes the large models;
- Main diff from spike-and-slab: $\mu_\gamma \in \Delta^{|\gamma|-1}$;

Summary

We propose a novel Bayesian SCM framework, based on Bayesian variable selection with a soft simplex constraint(BVS-SS). It

1. deals with the high-dimensional control group, while still keeping the simplex constraint;

2. learns the attitude of data on the constraint case by case.

Summary

$$\phi \sim \Gamma(\tfrac{\kappa_1}{2}, \tfrac{\kappa_1}{2}) \longrightarrow \epsilon \sim N(0, \phi^{-1}I_M)$$

$$\gamma_i \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(\theta) \longrightarrow \mu_\gamma \sim \text{sym-Dir}(\alpha) \longrightarrow w_\gamma \sim N(\mu_\gamma, \tfrac{\tau}{\phi}I_{|\gamma|}) \longrightarrow X_1 = X_0 w + \epsilon$$

$$\tau \sim \Gamma(a_1, a_2) \qquad w_{-\gamma} \sim \delta_0$$

Remark:

- Variance τ: quantifying the deviation from simplex (mean μ_γ).

Hindrance

- Once we have samples from the posterior $p(\mu_\gamma, \gamma, \tau, \phi \mid X_1)$ ▸ Details , we can predict the counterfactual by

$$\mathbb{E}_p[\tilde{X}_0 w] = \mathbb{E}_p\left[\mathbb{E}_p(\tilde{X}_0 w \mid \mu_\gamma, \gamma, \tau, \phi)\right]; \; \widehat{\mathrm{ATT}} = \frac{1}{\tilde{T}} \sum_{t=1}^{\tilde{T}} \left(\tilde{X}_{1,t} - \mathbb{E}_p[\tilde{X}_0 w]_t\right).$$

- How to generate the samples from the posterior distribution $p(\mu_\gamma, \gamma, \tau, \phi \mid X_1)$? The simplex constraint makes a Gibbs sampling non-trivial;

- What is the (non-asymptotic) property of the posterior distribution $p(\mu_\gamma, \gamma, \tau, \phi \mid X_1)$?

A Gibbs Scheme with the Simplex Constraint

How to generate the samples from the posterior distribution $p(\mu_\gamma, \gamma, \tau, \phi \mid X_1)$?

- Gibbs sampling: Fix μ_{-j}, update μ_j according to $p(\mu_j \mid Y, \mu_{-j}, \tau, \phi)$;
- $\mu_j = 1 - \sum_{i \neq j} \mu_i$ almost surely; $p(\mu_j \mid Y, \mu_{-j}, \tau, \phi)$ is degenerate;
- Standard methods like Geyer (1992); George and McCulloch (1993); Gilks et al. (1995); George and McCulloch (1997); Brooks (1998); Carlo (2004) don't work.

Update μ_i, μ_j (with $i \neq j$) simultaneously from the full conditional posterior $p(\mu_i, \mu_j \mid X_1, \mu_{-(i,j)}, \tau, \phi)$.

1. Selection based on $p(\gamma_i, \gamma_j \mid X_1, \mu_{-(i,j)}, \tau, \phi)$;

2. Sampling based on $p(\mu_i, \mu_j \mid X_1, \gamma_i, \gamma_j, \mu_{-(i,j)}, \tau, \phi)$

▸ Details ⟶

Non-asymptotic Bounds ($\tau = 0$)

Fix $\tau = 0$. The true model strictly satisfies the simplex constraint. We show the convergence rate of the posterior probability of selecting the true model γ^*.

Theorem 1

Under mild conditions, it is true that

$$p(\gamma^* \mid Y) \geq 1 - c_3 N^{-1},$$

with probability at least $1 - c_1 N^{-c_2 L}$.

Corollary 1

Under the setting of Theorem 1,

$$\mathbb{E}\left[\|\mu - \mu^*\|_2^2 \mid Y\right] = O_p\left(C_4 \frac{L \log N}{T_0}\right).$$

More Results

- We also show the property when the simplex constraint is substantially violated ($\tau \to \infty$);

- Several simulation studies show the advantages of BVS-SS;

- Revisit an empirical example, with 87 control units but only 35 pre-intervention periods. ▸ Details →

Concluding Remarks

- We construct a novel Bayesian synthetic control framework to solve the high dimension issue (BVS-SS);
  - The posterior of the weight shows an empirical attitude on the simplex constraint;
  - Li (2020); Chernozhukov et al. (2021); Martinez and Vives-i-Bastida (2022); Goh and Yu (2022);

- We propose a novel Markov chain Monte Carlo (MCMC) sampler for our BVS-SS posterior;
  - to select the "proper" control units and estimate the weights and counterfactuals through our posterior;
  - Geyer (1992); George and McCulloch (1993); Gilks et al. (1995); George and McCulloch (1997); Brooks (1998); Carlo (2004);

- We derive non-asymptotic bounds for our Bayesian estimations;
  - on control units selection and posterior mean of coeffs/weights;
  - George and McCulloch (1993, 1997); Brown et al. (1998); Li and Li (2008); Casella et al. (2009); Choi et al. (2010); Farcomeni (2010); Guan and Stephens (2011); Yang et al. (2016); Zhou and Guan (2019).

Thank You!

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Motivation 1: High Dimensional X (Contd)

To see why no frequentist implementations with simplex, consider the standard Lasso regression (Tibshirani, 1996).

$$\hat{w} = \underset{w}{\operatorname{argmin}} \frac{1}{T} \sum_{t=1}^{T} \left(X_{1,t} - X_{0,t}^{\top} w \right)^2 \ \text{ such that } \|w\|_1 \leq \lambda$$

where λ determines the degree of regularization.

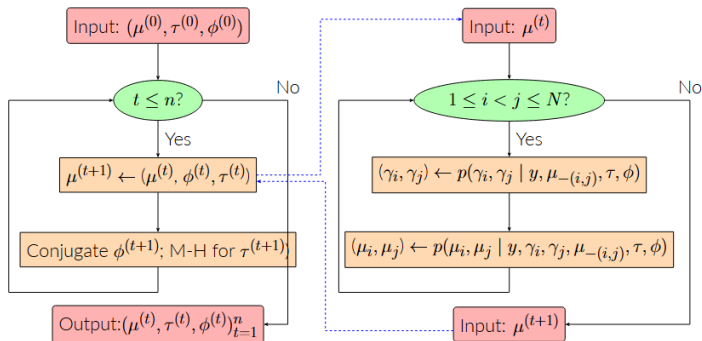The constraint in Lasso contradicts the simplex constraint Δ^{N-1}!

▸ Back ←

Posterior and ATT

A routine calculation using Woodbury identity yields the marginal likelihood of (μ, τ, ϕ):

$$p(X_1 \mid \mu, \tau, \phi) \propto \frac{\phi^{T/2}}{\tau^{|\gamma|/2}\det(V_{\gamma,\tau})^{1/2}} \exp\left\{-\frac{\phi}{2}(X_1 - X_{0,\gamma}\mu_\gamma)^\top \Sigma_{\gamma,\tau}(X_1 - X_{0,\gamma}\mu_\gamma)\right\},$$

where

$$V_{\gamma,\tau} = X_{0,\gamma}^\top X_{0,\gamma} + \tau^{-1}I, \quad \Sigma_{\gamma,\tau} = I - X_{0,\gamma}V_{\gamma,\tau}^{-1}X_{0,\gamma}^\top.$$

w's conditional posterior:

$$w \mid X_1, \mu, \tau, \phi \sim N\left(V_{\gamma,\tau}^{-1}(X_{0,\gamma}^\top X_1 + \tau^{-1}\mu_\gamma),\ \phi^{-1}V_{\gamma,\tau}^{-1}\right).$$

▶ Back ←

Metropolis-within-Gibbs

- When $\alpha = 1$, γ's conditional posterior can be expressed by CDF of standard normals; given $\gamma_i = \gamma_j = 1$, μ_i and μ_j's marginal posteriors degenerate to a **truncated Gaussian** distribution.

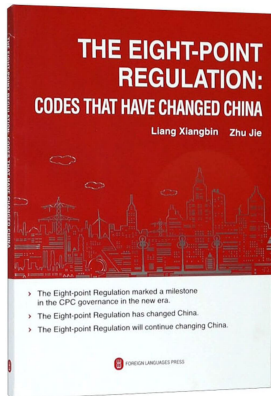▸ Back ←

China's anti-corruption campaign

Journal of Econometrics
Volume 234, Issue 2, June 2023, Pages S12-S35

Forward-selected panel data approach for program evaluation

Zhentao Shi [a], Jingyi Huang [b]

THE EIGHT-POINT REGULATION: CODES THAT HAVE CHANGED CHINA

Liang Xiangbin   Zhu Jie

> The Eight-point Regulation marked a milestone in the CPC governance in the new era.
> The Eight-point Regulation has changed China.
> The Eight-point Regulation will continue changing China.

FOREIGN LANGUAGES PRESS

- Shi and Huang (2023): China anti-corruption campaign in 2012;

- X_1: monthly growth rate of luxury watch imports;

- Category: "watches with case of, or clad with, precious metal";

- X_0: monthly growth rate of other 87 commodity categories;

- Only $T_0 = 35$ pre-treatment periods.

China's anti-corruption campaign

Table: Anti-corruption Campaign's Impact on Luxury Watches Importation

	ATT	τ	ϕ	$|\gamma|$
Mean	-0.021	0.069	20.86	5.09
95% credible interval	(-0.032, -0.008)	(0, 0.641)	(12.22, 32.76)	(1, 20)

Remark: The ratio τ/ϕ is very small $\rightarrow$ empirical support on the validity of SCM type estimator.

Control Units Selection

- Shi and Huang (2023):
  1. knitted or crocheted fabric;
  2. cork and articles of cork;
  3. salt, sulfur, earth, stone, plaster, lime and cement.
- BVS-SS:
  1. knitted or crocheted fabric (51.4%);
  2. special woven fabrics (lace, tapestries, trimmings, embroidery) (16.6%);
  3. man-made filaments (14.8%);
  4. wool, fine or coarse animal hair (12%);
  5. other fabric/textile;
  6. optical, photographic, cinematographic, measuring, checking, medical or surgical instruments and apparatus; parts and accessories"(10.2%).

Remarks: seems a more reasonable selection.

Counterfactual Prediction

Figure: Significant impact right after the Eight-point regulation