# HOMEWORK #6 SOLUTION

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## 1. Exercise 6.1 Illustrate local sensitivity analysis

**Problem:** Make an original example to illustrate the local-analysis concepts of this chapter.
**Solution:**

## 2. Exercise 6.2 Illustrate global sensitivity analysis

**Problem:** Using `AMPL`, make an original example, with at least three constraints, graphing the objective value of $(P)$, as a single `b[i]` is varied from $-\infty$ to $+\infty$. As you work on this, bear in mind Theorem 6.3.
**Solution:**

## 3. Exercise 6.3 *"I feel that I know the change that is needed."* – Mahatma Gandhi

**Problem:** We are given $2m$ numbers satisfying $L_i \leq 0 \leq U_i$, $i = 1, 2, ..., m$. Let $\beta$ be an optimal basis for all of the $m$ problems

$$\begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax = b + \Delta_i e^i; \\ & \quad x \geq 0. \end{array} \tag{3.1}$$

for all $\Delta_i$ satisfying $L_i \leq \Delta_i \leq U_i$. Lets be clear on what this means: For each $i$ individually, the basis $\beta$ is optimal when the $i$th right-hand side component is changed from $b_i$ to $b_i + \Delta_i$, as long as $\Delta_i$ is in the interval $[L_i, U_i]$.

The point of this problem is to be able to say something about *simultaneously* changing all of the $b_i$. Prove that we can simultaneously change $b_i$ to

$$\tilde{b}_i := b_i + \lambda_i \left\{ \begin{array}{c} L_i \\ U_i \end{array} \right\}$$

where $\lambda_i \geq 0$, when $\sum_{i=1}^m \lambda_i \leq 1$. [Note that in the formula above, for each $i$ we can $i = 1$ pick either $L_i$ (a decrease) or $U_i$ (an increase)].

**Proof:**

Let $D_i = \left\{ \begin{array}{c} L_i \\ U_i \end{array} \right\}$, so $\max_{k:h_k^i>0} -\frac{\overline{b}_k}{h_k^i} \leq D_i \leq \min_{k:h_k^i<0} -\frac{\overline{b}_k}{h_k^i}$, where $\overline{b} = A_\beta^{-1} b \geq 0$, $h^i = A_\beta^{-1} e^i$, and $k = 1...m$.

Thus, what we need to prove is that the basis $\beta$ is optimal when we simultaneously change $b_i$ to $\tilde{b}_i := b_i + \lambda_i D_i$, where $\sum_{i=1}^m \lambda_i = 1$. In the other word, we need to prove that the basis $\beta$ is optimal for (3.2), where $\sum_{i=1}^m \lambda_i = 1$.

$$\begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax = b + \sum_{i=1}^m \lambda_i D_i e^i; \\ & \quad x \geq 0. \end{array} \tag{3.2}$$

Consider a fixed and optimal basis $\beta$ for (3.3). For $\beta$ to be optimal for (3.2), we need $A_\beta^{-1}(b + \sum_{i=1}^m \lambda_i D_i e^i) = \overline{b} + \sum_{i=1}^m \lambda_i D_i h^i \geq 0$.

$$\begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax = b; \\ & \quad x \geq 0. \end{array} \tag{3.3}$$

For $k = 1...m$,
$\overline{b}_k + \sum_{i=1}^m \lambda_i D_i h_k^i$
$= \overline{b}_k + \sum_{i=1,h_k^i>0}^m \lambda_i D_i h_k^i + \sum_{i=1,h_k^i<0}^m \lambda_i D_i h_k^i$
$\geq \overline{b}_k + \sum_{i=1,h_k^i>0}^m \lambda_i (\max_{k':h_{k'}^i>0} -\frac{\overline{b}_{k'}}{h_{k'}^i}) h_k^i + \sum_{i=1,h_k^i<0}^m \lambda_i (\min_{k':h_{k'}^i<0} -\frac{\overline{b}_{k'}}{h_{k'}^i}) h_k^i$
$\geq \overline{b}_k + \sum_{i=1,h_k^i>0}^m \lambda_i (-\frac{\overline{b}_k}{h_k^i}) h_k^i + \sum_{i=1,h_k^i<0}^m \lambda_i (-\frac{\overline{b}_k}{h_k^i}) h_k^i$
$= \overline{b}_k + \sum_{i=1,h_k^i>0}^m \lambda_i (-\overline{b}_k) + \sum_{i=1,h_k^i<0}^m \lambda_i (-\overline{b}_k)$

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$= \overline{b}_k + \sum_{i=1}^m \lambda_i(-\overline{b}_k)$
$= \overline{b}_k + (-\overline{b}_k)\sum_{i=1}^m \lambda_i$
$= \overline{b}_k + (-\overline{b}_k)$
$= 0$
Thus, $\overline{b} + \sum_{i=1}^m \lambda_i D_i h^i \geq 0$, and $\beta$ is feasible and hence still optimal for (3.2).

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## 4. Exercise 6.4 Domain for objective variations

**Problem:** Prove Theorem 6.4: The domain of $g$ is a convex set.
**Proof:**
Suppose that $c^j$ is in the domain of $g$, for $j = 1, 2$. Therefore, there exist $x^j$ that are feasible for (4.1), for $j = 1, 2$.

$$\begin{array}{ll} \min & {c^j}'x \\ \text{s.t.} & Ax = b; \\ & x \geq 0. \end{array} \tag{4.1}$$

For any $0 < \lambda < 1$, consider $\widehat{c} := \lambda c^1 + (1-\lambda)c^2$, to prove the domain of $g$ is a convex set, we need to prove that (4.2) is feasible and not unbounded.

$$\begin{array}{ll} \min & \widehat{c}'x \\ \text{s.t.} & Ax = b; \\ & x \geq 0. \end{array} \tag{4.2}$$

Consider $\widehat{x} := \lambda x^1 + (1-\lambda)x^2$,
$\because$ $x^j$ that are feasible for (4.1), for $j = 1, 2$
$\therefore$ $x^j \geq 0$, and $Ax^j = b$, for $j = 1, 2$
$\therefore$ $A\widehat{x} = A(\lambda x^1 + (1-\lambda)x^2) = \lambda Ax^1 + (1-\lambda)Ax^2 = \lambda b + (1-\lambda)b = b$
$\because$ $\lambda > 0$
$\therefore$ $\widehat{x} = \lambda x^1 + (1-\lambda)x^2 \geq 0$
$\therefore$ $\widehat{x}$ is a feasible solution to (4.2)
$\therefore$ (4.2) is feasible
Consider the objective function $\widehat{c}'x$ of (4.2),
$\because$ $c^j$ is in the domain of $g$, for $j = 1, 2$
$\therefore$ (4.1) is not unbounded for $j = 1, 2$
$\therefore$ $\exists k^j \in \mathbb{R}$, for any $x$, such that ${c^j}'x \geq k^j$, for $j = 1, 2$
$\therefore$ for a feasible solution $\overline{x}$ of (4.2), $\widehat{c}'\overline{x} = (\lambda c^1 + (1-\lambda)c^2)'\overline{x} = \lambda {c^1}'\overline{x} + (1-\lambda){c^2}'\overline{x} \geq (\lambda k^1 + (1-\lambda)k^2)$
$\because$ $k^1, k^2 \in \mathbb{R}$, $0 < \lambda < 1$
$\therefore$ (4.2) is feasible and not unbounded
$\therefore$ $\widehat{c}$ is in the domain $g$
$\therefore$ the domain of $g$ is a convex set

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## 5. Exercise 6.5 Concave piecewise-linear function

**Problem:** Prove Theorem 6.5: $g$ is a concave piecewise-linear function on its domain.
**Proof:**
The function $g$ is

$$\begin{array}{lll} g(c) := \min & c'x \\ \text{s.t.} & Ax = b; \\ & x \geq 0. \end{array} \tag{5.1}$$

So a basis $\beta$ is feasible or not for (5.1), independent for $c_\beta$. So $g$ can be written as

$$g(c) = \min\left\{c_\beta'(A_\beta^{-1}b) : \beta \text{ is a feasible basis for (5.1)}\right\}$$

$\because$ $c_\beta'(A_\beta^{-1}b)$ are affine functions
$\therefore$ $g$ is the pointwise minimum of a finite number of affine functions
$\therefore$ $g$ is a concave piecewise-linear function

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