

Bayesian Synthetic Control with a Soft Simplex Constraint

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Overview

- The Synthetic Control Method (SCM) has at least two limitations;
 - High-dimensional problem: the number of control units is larger than the total observed time periods before the intervention;
 - Debate over the simplex constraint: all coefficients are restricted to be non-negative and sum-to-one.
- We propose a novel extension of SCM, using Bayesian variable selection (spike-and-slab prior) with a soft simplex constraint, BVS-SS;
 - Contribution 1: BVS-SS solves the high dimension problem while still preserves the simplex constraint;
 - Contribution 2: BVS-SS introduces a notion of “soft” simplex constraint, asking data's advice on whether the constraint is proper.

SCM Setup

- T_0 is # of pre-treatment periods; T_1 is # of post-treatment periods;
- $X_{1,t}$ is the outcome of the treated unit; $X_{0,t}$ is the $N \times 1$ outcome vector of the control units;

$$\hat{w} = \underset{w \in \Delta^{N-1}}{\operatorname{argmin}} \frac{1}{T_0} \sum_{t=1}^{T_0} (X_{1,t} - X_{0,t}^\top w)^2, \quad (1)$$

$$\Delta^{N-1} = \left\{ u \in \mathbb{R}^N : u_i \geq 0 \text{ for each } i, \text{ and } \sum_{i=1}^N u_i = 1 \right\}$$

where Δ^{N-1} denotes the $(N-1)$ -simplex.

- The average treatment on the treated:

$$\widehat{\text{ATT}} = \frac{1}{T_1} \sum_{t=T_0+1}^T (X_{1,t} - X_{0,t}^\top \hat{w})$$

Challenges

- How to solve the optimization (1) when X_0 is high-dimensional ($N \gg T_0$)?
 - Panel data approach: Elastic net ([Doudchenko and Imbens, 2016](#)), Lasso ([Carvalho et al., 2018](#); [Hollingsworth and Wing, 2020](#)), forward selection ([Shi and Huang, 2023](#)), spike-and-slab ([Kim et al., 2020](#)) etc.;
 - Trade-off: dropping the simplex constraint;

Challenges

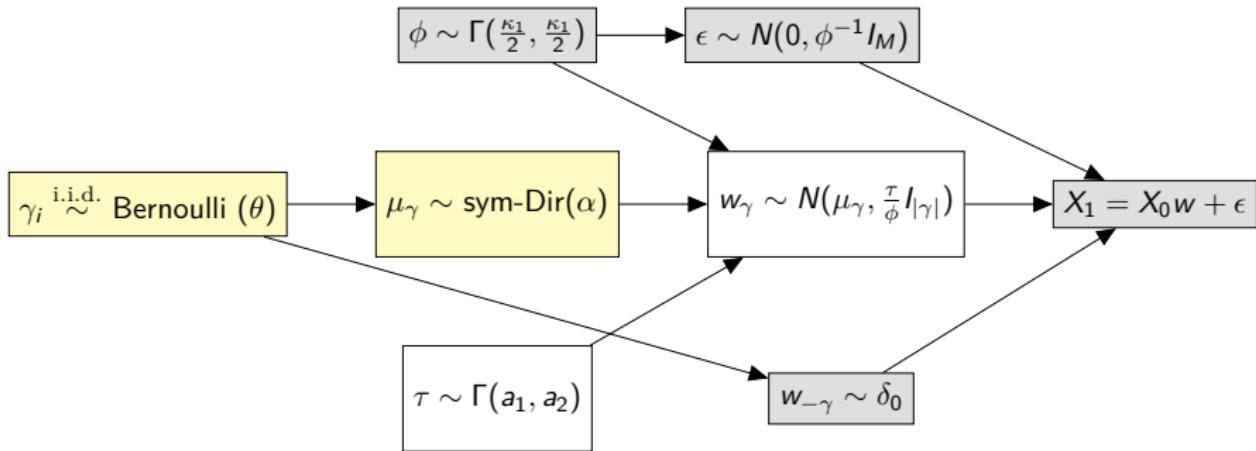
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 - Trade-off: dropping the simplex constraint;
- Is the simplex constraint necessary?
 - Pros: no hedging (e.g., $w_1 = 300\%$ and $w_2 = -250\%$); the counterfactual has economic interpretation;
 - Cons: too rigid (may have bad fitting); high-dimensional problem.

Summary

We propose a novel Bayesian SCM framework, based on Bayesian variable selection with a soft simplex constraint(BVS-SS). It

- ① deals with the high-dimensional control group, while still keeping the simplex constraint;

Summary



Remarks:

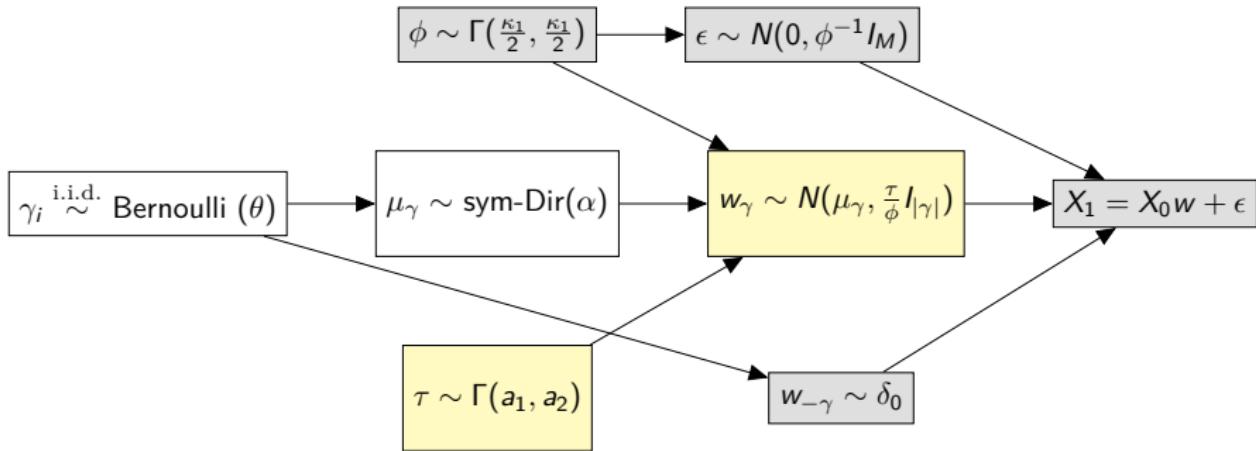
- A small θ penalizes the large models;
- Main diff from spike-and-slab: $\mu_\gamma \in \Delta^{|\gamma|-1}$;

Summary

We propose a novel Bayesian SCM framework, based on Bayesian variable selection with a soft simplex constraint(BVS-SS). It

- ① deals with the high-dimensional control group, while still keeping the simplex constraint;
- ② learns the attitude of data on the constraint case by case.

Summary



Remark:

- Variance τ : quantifying the deviation from simplex (mean μ_γ).

Hindrance

- Once we have samples from the posterior $p(\mu_\gamma, \gamma, \tau, \phi | X_1)$ ► Details, we can predict the counterfactual by

$$\mathbb{E}_p[\tilde{X}_0 w] = \mathbb{E}_p \left[\mathbb{E}_p(\tilde{X}_0 w | \mu_\gamma, \gamma, \tau, \phi) \right]; \widehat{\text{ATT}} = \frac{1}{\tilde{T}} \sum_{t=1}^{\tilde{T}} \left(\tilde{X}_{1,t} - \mathbb{E}_p[\tilde{X}_0 w]_t \right).$$

- How to generate the samples from the posterior distribution $p(\mu_\gamma, \gamma, \tau, \phi | X_1)$? A standard Gibbs sampling would fail due to the simplex constraint;
- What is the (non-asymptotic) property of the posterior distribution $p(\mu_\gamma, \gamma, \tau, \phi | X_1)$?

A Gibbs Scheme with the Simplex Constraint

How to generate the samples from the posterior distribution $p(\mu_\gamma, \gamma, \tau, \phi | X_1)$?

- Gibbs sampling: Fix μ_{-j} , update μ_j according to $p(\mu_j | Y, \mu_{-j}, \tau, \phi)$;
- $\mu_j = 1 - \sum_{i \neq j} \mu_i$ almost surely; $p(\mu_j | Y, \mu_{-j}, \tau, \phi)$ is degenerate;
- Standard methods like [Geyer \(1992\)](#); [George and McCulloch \(1993\)](#); [Gilks et al. \(1995\)](#); [George and McCulloch \(1997\)](#); [Brooks \(1998\)](#); [Carlo \(2004\)](#) don't work.

Update μ_i, μ_j (with $i \neq j$) simultaneously from the full conditional posterior $p(\mu_i, \mu_j | X_1, \mu_{-(i,j)}, \tau, \phi)$.

- ① Selection based on $p(\gamma_i, \gamma_j | X_1, \mu_{-(i,j)}, \tau, \phi)$;
- ② Sampling based on $p(\mu_i, \mu_j | X_1, \gamma_i, \gamma_j, \mu_{-(i,j)}, \tau, \phi)$

► Details →

Non-asymptotic Bounds ($\tau = 0$)

Fix $\tau = 0$. The true model strictly satisfies the simplex constraint. We show the convergence rate of the posterior probability of selecting the true model γ^* .

Theorem 1

Under mild conditions, it is true that

$$p(\gamma^* \mid Y) \geq 1 - c_3 N^{-1},$$

with probability at least $1 - c_1 N^{-c_2 L}$.

Corollary 1

Under the setting of Theorem 1,

$$\mathbb{E} [\|\mu - \mu^*\|_2^2 \mid Y] = O_p \left(C_4 \frac{L \log N}{T_0} \right).$$

More Results

- We also show the property when the simplex constraint is substantially violated ($\tau \rightarrow \infty$);
- Several simulation studies show the advantages of BVS-SS;
- Revisit an empirical example, with 87 control units but only 35 pre-intervention periods. [▶ Details →](#)

Concluding Remarks

- We construct a novel Bayesian synthetic control framework to solve the high dimension issue (BVS-SS);
 - The posterior of the weight shows an empirical attitude on the simplex constraint;
 - [Li \(2020\)](#); [Chernozhukov et al. \(2021\)](#); [Martinez and Vives-i-Bastida \(2022\)](#); [Goh and Yu \(2022\)](#);
- We propose a novel Markov chain Monte Carlo (MCMC) sampler for our BVS-SS posterior;
 - to select the “proper” control units and estimate the weights and counterfactuals through our posterior;
 - [Geyer \(1992\)](#); [George and McCulloch \(1993\)](#); [Gilks et al. \(1995\)](#); [George and McCulloch \(1997\)](#); [Brooks \(1998\)](#); [Carlo \(2004\)](#);
- We derive non-asymptotic bounds for our Bayesian estimations;
 - on control units selection and posterior mean of coeffs/weights;
 - [George and McCulloch \(1993, 1997\)](#); [Brown et al. \(1998\)](#); [Li and Li \(2008\)](#); [Casella et al. \(2009\)](#); [Choi et al. \(2010\)](#); [Farcomeni \(2010\)](#); [Guan and Stephens \(2011\)](#); [Yang et al. \(2016\)](#); [Zhou and Guan \(2019\)](#).

Thank You!

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Motivation 1: High Dimensional X (Contd)

To see why no frequentist implementations with simplex, consider the standard Lasso regression ([Tibshirani, 1996](#)).

$$\hat{w} = \underset{w}{\operatorname{argmin}} \frac{1}{T} \sum_{t=1}^T (X_{1,t} - X_{0,t}^\top w)^2 \text{ such that } \|w\|_1 \leq \lambda$$

where λ determines the degree of regularization.

The constraint in Lasso contradicts the simplex constraint Δ^{N-1} !

▶ Back ←

Posterior and ATT

A routine calculation using Woodbury identity yields the marginal likelihood of (μ, τ, ϕ) :

$$p(X_1 | \mu, \tau, \phi) \propto \frac{\phi^{T/2}}{\tau^{|\gamma|/2} \det(V_{\gamma, \tau})^{1/2}} \exp \left\{ -\frac{\phi}{2} (X_1 - X_{0, \gamma} \mu_\gamma)^\top \Sigma_{\gamma, \tau} (X_1 - X_{0, \gamma} \mu_\gamma) \right\},$$

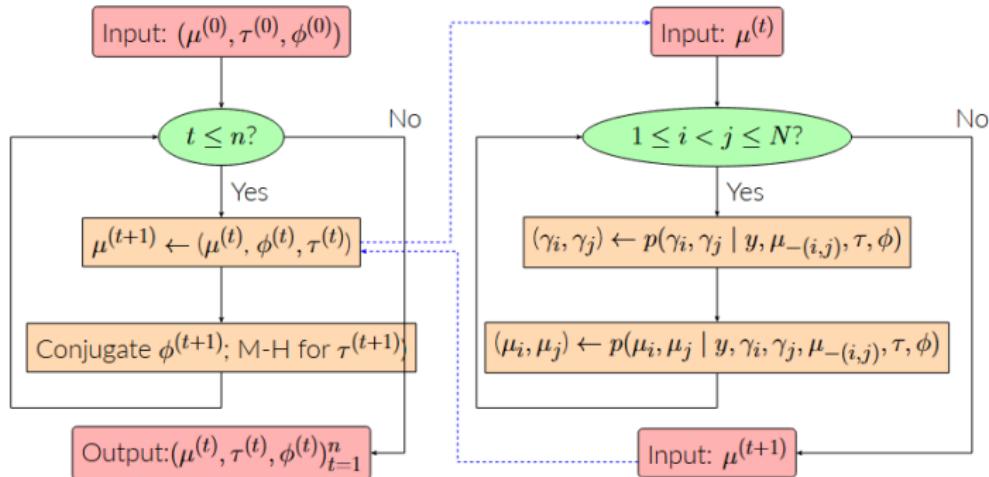
where

$$V_{\gamma, \tau} = X_{0, \gamma}^\top X_{0, \gamma} + \tau^{-1} I, \quad \Sigma_{\gamma, \tau} = I - X_{0, \gamma} V_{\gamma, \tau}^{-1} X_{0, \gamma}^\top.$$

w's conditional posterior:

$$w | X_1, \mu, \tau, \phi \sim N \left(V_{\gamma, \tau}^{-1} (X_{0, \gamma}^\top X_1 + \tau^{-1} \mu_\gamma), \phi^{-1} V_{\gamma, \tau}^{-1} \right).$$

Metropolis-within-Gibbs



- When $\alpha = 1$, γ 's conditional posterior can be expressed by CDF of standard normals; given $\gamma_i = \gamma_j = 1$, μ_i and μ_j 's marginal posteriors degenerate to a **truncated Gaussian** distribution.

China's anti-corruption campaign

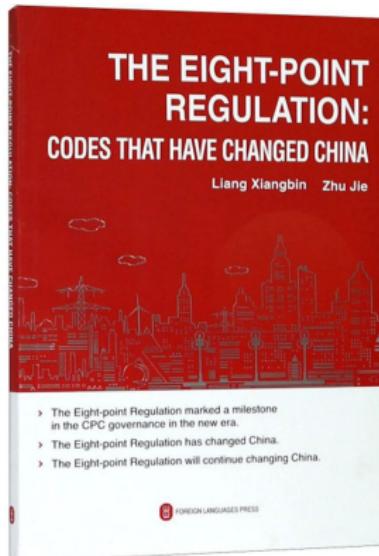


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Forward-selected panel data approach for program evaluation

Zhentao Shi^a , Jingyi Huang^b



- Shi and Huang (2023): China anti-corruption campaign in 2012;
- X_1 : monthly growth rate of luxury watch imports;
- Category: “watches with case of, or clad with, precious metal”;
- X_0 : monthly growth rate of other 87 commodity categories;
- Only $T_0 = 35$ pre-treatment periods.

China's anti-corruption campaign

Table: Anti-corruption Campaign's Impact on Luxury Watches Importation

	ATT	τ	ϕ	$ \gamma $
Mean	-0.021	0.069	20.86	5.09
95% credible interval	(-0.032, -0.008)	(0, 0.641)	(12.22, 32.76)	(1, 20)

Remark: The ratio τ/ϕ is very small → empirical support on the validity of SCM type estimator.

Control Units Selection

- Shi and Huang (2023):

- ① knitted or crocheted fabric;
- ② cork and articles of cork;
- ③ salt, sulfur, earth, stone, plaster, lime and cement.

- BVS-SS:

- ① knitted or crocheted fabric (51.4%);
- ② special woven fabrics (lace, tapestries, trimmings, embroidery) (16.6%);
- ③ man-made filaments (14.8%);
- ④ wool, fine or coarse animal hair (12%);
- ⑤ other fabric/textile;
- ⑥ optical, photographic, cinematographic, measuring, checking, medical or surgical instruments and apparatus; parts and accessories" (10.2%).

Remarks: seems a more reasonable selection.

Counterfactual Prediction

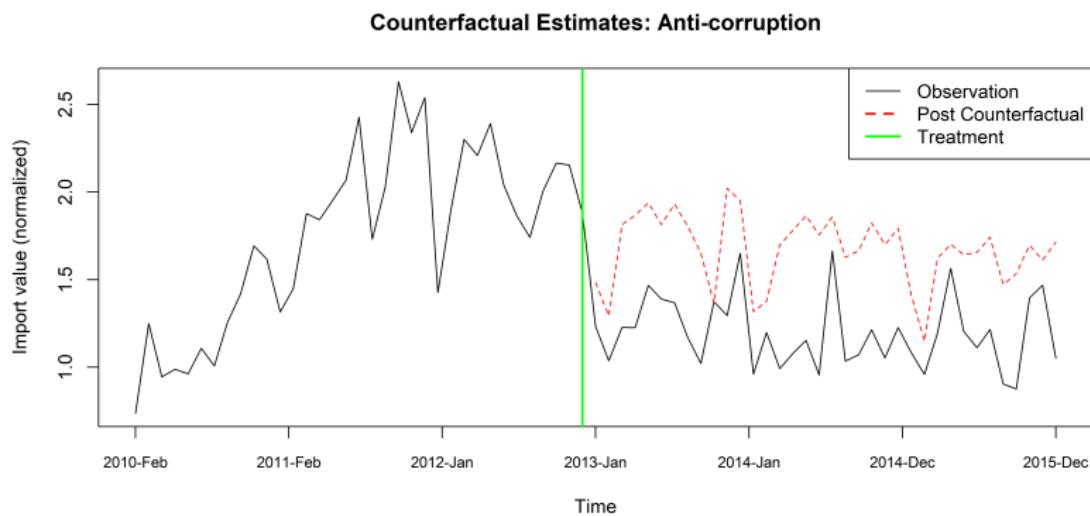


Figure: Significant impact right after the Eight-point regulation