

## HOMEWORK #8 SOLUTION

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### 1. EXERCISE 8.1 TASK SCHEDULING, CONTINUED

**Problem:** Consider again the “task scheduling” Exercise 2.4. Take the dual of the linear-optimization problem that you formulated. Explain how this dual can be interpreted as a kind of network problem. Using AMPL, solve the dual of the example that you created for Exercise 2.4 and interpret the solution.

**Solution:** Let  $p_{i,j}$  represent the precedence matrix, i.e.

$$(1.1) \quad p_{i,j} = \begin{cases} 1, & \text{if job } j \text{ precedes job } i; \\ 0, & \text{otherwise.} \end{cases}$$

Then the task scheduling problem can be formulated mathematically as follows:

$$(P') \quad \begin{aligned} \min \quad & t_{n+1} \\ & p_{i,j}(t_j + d_j) \leq t_i, \quad \text{for } p_{i,j} = 1; \\ & t_0 = 0, \\ & t_i \geq 0, \quad \forall i. \end{aligned}$$

Let  $m$  be the number of pair  $(i, j)$  such that  $p_{i,j} = 1$ . Transferring (P') into the standard-form problem (P):

$$(P) \quad \begin{aligned} \min \quad & t_{n+1} \\ & t_i - t_j - s_k = d_j, \quad \text{for } p_{i,j} = 1; \\ & t_0 = 0, \\ & t_i \geq 0, \quad \forall i, \\ & s_k \geq 0, \quad \forall k, \end{aligned}$$

which is in standard form of the following

$$(P) \quad \begin{aligned} \min \quad & c'x \\ & Ax = b; \\ & x \geq \mathbf{0}, \end{aligned}$$

where

$$\begin{aligned} x &= (t_0 \quad \cdots \quad t_n \quad t_{n+1} \quad s_1 \quad \cdots \quad s_m)' , \\ c &= (0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0)' , \\ b &= (b_1 \quad \cdots \quad b_m \quad 0)' , \end{aligned}$$

$A$  is a  $(m+1) \times (n+m+2)$  matrix. The first  $m$  lines of  $A$  are parameters for  $m$  constraints for  $p_{i,j} = 1$ , and the last line of  $A$  is for  $t_0 = 0$ . The dual of (P) is

$$(D) \quad \begin{aligned} \max \quad & y'b \\ & y'A \leq c', \end{aligned}$$

Let

$$y = (y_1 \quad \cdots \quad y_m \quad y_{m+1})'$$

the objective function can be written as

$$\sum_{i=1}^m y_i b_i$$

$y'A \leq c'$  represents  $(m + n + 2)$  constraints. Considering the structure of matrix  $A$ , the first  $(n + 1)$  constraints have the following form ( $l$  denote the column number of matrix  $A$  starting from 0, which also corresponds to  $t_0$  to  $t_n$ )

$$\sum_{(l,i) \text{ assoc. } j:p_{l,i}=1} y_j - \sum_{(i,l) \text{ assoc. } j:p_{i,l}=1} y_j \leq 0$$

For the  $(n + 2)$ th constraint (which corresponds to the column of  $t_{n+1}$ ):

$$\sum_{(n+1,i) \text{ assoc. } j:p_{n+1,i}=1} y_j - \sum_{(i,n+1) \text{ assoc. } j:p_{i,n+1}=1} y_j \leq 1$$

For last  $m$  constraints:

$$-y_i \leq 0.$$

Then the problem (D) is equivalent to

$$(D') \quad \begin{aligned} \max \quad & \sum_{i=1}^m y_i b_i \\ & \sum_{(i,l) \text{ assoc. } j:p_{i,l}=1} y_j - \sum_{(l,i) \text{ assoc. } j:p_{l,i}=1} y_j = u_l, \quad l = 0, \dots, n, \\ & \sum_{(i,n+1) \text{ assoc. } j:p_{i,n+1}=1} y_j - \sum_{(n+1,i) \text{ assoc. } j:p_{n+1,i}=1} y_j = u_{n+1} - 1, \\ & y_i \geq 0, \\ & u_j \geq 0. \end{aligned}$$

Based on the task dependency graph, we can actually see that  $\sum_{(i,l) \text{ assoc. } j:p_{i,l}=1} y_j$  can be viewed as all the outgoing net flow from node  $l$ , and  $\sum_{(l,i) \text{ assoc. } j:p_{l,i}=1} y_j$  can be viewed as all the incoming net flow to node  $l$ . The  $u_l$  can be viewed as the net supply at node  $l$ . For node 0 to  $n$ , the outgoing net flow, minus the incoming net flow, is equal to the net supply. However, node  $n + 1$  absorbs the net flow by 1. Each dependency arc has the cost and the objective is to find the possible maximum cost for the flow. This is how the dual can be interpreted as a kind of network problem. The dual actually gives the critical path of the task scheduling problem, the primal. We will next show a concrete example.

Let  $n = 5$ ,  $m = 8$ ,  $d(i = 0..5) = (0, 2, 1, 6, 4, 2)$ ,

$$(p_{i=0..6, j=0..6}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}'.$$

The primal has the following constraints for  $p_{i,j} = 1$  (in the same order for matrix  $A$ ):

$$\begin{aligned} t_1 - t_0 &\geq 0 \\ t_2 - t_1 &\geq 2 \\ t_4 - t_1 &\geq 2 \\ t_3 - t_2 &\geq 1 \\ t_5 - t_3 &\geq 6 \\ t_5 - t_4 &\geq 4 \\ t_6 - t_5 &\geq 2 \\ t_6 - t_4 &\geq 4 \end{aligned}$$

Then the dual problem (D)

$$(D) \quad \begin{aligned} \max \quad & 2y_2 + 2y_3 + y_4 + 6y_5 + 4y_6 + 2y_7 + 4y_8 \\ & -y_1 \leq 0, \\ & y_1 - y_2 - y_3 \leq 0, \\ & y_2 - y_4 \leq 0, \\ & y_4 - y_5 \leq 0, \\ & y_3 - y_6 - y_8 \leq 0, \\ & y_5 + y_6 - y_7 \leq 0, \\ & y_7 + y_8 \leq 1, \\ & y_i \geq 0. \end{aligned}$$

Through **AMPL**, we get the optimal solution for (D):  $\hat{y} = (0, 1, 0, 1, 1, 0, 1, 0)'$ , and the optimal objective value is 11. By examining the constraints, we can see that the '1's in the optimal solution correspond to the following task dependency edge:  $(1 \rightarrow 2), (2 \rightarrow 3), (3 \rightarrow 5), (5 \rightarrow 6)$ . By comparing this solution to the primal optimal solution, we see that the primal and the dual has the same optimal value, and the dual solution actually indicates the critical path of finishing the task. That is, it indicates the longest necessary path through the network of tasks.

## 2. EXERCISE 8.2 PIVOTING AND TOTAL UNIMODULARITY

**Problem:** A pivot in an  $m \times n$  matrix  $A$  means choosing a row  $i$  and column  $j$  with  $a_{ij} \neq 0$ , subtracting  $\frac{a_{kj}}{a_{ij}}$  times row  $i$  from all other rows  $k (\neq i)$ , and then dividing row  $i$  by  $a_{ij}$ . Note that after the pivot, column  $j$  becomes the  $i$ -th standard-unit column. Prove that if  $A$  is  $TU$ , then it is  $TU$  after a pivot.

**Proof:**

Let  $A_p$  be the matrix after applying the pivot to  $A$  by choosing a row  $i$  and column  $j$  with  $a_{ij} \neq 0$ . After

the pivot,  $A^p$  will be something like this: 
$$i \quad \left( \begin{array}{c|c|c} & j & \\ \hline & 0 & \\ \vdots & \vdots & \\ \hline & 0 & \\ \vdots & \vdots & \\ \hline & 1 & \\ \vdots & \vdots & \\ \hline & 0 & \\ \vdots & \vdots & \\ \hline & 0 & \end{array} \right).$$
 So our goal is to prove that  $A^p$  is also a

$TU$  when  $A$  is  $TU$ .

$\because A$  is  $TU$ , and according to Theorem 8.6 (iii), appending standard-unit columns to  $A$  leaves  $A$   $TU$

$\therefore A^1 = [A, \mathbf{I}_m]$  is  $TU$

$\therefore$  by definition every square nonsingular submatrix  $B$  of  $A^1$  has  $\det(B) = \pm 1$

$\therefore$  every square nonsingular submatrix  $B$  of  $A^1$  is unimodular

$\therefore$  every basis matrix of  $A^1$  is square nonsingular submatrix

$\therefore$  every basis matrix of  $A^1$  is unimodular

Then we apply the  $a_{ij}$  pivot to  $A^1$ , and get  $A^2 = \begin{pmatrix} A^p & D \end{pmatrix}$ , where  $D$  will be something like this:

$$i \quad \left( \begin{array}{c|c|c} & i & \\ \hline \mathbf{I}_{i-1} & \vdots & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & \vdots & \mathbf{I}_{m-i} \end{array} \right).$$

$\therefore$  the pivot is a set of elementary row operations on the basis matrix of  $A^1$ , and each element of  $A$  is either 1 or (-1)

$\therefore \frac{a_{kj}}{a_{ij}} = \pm 1$  and  $a_{ij} = \pm 1$  in these elementary row operations

$\therefore$  the determinants of every basis matrix of  $A^1$  and the corresponding basis matrix of  $A^2$  using the same basis are  $\pm 1$

$\therefore$  every basis matrix of  $A^2$  is also unimodular

We switch the  $j$ -th column of  $A^p$  and the  $i$ -th column of  $D$  in  $A^2$ .  $A_j = e^i$ , so  $D$  become  $\mathbf{I}_m$ , and  $A^p$  becomes  $\begin{pmatrix} A_1 & \dots & A_{j-1} & D_i & A_{j+1} & \dots & A_n \end{pmatrix}$ .

Now we consider the matrix  $A^3 = \begin{pmatrix} A^q & \mathbf{I}_m \end{pmatrix}$ , where  $A^q = \begin{pmatrix} A_1 & \dots & A_{j-1} & A_{j+1} & \dots & A_n \end{pmatrix}$ .

$\therefore A^3$  consists of all columns in  $A^2$  except  $D_i$

$\therefore$  every basis matrix of  $A^3$  is also basis matrix of  $A^2$

$\therefore$  every basis matrix of  $A^3$  is unimodular

$\therefore$  using Theorem 8.5,  $A^q$  is  $TU$

$\therefore A^{q1} = \begin{pmatrix} A_1 & \dots & A_{j-1} & A_{j+1} & \dots & A_n & e^i \end{pmatrix}$  is also  $TU$  using Theorem 8.6 (iii)

$\therefore A_j = e^i$

$\therefore A^{q1} = \begin{pmatrix} A_1 & \dots & A_{j-1} & A_{j+1} & \dots & A_n & A_j \end{pmatrix}$  is  $TU$

Note that by switching columns of  $A^{q1}$ , we can get  $A^p$ . So if switching columns does not change the  $TU$  property,  $A^p$  is then proved to be  $TU$ . So next we prove that switching columns does not change the  $TU$  property.

For  $TU$  matrix  $A = \begin{pmatrix} A_1 & \dots & A_i & \dots & A_j & \dots & A_n \end{pmatrix}$ , suppose that we switch column  $A_i$  and  $A_j$  and get  $A^1$ . For a square nonsingular submatrix  $B^1$  of  $A^1$ , there are 3 cases:

- (1)  $B^1$  does not have any overlap with column  $i$  or  $j$  in  $A^1$ . So after the switching,  $B^1$  also has the same determinant value as in  $A$ , thus  $\det(B^1) = \pm 1$ ;

- (2)  $B^1$  has overlap with one of column  $i$  or  $j$  in  $A^1$ . Let's say with column  $i$  in  $A^1$ . Suppose  $B^1$  is a  $r \times r$  matrix and column  $m$  of  $B^1$  overlaps with column  $i$  of  $A^1$ , so  $B_m^1$  was in column  $j$  of  $A$ . So  $B^1$  can be obtained by switching several columns of  $A$ 's square submatrix  $B = (B_1^1 \dots B_{m-1}^1 B_{m+1}^1 \dots B_r^1 B_m^1)$ . Since  $B^1$  is nonsingular and  $B$  just changes the order of columns compared to  $B^1$ ,  $B$  is also nonsingular. Considering that  $A$  is  $TU$ ,  $\det(B) = \pm 1$ . So  $\det(B^1) = (-1)^s \det(B) = (-1)^s (\pm 1) = \pm 1$ , where  $s$  is the time of switching.
- (3)  $B^1$  has overlap with both column  $i$  and  $j$  in  $A^1$ . So  $B^1$  can be obtained by switching two columns of a nonsingular submatrix  $B$  in  $A$ . Thus,  $\det(B^1) = \pm 1$ .

Thus, for all square nonsingular matrix  $B^1$  of  $A^1$ ,  $\det(B^1) = \pm 1$ , which means that all of them are unimodular. So we just prove that switching two columns of  $TU$  matrix  $A$  still results in a  $TU$  matrix.

Thus, after switching columns of  $TU$  matrix  $A^{q1}$ , we get  $A^p$ , which is also a  $TU$  matrix. ■

### 3. EXERCISE 8.3 COMPARING FORMULATIONS FOR A TOY PROBLEM

**Problem:** Consider the systems:

$$(3.1) \quad \begin{aligned} S_1 : \quad & 2x_1 + 2x_2 + x_3 + x_4 \leq 2; \\ & x_j \leq 1; \\ & -x_j \leq 0. \end{aligned}$$

$$(3.2) \quad \begin{aligned} S_2 : \quad & x_1 + x_2 + x_3 \leq 1; \\ & x_1 + x_2 + x_4 \leq 1; \\ & -x_j \leq 0. \end{aligned}$$

$$(3.3) \quad \begin{aligned} S_3 : \quad & x_1 + x_2 \leq 1; \\ & x_1 + x_3 \leq 1; \\ & x_1 + x_4 \leq 1; \\ & x_2 + x_3 \leq 1; \\ & x_2 + x_4 \leq 1; \\ & -x_j \leq 0. \end{aligned}$$

Notice that each system has precisely the same set of integer solutions. Compare the feasible regions  $S_i$  of the continuous relaxations, for each pair of these systems. Specifically, for each choice of pair  $i = j$ , demonstrate whether or not the solution set of  $S_i$  is contained in the solution set of  $S_j$ . HINT: To prove that the solution set of  $S_i$  is contained in the solution set of  $S_j$ , it suffices to demonstrate that every inequality of  $S_j$  is a nonnegative linear combination of the inequalities of  $S_i$ . To prove that the solution set of  $S_i$  is not contained in the solution set of  $S_j$ , it suffices to give a solution of  $S_i$  that is not a solution of  $S_j$ .

**Solution:** Comparing the feasible regions  $S_i$  of the continuous relaxations: (1) Comparing  $S_1$  and  $S_2$ .  $\forall x \in S_2$ , we have

$$x_1 + x_2 + x_3 \leq 1, \quad x_1 + x_2 + x_4 \leq 1, \quad -x_j \leq 0.$$

Then,

$$\begin{aligned} (x_1 + x_2 + x_3) + (x_1 + x_2 + x_4) &\leq 1 + 1, \\ (x_1 + x_2 + x_3) + (-x_2) + (-x_3) &\leq 1, \\ (x_1 + x_2 + x_3) + (-x_1) + (-x_3) &\leq 1, \\ (x_1 + x_2 + x_3) + (-x_1) + (-x_2) &\leq 1, \\ (x_1 + x_2 + x_4) + (-x_1) + (-x_2) &\leq 1, \\ -x_j &\leq 0, \end{aligned}$$

which are equivalent to

$$2x_1 + 2x_2 + x_3 + x_4 \leq 2, \quad x_j \leq 1, \quad -x_j \leq 0.$$

So  $x \in S_1$ , and  $S_2 \subset S_1$ .  $S_1$  is not contained in  $S_2$  because  $(0, 0.5, 1, 0)' \in S_1$  but  $(0, 0.5, 1, 0)' \notin S_2$ .

(2) Comparing  $S_2$  and  $S_3$ .  $\forall x \in S_2$ , we have

$$x_1 + x_2 + x_3 \leq 1, \quad x_1 + x_2 + x_4 \leq 1, \quad -x_j \leq 0.$$

Then,

$$(x_1 + x_2 + x_3) + (-x_3) \leq 1,$$

$$\begin{aligned}
(x_1 + x_2 + x_3) + (-x_2) &\leq 1, \\
(x_1 + x_2 + x_4) + (-x_2) &\leq 1, \\
(x_1 + x_2 + x_3) + (-x_1) &\leq 1, \\
(x_1 + x_2 + x_4) + (-x_1) &\leq 1, \\
-x_j &\leq 0,
\end{aligned}$$

which are equivalent to

$$x_1 + x_2 \leq 1, \quad x_1 + x_3 \leq 1, \quad x_1 + x_4 \leq 1, \quad x_2 + x_3 \leq 1, \quad x_2 + x_4 \leq 1, \quad -x_j \leq 0.$$

So  $x \in S_3$ , and  $S_2 \subset S_3$ .  $S_3$  is not contained in  $S_2$  because  $(0.5, 0.5, 0.5, 0.5)' \in S_3$  but  $(0.5, 0.5, 0.5, 0.5)' \notin S_2$ .

(3) Comparing  $S_1$  and  $S_3$ .  $S_3$  is not contained in  $S_1$  because  $(0.5, 0.5, 0.5, 0.5)' \in S_3$  but  $(0.5, 0.5, 0.5, 0.5)' \notin S_1$ .  $S_1$  is not contained in  $S_3$  because  $(0, 0.5, 1, 0)' \in S_1$  but  $(0, 0.5, 1, 0)' \notin S_3$ .

#### 4. EXERCISE 8.4 COMPARING FACILITY-LOCATION FORMULATIONS

**Problem:** We have seen two formulations of the forcing constraints for the uncapacitated facility-location problem. We have a choice of the  $mn$  constraints:  $y_i + x_{ij} \leq 0$ , for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ , or the  $m$  constraints:  $ny_i + nx_{ij} \leq 0$ , for  $i = 1, \dots, m$ . Which formulation is stronger? That is, compare (both *computationally* and *analytically*) the strength of the two associated continuous relaxations (i.e., when we relax  $y_i \in \{0, 1\}$  to  $0 \leq y_i \leq 1$ , for  $i = 1, \dots, m$ ). In Appendix A.3, there is AMPL code for trying computational experiments.

**Solution:**

#### 5. EXERCISE 8.5 COMPARING PIECEWISE-LINEAR FORMULATIONS

**Problem:** We have seen that the adjacency condition for piecewise-linear univariate functions can be modeled by

$$\begin{aligned}
(5.1) \quad &\lambda_1 \leq y_1; \\
&\lambda_j \leq y_{j-1} + y_j, \quad \text{for } j = 2, \dots, n-1; \\
&\lambda_n \leq y_{n-1}.
\end{aligned}$$

An alternative formulation is

$$\begin{aligned}
(5.2) \quad &\sum_{i=1}^j y_i \leq \sum_{i=1}^{j+1} \lambda_i, \quad \text{for } j = 1, \dots, n-2; \\
&\sum_{i=j}^{n-1} y_i \leq \sum_{i=j}^n \lambda_i, \quad \text{for } j = 2, \dots, n-1.
\end{aligned}$$

Explain why this alternative formulation is valid, and compare its strength to the original formulation, when we relax  $y_i \in \{0, 1\}$  to  $0 \leq y_i \leq 1$ , for  $i = 1, \dots, n-1$ . (Note that for both formulations, we require  $\lambda_i \geq 0$ , for  $i = 1, \dots, n$ ,  $\sum_{i=1}^n \lambda_i = 1$ , and  $\sum_{i=1}^{n-1} y_i = 1$ ).

**Solution:**

When we have  $y_i \in \{0, 1\}$ , the alternative formulation (5.2) is equivalent to formulation (5.1). In (5.1), the formulation means that if  $y_k$  is 1, for some  $k$  ( $1 \leq k \leq n$ ), and necessarily all of the other  $y_j$  are 0, then only  $\lambda_k$  and  $\lambda_{k+1}$  can be positive. It is easy to see that in (5.2), the meaning is the same: if  $y_k$  is 1,  $\sum_{i=1}^k y_i = 1 \leq \sum_{i=1}^{k+1} \lambda_i$ , and  $\sum_{i=k}^{n-1} y_i = 1 \leq \sum_{i=k}^n \lambda_i$ , so considering that  $\sum_{i=1}^n \lambda_i = 1$ , thus  $\sum_{i=1}^{k+1} \lambda_i = \sum_{i=k}^n \lambda_i = 1$ , thus  $\lambda_k + \lambda_{k+1} = \sum_{i=1}^{k+1} \lambda_i - (\sum_{i=1}^n \lambda_i - \sum_{i=k}^n \lambda_i) = 1 - (1 - 1) = 1$

#### 6. EXERCISE 8.6 GOMORY CUTS

**Problem:** Consider the standard form problem having

$$A := \begin{pmatrix} 7 & 1 & 1 & 0 & 0 \\ -1 & 3 & 0 & 1 & 0 \\ -8 & -9 & 0 & 0 & 1 \end{pmatrix}, \quad b := \begin{pmatrix} 28 \\ 7 \\ -32 \end{pmatrix}, \quad c := (-2, -1, 0, 0, 0)'.$$

An optimal basis is  $\beta := (1, 2, 5)$ ,  $\eta := (3, 4)$ . Suppose that all of the variables are constrained to be integer. Derive all (Gom) cuts for this basis and compare their strength on the continuous relaxation of the feasible region. HINT: View everything in the space of the variables  $x_\eta$ .

Can you get a stronger (GMI) cut (with respect to this basis) by ignoring the integrality on some variables?

**Solution:**

## 7. EXERCISE 8.7 MAKE AMENDS

**Problem:** Find an *interesting applied problem*, model it as a pure- or mixed-*integer* linear-optimization problem, and test your model with AMPL. Credit will be given for *deft* modeling, *sophisticated* use of AMPL, testing on meaningfully-*large* instances, and *insightful* analysis. Try to play with CPLEX *integer* solver options (they can be set through AMPL) to get better behavior of the solver.

**Solution:**