# 1. Beta-Binomial Conjugate.

**Variables**

$N$ = total number of trials

$m$ = number of success

$a, b$ = shape parameter of Beta

**function definition**

① Likelihood : Binomial function : $Bin(m \mid N, p) = \binom{N}{m} p^m (1-p)^{N-m}$

② gamma function : $\Gamma(x) = \int_0^\infty p^{x-1} e^{-p} \, dp$

③ beta function : $B(a,b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

④ beta distribution : $Beta(p \mid a, b) = \frac{1}{B(a,b)} p^{a-1} (1-p)^{b-1}$

$$\text{posterior} = \frac{\text{likelihood} \cdot \text{prior}}{\text{marginal}}$$

$$P(p \mid N, m, a, b) = \frac{Bin(m \mid N, p) \cdot Beta(p \mid a, b)}{P(m \mid N, a, b)}$$

$$= \frac{\binom{N}{m} p^m (1-p)^{N-m} \cdot \frac{1}{B(a,b)} p^{a-1} (1-p)^{b-1}}{\int_0^1 \binom{N}{m} x^m (1-x)^{N-m} \frac{1}{B(a,b)} x^{a-1} (1-x)^{b-1} dx}$$

$$= \frac{\cancel{\binom{N}{m}} \cdot \cancel{\frac{1}{B(a,b)}} \cdot p^m \cdot (1-p)^{N-m} \cdot p^{a-1} (1-p)^{b-1}}{\cancel{\binom{N}{m}} \cdot \cancel{\frac{1}{B(a,b)}} \int_0^1 x^m (1-x)^{N-m} x^{a-1} (1-x)^{b-1} dx}$$

$$= \frac{p^{m+a-1} \cdot (1-p)^{N-m+b-1}}{\int_0^1 x^{(m+a)-1} (1-x)^{(N-m+b)-1} dx}$$

$$= \frac{p^{(m+a)-1} \cdot (1-p)^{(N-m+b)-1}}{B(m+a, N-m+b)}$$

$$= Beta(p \mid m+a, N-m+b)$$

## 2. Gamma - Poisson conjugation

Function :

① Likelihood : Poisson $\Rightarrow$ $Poi(m|\lambda) = \dfrac{\lambda^m e^{-\lambda}}{m!}$

② Gamma function : $\Gamma(k) = \int_0^\infty p^{k-1} e^{-p}\, dp$

③ Prior : Gamma $\Rightarrow$ $Gamma(\lambda|\alpha,\beta) = \dfrac{\beta^\alpha}{\Gamma(\alpha)} \cdot \lambda^{\alpha-1} e^{-\beta\lambda}$

$$P(\lambda|m,\alpha,\beta) = \frac{Poi(m|\lambda)\cdot Gamma(\lambda|\alpha,\beta)}{P(m|\alpha,\beta)}$$

$$= \frac{\frac{\lambda^m e^{-\lambda}}{m!}\cdot\frac{\beta^\alpha}{\Gamma(\alpha)}\cdot\lambda^{\alpha-1}\cdot e^{-\beta\lambda}}{\int_0^\infty \frac{\theta^m e^{-\theta}}{m!}\cdot\frac{\beta^\alpha}{\Gamma(\alpha)}\cdot\theta^{\alpha-1}\cdot e^{-\beta\theta}\, d\theta}$$

$$= \frac{\cancel{\frac{1}{m!}}\cdot\cancel{\frac{\beta^\alpha}{\Gamma(\alpha)}}\cdot\lambda^{m+\alpha-1}\, e^{-(\beta+1)\lambda}}{\cancel{\frac{1}{m!}}\cdot\cancel{\frac{\beta^\alpha}{\Gamma(\alpha)}}\int_0^\infty \theta^{(m+\alpha)-1}\cdot e^{-(\beta+1)\theta}\, d\theta} \rightarrow$$

let $m+\alpha = k$ , $\beta+1 = t$

$$\int_0^\infty \theta^{(m+\alpha)-1}\cdot e^{-(\beta+1)\theta}\, d\theta$$

$$= \int_0^\infty \theta^{k-1}\cdot e^{-t\theta}\, d\theta$$

let $u = t\theta$ , $du = t\,d\theta \rightarrow d\theta = \frac{du}{t}$

$$= \int_0^\infty \frac{1}{t^{k-1}}\, u^{k-1}\cdot e^{-u}\cdot\frac{1}{t}\, du$$

$$= \frac{\Gamma(k)}{t^k} = \frac{\Gamma(m+\alpha)}{(\beta+1)^{(m+\alpha)}}$$

$$= \frac{(\beta+1)^{(m+\alpha)}}{\Gamma(m+\alpha)}\cdot\lambda^{(m+\alpha)-1}\cdot e^{-(\beta+1)\lambda}$$

$$= Gamma(\lambda\,|\,m+\alpha,\ \beta+1)$$