HOMEWORK #5 SOLUTION

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1. Exercise 5.1 Duality and complementarity with AMPL

Problem:
Solution:

2. Exercise 5.2 Complementary Slackness

Problem:
Solution: The goal of the solution is to construct an example in which the standard-form problem (P) has an optimal solution $\hat{x}$ and there exists $\hat{y}$ that is not feasible in its dual dual (D) and is complementary to $\hat{x}$. In this example, there exists multiple $\hat{y}$ that are complementary to $\hat{x}$, yet not all of them are feasible. Let's first take a look at the definition of **complementarity**.

Definition 1. With respect to the standard-form problem (P) and its dual (D), the solutions $\hat{x}$ and $\hat{y}$ are **complementary** if

$$\begin{aligned} (c_j - \hat{y}'A_{\cdot j})\hat{x}_j &= 0\ , \quad \text{for } j = 1,2,...,n\ ; \\ \hat{y}_i(A_{i\cdot}\hat{x} - b_i) &= 0\ , \quad \text{for } i = 1,2,...,m\ . \end{aligned}$$

Given that $\hat{x}$ is optimal, $\hat{x}$ is the basic feasible solution of (P). So

$$\begin{aligned} A\hat{x} &= b\ , \\ \hat{x} &\geq \mathbf{0}\ . \end{aligned}$$

which means $A_{i\cdot}\hat{x} - b_i = 0, \text{for } i = 1,2,...,m$, so the second equation for complementarity is satisfied: $\hat{y}_i(A_{i\cdot}\hat{x} - b_i) = 0$. Since $\hat{x}$ is a basic solution, $\hat{x}_\eta = 0$, so $(c_{\eta_j} - \hat{y}'A_{\cdot\eta_j})\hat{x}_{\eta_j} = 0$, for $j = 1,2,...,n-m$. In order to solve $\hat{y}$ given $\hat{x}$, we need to solve the following equations

$$(c_{\beta_j} - \hat{y}'A_{\cdot\beta_j})\hat{x}_{\beta_j} = 0 \text{ , for } j = 1,2,...,m\ .$$

Given that $\hat{y}$ is m by 1, if none of the $\hat{x}_{\beta_j}$, for $j = 1,2,...,m$ is zero, there is unique $\hat{y}$ complementary to $\hat{x}$, which is also the feasible solution of (D). However, if there exists $\hat{x}_{\beta_j}$ that equals zero, there are more than one $\hat{y}$ that complementary to $\hat{x}$, and it is possible to construct one of the $\hat{y}$ to be not feasible of (D) as we can add one more constraint of $c_{\beta_j} - \hat{y}'A_{\cdot\beta_j} < 0$ so that $\hat{y}'A_{\cdot\beta_j} > c_{\beta_j}$ for β_j that $\hat{x}_{\beta_j} = 0$. In this way, $\hat{y}'A \leq c'$ does not hold and $\hat{y}$ is not a feasible solution of (D).

To construct an problem (P) with optimal solution $\hat{x}$ such that $\hat{x}_{\beta_j} = 0$ for some j, we leverage the geometric explanation of basic and nonbasic variables. Suppose (β, η) is the partition for the optimal solution, then on the space of nonbasic variables, for the projection of the lines $\left(A_\beta^{-1}A_\eta\right)x_\eta = A_\beta^{-1}b$, there exists one line that is across the origin $(0,0)$. The algebraic explanation of this is that there exists $\hat{x}_{\beta_j} = 0$ for the basic variable. We leverage this observation to construct the example. We first construct an standard-form problem (P) with $m = 3$ and $n = 5$ and found the optimal solution through `AMPL`:

$$\begin{aligned} \tilde{A} &:= \begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & -4 & 2 & 2 & 0 \\ 0 & -9 & 0 & 6 & 3 \end{pmatrix}, \\ \tilde{b} &:= (1,2,18)'\ , \\ \tilde{c} &:= (16,7,20,10,4)'\ . \end{aligned}$$

The optimal solution $\hat{\tilde{x}} = (2,0,0,1,4)'$ and the associated partition is $\tilde{\beta} = \{1,4,5\}, \tilde{\eta} = \{2,3\}$. We show the projection onto the space of nonbasic variables in Figure 1. We can know that

$$\tilde{A}_\beta^{-1} = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1 & 1/3 \end{pmatrix}, \quad \tilde{A}_\beta^{-1}\tilde{A}_\eta = \begin{pmatrix} -3 & 1 \\ -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad \tilde{A}_\beta^{-1}\tilde{b} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix},$$

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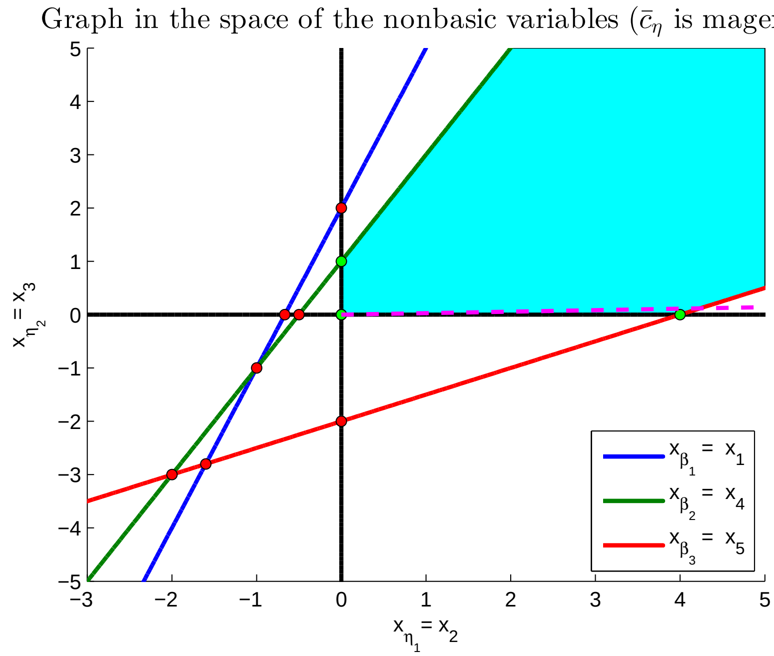

FIGURE 1. Feasible region projected into the space of (x_2, x_3) with $\tilde{A}, \tilde{c}, \tilde{b}$

Then we add one more basic variable (x_6), and corresponding add one more line in the space of nonbasic variables. This line should not affect the feasible region and should be across the origin. We also need to guarantee that other lines are not affected and the optimal solution is kept regarding the original variables. We achieve this by changing the following matrix (the geometric explanation is shown in Figure 2):

$$A_\beta^{-1} = \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & -1 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_\beta^{-1} A_\eta = \begin{pmatrix} -3 & 1 \\ -2 & 1 \\ 1 & -2 \\ -1 & -1 \end{pmatrix}, \quad A_\beta^{-1} b = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 0 \end{pmatrix},$$

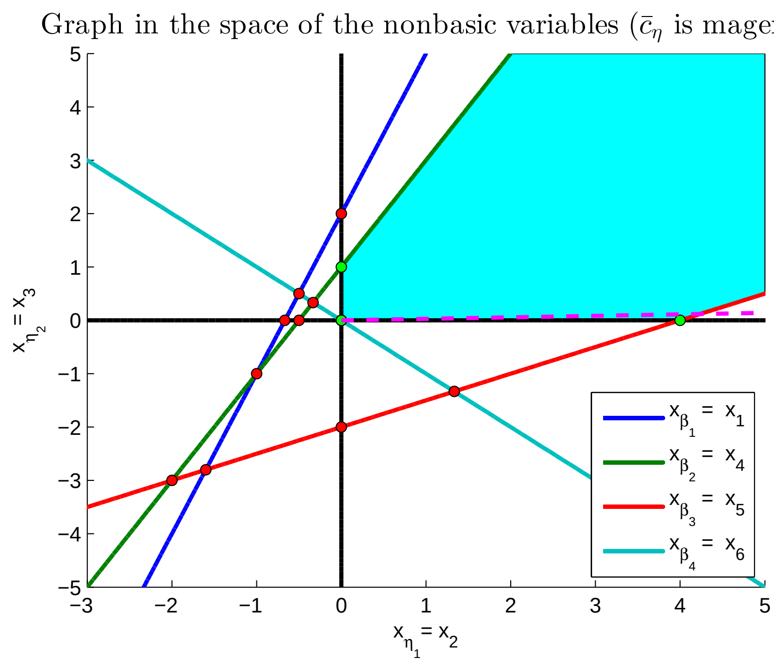

FIGURE 2. Feasible region projected into the space of (x_2, x_3) with A, c, b

We then solve A_η and b from the above equations and get A:

$$A_\eta = \begin{pmatrix} -1 & 0 \\ -4 & 2 \\ -9 & 0 \\ -1 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & -4 & 2 & 2 & 0 & 0 \\ 0 & -9 & 0 & 6 & 3 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 18 \\ 0 \end{pmatrix}, \quad c = (16, 7, 20, 10, 4, 0)'$$

We verified that $\hat{x} = (2, 0, 0, 1, 4, 0)$ is a feasible solution and $\bar{c}_\eta = (71, 2)' \geq \mathbf{0}$. So $\hat{x}$ is an optimal solution. Following the method we describe above, we get two $\hat{y}$ that is complemenatary to $\hat{x}$ (and we confirm that they actually are): $\hat{y}^1 = (16, 9, 4/3, 1)'$ and $\hat{y}^2 = (16, 9, 4/3, 0)'$. $\hat{y}^1$ is not a feasible solution of (D) while $\hat{y}^2$ is. Still, we can show $\hat{x}$ is optimal solution as $\hat{x}$ and $\hat{y}^2$ are feasible and complementary with repect to (P) and (D), then $\hat{x}$ and $\hat{y}^2$ are optimal.

3. Exercise 5.4 Another Proof of a Theorem of the Alternative

Problem: Prove the Theorem of the Alternative for Linear Inequalities directly from the Farkas Lemma, without appealing to linear-optimization duality. HINT: Transform (I) of the Theorem of the Alternative for Linear Inequalities to a system of the form of (I) of the Farkas Lemma.

Solution:

In the Theorem of the Alternative for Linear Inequalities, we have:

$$Ax \geq b. \tag{3.1}$$

$$\begin{aligned} y'b &> 0; \\ y'A &= 0; \\ y &\geq 0. \end{aligned} \tag{3.2}$$

In (3.1), we replace unrestricted variable x with $x^+ - x^(-)$, where x^+ and $x^(-)$ are a pair of non-negative variables. Next we introduce slack variable t to replace inequality with equality. After these transformation we have a new form of (3.1) as follows.

$$\begin{aligned} Ax^+ - Ax^- - t &= b; \\ x^+ \geq 0, x^- \geq 0, t &\geq 0; \end{aligned} \tag{3.3}$$

Let $A_1 = \begin{pmatrix} A & -A & -I \end{pmatrix}$, and $x_1 = \begin{pmatrix} x^+ \\ x^- \\ t \end{pmatrix}$, the (3.3) is equivalent to (3.4) as follows.

$$\begin{aligned} A_1 x_1 &= b; \\ x_1 &\geq 0; \end{aligned} \tag{3.4}$$

Then we can apply the Farkas Lemma, which states that exactly one of the two systems (3.4) and (3.5) as follows has a solution.

$$\begin{aligned} y'b &> 0; \\ y'A_1 &\leq 0; \end{aligned} \tag{3.5}$$

(3.5) is equivalent to (3.6) as follows,

$$\begin{aligned} y'b &> 0; \\ y'A &\leq 0; \\ -y'A &\leq 0; \\ -y' &\leq 0; \end{aligned} \tag{3.6}$$

And (3.6) is equivalent to (3.7) as follows,

$$\begin{aligned} y'b &> 0; \\ y'A &= 0; \\ y' &\geq 0; \end{aligned} \tag{3.7}$$

(3.7) is exactly the same as (3.2).

Since exactly one of the two systems (3.4) and (3.5) has a solution, (3.4) is equivalent to (3.1) and (3.5) is equivalent to (3.2), we have the conclusion that exactly one of the two systems (3.1) and (3.2) has a solution, which proves Theorem of the Alternative for Linear Inequalities.

■

4. Exercise 5.5 A General Theorem of the Alternative

Problem: State and prove a "Theorem of the Alternative" for the system:

$$
\begin{array}{r}
A_P^G x_P + A_N^G x_N + A_U^G x_U \ge b^G; \\
A_P^L x_P + A_N^L x_N + A_U^L x_U \le b^L; \\
A_P^E x_P + A_N^E x_N + A_U^E x_U = b^E; \\
x_P \ge 0, x_N \le 0.
\end{array} \tag{4.1}
$$

Solution: We first state the alternative of system (4.1) as follows,

$$
\begin{array}{r}
y_G' b^G + y_L' b^L + y_E' b^E > 0; \\
y_G' A_P^G + y_L' A_P^L + y_E' A_P^E \le 0; \\
y_G' A_N^G + y_L' A_N^L + y_E' A_N^E \ge 0; \\
y_G' A_U^G + y_L' A_U^L + y_E' A_U^E = 0; \\
y_G' \ge 0, y_L' \le 0.
\end{array} \tag{4.2}
$$

Then we state the "Theorem of the Alternative" for the system (4.1) and (4.2),

Theorem 4.1. *Let A_P^G, A_N^G, A_U^G, A_P^L, A_N^L, A_U^L, A_P^E, A_N^E, A_U^E, b^G, b^L, and b^E be given. Then exactly one the the system (4.1) and (4.2) has a solution.*

To prove the theorem, we first claim that there cannot simultaneously be a solution $\widehat{x}^P$, $\widehat{x}^N$ and $\widehat{x}^U$ to (4.1) and $\widehat{y}^G$, $\widehat{y}^L$ and $\widehat{y}^E$ to (4.2). To see that, we have:

$\because \widehat{y}_G' A_P^G + \widehat{y}_L' A_P^L + \widehat{y}_E' A_P^E \le 0$, $\widehat{y}_G' A_N^G + \widehat{y}_L' A_N^L + \widehat{y}_E' A_N^E \ge 0$, and $\widehat{y}_G' A_U^G + \widehat{y}_L' A_U^L + \widehat{y}_E' A_U^E = 0$, $\widehat{x}_P \ge 0$, $\widehat{x}_N \le 0$

$\therefore \widehat{y}_G' A_P^G \widehat{x}_P + \widehat{y}_L' A_P^L \widehat{x}_P + \widehat{y}_E' A_P^E \widehat{x}_P \le 0$, $\widehat{y}_G' A_N^G \widehat{x}_N + \widehat{y}_L' A_N^L \widehat{x}_N + \widehat{y}_E' A_N^E \widehat{x}_N \le 0$, and $\widehat{y}_G' A_U^G \widehat{x}_U + \widehat{y}_L' A_U^L \widehat{x}_U + \widehat{y}_E' A_U^E \widehat{x}_U = 0$

$\therefore \widehat{y}_G' A_P^G \widehat{x}_P + \widehat{y}_L' A_P^L \widehat{x}_P + \widehat{y}_E' A_P^E \widehat{x}_P + \widehat{y}_G' A_N^G \widehat{x}_N + \widehat{y}_L' A_N^L \widehat{x}_N + \widehat{y}_E' A_N^E \widehat{x}_N + \widehat{y}_G' A_U^G \widehat{x}_U + \widehat{y}_L' A_U^L \widehat{x}_U + \widehat{y}_E' A_U^E \widehat{x}_U \le 0$

$\therefore \widehat{y}_G' (A_P^G \widehat{x}_P + A_N^G \widehat{x}_N + A_U^G \widehat{x}_U) + \widehat{y}_L' (A_P^L \widehat{x}_P + A_N^L \widehat{x}_N + A_U^L \widehat{x}_U) + \widehat{y}_E' (A_P^E \widehat{x}_P + A_N^E \widehat{x}_N + A_U^E \widehat{x}_U) \le 0$

$\because A_P^G \widehat{x}_P + A_N^G \widehat{x}_N + A_U^G \widehat{x}_U \ge b^G$, $A_P^L \widehat{x}_P + A_N^L \widehat{x}_N + A_U^L \widehat{x}_U \le b^L$, $A_P^E \widehat{x}_P + A_N^E \widehat{x}_N + A_U^E \widehat{x}_U = b^E$, and $\widehat{y}_G' \ge 0$, $\widehat{y}_L' \le 0$

$\therefore \widehat{y}_G' (A_P^G \widehat{x}_P + A_N^G \widehat{x}_N + A_U^G \widehat{x}_U) \ge \widehat{y}_G' b^G$,$\widehat{y}_L' (A_P^L \widehat{x}_P + A_N^L \widehat{x}_N + A_U^L \widehat{x}_U) \ge \widehat{y}_L' b^L$, and $\widehat{y}_E' (A_P^E \widehat{x}_P + A_N^E \widehat{x}_N + A_U^E \widehat{x}_U) = \widehat{y}_E' b^E$

$\therefore \widehat{y}_G' (A_P^G \widehat{x}_P + A_N^G \widehat{x}_N + A_U^G \widehat{x}_U) + \widehat{y}_L' (A_P^L \widehat{x}_P + A_N^L \widehat{x}_N + A_U^L \widehat{x}_U) + \widehat{y}_E' (A_P^E \widehat{x}_P + A_N^E \widehat{x}_N + A_U^E \widehat{x}_U) \ge \widehat{y}_G' b^G + \widehat{y}_L' b^L + \widehat{y}_E' b^E$

$\therefore 0 \ge \widehat{y}_G' (A_P^G \widehat{x}_P + A_N^G \widehat{x}_N + A_U^G \widehat{x}_U) + \widehat{y}_L' (A_P^L \widehat{x}_P + A_N^L \widehat{x}_N + A_U^L \widehat{x}_U) + \widehat{y}_E' (A_P^E \widehat{x}_P + A_N^E \widehat{x}_N + A_U^E \widehat{x}_U) \ge \widehat{y}_G' b^G + \widehat{y}_L' b^L + \widehat{y}_E' b^E$

$\therefore$ inconsistent with $\widehat{y}_G' b^G + \widehat{y}_L' b^L + \widehat{y}_E' b^E > 0$ in (4.2).

Thus, there cannot simultaneously be a solution to (4.1) and (4.2).

Next, suppose that (4.1) has no solution. Then the following problem is infeasible:

$$
\begin{array}{ll}
\min & 0' x_P + 0' x_N + 0' x_U \\
\text{s.t.} & A_P^G x_P + A_N^G x_N + A_U^G x_U \ge b^G; \\
& A_P^L x_P + A_N^L x_N + A_U^L x_U \le b^L; \\
& A_P^E x_P + A_N^E x_N + A_U^E x_U = b^E; \\
& x_P \ge 0, x_N \le 0.
\end{array} \tag{4.3}
$$

Its dual is

$$
\begin{array}{ll}
\max & y_G' b^G + y_L' b^L + y_E' b^E \\
\text{s.t.} & y_G' A_P^G + y_L' A_P^L + y_E' A_P^E \le 0; \\
& y_G' A_N^G + y_L' A_N^L + y_E' A_N^E \ge 0; \\
& y_G' A_U^G + y_L' A_U^L + y_E' A_U^E = 0; \\
& y_G' \ge 0, y_L' \le 0.
\end{array} \tag{4.4}
$$

Since (4.3) is infeasible, then (4.4) is either infeasible or unbounded. But $y_G' = 0$, $y_L' = 0$, $y_E' = 0$ is a feasible solution to (4.4), therefore (4.4) must be unbounded. Therefore, there exist a feasible solution $\widehat{y}^G$, $\widehat{y}^L$ and $\widehat{y}^E$ to (4.2) having objective value greater than zero (or even any fixed constant). Such $\widehat{y}^G$, $\widehat{y}^L$ and $\widehat{y}^E$ are a solution to (4.2).

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5. Exercise 5.6 Dual Ray

Problem: Consider the linear-optimization problem

$$\begin{aligned} \min \quad & c'x \\ \text{s.t.} \quad & Ax \geq b; \\ & x \geq 0. \end{aligned} \tag{5.1}$$

a) Suppose that (5.1) is infeasible. Then, by a 'Theorem of the Alternative' there is a solution to what system?

b) Suppose, further, that the dual of (5.1) is feasible. Take a feasible solution $\widehat{y}$ of dual and a solution $\tilde{y}$ to your system of part (a) and combine them appropriately to prove that the dual is unbounded.

Solution:

a) The system for (5.1) is as follows,

$$\begin{aligned} Ax &\geq b; \\ x &\geq 0. \end{aligned} \tag{5.2}$$

By the 'Theorem of the Alternative' the following system has a solution.

$$\begin{aligned} y'b &> 0; \\ y'A &\leq 0; \\ y' &\geq 0. \end{aligned} \tag{5.3}$$

b) The dual of (5.1) is as follows,

$$\begin{aligned} \max \quad & y'b \\ \text{s.t.} \quad & y'A \leq c'; \\ & y \geq 0. \end{aligned} \tag{5.4}$$

We combine the feasible solution $\widehat{y}$ for (5.4) and a solution $\tilde{y}$ to (5.3) to $\overline{y} = \widehat{y} + \alpha\tilde{y}$, $\alpha > 0$.

$\because \overline{y}'A = (\widehat{y} + \alpha\tilde{y})'A = \widehat{y}'A + \alpha\tilde{y}'A$, $\tilde{y}'A \leq 0$, $\alpha > 0$, and $\widehat{y}'A \leq c'$

$\therefore \overline{y}'A \leq c'$

$\because \widehat{y} \geq 0$, $\tilde{y} \geq 0$, $\alpha > 0$

$\therefore \overline{y} \geq 0$

$\therefore$ $\overline{y}$ is a feasible solution for (5.4)

$\because \overline{y}'b = (\widehat{y} + \alpha\tilde{y})'b = \widehat{y}'b + \alpha\tilde{y}'b$, and $\tilde{y}'b > 0$

$\therefore$ When $\alpha \to +\infty$, $\overline{y}'b = \widehat{y}'b + \alpha\tilde{y}'b \to +\infty$

$\therefore$ The dual (5.4) is unbounded.

$\blacksquare$