

HOMEWORK #1 SOLUTION

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1. EXERCISE 1.1 CONVERT TO STANDARD FORM

Let's take the following problem as an example. Suppose Tom wants to make money and he decides to open several restaurants in the city he lives. There are two types of restaurants, Café which needs 10 employees each and has a profit of \$100 per day, and Mongolian barbecue which needs 20 employees each and has a profit of \$300 per day. Tom would like to open at least 10 restaurants but he only has 150 people to serve in restaurants. The goal is to maximize the profit of all restaurants. This is an optimization problem and can be formalized as follows:

$$\begin{aligned} \max \quad & 100x_1 + 300x_2 \\ & x_1 + x_2 \geq 10 ; \\ & 10x_1 + 20x_2 \leq 150 ; \\ & x_1 \geq 0 ; \\ & x_2 \geq 0 . \end{aligned}$$

To transform this general linear-optimization problem to one in standard form, first we change the maximum of $100x_1 + 300x_2$ to the negative of the minimum of $-100x_1 - 300x_2$. We then change inequalities to equalities: $x_1 + x_2 \geq 10$ to $x_1 + x_2 - t_1 = 10$, and $10x_1 + 20x_2 \leq 150$ to $10x_1 + 20x_2 + t_2 = 150$, where $t_1, t_2 \geq 0$. Now we have

$$\begin{aligned} \min \quad & -100x_1 - 300x_2 \\ & x_1 + x_2 - t_1 = 10 ; \\ & 10x_1 + 20x_2 + t_2 = 150 ; \\ & t_1 \geq 0 ; \\ & t_2 \geq 0 ; \\ & x_1 \geq 0 ; \\ & x_2 \geq 0 , \end{aligned}$$

which is in standard form of the following

$$(P) \quad \begin{aligned} \min \quad & c'x \\ & Ax = b ; \\ & x \geq \mathbf{0} , \end{aligned}$$

where

$$c = \begin{pmatrix} -100 \\ -300 \\ 0 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 10 & 20 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 10 \\ 150 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ t_1 \\ t_2 \end{pmatrix}.$$

2. EXERCISE 1.2 WEAK DUALITY EXAMPLE

We still take the optimization problem (P) in Exercise 1.1 as an example. Its Dual is

$$(D) \quad \begin{aligned} \max \quad & y'b \\ & y'A \leq c' , \end{aligned}$$

where y is a vector of variables in \mathbb{R}^2 . Let $y = (y_1 \ y_2)'$. (D) is equivalent to

$$\begin{aligned}
\max \quad & 10y_1 + 150y_2 \\
& y_1 + 10y_2 \leq -100 ; \\
& y_1 + 20y_2 \leq -300 ; \\
& y_1 \geq 0 ; \\
& y_2 \leq 0 .
\end{aligned}$$

Weak Duality Theorem gives that if \hat{x} is feasible in (P) and \hat{y} is feasible in (D), then $c'\hat{x} \geq \hat{y}'b$, as $c'\hat{x} \geq \hat{y}'A\hat{x} = \hat{y}'b$. In our example,

$$c'\hat{x} = -100\hat{x}_1 - 300\hat{x}_2 ,$$

$$\hat{y}'b = 10\hat{y}_1 + 150\hat{y}_2 ,$$

So

$$-100\hat{x}_1 - 300\hat{x}_2 \geq 10\hat{y}_1 + 150\hat{y}_2 .$$

Take

$$\hat{x} = \begin{pmatrix} 6 \\ 4 \\ 0 \\ 0 \end{pmatrix}, \hat{y} = \begin{pmatrix} 10 \\ -20 \end{pmatrix} ,$$

we can see that $c'\hat{x} = -1800$, $\hat{y}'b = -2900$ and $c'\hat{x} \geq \hat{y}'b$.

3. EXERCISE 1.3 CONVERT TO \leq FORM

First follow the transformation to transform the constraints on every variable x_j (we only consider the comparison to zero as comparison to other real number can be transformed to what we discussed):

- (1) if the constraint is in form $x_j \leq 0$, then it is the \leq form;
- (2) if the constraint is in form $x_j \geq 0$, then let $x_j^- = -x_j$. Replace x_j with x_j^- in the objective and all constraints. So we get $x_j^- \leq 0$;
- (3) if x_j is unrestricted, then replace x_j with the difference of a pair of non-positive variables x_j^+ and x_j^- in the objective and all constraints, that is $x_j^+ - x_j^- = x_j$.

After transforming the constraints on only variables, let's now consider other constraints:

- (1) if the constraint is in form $Ax \leq b$, then it is the \leq form;
- (2) if the constraint is in form $Ax \geq b$, then change it to the equivalent constraint $(-A)x \leq -b$;
- (3) if the constraint is in form $Ax = b$, then replace this with $Ax + It \leq b$, where $t \leq 0$. Let $x^* = \begin{pmatrix} x' & t' \end{pmatrix}'$, replace x with x^* in other constraints.

Now we are done with all constraints and they are all in \leq form.

4. EXERCISE 1.4 $m + 1$ INEQUALITIES

Problem: Prove that the system of m equations in n variables $Ax = b$ is equivalent to the system $Ax \leq b$ augmented by only *one* additional linear inequality – that is, a total of only $m+1$ inequalities. The characteristic polynomial $\chi(\lambda)$ of the 3×3 matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

is given by the formula

$$(4.1) \quad \chi(\lambda) = \begin{vmatrix} \lambda - a & -b & -c \\ -d & \lambda - e & -f \\ -g & -h & \lambda - i \end{vmatrix}$$

5. EXERCISE 1.5 WEAK DUALITY FOR ANOTHER FORM

6. EXERCISE 1.6 WEAK DUALITY FOR A COMPLICATED FORM

7. EXERCISE 1.7 WEAK DUALITY FOR A COMPLICATED FORM — WITH MATLAB

The MATLAB code is as follows. We use `linprog` function to solve the dual. We compare the optimal value of the original problem and its dual. Since there is computation deviation, we consider the difference less than 10^{-8} be negligible and the two optimal values are the same.

```
% DualityWithMatlab1.m // Jon Lee
%
n1=7
n2=15
m1=2
m2=4
rng('default');
rng(1); % set seed
A = rand(m1,n1);
B = rand(m1,n2);
D = rand(m2,n1);
% Organize the situation
% so that the problem has a feasible solution
x = rand(n1,1);
w = -rand(n2,1);
b = A*x + B*w + 0.01 * rand(m1,1);
g = D*x;
% Organize the situation
% so that the dual problem has a feasible solution
y = -rand(m1,1);
pi = rand(m2,1) - rand(m2,1);
c = A'*y + D'*pi + 0.01 * rand(n1,1);
f = B'*y - 0.01 * rand(n2,1);
% Here is how the 'linprog' function works:
%
% [v,z,exitflag] = linprog(c,A,b,Aeq,beq,lb,ub)
% minimizes c'v ,
% with constraints A v <= b, Aeq v = beq, and
% variable bounds lb <= v <=ub.
% Some parts of the model can be null: for example,
% set Aeq = [] and beq = [] if no equalities exist.
% set ub = [] if no upper bounds exist.
%
% [v,z,exitflag] = linprog(...) returns values:
% v = solution vector
% z = minimum objective value
% exitflag = describes the exit condition:
% 1 optimal solution found.
% ?2 problem is infeasible.
% ?3 problem is unbounded.
% ?5 problem and its dual are both infeasible
% Study the following part carefully. This is how we set up
% a block problem. [c ; f] stacks vertically the vectors c and f.
% [A B] concatenates A and B horizontally. zeros(m2,n2) is an
% m2-by-n2 matrix of 0's. ones(n1,1) is an n1-vector of 1's, and
% Inf is +infinity, so Inf*ones(n1,1) is a vector of length n1
% with all components equal to +infinity. In this way we have
% infinite upper bounds on some variables.
[v,z,exitflag] = linprog([c ; f],[A B],b,[D zeros(m2,n2)],g, ...
[zeros(n1,1);-Inf*ones(n2,1)],[Inf*ones(n1,1);zeros(n2,1)])
if (exitflag < 1)
disp('fail 1: LP did not have an optimal solution');
return;
end;
% Extract the parts of the solution v to the formulation vectors
x = v(1:n1,1)
```

```
w = v(n1+1:n1+n2,1)

[v2,z2,exitflag] = linprog([-b;-g], [A' D'; -B' zeros(n2,m2)], [c; -f],[],[],[],[zeros(m1,1);Inf*ones(m2,1)])

if (abs(z2 + z) < 1e-8)
    disp('The problem and its dual have the same optimal value')
end
```
