Bayesian Synthetic Control with a Soft Simplex Constraint

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Overview

- The Synthetic Control Method (SCM) has at least two limitations;
  - High-dimensional problem: the number of control units needs to be much smaller than the total observed time periods before the intervention;
  - Debate over the simplex constraint: all coefficients are restricted to be non-negative and sum-to-one.
- We propose a novel extension of SCM, using Bayesian variable selection (spike-and-slab prior) with a soft simplex constraint, BVS-SS;
  - Contribution **1**: BVS-SS solves the high dimension problem while still preserves the simplex constraint;
  - Contribution **2**: BVS-SS introduces a notion of "**soft**" simplex constraint, asking data's advice on whether the constraint is proper.

SCM Setup

- T_0 is # of pre-treatment periods; T_1 is # of post-treatment periods;
- $X_{1,t}$ is the outcome of the treated unit; $X_{0,t}$ is the $N \times 1$ outcome vector of the control units;

$$\hat{w} = \underset{w \in \Delta^{N-1}}{\operatorname{argmin}} \frac{1}{T_0} \sum_{t=1}^{T_0} \left(X_{1,t} - X_{0,t}^{\top} w \right)^2, \tag{1}$$

$$\Delta^{N-1} = \left\{ u \in \mathbb{R}^N : \; u_i \geq 0 \text{ for each } i, \text{ and } \sum\nolimits_{i=1}^{N} u_i = 1 \right\}$$

where Δ^{N-1} denotes the $(N-1)$-simplex.

- The average treatment on the treated:

$$\widehat{\mathrm{ATT}} = \frac{1}{T_1} \sum_{t=T_0+1}^{T} \left(X_{1,t} - X_{0,t}^{\top} \hat{w} \right)$$

Challenges

- How to solve the optimization (1) when X_0 is high-dimensional ($N \gg T_0$)?
  - Panel data approach: Elastic net (Doudchenko and Imbens, 2016), Lasso (Carvalho et al., 2018; Hollingsworth and Wing, 2020), forward selection (Shi and Huang, 2023), spike-and-slab (Kim et al., 2020) etc.;
  - Trade-off: dropping the simplex constraint;

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  - Trade-off: dropping the simplex constraint;
- Is the simplex constraint necessary?
  - Pros: no hedging (e.g., $w_1 = 300\%$ and $w_2 = -250\%$); the counterfactual has economic interpretation;
  - Cons: too rigid (may have bad fitting); high-dimensional problem.

Motivations

We want to propose a novel SCM framework that

1. deals with the high-dimensional control group, while still keeping the simplex constraint (**NOT TRIVIAL!**);
2. learns the attitude of data on the constraint case by case.

An intuitive way is to make a "**soft**" constraint for the selected control units, that is, the expectation of the selected weights satisfy the simplex constraint.

Problem Setup

- Pre: outcome of the treated unit $X_1 \in \mathbb{R}^{T_0}$; Outcome of the control units $X_0 \in \mathbb{R}^{T_0 \times N}$ $(N \gg T_0)$;
- Post: $\tilde{X}_1 \in \mathbb{R}^{T_1}$; $\tilde{X}_0 \in \mathbb{R}^{T_0 \times N}$;
- Treatment: δ;

$$
\begin{aligned}
X_1 &= X_0 w + \epsilon; \\
\tilde{X}_1 &= \tilde{X}_0 w + \delta + \tilde{\epsilon}.
\end{aligned}
$$

Problem Setup

Suppose the true model of the counterfactual is,

$$X_1 = X_{0,\gamma^*} w_{\gamma^*} + \epsilon.$$

- $\gamma \in \{0,1\}^N$: $\gamma_i = 1$ if and only if $w_i \neq 0$;
- w_γ: the subvector of w with entries indexed by $\{i : \gamma_i = 1\}$;

Two concerns:

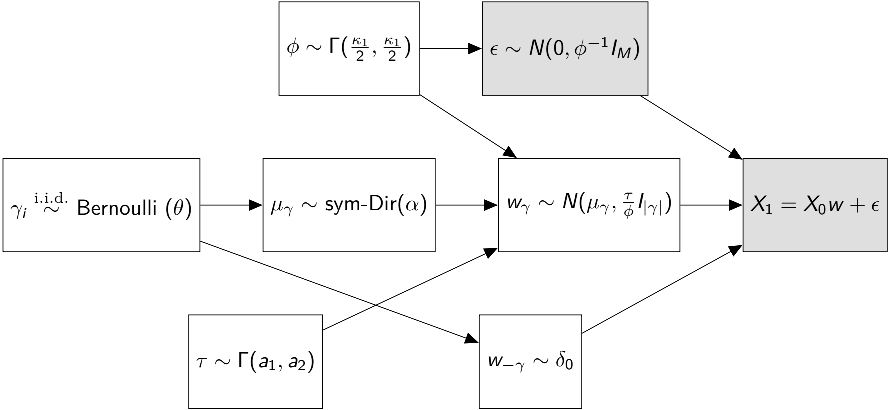
- **Variable selection:** estimate unknown $\gamma^* \in \{0,1\}^N$;
- **Soft constraint:**
  - Hard: $w_\gamma \in \Delta^{|\gamma|-1}$;
  - Soft: $\mathbb{E}[w_\gamma] \in \Delta^{|\gamma|-1}$; w_γ is random.

Bayesian estimation produces not just a single "point estimate" but an entire probability distribution for the parameter.

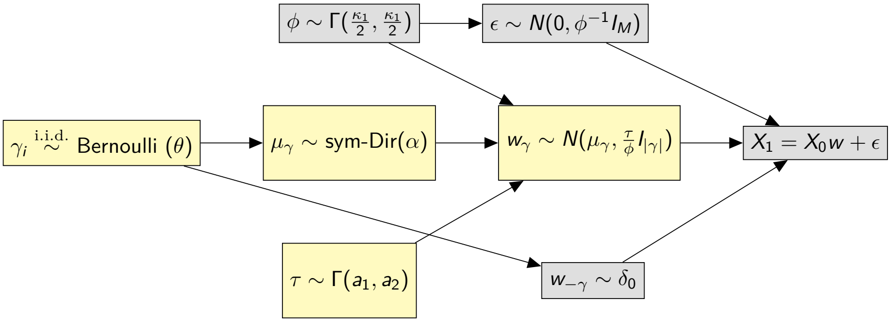
$$\underbrace{p(w_\gamma, \gamma, \phi \mid X_1)}_{\text{Posterior}} \propto \underbrace{f(X_1; w_\gamma, \gamma, \phi)}_{\text{Likelihood}} \underbrace{p(w_\gamma, \gamma, \phi)}_{\text{Prior}}$$

Model and Priors: BVS-SS

BVS-SS: Bayesian variable selection with a soft simplex constraint.



Model and Priors: BVS-SS



Remarks:

- A small θ penalizes the large models;
- Main diff from spike-and-slab: $\mu_\gamma \in \Delta^{|\gamma|-1}$;
- Variance τ: quantifying the deviation from simplex (mean μ_γ).

Hindrance

- Once we have samples from the posterior $p(\mu_\gamma, \gamma, \tau, \phi \mid X_1)$ ▸ Details, we can predict the counterfactual by

$$\mathbb{E}_p[\tilde{X}_0 w] = \mathbb{E}_p\left[\mathbb{E}_p(\tilde{X}_0 w \mid \mu_\gamma, \gamma, \tau, \phi)\right]; \ \widehat{\mathrm{ATT}} = \frac{1}{\tilde{T}} \sum_{t=1}^{\tilde{T}} \left(\tilde{X}_{1,t} - \mathbb{E}_p[\tilde{X}_0 w]_t\right).$$

- How to generate the samples from the posterior distribution $p(\mu_\gamma, \gamma, \tau, \phi \mid X_1)$? The simplex constraint makes a Gibbs sampling non-trivial;

- What is the (non-asymptotic) property of the posterior distribution $p(\mu_\gamma, \gamma, \tau, \phi \mid X_1)$?

A Gibbs Scheme with the Simplex Constraint

How to generate the samples from the posterior distribution $p(\mu_\gamma, \gamma, \tau, \phi \mid X_1)$?

- Gibbs sampling: Fix μ_{-j}, update μ_j according to $p(\mu_j \mid Y, \mu_{-j}, \tau, \phi)$;
- $\mu_j = 1 - \sum_{i \neq j} \mu_i$ almost surely; $p(\mu_j \mid Y, \mu_{-j}, \tau, \phi)$ is degenerate;
- Standard methods like Geyer (1992); George and McCulloch (1993); Gilks et al. (1995); George and McCulloch (1997); Brooks (1998); Carlo (2004) don't work.

Update μ_i, μ_j (with $i \neq j$) simultaneously from the full conditional posterior $p(\mu_i, \mu_j \mid X_1, \mu_{-(i,j)}, \tau, \phi)$.

1. Selection based on $p(\gamma_i, \gamma_j \mid X_1, \mu_{-(i,j)}, \tau, \phi)$;
2. Sampling based on $p(\mu_i, \mu_j \mid X_1, \gamma_i, \gamma_j, \mu_{-(i,j)}, \tau, \phi)$

Details ⟶

Non-asymptotic Bounds ($\tau = 0$)

Fix $\tau = 0$. The true model strictly satisfies the simplex constraint. We show the convergence rate of the posterior probability of selecting the true model γ^*.

Theorem 1

Under mild conditions, it is true that

$$p(\gamma^* \mid Y) \geq 1 - c_3 N^{-1},$$

with probability at least $1 - c_1 N^{-c_2 L}$.

Corollary 1

Under the setting of Theorem 1,

$$\mathbb{E}\left[\|\mu - \mu^*\|_2^2 \mid Y\right] = O_p\left(C_4 \frac{L \log N}{T_0}\right).$$

More Results

- We also show the property when the simplex constraint is substantially violated ($\tau \to \infty$);
- Several simulation studies show the advantages of BVS-SS;
- Revisit an empirical example, with 87 control units but only 35 pre-intervention periods. Details $\longrightarrow$

Concluding Remarks

- We construct a novel Bayesian synthetic control framework to solve the high dimension issue (BVS-SS);
  - The posterior of the weight shows an empirical attitude on the simplex constraint;
  - Li (2020); Chernozhukov et al. (2021); Martinez and Vives-i-Bastida (2022); Goh and Yu (2022);
- We propose a novel Markov chain Monte Carlo (MCMC) sampler for our BVS-SS posterior;
  - to select the "proper" control units and estimate the weights and counterfactuals through our posterior;
  - Geyer (1992); George and McCulloch (1993); Gilks et al. (1995); George and McCulloch (1997); Brooks (1998); Carlo (2004);
- We derive non-asymptotic bounds for our Bayesian estimations;
  - on control units selection and posterior mean of coeffs/weights;
  - George and McCulloch (1993, 1997); Brown et al. (1998); Li and Li (2008); Casella et al. (2009); Choi et al. (2010); Farcomeni (2010); Guan and Stephens (2011); Yang et al. (2016); Zhou and Guan (2019).

Thank You!

References I

S. Brooks. Markov chain monte carlo method and its application. *Journal of the royal statistical society: series D (the Statistician)*, 47(1):69–100, 1998.

P. J. Brown, M. Vannucci, and T. Fearn. Multivariate Bayesian variable selection and prediction. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 60(3):627–641, 1998.

C. M. Carlo. Markov chain monte carlo and gibbs sampling. *Lecture notes for EEB*, 581(540):3, 2004.

C. Carvalho, R. Masini, and M. C. Medeiros. Arco: An artificial counterfactual approach for high-dimensional panel time-series data. *Journal of econometrics*, 207(2):352–380, 2018.

G. Casella, F. J. Girón, M. L. Martínez, and E. Moreno. Consistency of Bayesian procedures for variable selection. *The Annals of Statistics*, 37(3):1207–1228, 2009.

V. Chernozhukov, K. Wüthrich, and Y. Zhu. An exact and robust conformal inference method for counterfactual and synthetic controls. *Journal of the American Statistical Association*, 116(536):1849–1864, 2021.

References II

N. H. Choi, W. Li, and J. Zhu. Variable selection with the strong heredity constraint and its oracle property. *Journal of the American Statistical Association*, 105(489):354–364, 2010.

N. Doudchenko and G. W. Imbens. Balancing, regression, difference-in-differences and synthetic control methods: A synthesis. Technical report, National Bureau of Economic Research, 2016.

A. Farcomeni. Bayesian constrained variable selection. *Statistica Sinica*, 20(3): 1043–1062, 2010. URL https://www.jstor.org/stable/24309479.

E. I. George and R. E. McCulloch. Variable selection via Gibbs sampling. *Journal of the American Statistical Association*, 88(423):881–889, 1993.

E. I. George and R. E. McCulloch. Approaches for Bayesian variable selection. *Statistica sinica*, pages 339–373, 1997.

C. J. Geyer. Practical markov chain monte carlo. *Statistical science*, pages 473–483, 1992.

W. R. Gilks, S. Richardson, and D. Spiegelhalter. *Markov chain Monte Carlo in practice*. CRC press, 1995.

References III

G. Goh and J. Yu. Synthetic control method with convex hull restrictions: a bayesian maximum a posteriori approach. *The Econometrics Journal*, 25(1): 215–232, 2022.

Y. Guan and M. Stephens. Bayesian variable selection regression for genome-wide association studies and other large-scale problems. *The Annals of Applied Statistics*, 5(3):1780, 2011.

A. Hollingsworth and C. Wing. Tactics for design and inference in synthetic control studies: An applied example using high-dimensional data. *Available at SSRN 3592088*, 2020.

S. Kim, C. Lee, and S. Gupta. Bayesian synthetic control methods. *Journal of Marketing Research*, 57(5):831–852, 2020.

C. Li and H. Li. Network-constrained regularization and variable selection for analysis of genomic data. *Bioinformatics*, 24(9):1175–1182, 2008. doi: 10.1093/bioinformatics/btn081.

References IV

K. T. Li. Statistical inference for average treatment effects estimated by synthetic control methods. *Journal of the American Statistical Association*, 115(532):2068–2083, 2020.

I. Martinez and J. Vives-i-Bastida. Bayesian and frequentist inference for synthetic controls. *arXiv preprint arXiv:2206.01779*, 2022. URL https://arxiv.org/abs/2206.01779.

Z. Shi and J. Huang. Forward-selected panel data approach for program evaluation. *Journal of Econometrics*, 234(2):512–535, 2023.

R. Tibshirani. Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 58(1):267–288, 1996.

Y. Yang, M. J. Wainwright, and M. I. Jordan. On the computational complexity of high-dimensional Bayesian variable selection. *The Annals of Statistics*, 44 (6):2497–2532, 2016.

Q. Zhou and Y. Guan. Fast model-fitting of Bayesian variable selection regression using the iterative complex factorization algorithm. *Bayesian analysis*, 14(2):573, 2019.

Motivation 1: High Dimensional X (Contd)

To see why no frequentist implementations with simplex, consider the standard Lasso regression (Tibshirani, 1996).

$$\hat{w} = \operatorname*{argmin}_{w} \frac{1}{T} \sum_{t=1}^{T} \left(X_{1,t} - X_{0,t}^{\top} w \right)^2 \quad \text{such that } \|w\|_1 \leq \lambda$$

where λ determines the degree of regularization.

The constraint in Lasso contradicts the simplex constraint Δ^{N-1}!

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Posterior and ATT

A routine calculation using Woodbury identity yields the marginal likelihood of (μ, τ, ϕ):

$$p(X_1 \mid \mu, \tau, \phi) \propto \frac{\phi^{T/2}}{\tau^{|\gamma|/2}\det(V_{\gamma,\tau})^{1/2}} \exp\left\{-\frac{\phi}{2}(X_1 - X_{0,\gamma}\mu_\gamma)^\top \Sigma_{\gamma,\tau}(X_1 - X_{0,\gamma}\mu_\gamma)\right\},$$

where

$$V_{\gamma,\tau} = X_{0,\gamma}^\top X_{0,\gamma} + \tau^{-1} I, \quad \Sigma_{\gamma,\tau} = I - X_{0,\gamma} V_{\gamma,\tau}^{-1} X_{0,\gamma}^\top.$$

w's conditional posterior:

$$w \mid X_1, \mu, \tau, \phi \sim N\left(V_{\gamma,\tau}^{-1}(X_{0,\gamma}^\top X_1 + \tau^{-1}\mu_\gamma),\, \phi^{-1} V_{\gamma,\tau}^{-1}\right).$$

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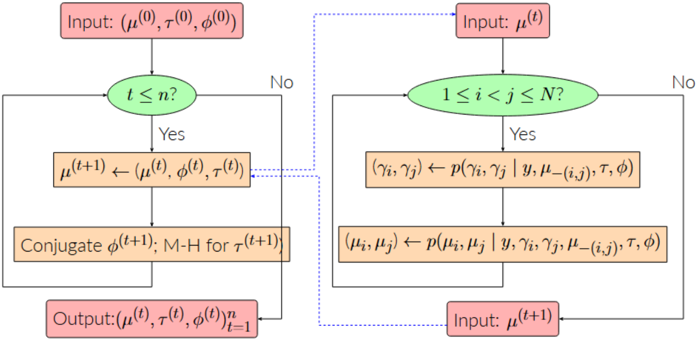
Metropolis-within-Gibbs



- When $\alpha = 1$, γ's conditional posterior can be expressed by CDF of standard normals; given $\gamma_i = \gamma_j = 1$, μ_i and μ_j's marginal posteriors degenerate to a **truncated Gaussian** distribution.

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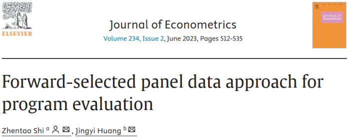
China's anti-corruption campaign



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Forward-selected panel data approach for program evaluation

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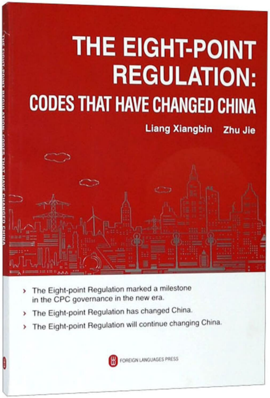

- Shi and Huang (2023): China anti-corruption campaign in 2012;
- X_1: monthly growth rate of luxury watch imports;
- Category: "watches with case of, or clad with, precious metal";
- X_0: monthly growth rate of other 87 commodity categories;
- Only $T_0 = 35$ pre-treatment periods.

China's anti-corruption campaign

Table: Anti-corruption Campaign's Impact on Luxury Watches Importation

	ATT	τ	ϕ	$\lvert\gamma\rvert$
Mean	-0.021	0.069	20.86	5.09
95% credible interval	(-0.032, -0.008)	(0, 0.641)	(12.22, 32.76)	(1, 20)

Remark: The ratio τ/ϕ is very small $\rightarrow$ empirical support on the validity of SCM type estimator.

Control Units Selection

- Shi and Huang (2023):
  1. knitted or crocheted fabric;
  2. cork and articles of cork;
  3. salt, sulfur, earth, stone, plaster, lime and cement.
- BVS-SS:
  1. knitted or crocheted fabric (51.4%);
  2. special woven fabrics (lace, tapestries, trimmings, embroidery) (16.6%);
  3. man-made filaments (14.8%);
  4. wool, fine or coarse animal hair (12%);
  5. other fabric/textile;
  6. optical, photographic, cinematographic, measuring, checking, medical or surgical instruments and apparatus; parts and accessories"(10.2%).

Remarks: seems a more reasonable selection.

Counterfactual Prediction

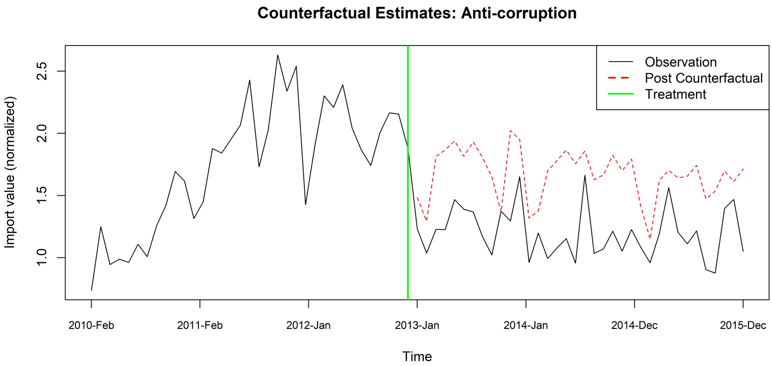

Figure: Significant impact right after the Eight-point regulation

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