HOMEWORK #6 SOLUTION

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1. Exercise 6.1 Illustrate local sensitivity analysis

Problem: Make an original example to illustrate the local-analysis concepts of this chapter.
Solution:

2. Exercise 6.2 Illustrate global sensitivity analysis

Problem: Using `AMPL`, make an original example, with at least three constraints, graphing the objective value of (P), as a single `b[i]` is varied from $-\infty$ to $+\infty$. As you work on this, bear in mind Theorem 6.3.
Solution:

3. Exercise 6.3 *"I feel that I know the change that is needed."* – Mahatma Gandhi

Problem: We are given $2m$ numbers satisfying $L_i \le 0 \le U_i$, $i = 1, 2, ..., m$. Let β be an optimal basis for all of the m problems

$$\begin{array}{lr} \min & c'x \\ \text{s.t.} & Ax = b + \Delta_i e^i; \\ & x \ge 0. \end{array} \tag{3.1}$$

for all Δ_i satisfying $L_i \le \Delta_i \le U_i$. Lets be clear on what this means: For each i individually, the basis β is optimal when the ith right-hand side component is changed from b_i to $b_i + \Delta_i$, as long as Δ_i is in the interval $[L_i, U_i]$.

The point of this problem is to be able to say something about *simultaneously* changing all of the b_i. Prove that we can simultaneously change b_i to

$$\tilde{b}_i := b_i + \lambda_i \left\{ \begin{array}{c} L_i \\ U_i \end{array} \right\}$$

where $\lambda_i \ge 0$, when $\sum_{i=1}^{m} \lambda_i \le 1$. [Note that in the formula above, for each i we can $i = 1$ pick either L_i (a decrease) or U_i (an increase)].
Solution:

4. Exercise 6.4 Domain for objective variations

Problem: Prove Theorem 6.4: The domain of g is a convex set.
Proof:
Suppose that c^j is in the domain of g, for $j = 1, 2$. Therefore, there exist x^j that are feasible for (4.1), for $j = 1, 2$.

$$\begin{array}{lr} \min & {c^j}'x \\ \text{s.t.} & Ax = b; \\ & x \ge 0. \end{array} \tag{4.1}$$

For any $0 < \lambda < 1$, consider $\widehat{c} := \lambda c^1 + (1-\lambda)c^2$, to prove the domain of g is a convex set, we need to prove that (4.2) is feasible and not unbounded.

$$\begin{array}{lr} \min & \widehat{c}'x \\ \text{s.t.} & Ax = b; \\ & x \ge 0. \end{array} \tag{4.2}$$

Consider $\widehat{x} := \lambda x^1 + (1-\lambda)x^2$,
$\because x^j$ that are feasible for (4.1), for $j = 1, 2$
$\therefore x^j \ge 0$, and $Ax^j = b$, for $j = 1, 2$
$\therefore A\widehat{x} = A(\lambda x^1 + (1-\lambda)x^2) = \lambda Ax^1 + (1-\lambda)Ax^2 = \lambda b + (1-\lambda)b = b$
$\because \lambda > 0$
$\therefore \widehat{x} = \lambda x^1 + (1-\lambda)x^2 \ge 0$
$\therefore \widehat{x}$ is a feasible solution to (4.2)

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$\therefore$ (4.2) is feasible
Consider the objective function $\widehat{c}'x$ of (4.2),
$\because$ c^j is in the domain of g, for $j = 1, 2$
$\therefore$ (4.1) is not unbounded for $j = 1, 2$
$\therefore$ $\exists k^j \in \mathbb{R}$, for any x, such that ${c^j}'x \geq k^j$, for $j = 1, 2$
$\therefore$ for a feasible solution $\overline{x}$ of (4.2), $\widehat{c}'\overline{x} = (\lambda c^1 + (1-\lambda)c^2)'\overline{x} = \lambda {c^1}'\overline{x} + (1-\lambda){c^2}'\overline{x} \geq (\lambda k^1 + (1-\lambda)k^2)$
$\because$ $k^1, k^2 \in \mathbb{R}$, $0 < \lambda < 1$
$\therefore$ (4.2) is feasible and not unbounded
$\therefore$ $\widehat{c}$ is in the domain g
$\therefore$ the domain of g is a convex set

$\blacksquare$

5. Exercise 6.5 Concave piecewise-linear function

Problem: Prove Theorem 6.5: g is a concave piecewise-linear function on its domain.
Proof:
The function g is

$$
\begin{array}{lr}
g(c) := \min & c'x \\
\text{s.t.} & Ax = b; \\
 & x \geq 0.
\end{array}
\tag{5.1}
$$

So a basis β is feasible or not for (5.1), independent for c_β. So g can be written as

$$g(c) = \min \left\{ c'_\beta (A_\beta^{-1} b) : \beta \text{ is a feasible basis for (5.1)} \right\}$$

$\because$ $c'_\beta(A_\beta^{-1}b)$ are affine functions
$\therefore$ g is the pointwise minimum of a finite number of affine functions
$\therefore$ g is a concave piecewise-linear function

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