HOMEWORK #5 SOLUTION

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1. Exercise 5.1 Duality and complementarity with AMPL

Problem:
Solution:

2. Exercise 5.4 Another Proof of a Theorem of the Alternative

Problem: Prove the Theorem of the Alternative for Linear Inequalities directly from the Farkas Lemma, without appealing to linear-optimization duality. HINT: Transform (I) of the Theorem of the Alternative for Linear Inequalities to a system of the form of (I) of the Farkas Lemma.

Solution:
In the Theorem of the Alternative for Linear Inequalities, we have:

$$Ax \geq b. \tag{2.1}$$

$$\begin{aligned} y'b &> 0; \\ y'A &= 0; \\ y &\geq 0. \end{aligned} \tag{2.2}$$

In (2.1), we replace unrestricted variable x with $x^+ - x^(-)$, where x^+ and $x^(-)$ are a pair of non-negative variables. Next we introduce slack variable t to replace inequality with equality. After these transformation we have a new form of (2.1) as follows.

$$\begin{aligned} Ax^+ - Ax^- - t = b; \\ x^+ \geq 0, x^- \geq 0, t \geq 0; \end{aligned} \tag{2.3}$$

Let $A_1 = \begin{pmatrix} A & -A & -I \end{pmatrix}$, and $x_1 = \begin{pmatrix} x^+ \\ x^- \\ t \end{pmatrix}$, the (2.3) is equivalent to (2.4) as follows.

$$\begin{aligned} A_1 x_1 &= b; \\ x_1 &\geq 0; \end{aligned} \tag{2.4}$$

Then we can apply the Farkas Lemma, which states that exactly one of the two systems (2.4) and (2.5) as follows has a solution.

$$\begin{aligned} y'b &> 0; \\ y'A_1 &\leq 0; \end{aligned} \tag{2.5}$$

(2.5) is equivalent to (2.6) as follows,

$$\begin{aligned} y'b &> 0; \\ y'A &\leq 0; \\ -y'A &\leq 0; \\ -y' &\leq 0; \end{aligned} \tag{2.6}$$

And (2.6) is equivalent to (2.7) as follows,

$$\begin{aligned} y'b &> 0; \\ y'A &= 0; \\ y' &\geq 0; \end{aligned} \tag{2.7}$$

(2.7) is exactly the same as (2.2).

Since exactly one of the two systems (2.4) and (2.5) has a solution, (2.4) is equivalent to (2.1) and (2.5) is equivalent to (2.2), we have the conclusion that exactly one of the two systems (2.1) and (2.2) has a solution, which proves Theorem of the Alternative for Linear Inequalities.

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3. Exercise 5.5 A General Theorem of the Alternative

Problem:
Solution:

4. Exercise 5.6 Dual Ray

Problem:
Solution: