HOMEWORK #6 SOLUTION

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1. Exercise 6.1 Illustrate local sensitivity analysis

Problem: Make an original example to illustrate the local-analysis concepts of this chapter.
Solution:

2. Exercise 6.2 Illustrate global sensitivity analysis

Problem: Using `AMPL`, make an original example, with at least three constraints, graphing the objective value of (P), as a single `b[i]` is varied from $-\infty$ to $+\infty$. As you work on this, bear in mind Theorem 6.3.
Solution: Consider the standard-form problem (P) with $m = 3$ and $n = 5$

$$(\mathrm{P}) \qquad \begin{array}{lrcl} \min & c'x & & \\ & Ax & = & b\ ; \\ & x & \geq & \mathbf{0}\ , \end{array}$$

where

$$A := \begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & -4 & 2 & 2 & 0 \\ 0 & -9 & 0 & 6 & 3 \end{pmatrix} \ , \quad b := (1, 2, 18)'\ , \quad c := (16, 7, 20, 10, 4)'\ .$$

Using `AMPL`, at each time we vary only one b_i from b and check the range of b_i in which the optimal basis for (P) keeps the same. We show the optimal basis as follows:

b_1	$(-\infty, -13)$	$(-13, -1)$	$(-1, +\infty)$
optimal basis β	$\{2, 3, 4\}$	$\{2, 4, 5\}$	$\{1, 4, 5\}$
$c'_\beta A_\beta^{-1}$	$(-12.8\ 10\ -3.8)$	$(-23/3\ -17/6\ 4/3)$	$(16\ 9\ 4/3)$
objective value	$-12.8b_1 - 48.4$	$(-23/3)b_1 + 55/3$	$16b_1 + 42$

b_2	$(-\infty, 0)$	$(0, 6)$	$(6, +\infty)$
optimal basis β	$\{1, 2, 5\}$	$\{1, 4, 5\}$	$\{1, 3, 4\}$
$c'_\beta A_\beta^{-1}$	$(16\ -8.75\ 4/3)$	$(16\ 9\ 4/3)$	$(16\ 10\ 18)$
objective value	$-8.75b_2 + 40$	$9b_2 + 40$	$10b_2 + 34$

b_3	$(-\infty, 0)$	$(0, 6)$	$(6, +\infty)$
optimal basis β	$\{1, 2, 3\}$	$\{1, 3, 4\}$	$\{1, 4, 5\}$
$c'_\beta A_\beta^{-1}$	$(16\ 10\ -7)$	$(16\ 10\ 1)$	$(16\ 9\ 4/3)$
objective value	$-7b_3 + 36$	$b_3 + 36$	$(4/3)b_3 + 34$

Table 1. Changing single b_i from $-\infty$ to $+\infty$

As we get the above data from `AMPL`, we graph the objective value of (P) as a single b_i is varied from $-\infty$ to $+\infty$ in Figure 1.

3. Exercise 6.3 *"I feel that I know the change that is needed."* – Mahatma Gandhi

Problem: We are given $2m$ numbers satisfying $L_i \leq 0 \leq U_i$, $i = 1, 2, ..., m$. Let β be an optimal basis for all of the m problems

$$\begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax = b + \Delta_i e^i; \\ & \qquad x \geq 0. \end{array} \tag{3.1}$$

for all Δ_i satisfying $L_i \leq \Delta_i \leq U_i$. Lets be clear on what this means: For each i individually, the basis β is optimal when the ith right-hand side component is changed from b_i to $b_i + \Delta_i$, as long as Δ_i is in the interval $[L_i, U_i]$.

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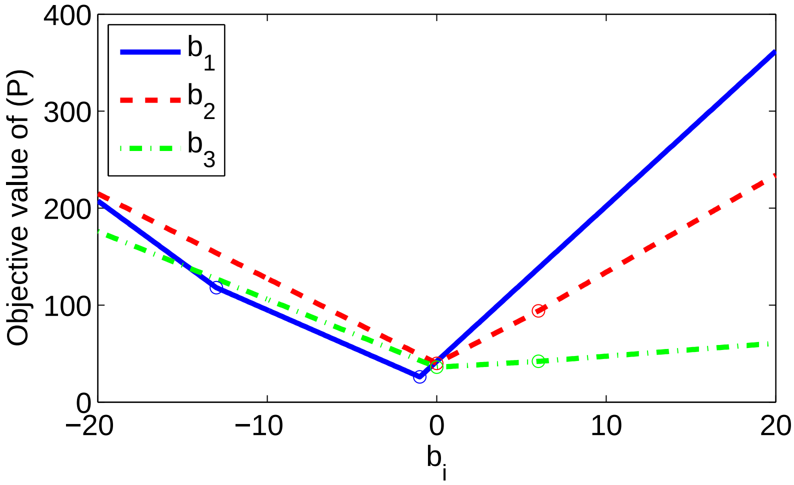

FIGURE 1. Objective value of (P) as a single b_i is varied from $-\infty$ to $+\infty$

The point of this problem is to be able to say something about *simultaneously* changing all of the b_i. Prove that we can simultaneously change b_i to

$$\tilde{b}_i := b_i + \lambda_i \left\{ \begin{array}{c} L_i \\ U_i \end{array} \right\}$$

where $\lambda_i \geq 0$, when $\sum_{i=1}^m \lambda_i \leq 1$. [Note that in the formula above, for each i we can $i = 1$ pick either L_i (a decrease) or U_i (an increase)].

Proof:

Let $D_i = \left\{ \begin{array}{c} L_i \\ U_i \end{array} \right\}$, so $\max_{k:h_k^i>0} -\frac{\overline{b}_k}{h_k^i} \leq D_i \leq \min_{k:h_k^i<0} -\frac{\overline{b}_k}{h_k^i}$, where $\overline{b} = A_\beta^{-1}b \geq 0$, $h^i = A_\beta^{-1}e^i$, and $k = 1...m$.

Thus, what we need to prove is that the basis β is optimal when we simultaneously change b_i to $\tilde{b}_i := b_i + \lambda_i D_i$, where $\sum_{i=1}^m \lambda_i = 1$. In the other word, we need to prove that the basis β is optimal for (3.2), where $\sum_{i=1}^m \lambda_i = 1$.

$$\begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax = b + \sum_{i=1}^m \lambda_i D_i e^i; \\ & \qquad x \geq 0. \end{array} \tag{3.2}$$

Consider a fixed and optimal basis β for (3.3). For β to be optimal for (3.2), we need $A_\beta^{-1}(b + \sum_{i=1}^m \lambda_i D_i e^i) = \overline{b} + \sum_{i=1}^m \lambda_i D_i h^i \geq 0$.

$$\begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax = b; \\ & \quad x \geq 0. \end{array} \tag{3.3}$$

For $k = 1...m$,

$\overline{b}_k + \sum_{i=1}^m \lambda_i D_i h_k^i$

$= \overline{b}_k + \sum_{i=1,h_k^i>0}^m \lambda_i D_i h_k^i + \sum_{i=1,h_k^i<0}^m \lambda_i D_i h_k^i$

$\geq \overline{b}_k + \sum_{i=1,h_k^i>0}^m \lambda_i (\max_{k':h_{k'}^i>0} -\frac{\overline{b}_{k'}}{h_{k'}^i}) h_k^i + \sum_{i=1,h_k^i<0}^m \lambda_i (\min_{k':h_{k'}^i<0} -\frac{\overline{b}_{k'}}{h_{k'}^i}) h_k^i$

$\geq \overline{b}_k + \sum_{i=1,h_k^i>0}^m \lambda_i (-\frac{\overline{b}_k}{h_k^i}) h_k^i + \sum_{i=1,h_k^i<0}^m \lambda_i (-\frac{\overline{b}_k}{h_k^i}) h_k^i$

$= \overline{b}_k + \sum_{i=1,h_k^i>0}^m \lambda_i (-\overline{b}_k) + \sum_{i=1,h_k^i<0}^m \lambda_i (-\overline{b}_k)$

$= \overline{b}_k + \sum_{i=1}^m \lambda_i (-\overline{b}_k)$

$= \overline{b}_k + (-\overline{b}_k) \sum_{i=1}^m \lambda_i$

$\geq \overline{b}_k + (-\overline{b}_k)$

$= 0$

Thus, $\overline{b} + \sum_{i=1}^m \lambda_i D_i h^i \geq 0$, and β is feasible and hence still optimal for (3.2).

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4. Exercise 6.4 Domain for objective variations

Problem: Prove Theorem 6.4: The domain of g is a convex set.
Proof:
Suppose that c^j is in the domain of g, for $j = 1, 2$. Therefore, there exist x^j that are feasible for (4.1), for $j = 1, 2$.

$$\begin{array}{ll} \min & {c^j}'x \\ \text{s.t.} & Ax = b; \\ & x \geq 0. \end{array} \tag{4.1}$$

For any $0 < \lambda < 1$, consider $\widehat{c} := \lambda c^1 + (1 - \lambda)c^2$, to prove the domain of g is a convex set, we need to prove that (4.2) is feasible and not unbounded.

$$\begin{array}{ll} \min & \widehat{c}'x \\ \text{s.t.} & Ax = b; \\ & x \geq 0. \end{array} \tag{4.2}$$

Consider $\widehat{x} := \lambda x^1 + (1 - \lambda)x^2$,
$\because$ x^j that are feasible for (4.1), for $j = 1, 2$
$\therefore$ $x^j \geq 0$, and $Ax^j = b$, for $j = 1, 2$
$\therefore$ $A\widehat{x} = A(\lambda x^1 + (1 - \lambda)x^2) = \lambda Ax^1 + (1 - \lambda)Ax^2 = \lambda b + (1 - \lambda)b = b$
$\because$ $\lambda > 0$
$\therefore$ $\widehat{x} = \lambda x^1 + (1 - \lambda)x^2 \geq 0$
$\therefore$ $\widehat{x}$ is a feasible solution to (4.2)
$\therefore$ (4.2) is feasible
Consider the objective function $\widehat{c}'x$ of (4.2),
$\because$ c^j is in the domain of g, for $j = 1, 2$
$\therefore$ (4.1) is not unbounded for $j = 1, 2$
$\therefore$ $\exists k^j \in \mathbb{R}$, for any x, such that ${c^j}'x \geq k^j$, for $j = 1, 2$
$\therefore$ for a feasible solution $\overline{x}$ of (4.2), $\widehat{c}'\overline{x} = (\lambda c^1 + (1 - \lambda)c^2)'\overline{x} = \lambda {c^1}'\overline{x} + (1 - \lambda){c^2}'\overline{x} \geq (\lambda k^1 + (1 - \lambda)k^2)$
$\because$ $k^1, k^2 \in \mathbb{R}$, $0 < \lambda < 1$
$\therefore$ (4.2) is feasible and not unbounded
$\therefore$ $\widehat{c}$ is in the domain g
$\therefore$ the domain of g is a convex set

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5. Exercise 6.5 Concave piecewise-linear function

Problem: Prove Theorem 6.5: g is a concave piecewise-linear function on its domain.
Proof:
The function g is

$$\begin{array}{ll} g(c) := \min & c'x \\ \text{s.t.} & Ax = b; \\ & x \geq 0. \end{array} \tag{5.1}$$

So a basis β is feasible or not for (5.1), independent for c_β. So g can be written as

$$g(c) = \min \left\{ c'_\beta(A_\beta^{-1}b) : \beta \text{ is a feasible basis for (5.1)} \right\}$$

$\because$ $c'_\beta(A_\beta^{-1}b)$ are affine functions
$\therefore$ g is the pointwise minimum of a finite number of affine functions
$\therefore$ g is a concave piecewise-linear function

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