Demo1: Simple Harmonic Oscillator

In this jupyter notebook, we will learn how to use the Euler's method to solve for the motions of a simple harmonic oscillaotr.

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Governing equations

The governing equations are

$$a^{t^n} = -\omega_0^2 x^{t^n},$$

$$x^{t^{n+1}} = x^{t^n} + v^{t^n} \times \Delta t,$$

and

$$v^{t^{n+1}} = v^{t^n} + a^{t^n} \times \Delta t.$$

Initial Conditions

At time $t = 0$, position $x = 1$ and velocity $v = 0$.\ Set $A = k = m = 1$, and $\omega_0 = \sqrt{k/m} = 1$ as well.

Exercie 1:

Use a small time step $\Delta t = 0.01$ and solve for the solution at $t = 20$.

In [2]:

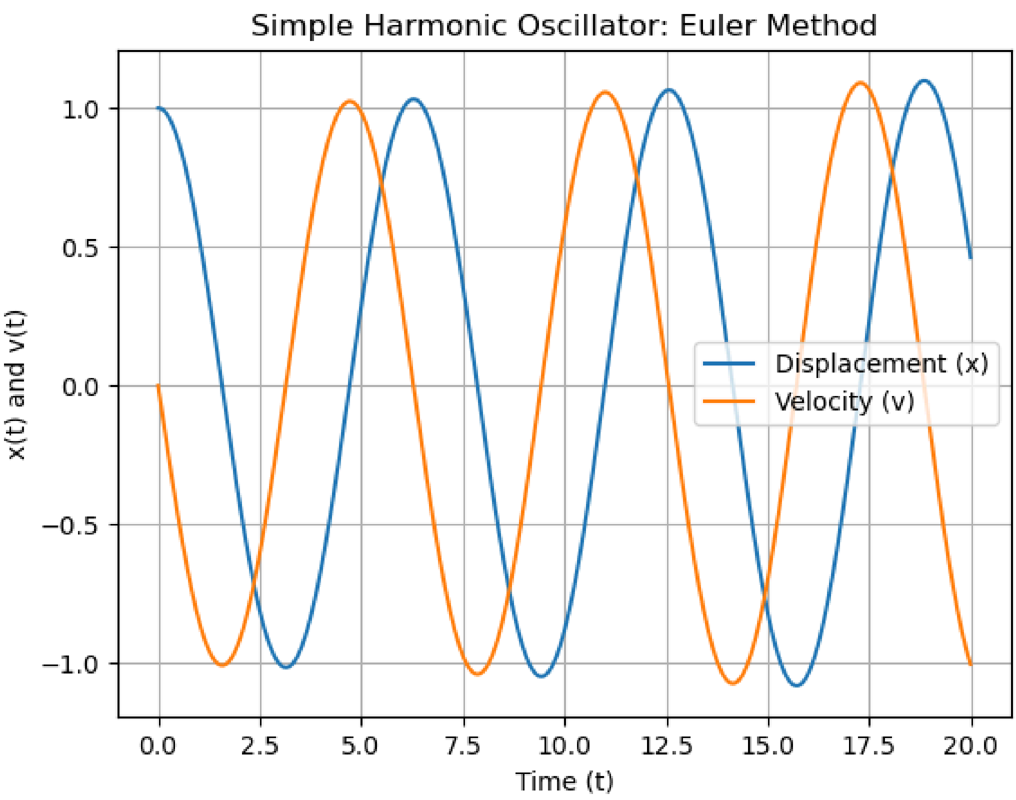
```
# import required libraries
import numpy as np
import matplotlib.pyplot as plt
```

In [9]:

```
#
# This is a simple example of how to solve a simple harmonic oscillator using the Euler method
#

# Step 1: set up the parameters of the problem
A=1
k=1
m=1
omega0 = np.sqrt(k/m)
dt = 0.01
# t range
t_max = 20

def sho_euler(A, k, m, dt, t_max):
# Step 2: set up the time and solution arrays
    t = np.arange(0, t_max, dt)  # Time array
    x = np.zeros_like(t)         # Displacement array
    v = np.zeros_like(t)         # Velocity array

    # Step 3: set up the initial conditions
    x[0] = A       # Initial displacement
    v[0] = 0       # Initial velocity

    # Step 4: solve the difference equation using the Euler method
    for i in range(1, len(t)):
        # Update position and velocity using Euler's method
        v[i] = v[i-1] - (k/m) * x[i-1] * dt  # Velocity update
        x[i] = x[i-1] + v[i-1] * dt          # Position update

    return t, x, v
```

```
t, x, v = sho_euler(A, k, m, dt, t_max)
```

In [10]:
```
# Step 5: plot the solution
#TODO
plt.plot(t, x, label="Displacement (x)")
plt.plot(t, v, label="Velocity (v)")
plt.xlabel("Time (t)")
plt.ylabel("x(t) and v(t)")
plt.title("Simple Harmonic Oscillator: Euler Method")
plt.legend()
plt.grid(True)
plt.show()
```



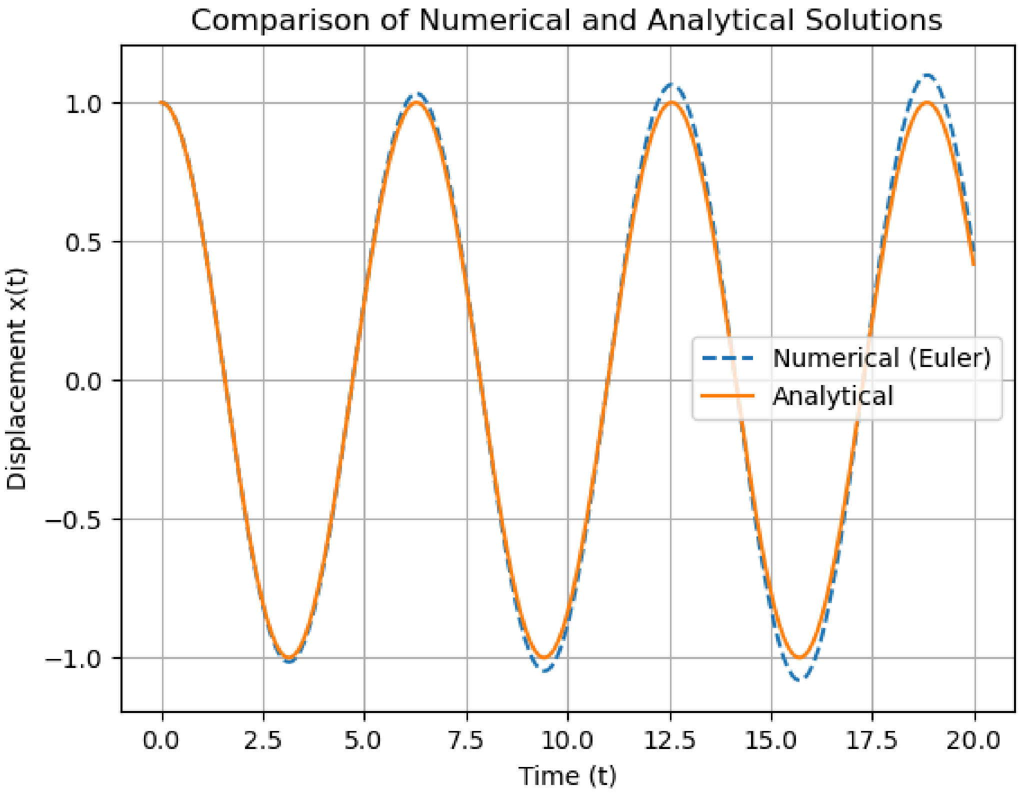
We could verify our numerical solution be comparing it with the analytical solutions. The analytical solutions are:

$$x = A\cos(\omega_0 t + \phi),$$

and

$$ v = -A \omega_0 \sin(\omega_0 t + \delta).

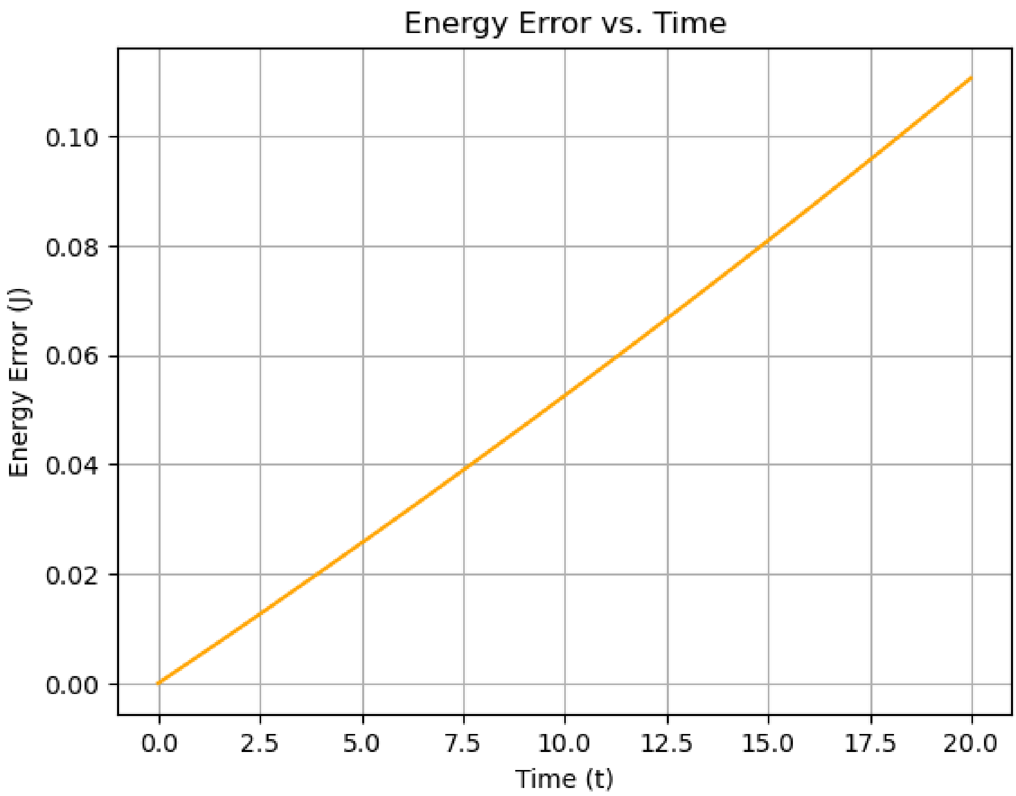
In [11]:
```
# Step 6: evaluate the analytical solution and plot it

# TODO
# Analytical solution for displacement
x_analytical = A * np.cos(omega0 * t)

# Plot numerical and analytical solutions
plt.plot(t, x, label="Numerical (Euler)", linestyle='--')
plt.plot(t, x_analytical, label="Analytical", linestyle='-')
plt.xlabel("Time (t)")
plt.ylabel("Displacement x(t)")
plt.title("Comparison of Numerical and Analytical Solutions")
plt.legend()
plt.grid(True)

plt.show()
```



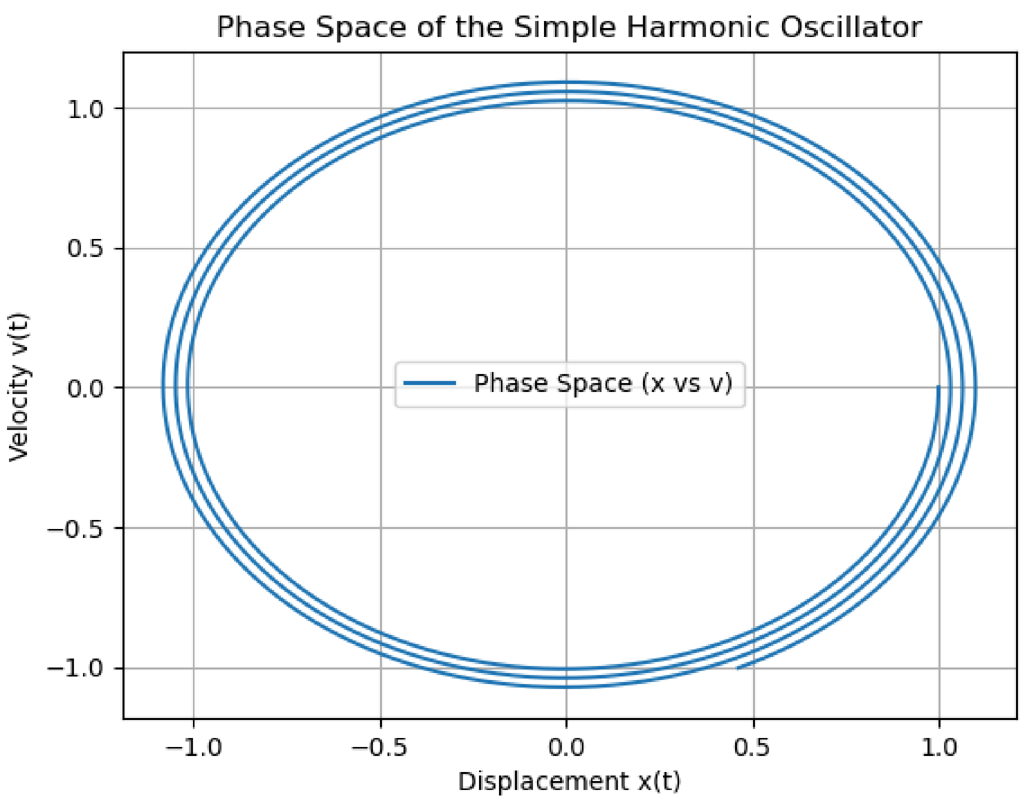
Another way to chekc the accuray of our numerical solution is to check the energy conservation and the phase-sapce diagram.

In [12]:

```
# Step 7: evaluate the energy (error) of the system

# TODO
# Calculate Kinetic Energy (KE) and Potential Energy (PE)
KE = 0.5 * m * v**2      # Kinetic energy
PE = 0.5 * k * x**2      # Potential energy

# Total energy
E_total = KE + PE

# Theoretical total energy
E_theoretical = 0.5 * k * A**2  # Initial energy (when x = A and v = 0)

# Evaluate the error (difference between total energy and theoretical energy)
energy_error = E_total - E_theoretical

# Plot the energy error over time
plt.plot(t, energy_error, label="Energy Error", color='orange')
plt.xlabel("Time (t)")
plt.ylabel("Energy Error (J)")
plt.title("Energy Error vs. Time")
plt.grid(True)

plt.show()
```



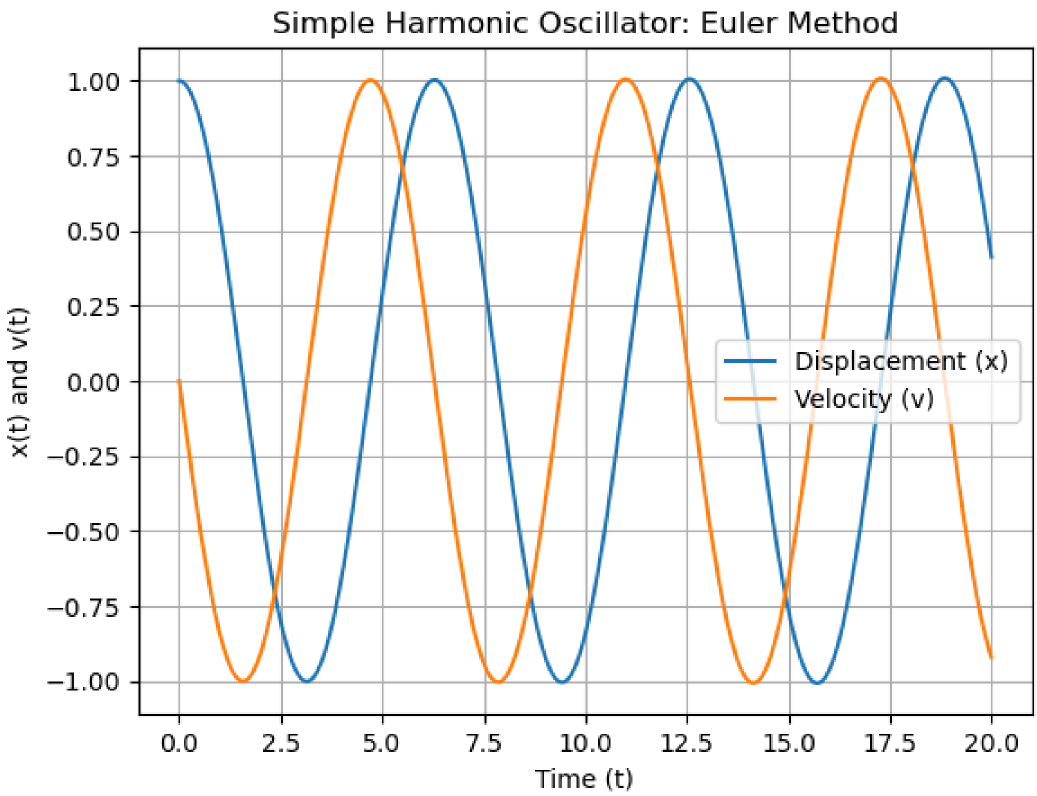
In [13]:
```
# Step 8: evaluate the phase space

# TODO
# Plot the phase space (x vs v)
plt.plot(x, v, label="Phase Space (x vs v)")
plt.xlabel("Displacement x(t)")
plt.ylabel("Velocity v(t)")
plt.title("Phase Space of the Simple Harmonic Oscillator")
plt.grid(True)
plt.legend()
plt.show()
```

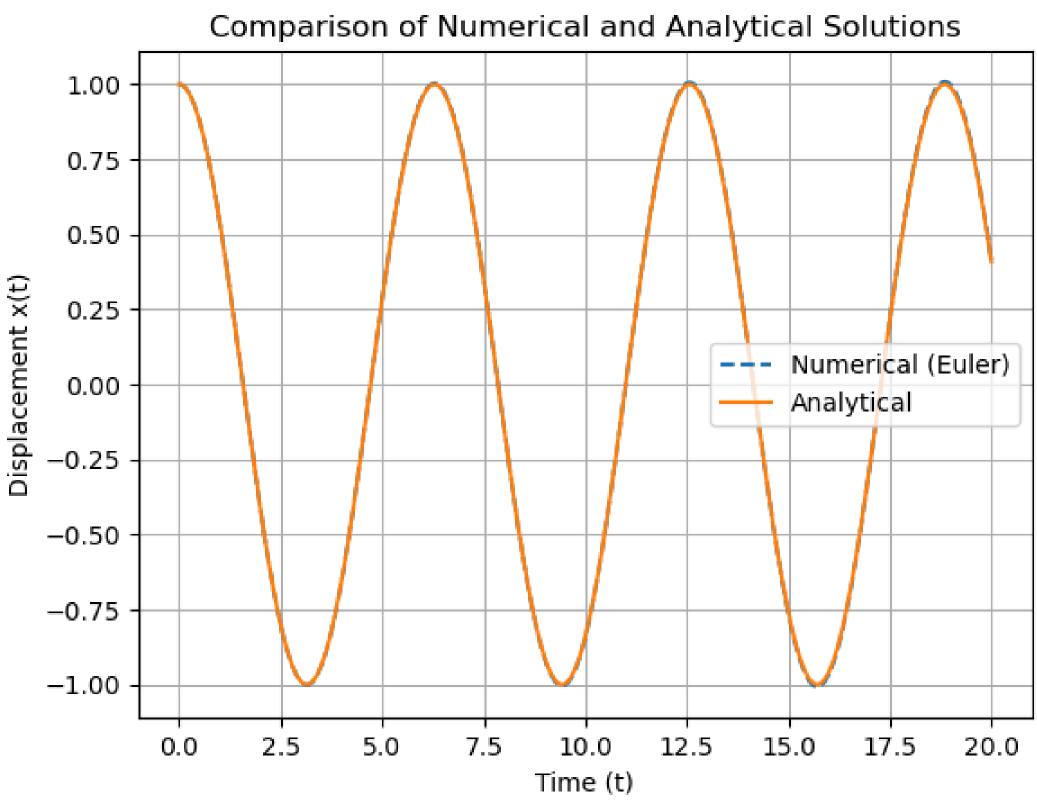


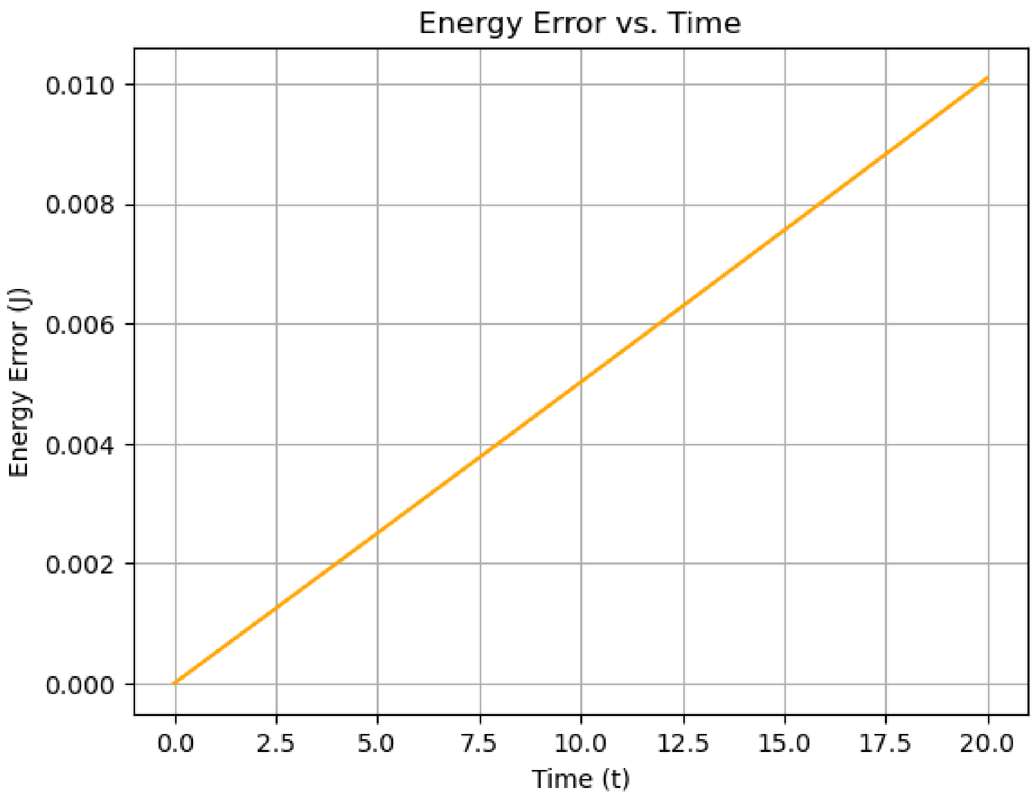
Exercise 2:

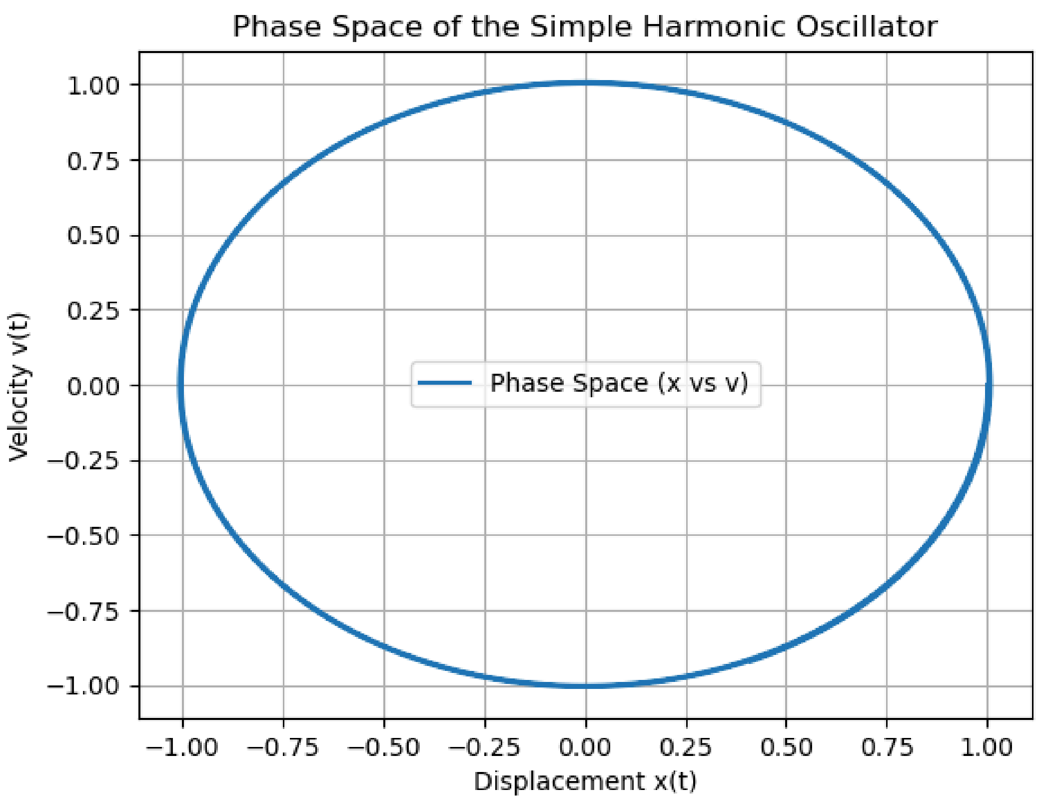
Check if the accuracy can be improved by reducing the time step to $\Delta t = 0.001$.

In [14]:
```
# Step 1: set up the parameters of the problem
A=1
k=1
m=1
omega0 = np.sqrt(k/m)
dt = 0.001
# t range
t_max = 20
t, x, v = sho_euler(A, k, m, dt, t_max)

# Step 2: plot the solution
plt.plot(t, x, label="Displacement (x)")
plt.plot(t, v, label="Velocity (v)")
plt.xlabel("Time (t)")
plt.ylabel("x(t) and v(t)")
plt.title("Simple Harmonic Oscillator: Euler Method")
plt.legend()
plt.grid(True)
plt.show()

# Step 3: evaluate the analytical solution and plot it
# Analytical solution for displacement
x_analytical = A * np.cos(omega0 * t)

# Plot numerical and analytical solutions
plt.plot(t, x, label="Numerical (Euler)", linestyle='--')
plt.plot(t, x_analytical, label="Analytical", linestyle='-')
plt.xlabel("Time (t)")
plt.ylabel("Displacement x(t)")
plt.title("Comparison of Numerical and Analytical Solutions")
plt.legend()
plt.grid(True)
plt.show()

# Step 3: evaluate the energy (error) of the system
# Calculate Kinetic Energy (KE) and Potential Energy (PE)
KE = 0.5 * m * v**2      # Kinetic energy
PE = 0.5 * k * x**2      # Potential energy
# Total energy
E_total = KE + PE
# Theoretical total energy
E_theoretical = 0.5 * k * A**2  # Initial energy (when x = A and v = 0)
# Evaluate the error (difference between total energy and theoretical energy)
energy_error = E_total - E_theoretical
# Plot the energy error over time
plt.plot(t, energy_error, label="Energy Error", color='orange')
plt.xlabel("Time (t)")
plt.ylabel("Energy Error (J)")
plt.title("Energy Error vs. Time")
plt.grid(True)
plt.show()

# Step 4: evaluate the phase space
# Plot the phase space (x vs v)
plt.plot(x, v, label="Phase Space (x vs v)")
plt.xlabel("Displacement x(t)")
plt.ylabel("Velocity v(t)")
plt.title("Phase Space of the Simple Harmonic Oscillator")
plt.grid(True)
plt.legend()
plt.show()
```


Simple Harmonic Oscillator: Euler Method
x(t) and v(t)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
Time (t)
Displacement (x)
Velocity (v)


Comparison of Numerical and Analytical Solutions
Displacement x(t)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
Time (t)
Numerical (Euler)
Analytical





Note

Reducing the time step is not the best solution. The better solution is to use higher-order schemes. Do NOT use Eulter's method in any production runs.

Demo2: Simple Harmonic Oscillator with RK2/RK4

In this jupyter notebook, we will learn how to use the Runge-Kutta 2/4 methods to solve for the motions of a simple harmonic oscillaotr.

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Governing equations

The governing equations are

$$a^{t^n} = -\omega_0^2 x^{t^n},$$

$$\frac{dx}{dt} = v,$$

and

$$\frac{dv}{dt} = a.$$

Runge-Jutta methods

Higher-order explicit schemes.

- RK2

$$y_{k+1} = y_k + \frac{h_k}{2}(k_1 + k2),$$

where $k_1 = f(t_k, y_k)$ and $k_2 = f(t_k + h_k, y_k + h_k k1)$.

- RK4

$$y_{k+1} = y_k + \frac{h_k}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where $k_1 = f(t_k, y_k)$, $k_2 = f(t_k + h_k/2, y_k + (h_k/2)k_1)$, $k_3 = f(t_k + h_k/2, y_k + (h_k/2)k_2)$, and $k_4 = f(t_k + h_k, y_k + h_k k3)$.

Initial Conditions

At time $t = 0$, position $x = 1$ and velocity $v = 0$.\ Set $A = k = m = 1$, and $\omega_0 = \sqrt{k/m} = 1$ as well.

Exercie 1: Redo demo1 with the RK2 method

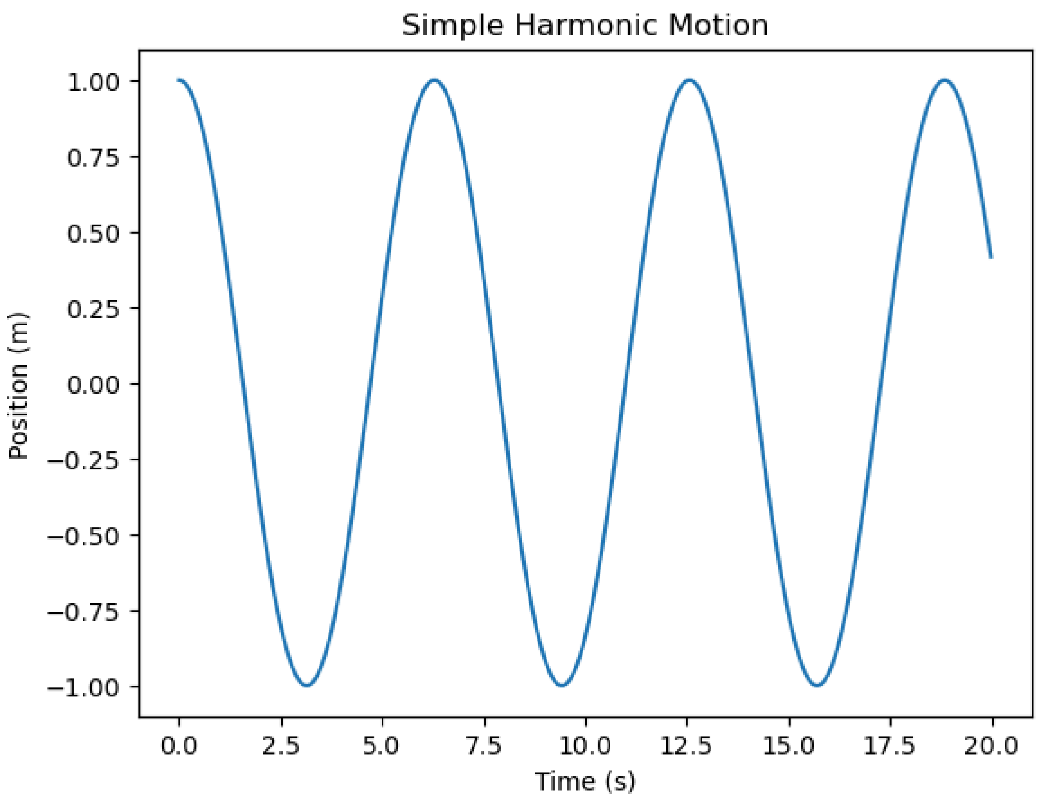
Use a small time step $\Delta t = 0.01$ and solve for the solution at $t = 20$.

In [2]:
```
# import required libraries
import numpy as np
import matplotlib.pyplot as plt
```

In [3]:
```
#
# This is a simple example of how to solve a simple harmonic oscillator using the RK2 method
#

# Step 1: set up the parameters of the problem
```

```python
A=1
k=1
m=1
dt = 0.01
omega0 = np.sqrt(k/m)

def sho_rk2(A, k, m, dt, t_max):
    # Step 2: set up the time and solution arrays
    times = np.arange(0, t_max, dt)
    x = np.zeros_like(times)
    v = np.zeros_like(times)
    # Step 3: set up the initial conditions
    x[0] = 1
    v[0] = 0
    # Step 4: solve the difference equation using the RK2 method
    for i in range(1,len(times)):
        y = np.array([x[i-1], v[i-1]])  # y = (x , v)
        t = times[i-1]
        h = dt

        def f(t, y):
            return np.array([y[1], -omega0**2*y[0]])

        k1 = f(t, y)

        k2 = f(t+dt, y+k1*dt)

        ynext = y + (k1+k2)*dt/2

        x[i] = ynext[0]
        v[i] = ynext[1]

    return times, x, v

times, x, v = sho_rk2(A, k, m, dt, 20)
```

In [4]:
```python
# Step 5: plot the solution
plt.plot(times,x)
plt.xlabel('Time (s)')
plt.ylabel('Position (m)')
plt.title('Simple Harmonic Motion')
plt.show()
```



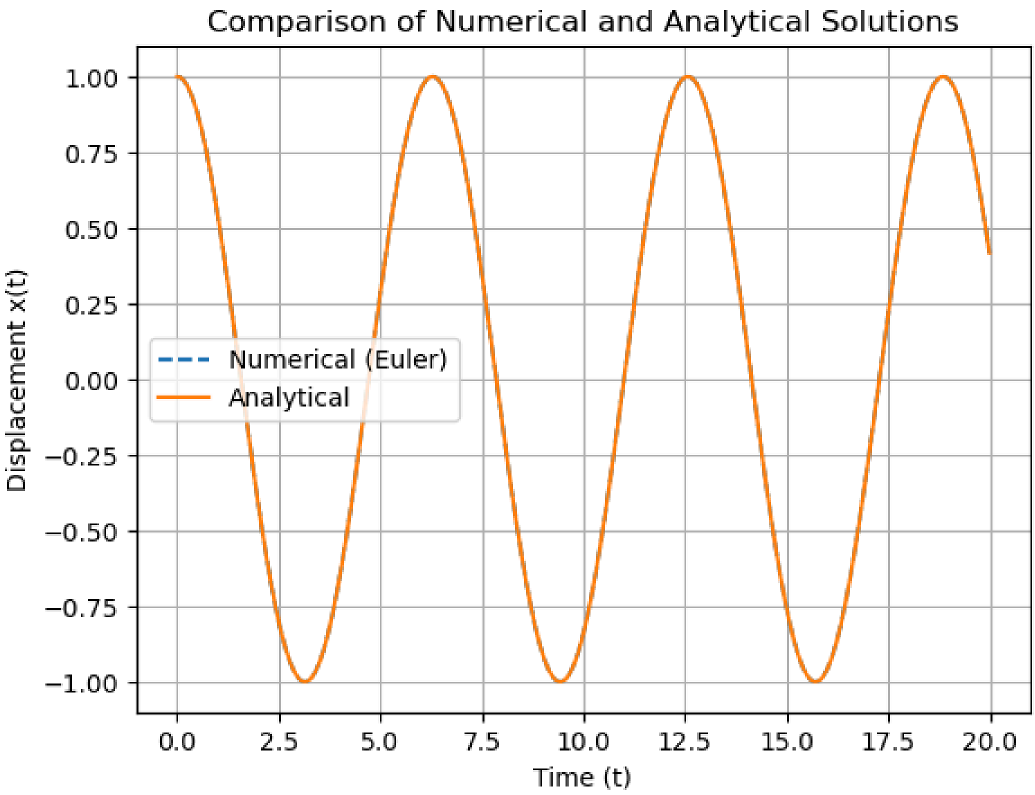
We could verify our numerical solution be comparing it with the analytical solutions. The analytical solutions are:

$$x = A\cos(\omega_0 t + \phi),$$

and

$$ v = -A \omega_0 \sin(\omega_0 t + \delta).

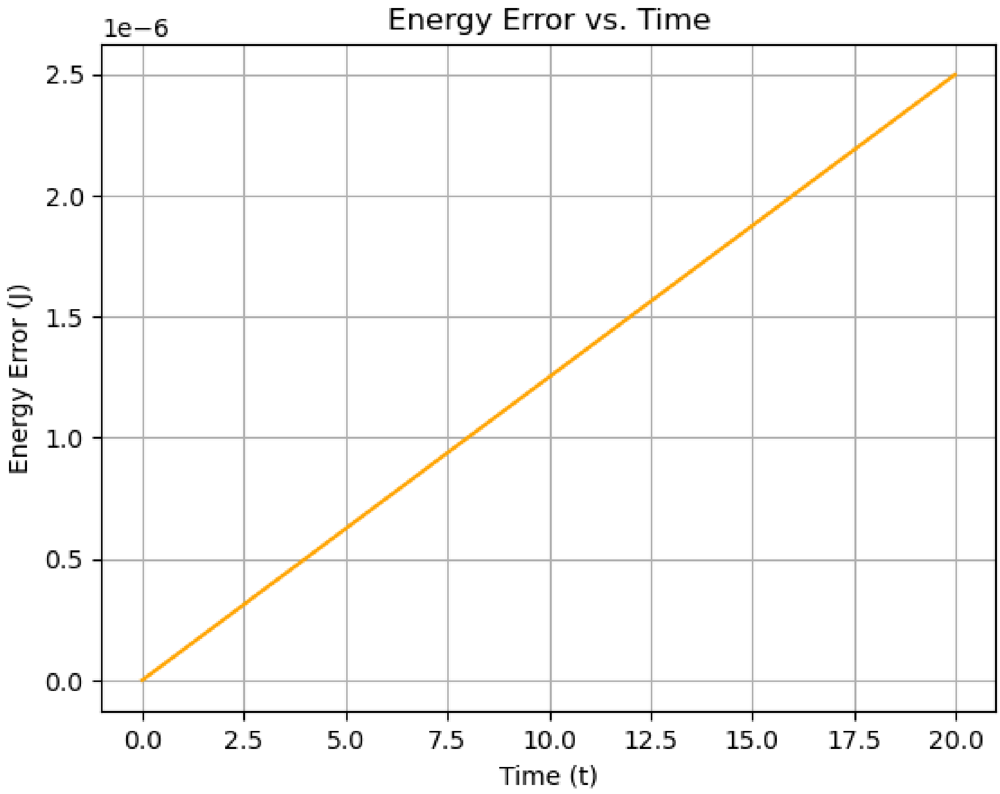
In [5]:
```python
# Step 6: evaluate the analytical solution

# TODO:
x_analytical = A * np.cos(omega0 * times)

# Plot numerical and analytical solutions
plt.plot(times, x, label="Numerical (Euler)", linestyle='--')
plt.plot(times, x_analytical, label="Analytical", linestyle='-')
plt.xlabel("Time (t)")
plt.ylabel("Displacement x(t)")
plt.title("Comparison of Numerical and Analytical Solutions")
plt.legend()
plt.grid(True)
```



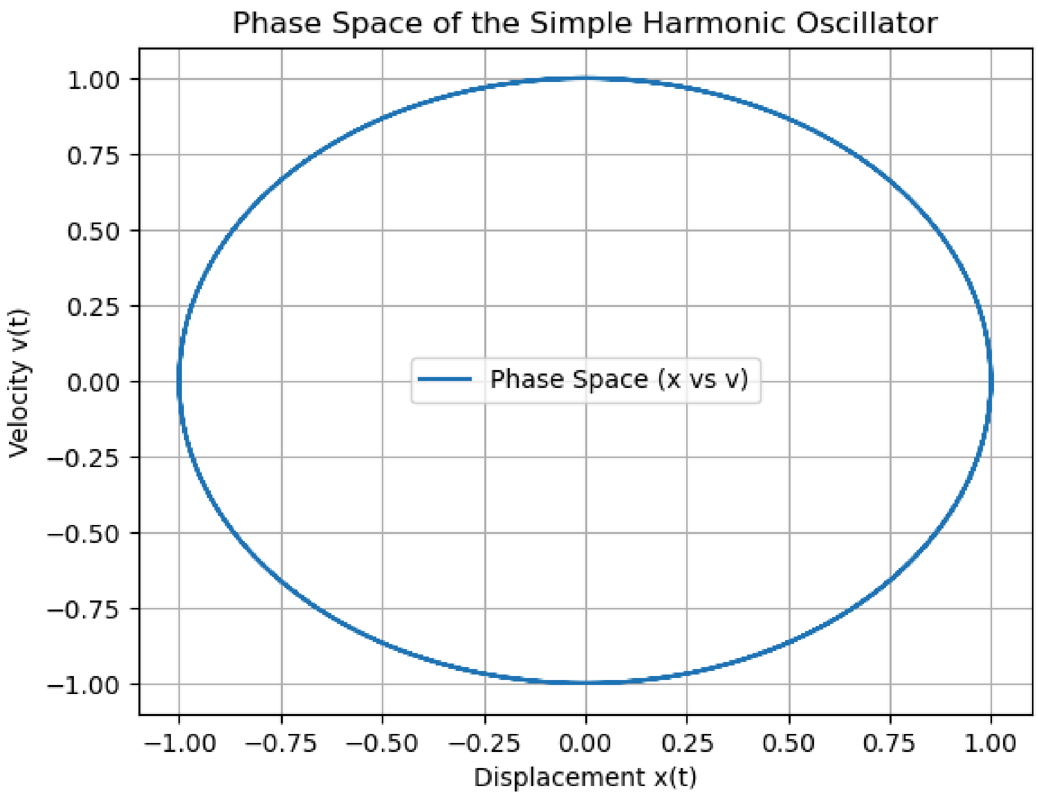
Another way to chekc the accuray of our numerical solution is to check the energy conservation and the phase-sapce diagram.

In [6]:
```python
# Step 7: evaluate the energy (error) of the system

# TODO:
# Calculate Kinetic Energy (KE) and Potential Energy (PE)
KE = 0.5 * m * v**2      # Kinetic energy
PE = 0.5 * k * x**2      # Potential energy

# Total energy
E_total = KE + PE

# Theoretical total energy
E_theoretical = 0.5 * k * A**2  # Initial energy (when x = A and v = 0)

# Evaluate the error
```

```
energy_error = E_total - E_theoretical

# Plot the energy error over time
plt.plot(times, energy_error, label="Energy Error", color='orange')
plt.xlabel("Time (t)")
plt.ylabel("Energy Error (J)")
plt.title("Energy Error vs. Time")
plt.grid(True)
plt.show()
```



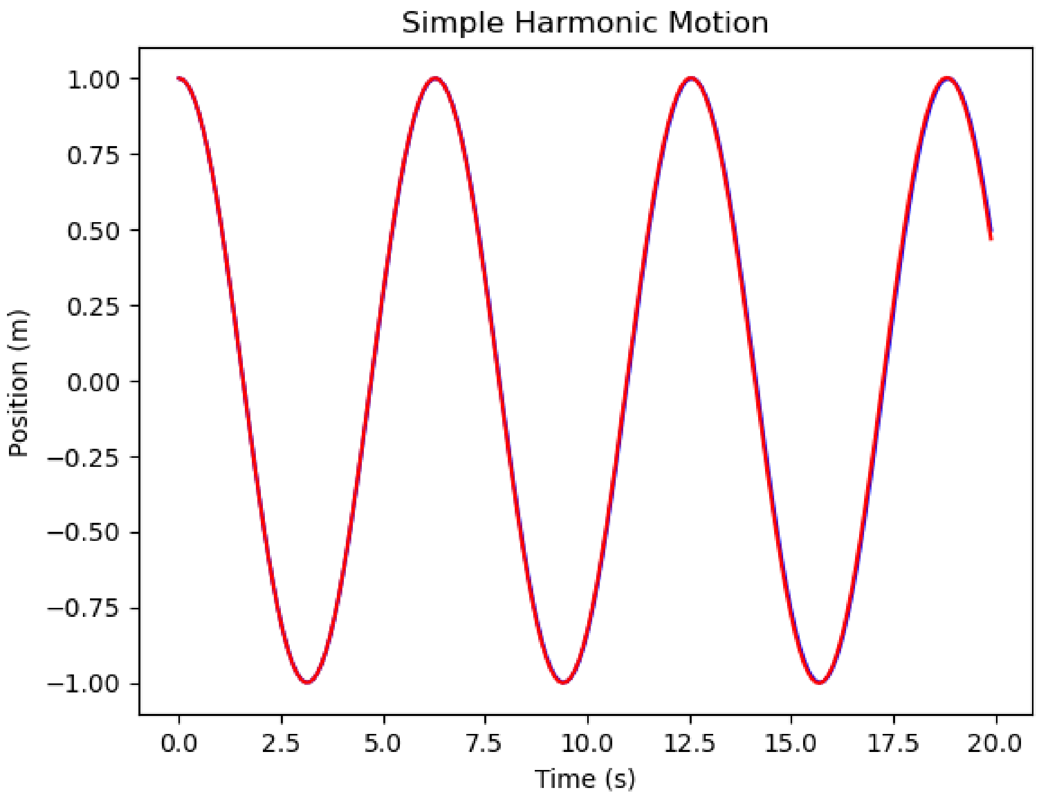
In [7]:
```
# Step 8: evaluate the phase space

# TODO:
# Plot the phase space (x vs v)
plt.plot(x, v, label="Phase Space (x vs v)")
plt.xlabel("Displacement x(t)")
plt.ylabel("Velocity v(t)")
plt.title("Phase Space of the Simple Harmonic Oscillator")
plt.grid(True)
plt.legend()
plt.show()
```



Exercise 2: Repeat with RK4

In [8]:
```
#
# This is a simple example of how to solve a simple harmonic oscillator using the RK4 method
#

# TODO:

def sho_rk4(A, k, m, dt, t_max):

    # Step 2: set up the time and solution arrays
    times = np.arange(0, t_max, dt)
    x = np.zeros_like(times)
    v = np.zeros_like(times)

    # Step 3: set up the initial conditions
    x[0] = 1
    v[0] = 0

    # Step 4: solve the difference equation using the RK4 method
    for i in range(1,len(times)):

        y = np.array([x[i-1], v[i-1]])  # y = (x , v)
        t = times[i-1]
        h = dt

        def f(t, y):
            return np.array([y[1], -omega0**2*y[0]])

        k1 = f(t, y)
        k2 = f(t+dt/2, y+k1*dt/2)
        k3 = f(t+dt/2, y+k2*dt/2)
        k4 = f(t+dt/2, y+k3*dt)

        ynext = y + (k1+2*k2+2*k3+k4)*dt/6

        x[i] = ynext[0]
        v[i] = ynext[1]

    return times, x, v
```

```python
# Example usage
A = 1
k = 1
m = 1
dt = 0.1
t_max = 20

# Call the RK4 function
t, x_rk4, v_rk4 = sho_rk4(A, k, m, dt, t_max)
t, x_rk2, v_rk2 = sho_rk2(A, k, m, dt, t_max)
plt.plot(t, x_rk4, color = 'blue')
plt.plot(t, x_rk2, color = 'red')
plt.xlabel('Time (s)')
plt.ylabel('Position (m)')
plt.title('Simple Harmonic Motion')
plt.show()
```

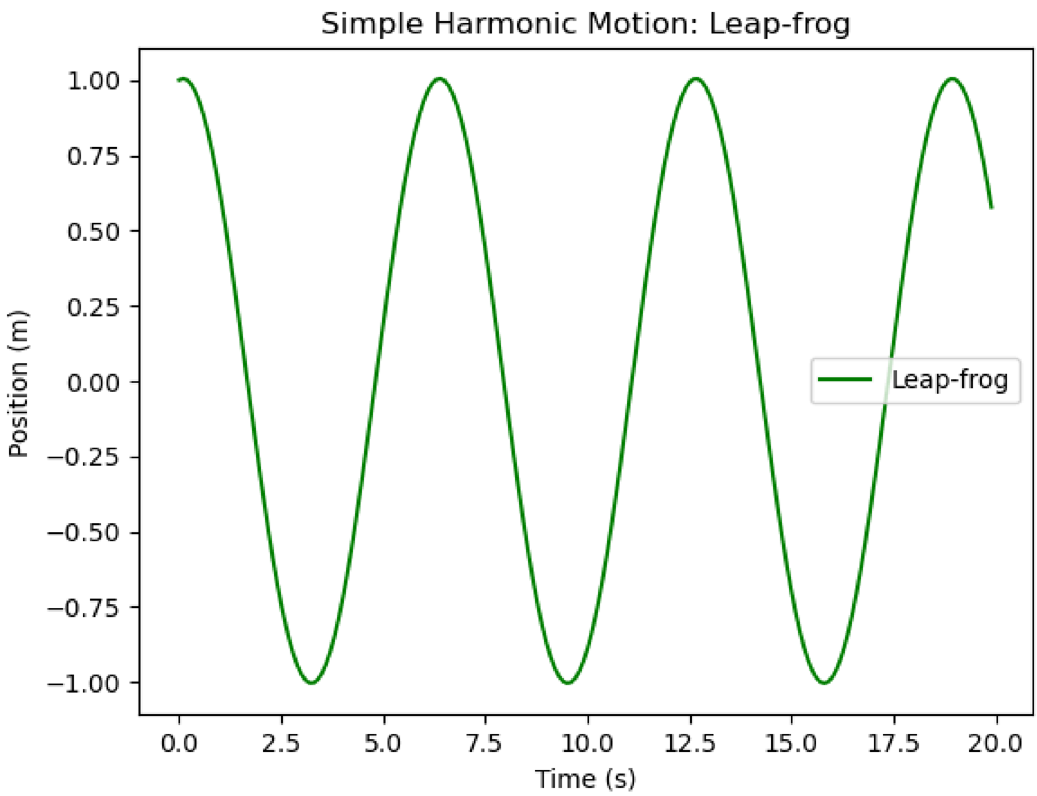


Mini homework: Repeat with Leap-frog

In [12]:

```python
# Leap-frog method for Simple Harmonic Oscillator
def sho_leapfrog(A, k, m, dt, t_max):

    # Step 2: set up the time and solution arrays
    times = np.arange(0, t_max, dt)
    x = np.zeros_like(times)
    v = np.zeros_like(times)

    # Step 3: set up the initial conditions
    x[0] = 1   # Initial position
    v_half = 0 - (dt / 2) * (-k/m * x[0])  # First half-step for velocity

    # Step 4: solve the difference equation using the Leap-frog method
    for i in range(1, len(times)):
        # Full step for position
        x[i] = x[i-1] + v_half * dt

        # Full step for velocity at the half-step
        v_half = v_half + (-k/m * x[i]) * dt

    return times, x
```

```
# Example usage
A = 1      # Amplitude
k = 1      # Spring constant
m = 1      # Mass
dt = 0.1   # Time step
t_max = 20 # Maximum time

# Call the Leap-frog function
t, x_leapfrog = sho_leapfrog(A, k, m, dt, t_max)


# Plotting the results
plt.plot(t, x_leapfrog, label='Leap-frog', color='green')
plt.xlabel('Time (s)')
plt.ylabel('Position (m)')
plt.title('Simple Harmonic Motion: Leap-frog')
plt.legend()
plt.show()
```



do convergence test for 4 methods

1. Euler Method (First-order method):
2. RK2 Method (Second-order method):
3. Leap-frog Method (Second-order method):The Leap-frog method, like RK2, is a second-order method. Halving the time step size should reduce the error by a factor of four.
4. RK4 Method (Fourth-order method):

In [10]:
```
import solver as solver
def derive_func(y,K,M):
    f = np.zeros(len(y))
    f[0] = y[1]            # y'[0] = v
    f[1] = -K * y[0]/M        # y'[1] = a = F/M
    return f

def true_solution(t, A, omega):
    return A * np.cos(omega * t)
```

```python
def calculate_error(numerical_sol, t_eval, A, omega):
    true_sol = true_solution(t_eval, A, omega)
    # Calculate the L2 norm of the error
    error = np.linalg.norm(numerical_sol[0] - true_sol)
    return error

def convergence_test():
    t_span = (0, 20) # from t=0 to t=20
    y0 = np.array([1, 0])  # initial condition : x(0) = 1, v(0) = 0
    K = 1
    M = 1
    omega = np.sqrt(K / M)
    methods = ["Euler", "RK2", "RK4", "Leapfrog"]
    time_steps = [0.1, 0.05, 0.01, 0.005, 0.001]  # different time step sizes
    errors = {method: [] for method in methods}

    for dt in time_steps:
        t_eval = np.arange(t_span[0], t_span[1], dt)
        for method in methods:
            sol = solver.solve_ivp(derive_func, t_span, y0, method=method, t_eval=t_eval, args
            error = calculate_error(sol, t_eval, 1, omega)
            errors[method].append(error)

    # Plot the error vs time step size
    plt.figure(figsize=(8, 6))
    for method in methods:
        plt.loglog(time_steps, errors[method], label=f'{method} method')

    plt.xlabel('Time step size (dt)')
    plt.ylabel('Error (L2 norm)')
    plt.title('Convergence Test: Error vs Time Step Size')
    plt.legend()
    plt.grid(True, which="both", ls="--")
    plt.show()


# do the convergence test
convergence_test()
```