# HOMEWORK #6 SOLUTION

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## 1. Exercise 6.1 Illustrate local sensitivity analysis

**Problem:** Make an original example to illustrate the local-analysis concepts of this chapter.
**Solution:** Consider the standard-form problem $(P_b)$ with $m = 3$ and $n = 5$

$$(P_b) \qquad \begin{array}{rrrcl} f(b) & := & \min & c'x & \\ & & & Ax & = & b\;; \\ & & & x & \geq & \mathbf{0}\,, \end{array}$$

where

$$A := \begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & -4 & 2 & 2 & 0 \\ 0 & -9 & 0 & 6 & 3 \end{pmatrix}, \quad b := (1, 2, 18)', \quad c := (16, 7, 20, 10, 4)'\,.$$

Consider a fixed basis $\beta = \{1, 4, 5\}$ for $(P_b)$. Associated with that basis is the basic solution $\bar{x}_\beta = A_\beta^{-1} b = (2, 1, 4)'$ and the corresponding dual solution $\bar{y}' = c'_\beta A_\beta^{-1} = (16, 9, 4/3)$, which is feasible for the dual of $(P_b)$. We have

$$f(b) = \bar{y}'b = 16b_1 + 9b_2 + (4/3)b_3,$$

for $b \in \mathcal{B}$ where $\mathcal{B} \subset \mathbb{R}^m$ is the solution set of m linear inequalities (more details below). We have

$$\frac{\partial f}{\partial b_i} = \bar{y}_i, \ i = 1, ..., m.$$

The $\mathcal{B}$ represents that the change to a single right-hand side element $b_i$ should keep $\bar{b}_i > 0$ for $i = 1, ..., m$. Let the right hand change be $b + \Delta_i \boldsymbol{e^i}$. Then $A_\beta^{-1}(b + \Delta_i \boldsymbol{e^i}) \geq \mathbf{0}$. Let $h^i := A_\beta^{-1} \boldsymbol{e^i}$. This means that $\Delta_i$ must be in the interval $[L_i, U_i]$, where

$$L_i := \max_{k : h^i_k > 0} -\bar{b}_k / h^i_k$$

, and

$$U_i := \min_{k : h^i_k < 0} -\bar{b}_k / h^i_k$$

.

We can confirm through `AMPL` that $\Delta_1 \in [-2, +\infty)$ , $\Delta_2 \in [-2, 4]$, and $\Delta_3 \in [-12, +\infty)$ (consistent with Table 1).

Next, we define a function $g : \mathbb{R}^n \to \mathbb{R}$ via

$$(P^c) \qquad \begin{array}{rrrcl} g(c) & := & \min & c'x & \\ & & & Ax & = & b\;; \\ & & & x & \geq & \mathbf{0}\,, \end{array}$$

with the same parameters. Consider a fixed basis $\beta = \{1, 4, 5\}$ for $(P^c)$. Similarly, we have $g(c) = c'_\beta \bar{x}_\beta = 2c_1 + c_4 + 4c_5$, which is a linear function on $c \in \mathcal{C}$. If only changing a single $c_i$ and the optimal basis keeps, the interval of $c_i$ are: $c_1 \in [-23/3, 18]$, $c_2 \in [-64, +\infty)$, $c_3 \in [18, +\infty)$, $c_4 \in [-25.5, 12]$, and $c_5 \in [3, 75]$.

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## 2. Exercise 6.2 Illustrate global sensitivity analysis

**Problem:** Using `AMPL`, make an original example, with at least three constraints, graphing the objective value of $(P)$, as a single `b[i]` is varied from $-\infty$ to $+\infty$. As you work on this, bear in mind Theorem 6.3.
**Solution:** Consider the standard-form problem (P) with $m = 3$ and $n = 5$

$$\text{(P)} \qquad \begin{array}{rrcl} \min & c'x & & \\ & Ax & = & b\,; \\ & x & \geq & \mathbf{0}\,, \end{array}$$

where

$$A := \begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & -4 & 2 & 2 & 0 \\ 0 & -9 & 0 & 6 & 3 \end{pmatrix}, \quad b := (1, 2, 18)', \quad c := (16, 7, 20, 10, 4)'.$$

Using `AMPL`, at each time we vary only one $b_i$ from $b$ and check the range of $b_i$ in which the optimal basis for (P) keeps the same. We show the optimal basis as follows:

| $b_1$ | $(-\infty, -13)$ | $(-13, -1)$ | $(-1, +\infty)$ |
|---|---|---|---|
| optimal basis $\beta$ | $\{2, 3, 4\}$ | $\{2, 4, 5\}$ | $\{1, 4, 5\}$ |
| $c'_\beta A_\beta^{-1}$ | $(-12.8\ 10\ \ -3.8)$ | $(-23/3\ \ -17/6\ 4/3)$ | $(16\ 9\ 4/3)$ |
| objective value | $-12.8b_1 - 48.4$ | $(-23/3)b_1 + 55/3$ | $16b_1 + 42$ |

| $b_2$ | $(-\infty, 0)$ | $(0, 6)$ | $(6, +\infty)$ |
|---|---|---|---|
| optimal basis $\beta$ | $\{1, 2, 5\}$ | $\{1, 4, 5\}$ | $\{1, 3, 4\}$ |
| $c'_\beta A_\beta^{-1}$ | $(16\ \ -8.75\ 4/3)$ | $(16\ 9\ 4/3)$ | $(16\ 10\ 18)$ |
| objective value | $-8.75b_2 + 40$ | $9b_2 + 40$ | $10b_2 + 34$ |

| $b_3$ | $(-\infty, 0)$ | $(0, 6)$ | $(6, +\infty)$ |
|---|---|---|---|
| optimal basis $\beta$ | $\{1, 2, 3\}$ | $\{1, 3, 4\}$ | $\{1, 4, 5\}$ |
| $c'_\beta A_\beta^{-1}$ | $(16\ 10\ \ -7)$ | $(16\ 10\ 1)$ | $(16\ 9\ 4/3)$ |
| objective value | $-7b_3 + 36$ | $b_3 + 36$ | $(4/3)b_3 + 34$ |

Table 1. Changing single $b_i$ from $-\infty$ to $+\infty$

As we get the above data from `AMPL`, we graph the objective value of (P) as a single $b_i$ is varied from $-\infty$ to $+\infty$ in Figure 1.

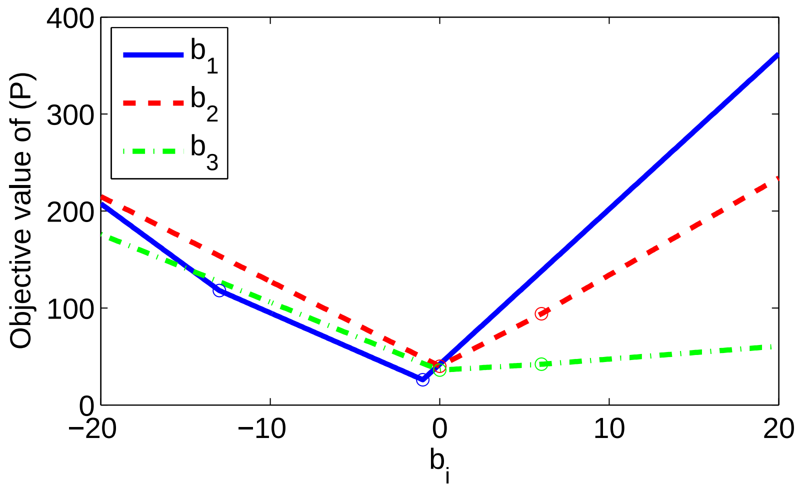


Figure 1. Objective value of (P) as a single $b_i$ is varied from $-\infty$ to $+\infty$

## 3. Exercise 6.3 *"I feel that I know the change that is needed."* – Mahatma Gandhi

**Problem:** We are given $2m$ numbers satisfying $L_i \leq 0 \leq U_i$, $i = 1, 2, ..., m$. Let $\beta$ be an optimal basis for all of the $m$ problems

$$\text{(3.1)} \qquad \begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax = b + \Delta_i e^i; \\ & x \geq 0. \end{array}$$

for all $\Delta_i$ satisfying $L_i \leq \Delta_i \leq U_i$. Lets be clear on what this means: For each $i$ individually, the basis $\beta$ is optimal when the $i$th right-hand side component is changed from $b_i$ to $b_i + \Delta_i$, as long as $\Delta_i$ is in the interval $[L_i, U_i]$.

The point of this problem is to be able to say something about *simultaneously* changing all of the $b_i$. Prove that we can simultaneously change $b_i$ to

$$\tilde{b}_i := b_i + \lambda_i \left\{ \begin{array}{c} L_i \\ U_i \end{array} \right\}$$

where $\lambda_i \geq 0$, when $\sum_{i=1}^m \lambda_i \leq 1$. [Note that in the formula above, for each $i$ we can $i = 1$ pick either $L_i$ (a decrease) or $U_i$ (an increase)].

**Proof:**

Let $D_i = \left\{ \begin{array}{c} L_i \\ U_i \end{array} \right\}$, so $\max_{k:h_k^i>0} -\frac{\overline{b}_k}{h_k^i} \leq D_i \leq \min_{k:h_k^i<0} -\frac{\overline{b}_k}{h_k^i}$, where $\overline{b} = A_\beta^{-1} b \geq 0$, $h^i = A_\beta^{-1} e^i$, and $k = 1...m$.

Thus, what we need to prove is that the basis $\beta$ is optimal when we simultaneously change $b_i$ to $\tilde{b}_i := b_i + \lambda_i D_i$, where $\sum_{i=1}^m \lambda_i = 1$. In the other word, we need to prove that the basis $\beta$ is optimal for (3.2), where $\sum_{i=1}^m \lambda_i = 1$.

$$\begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax = b + \sum_{i=1}^m \lambda_i D_i e^i; \\ & \qquad x \geq 0. \end{array} \tag{3.2}$$

Consider a fixed and optimal basis $\beta$ for (3.3). For $\beta$ to be optimal for (3.2), we need $A_\beta^{-1}(b + \sum_{i=1}^m \lambda_i D_i e^i) = \overline{b} + \sum_{i=1}^m \lambda_i D_i h^i \geq 0$.

$$\begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax = b; \\ & \quad x \geq 0. \end{array} \tag{3.3}$$

For $k = 1...m$,
$\overline{b}_k + \sum_{i=1}^m \lambda_i D_i h_k^i$
$= \overline{b}_k + \sum_{i=1,h_k^i>0}^m \lambda_i D_i h_k^i + \sum_{i=1,h_k^i<0}^m \lambda_i D_i h_k^i$
$\geq \overline{b}_k + \sum_{i=1,h_k^i>0}^m \lambda_i (\max_{k':h_{k'}^i>0} -\frac{\overline{b}_{k'}}{h_{k'}^i}) h_k^i + \sum_{i=1,h_k^i<0}^m \lambda_i (\min_{k':h_{k'}^i<0} -\frac{\overline{b}_{k'}}{h_{k'}^i}) h_k^i$
$\geq \overline{b}_k + \sum_{i=1,h_k^i>0}^m \lambda_i (-\frac{\overline{b}_k}{h_k^i}) h_k^i + \sum_{i=1,h_k^i<0}^m \lambda_i (-\frac{\overline{b}_k}{h_k^i}) h_k^i$
$= \overline{b}_k + \sum_{i=1,h_k^i>0}^m \lambda_i (-\overline{b}_k) + \sum_{i=1,h_k^i<0}^m \lambda_i (-\overline{b}_k)$
$= \overline{b}_k + \sum_{i=1}^m \lambda_i (-\overline{b}_k)$
$= \overline{b}_k + (-\overline{b}_k) \sum_{i=1}^m \lambda_i$
$\geq \overline{b}_k + (-\overline{b}_k)$
$= 0$
Thus, $\overline{b} + \sum_{i=1}^m \lambda_i D_i h^i \geq 0$, and $\beta$ is feasible and hence still optimal for (3.2).

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## 4. Exercise 6.4 Domain for objective variations

**Problem:** Prove Theorem 6.4: The domain of $g$ is a convex set.

**Proof:**

Suppose that $c^j$ is in the domain of $g$, for $j = 1, 2$. Therefore, there exist $x^j$ that are feasible for (4.1), for $j = 1, 2$.

$$\begin{array}{ll} \min & {c^j}'x \\ \text{s.t.} & Ax = b; \\ & \quad x \geq 0. \end{array} \tag{4.1}$$

For any $0 < \lambda < 1$, consider $\widehat{c} := \lambda c^1 + (1 - \lambda) c^2$, to prove the domain of $g$ is a convex set, we need to prove that (4.2) is feasible and not unbounded.

$$\begin{array}{ll} \min & \widehat{c}'x \\ \text{s.t.} & Ax = b; \\ & \quad x \geq 0. \end{array} \tag{4.2}$$

Consider $\widehat{x} := \lambda x^1 + (1 - \lambda) x^2$,
$\because$ $x^j$ that are feasible for (4.1), for $j = 1, 2$
$\therefore$ $x^j \geq 0$, and $Ax^j = b$, for $j = 1, 2$

$\therefore A\widehat{x} = A(\lambda x^1 + (1-\lambda)x^2) = \lambda Ax^1 + (1-\lambda)Ax^2 = \lambda b + (1-\lambda)b = b$
$\because \lambda > 0$
$\therefore \widehat{x} = \lambda x^1 + (1-\lambda)x^2 \geq 0$
$\therefore \widehat{x}$ is a feasible solution to (4.2)
$\therefore$ (4.2) is feasible
Consider the objective function $\widehat{c}'x$ of (4.2),
$\because c^j$ is in the domain of $g$, for $j = 1, 2$
$\therefore$ (4.1) is not unbounded for $j = 1, 2$
$\therefore \exists k^j \in \mathbb{R}$, for any $x$, such that ${c^j}'x \geq k^j$, for $j = 1, 2$
$\therefore$ for a feasible solution $\overline{x}$ of (4.2), $\widehat{c}'\overline{x} = (\lambda c^1 + (1-\lambda)c^2)'\overline{x} = \lambda {c^1}'\overline{x} + (1-\lambda){c^2}'\overline{x} \geq (\lambda k^1 + (1-\lambda)k^2)$
$\because k^1, k^2 \in \mathbb{R}$, $0 < \lambda < 1$
$\therefore$ (4.2) is feasible and not unbounded
$\therefore \widehat{c}$ is in the domain $g$
$\therefore$ the domain of $g$ is a convex set

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## 5. Exercise 6.5 Concave piecewise-linear function

**Problem:** Prove Theorem 6.5: $g$ is a concave piecewise-linear function on its domain.
**Proof:**
The function $g$ is

$$
\begin{aligned}
g(c) := \min \quad & c'x \\
\text{s.t.} \quad & Ax = b; \\
& x \geq 0.
\end{aligned}
\tag{5.1}
$$

So a basis $\beta$ is feasible or not for (5.1), independent for $c_\beta$. So $g$ can be written as

$$g(c) = \min\left\{c'_\beta(A_\beta^{-1}b) : \beta \text{ is a feasible basis for (5.1)}\right\}$$

$\because c'_\beta(A_\beta^{-1}b)$ are affine functions
$\therefore g$ is the pointwise minimum of a finite number of affine functions
$\therefore g$ is a concave piecewise-linear function

$\blacksquare$