HOMEWORK #6 SOLUTION

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1. Exercise 6.1 Illustrate local sensitivity analysis

Problem: Make an original example to illustrate the local-analysis concepts of this chapter.
Solution: Consider the standard-form problem (P_b) with $m = 3$ and $n = 5$

$$(P_b) \qquad \begin{array}{rcl} f(b) := \min & c'x & \\ & Ax & = b\ ; \\ & x & \geq \mathbf{0}\ , \end{array}$$

where

$$A := \begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & -4 & 2 & 2 & 0 \\ 0 & -9 & 0 & 6 & 3 \end{pmatrix}\ , \quad b := (1, 2, 18)'\ , \quad c := (16, 7, 20, 10, 4)'\ .$$

Consider a fixed basis $\beta = \{1, 4, 5\}$ for (P_b). Associated with that basis is the basic solution $\bar{x}_\beta = A_\beta^{-1} b = (2, 1, 4)'$ and the corresponding dual solution $\bar{y}' = c'_\beta A_\beta^{-1} = (16, 9, 4/3)$, which is feasible for the dual of (P_b). We have

$$f(b) = \bar{y}'b = 16b_1 + 9b_2 + (4/3)b_3,$$

for $b \in \mathcal{B}$ where $\mathcal{B} \subset \mathbb{R}^m$ is the solution set of m linear inequalities (more details below). We have

$$\frac{\partial f}{\partial b_i} = \bar{y}_i,\ i = 1, ..., m.$$

The $\mathcal{B}$ represents that the change to a single right-hand side element b_i should keep $\bar{b}_i > 0$ for $i = 1, ..., m$. Let the right hand change be $b + \Delta_i \boldsymbol{e^i}$. Then $A_\beta^{-1}(b + \Delta_i \boldsymbol{e^i}) \geq \mathbf{0}$. Let $h^i := A_\beta^{-1} \boldsymbol{e^i}$. This means that Δ_i must be in the interval $[L_i, U_i]$, where

$$L_i := \max_{k: h_k^i > 0} -\bar{b}_k / h_k^i$$

, and

$$U_i := \min_{k: h_k^i < 0} -\bar{b}_k / h_k^i$$

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We can confirm through `AMPL` that $\Delta_1 \in [-2, +\infty)$, $\Delta_2 \in [-2, 4]$, and $\Delta_3 \in [-12, +\infty)$ (consistent with Table 1).

Next, we define a function $g\ :\ \mathbb{R}^n \to \mathbb{R}$ via

$$(P^c) \qquad \begin{array}{rcl} g(c) := \min & c'x & \\ & Ax & = b\ ; \\ & x & \geq \mathbf{0}\ , \end{array}$$

with the same parameters. Consider a fixed basis $\beta = \{1, 4, 5\}$ for (P^c). Similarly, we have $g(c) = c'_\beta \bar{x}_\beta = 2c_1 + c_4 + 4c_5$, which is a linear function on $c \in \mathcal{C}$. If only changing a single c_i and the optimal basis keeps, the interval of c_i are: $c_1 \in [-23/3, 18]$, $c_2 \in [-64, +\infty)$, $c_3 \in [18, +\infty)$, $c_4 \in [-25.5, 12]$, and $c_5 \in [3, 75]$.

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2. Exercise 6.2 Illustrate global sensitivity analysis

Problem: Using `AMPL`, make an original example, with at least three constraints, graphing the objective value of (P), as a single `b[i]` is varied from $-\infty$ to $+\infty$. As you work on this, bear in mind Theorem 6.3.

Solution: Consider the standard-form problem (P) with $m = 3$ and $n = 5$

$$\text{(P)} \qquad \begin{array}{rrcl} \min & c'x & & \\ & Ax & = & b\,; \\ & x & \geq & \mathbf{0}\,, \end{array}$$

where

$$A := \begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & -4 & 2 & 2 & 0 \\ 0 & -9 & 0 & 6 & 3 \end{pmatrix}, \quad b := (1, 2, 18)', \quad c := (16, 7, 20, 10, 4)'.$$

Using `AMPL`, at each time we vary only one b_i from b and check the range of b_i in which the optimal basis for (P) keeps the same. We show the optimal basis as follows:

b_1	$(-\infty, -13)$	$(-13, -1)$	$(-1, +\infty)$
optimal basis β	$\{2, 3, 4\}$	$\{2, 4, 5\}$	$\{1, 4, 5\}$
$c'_\beta A_\beta^{-1}$	$(-12.8\ 10\ -3.8)$	$(-23/3\ -17/6\ 4/3)$	$(16\ 9\ 4/3)$
objective value	$-12.8b_1 - 48.4$	$(-23/3)b_1 + 55/3$	$16b_1 + 42$

b_2	$(-\infty, 0)$	$(0, 6)$	$(6, +\infty)$
optimal basis β	$\{1, 2, 5\}$	$\{1, 4, 5\}$	$\{1, 3, 4\}$
$c'_\beta A_\beta^{-1}$	$(16\ -8.75\ 4/3)$	$(16\ 9\ 4/3)$	$(16\ 10\ 18)$
objective value	$-8.75b_2 + 40$	$9b_2 + 40$	$10b_2 + 34$

b_3	$(-\infty, 0)$	$(0, 6)$	$(6, +\infty)$
optimal basis β	$\{1, 2, 3\}$	$\{1, 3, 4\}$	$\{1, 4, 5\}$
$c'_\beta A_\beta^{-1}$	$(16\ 10\ -7)$	$(16\ 10\ 1)$	$(16\ 9\ 4/3)$
objective value	$-7b_3 + 36$	$b_3 + 36$	$(4/3)b_3 + 34$

Table 1. Changing single b_i from $-\infty$ to $+\infty$

As we get the above data from `AMPL`, we graph the objective value of (P) as a single b_i is varied from $-\infty$ to $+\infty$ in Figure 1.

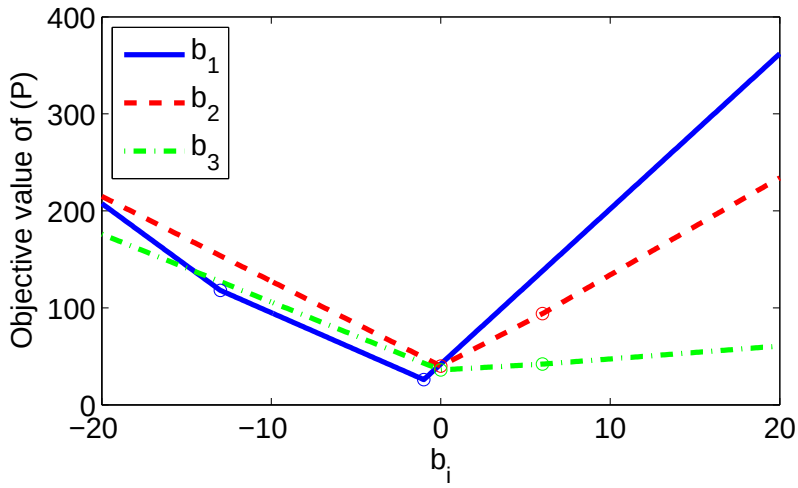

Figure 1. Objective value of (P) as a single b_i is varied from $-\infty$ to $+\infty$

3. Exercise 6.3 *"I feel that I know the change that is needed."* – Mahatma Gandhi

Problem: We are given $2m$ numbers satisfying $L_i \leq 0 \leq U_i$, $i = 1, 2, ..., m$. Let β be an optimal basis for all of the m problems

$$\text{(3.1)} \qquad \begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax = b + \Delta_i e^i; \\ & \qquad\qquad x \geq 0. \end{array}$$

for all Δ_i satisfying $L_i \le \Delta_i \le U_i$. Lets be clear on what this means: For each i individually, the basis β is optimal when the ith right-hand side component is changed from b_i to $b_i + \Delta_i$, as long as Δ_i is in the interval $[L_i, U_i]$.

The point of this problem is to be able to say something about *simultaneously* changing all of the b_i. Prove that we can simultaneously change b_i to

$$\tilde{b}_i := b_i + \lambda_i \left\{ \begin{array}{c} L_i \\ U_i \end{array} \right\}$$

where $\lambda_i \ge 0$, when $\sum_{i=1}^m \lambda_i \le 1$. [Note that in the formula above, for each i we can $i = 1$ pick either L_i (a decrease) or U_i (an increase)].

Proof:

Let $D_i = \left\{ \begin{array}{c} L_i \\ U_i \end{array} \right\}$, so $\max_{k:h_k^i>0} -\frac{\overline{b}_k}{h_k^i} \le D_i \le \min_{k:h_k^i<0} -\frac{\overline{b}_k}{h_k^i}$, where $\overline{b} = A_\beta^{-1} b \ge 0$, $h^i = A_\beta^{-1} e^i$, and $k = 1...m$.

Thus, what we need to prove is that the basis β is optimal when we simultaneously change b_i to $\tilde{b}_i := b_i + \lambda_i D_i$, where $\sum_{i=1}^m \lambda_i = 1$. In the other word, we need to prove that the basis β is optimal for (3.2), where $\sum_{i=1}^m \lambda_i = 1$.

$$\begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax = b + \sum_{i=1}^m \lambda_i D_i e^i; \\ & \qquad\qquad x \ge 0. \end{array} \tag{3.2}$$

Consider a fixed and optimal basis β for (3.3). For β to be optimal for (3.2), we need $A_\beta^{-1}(b + \sum_{i=1}^m \lambda_i D_i e^i) = \overline{b} + \sum_{i=1}^m \lambda_i D_i h^i \ge 0$.

$$\begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax = b; \\ & \quad x \ge 0. \end{array} \tag{3.3}$$

For $k = 1...m$,
$\overline{b}_k + \sum_{i=1}^m \lambda_i D_i h_k^i$
$= \overline{b}_k + \sum_{i=1,h_k^i>0}^m \lambda_i D_i h_k^i + \sum_{i=1,h_k^i<0}^m \lambda_i D_i h_k^i$
$\ge \overline{b}_k + \sum_{i=1,h_k^i>0}^m \lambda_i (\max_{k':h_{k'}^i>0} -\frac{\overline{b}_{k'}}{h_{k'}^i}) h_k^i + \sum_{i=1,h_k^i<0}^m \lambda_i (\min_{k':h_{k'}^i<0} -\frac{\overline{b}_{k'}}{h_{k'}^i}) h_k^i$
$\ge \overline{b}_k + \sum_{i=1,h_k^i>0}^m \lambda_i (-\frac{\overline{b}_k}{h_k^i}) h_k^i + \sum_{i=1,h_k^i<0}^m \lambda_i (-\frac{\overline{b}_k}{h_k^i}) h_k^i$
$= \overline{b}_k + \sum_{i=1,h_k^i>0}^m \lambda_i (-\overline{b}_k) + \sum_{i=1,h_k^i<0}^m \lambda_i (-\overline{b}_k)$
$= \overline{b}_k + \sum_{i=1}^m \lambda_i (-\overline{b}_k)$
$= \overline{b}_k + (-\overline{b}_k) \sum_{i=1}^m \lambda_i$
$\ge \overline{b}_k + (-\overline{b}_k)$
$= 0$
Thus, $\overline{b} + \sum_{i=1}^m \lambda_i D_i h^i \ge 0$, and β is feasible and hence still optimal for (3.2).

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4. Exercise 6.4 Domain for objective variations

Problem: Prove Theorem 6.4: The domain of g is a convex set.

Proof:

Suppose that c^j is in the domain of g, for $j = 1, 2$. Therefore, there exist x^j that are feasible for (4.1), for $j = 1, 2$.

$$\begin{array}{ll} \min & {c^j}'x \\ \text{s.t.} & Ax = b; \\ & \quad x \ge 0. \end{array} \tag{4.1}$$

For any $0 < \lambda < 1$, consider $\widehat{c} := \lambda c^1 + (1-\lambda) c^2$, to prove the domain of g is a convex set, we need to prove that (4.2) is feasible and not unbounded.

$$\begin{array}{ll} \min & \widehat{c}'x \\ \text{s.t.} & Ax = b; \\ & \quad x \ge 0. \end{array} \tag{4.2}$$

Consider $\widehat{x} := \lambda x^1 + (1-\lambda) x^2$,
$\because x^j$ that are feasible for (4.1), for $j = 1, 2$
$\therefore x^j \ge 0$, and $Ax^j = b$, for $j = 1, 2$

$\therefore A\widehat{x} = A(\lambda x^1 + (1-\lambda)x^2) = \lambda Ax^1 + (1-\lambda)Ax^2 = \lambda b + (1-\lambda)b = b$
$\because \lambda > 0$
$\therefore \widehat{x} = \lambda x^1 + (1-\lambda)x^2 \geq 0$
$\therefore \widehat{x}$ is a feasible solution to (4.2)
$\therefore$ (4.2) is feasible
Consider the objective function $\widehat{c}'x$ of (4.2),
$\because c^j$ is in the domain of g, for $j = 1, 2$
$\therefore$ (4.1) is not unbounded for $j = 1, 2$
$\therefore \exists k^j \in \mathbb{R}$, for any x, such that $c^{j\prime}x \geq k^j$, for $j = 1, 2$
$\therefore$ for a feasible solution $\overline{x}$ of (4.2), $\widehat{c}'\overline{x} = (\lambda c^1 + (1-\lambda)c^2)'\overline{x} = \lambda c^{1\prime}\overline{x} + (1-\lambda)c^{2\prime}\overline{x} \geq (\lambda k^1 + (1-\lambda)k^2)$
$\because k^1, k^2 \in \mathbb{R}$, $0 < \lambda < 1$
$\therefore$ (4.2) is feasible and not unbounded
$\therefore \widehat{c}$ is in the domain g
$\therefore$ the domain of g is a convex set

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5. Exercise 6.5 Concave piecewise-linear function

Problem: Prove Theorem 6.5: g is a concave piecewise-linear function on its domain.
Proof:
The function g is

$$\begin{aligned} g(c) := \min \quad & c'x \\ \text{s.t.} \quad & Ax = b; \\ & x \geq 0. \end{aligned} \tag{5.1}$$

So a basis β is feasible or not for (5.1), independent for c_β. So g can be written as

$$g(c) = \min \left\{ c'_\beta (A_\beta^{-1} b) : \beta \text{ is a feasible basis for (5.1)} \right\}$$

$\because c'_\beta(A_\beta^{-1}b)$ are affine functions
$\therefore g$ is the pointwise minimum of a finite number of affine functions
$\therefore g$ is a concave piecewise-linear function

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