HOMEWORK #1 SOLUTION

Yihua Guo (yhguo@umich.edu), Qi Chen (alfchen@umich.edu), Tianyi Ma (mtianyi@umich.edu)

1. Exercise 1.1 Convert to standard form

Let's take the following problem as an example. Suppose Tom wants to make money and he decides to open several restaurants in the city he lives. There are two types of restaurants, Café which needs 10 employees each and has a profit of \$100 per day, and Mongolian barbecue which needs 20 employees each and has a profit of \$300 per day. Tom would like to open at least 10 restaurants but he only has 150 people to serve in restaurants. The goal is to maximize the profit of all restaurants. This is an optimization problem and can be formalized as follows:

$$\begin{array}{rrcl} \max & 100x_1 + 300x_2 & & \\ & x_1 + x_2 & \geq & 10\ ; \\ & 10x_1 + 20x_2 & \leq & 150\ ; \\ & x_1 & \geq & 0\ ; \\ & x_2 & \geq & 0\ . \end{array}$$

To transform this general linear-optimization problem to one in standard form, first we change the maximum of $100x_1 + 300x_2$ to the negative of the minimum of $-100x_1 - 300x_2$. We then change inequalities to equalities: $x_1 + x_2 \geq 10$ to $x_1 + x_2 - t_1 = 10$, and $10x_1 + 20x_2 \leq 150$ to $10x_1 + 20x_2 + t_2 = 150$, where $t_1, t_2 \geq 0$. Now we have

$$\begin{array}{rrcl} \min & -100x_1 - 300x_2 & & \\ & x_1 + x_2 - t_1 & = & 10\ ; \\ & 10x_1 + 20x_2 + t_2 & = & 150\ ; \\ & t_1 & \geq & 0\ ; \\ & t_2 & \geq & 0\ ; \\ & x_1 & \geq & 0\ ; \\ & x_2 & \geq & 0\ , \end{array}$$

which is in standard form of the following

$$\text{(P)} \qquad \begin{array}{rrcl} \min & c'x & & \\ & Ax & = & b\ ; \\ & x & \geq & \mathbf{0}\ , \end{array}$$

where

$$c = \begin{pmatrix} -100 \\ -300 \\ 0 \\ 0 \end{pmatrix},\ A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 10 & 20 & 0 & 1 \end{pmatrix},\ b = \begin{pmatrix} 10 \\ 150 \end{pmatrix},\ x = \begin{pmatrix} x_1 \\ x_2 \\ t_1 \\ t_2 \end{pmatrix}.$$

2. Exercise 1.2 Weak Duality example

We still take the optimization problem (P) in Exercise 1.1 as an example. Its Dual is

$$\text{(D)} \qquad \begin{array}{rrcl} \max & y'b & & \\ & y'A & \leq & c'\ , \end{array}$$

where y is a vector of variables in $\mathbb{R}^2$. Let $y = \begin{pmatrix} y_1 & y_2 \end{pmatrix}'$. (D) is equivalent to

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$$\begin{array}{rrcl} \max & 10y_1+150y_2 & & \\ & y_1+10y_2 & \le & -100\ ; \\ & y_1+20y_2 & \le & -300\ ; \\ & y_1 & \ge & 0\ ; \\ & y_2 & \le & 0\ . \end{array}$$

Weak Duality Theorem gives that if $\hat{x}$ is feasible in (P) and $\hat{y}$ is feasible in (D), then $c'\hat{x} \ge \hat{y}'b$, as $c'\hat{x} \ge \hat{y}'A\hat{x} = \hat{y}'b$. In our example,

$$c'\hat{x} = -100\hat{x}_1 - 300\hat{x}_2\ ,$$

$$\hat{y}'b = 10\hat{y}_1 + 150\hat{y}_2,$$

So

$$-100\hat{x}_1 - 300\hat{x}_2 \ge 10\hat{y}_1 + 150\hat{y}_2.$$

Take

$$\hat{x} = \begin{pmatrix} 6 \\ 4 \\ 0 \\ 0 \end{pmatrix}, \hat{y} = \begin{pmatrix} 10 \\ -20 \end{pmatrix},$$

we can see that $c'\hat{x} = -1800$, $\hat{y}'b = -2900$ and $c'\hat{x} \ge \hat{y}'b$.

3. Exercise 1.3 Convert to $\le$ form

First follow the transformation to transform the constraints on every variable x_j (we only consider the comparison to zero as comparison to other real number can be transformed to what we discussed):

(1) if the constraint is in form $x_j \le 0$, then it is the $\le$ form;
(2) if the constraint is in form $x_j \ge 0$, then let $x_j^- = -x_j$. Replace x_j with x_j^- in the objective and all constraints. So we get $x_j^- \le 0$;
(3) if x_j is unrestricted, then replace x_j with the difference of a pair of non-positive variables x_j^+ and x_j^- in the objective and all constraints, that is $x_j^+ - x_j^- = x_j$.

After transforing the constraints on only variables, let's now consider other constrainsts:

(1) if the constraint is in form $Ax \le b$, then it is the $\le$ form;
(2) if the constraint is in form $Ax \ge b$, then change it to the equivalent constraint $(-A)x \le -b$;
(3) if the constraint is in form $Ax = b$, then replace this with $Ax + It \le b$, where $t \le \mathbf{0}$. Let $x^* = \begin{pmatrix} x' & t' \end{pmatrix}'$, replace x with x^* in other constraints.

Now we are done with all constraints and they are all in $\le$ form.

4. Exercise 1.4 $m+1$ inequalities

Problem: Prove that the system of m equations in n variables $Ax = b$ is equivalent to the system $Ax \le b$ augmented by only *one* additional linear inequality – that is, a total of only m+1 inequalities.

Proof:

Let $A = \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_m \end{pmatrix}$ where each element is a $1 \times n$ vector, and $b = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix}$.

Thus $Ax = b$ has the form of m equations as follows,

$$\begin{cases} a_1x = b_1 \\ a_1x = b_1 \\ \cdots \\ a_mx = b_m \end{cases} \tag{4.1}$$

We add one constraint $\sum_{i=1}^{m} a_i x \geq \sum_{i=1}^{m} b_i$ to $Ax \leq b$, and construct a system with $m+1$ inequalities as follows,

$$
\left\{
\begin{array}{c}
a_1 x \leq b_1 \\
a_2 x \leq b_2 \\
\dots \\
a_m x \leq b_m \\
\sum_{i=1}^{m} a_i x \geq \sum_{i=1}^{m} b_i
\end{array}
\right.
\tag{4.2}
$$

We claim that (4.1) is equivalent to (4.2). To prove that, we will show any feasible solution $\widehat{x}$ for (4.1) is also a feasible solution for (4.2), and vice versa.

"$\Rightarrow$": if $\widehat{x}$ is a feasible solution for (4.1), then it is also a feasible solution for (4.2).

$\because$ $\widehat{x}$ is a feasible solution for (4.1)

$\therefore$ $a_i \widehat{x} = b_i, i \in 1...m$, and $\sum_{i=1}^{m} a_i x = \sum_{i=1}^{m} b_i$

$\therefore$ $\widehat{x}$ also satisfy $a_i \widehat{x} \leq b_i, i \in 1...m$ and $\sum_{i=1}^{m} a_i x \geq \sum_{i=1}^{m} b_i$

$\therefore$ $\widehat{x}$ is also a feasible solution for (4.2)

"$\Leftarrow$": if $\widehat{x}$ is a feasible solution for (4.2), then it is also a feasible solution for (4.1).

$\because$ $\widehat{x}$ is a feasible solution for (4.2)

$\therefore$ $a_i \widehat{x} \leq b_i, i \in 1...m$, and $\sum_{i=1}^{m} a_i x \geq \sum_{i=1}^{m} b_i$

We continue the proof by contradiction. Assume one of the equalities does not hold, i.e. $a_j \widehat{x} < b_j$ for j, $j \in 1...m$.

$\because$ $a_j \widehat{x} < b_j$

$$
\begin{aligned}
\therefore\ & \sum_{i=1}^{m} a_i \widehat{x} \\
= & \sum_{i=1}^{j-1} a_i \widehat{x} + a_j \widehat{x} + \sum_{i=j+1}^{m} a_i \widehat{x} \\
\leq & \sum_{i=1}^{j-1} b_i + a_j \widehat{x} + \sum_{i=j+1}^{m} b_i \\
< & \sum_{i=1}^{j-1} b_i + b_j + \sum_{i=j+1}^{m} b_i \\
= & \sum_{i=1}^{m} b_i
\end{aligned}
$$

$\therefore$ contradict to $\sum_{i=1}^{m} a_i \widehat{x} \geq \sum_{i=1}^{m} b_i$ in (4.2)

$\therefore$ all of the equalities for (4.1)should hold if $\widehat{x}$ is a feasible solution for (4.2).

$\therefore$ $\widehat{x}$ is also a feasible solution for (4.1)

$\blacksquare$

5. Exercise 1.5 Weak Duality for another form

Problem: Give and prove a Weak Duality Theorem for

$$
\begin{array}{ll}
\max & c'x \\
\text{s.t.} & Ax \leq b \\
 & x \geq 0
\end{array}
\tag{5.1}
$$

Solution:

The dual of (5.1) is

$$
\begin{array}{ll}
\min & y'b \\
\text{s.t.} & y'A \geq c' \\
 & y' \geq 0
\end{array}
\tag{5.2}
$$

We give the Weak Duality Theorem for (5.1) as follows,

Theorem 5.1. *If $\widehat{x}$ is feasible in 5.1 and $\widehat{y}$ is feasible in (5.2), then $c'\widehat{x} \leq \widehat{y}'b$.*

Proof:
We first transform (5.1) to standard form as follows,

$$\begin{array}{ll} \min & c_1'x_1 \\ \text{s.t.} & A_1x_1 = b \\ & x_1 \geq 0 \end{array} \tag{5.3}$$

In (5.3), $x_1 = \begin{pmatrix} x \\ t \end{pmatrix}$, $c_1' = \begin{pmatrix} -c' & 0 \end{pmatrix}$, and $A_1 = \begin{pmatrix} A & I \end{pmatrix}$.
The dual of (5.3) is

$$\begin{array}{ll} \max & y_1'b \\ \text{s.t.} & y_1'A_1 \leq c_1' \end{array} \tag{5.4}$$

In (5.4),
$\because y_1'A_1 \leq c_1'$
$\therefore y_1' \begin{pmatrix} A & I \end{pmatrix} \leq \begin{pmatrix} -c' & 0 \end{pmatrix}$
$\therefore y_1'A \leq -c'$ and $y_1' \leq 0$
let $y_1 = -y_2$, then (5.4) becomes

$$\begin{array}{ll} \max & -y_2'b \\ \text{s.t.} & -y_2'A \leq -c' \\ & -y_2' \leq 0 \end{array} \tag{5.5}$$

which is equivalent to

$$\begin{array}{ll} \min & y_2'b \\ \text{s.t.} & y_2'A \geq c' \\ & y_2' \geq 0 \end{array} \tag{5.6}$$

We apply Weak Duality Theorem to primal (5.3) and dual (5.4), we have $c_1'\widehat{x_1} \geq \widehat{y_1}'b$, where $\widehat{x_1}$ is a feasible solution for (5.3) and $\widehat{y_1}$ is a feasible solution for (5.4).
$\because c_1'\widehat{x_1} \geq \widehat{y_1}'b$
$\therefore \begin{pmatrix} -c' & 0 \end{pmatrix} \begin{pmatrix} \widehat{x} \\ \widehat{t} \end{pmatrix} \geq \widehat{y_1}'b$, where $\widehat{x}$ is a feasible solution for (5.1)
$\therefore -c'\widehat{x} \geq \widehat{y_1}'b$, where $\widehat{x}$ is a feasible solution for (5.1)
$\therefore c'\widehat{x} \leq -\widehat{y_1}'b$, where $\widehat{x}$ is a feasible solution for (5.1)
$\therefore c'\widehat{x} \leq \widehat{y_2}'b$, where $\widehat{x}$ is a feasible solution for (5.1) and $\widehat{y_2}$ is a feasible solution for (5.6)
$\because$ (5.6) is the same system as (5.2)
$\therefore$ we have $c'\widehat{x} \leq \widehat{y}'b$, where $\widehat{x}$ is a feasible solution for (5.1) and $\widehat{y}$ is a feasible solution for (5.2)

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6. Exercise 1.6 Weak Duality for a complicated form

Problem: Give and prove a Weak Duality Theorem for

$$\begin{array}{ll} \min & c'x + f'w \\ \text{s.t.} & Ax + Bw \leq b \\ & Dx = g \\ & x \geq 0, w \leq 0 \end{array} \tag{6.1}$$

Solution:
The dual of (6.1) is

$$\begin{array}{ll} \max & y'b + \pi'g \\ \text{s.t.} & y'A + \pi'D \leq c' \\ & y'B \geq f' \\ & y' \leq 0 \end{array} \tag{6.2}$$

We give the Weak Duality Theorem for (6.1) as follows,

Theorem 6.1. *If $\widehat{x}$, $\widehat{w}$ is feasible in 6.1, and $\widehat{y}$, $\widehat{\pi}$ is feasible in (6.2), then $c'\widehat{x} + f'\widehat{w} \geq \widehat{y}'b + \widehat{\pi}'g$.*

Proof:
We first transform (6.1) to standard form as follows,

$$\begin{array}{ll} \min & c_1'x_1 \\ \text{s.t.} & A_1x_1 = b_1 \\ & x_1 \geq 0 \end{array} \tag{6.3}$$

In (6.3), $x_1 = \begin{pmatrix} x \\ -w \\ t \end{pmatrix}$, $c_1' = \begin{pmatrix} c' & -f' & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} A & -B & I \\ D & 0 & 0 \end{pmatrix}$, and $b_1 = \begin{pmatrix} b \\ g \end{pmatrix}$.

The dual of (6.3) is

$$\begin{array}{ll} \max & y_1' b_1 \\ \text{s.t.} & y_1' A_1 \le c_1' \end{array} \tag{6.4}$$

In (6.4), let $y_1' = \begin{pmatrix} y_2' & \pi_2' \end{pmatrix}$, then we have

$\because y_1' A_1 \le c_1'$

$\therefore \begin{pmatrix} y_2' & \pi_2' \end{pmatrix} \begin{pmatrix} A & -B & I \\ D & 0 & 0 \end{pmatrix} \le \begin{pmatrix} c' & -f' & 0 \end{pmatrix}$

$\therefore y_2' A + \pi_2' D \le c'$, $-y_2' B \le -f'$, $y_2' \le 0$

then (6.4) becomes

$$\begin{array}{ll} \max & y_2' b + \pi_2' g \\ \text{s.t.} & y_2' A + \pi_2' D \le c' \\ & y_2' B \ge f' \\ & y_2' \le 0 \end{array} \tag{6.5}$$

We apply Weak Duality Theorem to primal (6.3) and dual (6.4), we have $c_1' \widehat{x_1} \ge \widehat{y_1}' b_1$, where $\widehat{x_1}$ is a feasible solution for (6.3) and $\widehat{y_1}$ is a feasible solution for (6.4).

$\because c_1' \widehat{x_1} \ge \widehat{y_1}' b_1$

$\therefore \begin{pmatrix} c' & -f' & 0 \end{pmatrix} \begin{pmatrix} \widehat{x} \\ -\widehat{w} \\ \widehat{t} \end{pmatrix} \ge \widehat{y_1}' \begin{pmatrix} b \\ g \end{pmatrix}$, where $\widehat{x}$, $\widehat{w}$ is a feasible solution for (6.1)

$\therefore c'\widehat{x} + f'\widehat{w} \ge \widehat{y_1}' \begin{pmatrix} b \\ g \end{pmatrix}$, where $\widehat{x}$, $\widehat{w}$ is a feasible solution for (6.1)

$\therefore c'\widehat{x} + f'\widehat{w} \ge \begin{pmatrix} \widehat{y_2}' & \widehat{\pi_2}' \end{pmatrix} \begin{pmatrix} b \\ g \end{pmatrix}$, where $\widehat{x}$, $\widehat{w}$ is a feasible solution for (6.1), and $\widehat{y_2}$, $\widehat{\pi_2}$ is a feasible solution for (6.5)

$\therefore c'\widehat{x} + f'\widehat{w} \ge \widehat{y_2}' b + \widehat{\pi_2}' g$, where $\widehat{x}$, $\widehat{w}$ is a feasible solution for (6.1), and $\widehat{y_2}$, $\widehat{\pi_2}$ is a feasible solution for (6.5)

$\because$ (6.5) is the same system as (6.2)

$\therefore$ we have $c'\widehat{x} + f'\widehat{w} \ge \widehat{y}' b + \widehat{\pi}' g$, where $\widehat{x}$, $\widehat{w}$ is a feasible solution for (6.1), and $\widehat{y}$, $\widehat{\pi}$ is a feasible solution for (6.4)

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7. Exercise 1.7 Weak Duality for a complicated form — with MATLAB

The MATLAB code is as follows. We use `linprog` function to solve the dual. We compare the optimal value of the original problem and its dual. Since there is computation deviation, we consider the difference less than 10^{-8} be negligible and the two optimal values are the same.

```
% DualityWithMatlab1.m // Jon Lee
%
n1=7
n2=15
m1=2
m2=4
rng('default');
rng(1); % set seed
A = rand(m1,n1);
B = rand(m1,n2);
D = rand(m2,n1);
% Organize the situation
% so that the problem has a feasible solution
x = rand(n1,1);
w = -rand(n2,1);
b = A*x + B*w + 0.01 * rand(m1,1);
g = D*x;
% Organize the situation
% so that the dual problem has a feasible solution
y = -rand(m1,1);
```

```
pi = rand(m2,1) - rand(m2,1);
c = A'*y + D'*pi + 0.01 * rand(n1,1);
f = B'*y - 0.01 * rand(n2,1);
% Here is how the 'linprog' function works:
%
% [v,z,exitflag] = linprog(c,A,b,Aeq,beq,lb,ub)
% minimizes c'v ,
% with constraints A v <= b, Aeq v = beq, and
% variable bounds lb <= v <=ub.
% Some parts of the model can be null: for example,
% set Aeq = [] and beq = [] if no equalities exist.
% set ub = [] if no upper bounds exist.
%
% [v,z,exitflag] = linprog(...) returns values:
% v = solution vector
% z = minimum objective value
% exitflag = describes the exit condition:
% 1 optimal solution found.
% ?2 problem is infeasible.
% ?3 problem is unbounded.
% ?5 problem and its dual are both infeasible
% Study the following part carefully. This is how we set up
% a block problem. [c ; f] stacks vertically the vectors c and f.
% [A B] concatenates A and B horizontally. zeros(m2,n2) is an
% m2?by?n2 matrix of 0's. ones(n1,1) is an n1?vector of 1's, and
% Inf is +infinity, so Inf*ones(n1,1) is a vector of length n1
% with all components equal to +infinity. In this way we have
% infinite upper bounds on some variables.
[v,z,exitflag] = linprog([c ; f],[A B],b,[D zeros(m2,n2)],g, ...
[zeros(n1,1);-Inf*ones(n2,1)],[Inf*ones(n1,1);zeros(n2,1)])
if (exitflag < 1)
disp('fail 1: LP did not have an optimal solution');
return;
end;
% Extract the parts of the solution v to the formulation vectors
x = v(1:n1,1)
w = v(n1+1:n1+n2,1)

[v2,z2,exitflag] = linprog([-b;-g], [A' D'; -B' zeros(n2,m2)], [c; -f],[],[],[],[zeros(m1,1);Inf*ones(m2,1)])

if (abs(z2 + z) < 1e-8)
    disp('The problem and its dual have the same optimal value')
end
```