

HOMEWORK #6 SOLUTION

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1. EXERCISE 6.1 ILLUSTRATE LOCAL SENSITIVITY ANALYSIS

Problem: Make an original example to illustrate the local-analysis concepts of this chapter.

Solution: Consider the standard-form problem (P_b) with $m = 3$ and $n = 5$

$$(P_b) \quad \begin{aligned} f(b) &:= \min && c'x \\ &&& Ax = b; \\ &&& x \geq \mathbf{0}, \end{aligned}$$

where

$$A := \begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & -4 & 2 & 2 & 0 \\ 0 & -9 & 0 & 6 & 3 \end{pmatrix}, \quad b := (1, 2, 18)', \quad c := (16, 7, 20, 10, 4)'.$$

Consider a fixed basis $\beta = \{1, 4, 5\}$ for (P_b) . Associated with that basis is the basic solution $\bar{x}_\beta = A_\beta^{-1}b = (2, 1, 4)'$ and the corresponding dual solution $\bar{y}' = c'_\beta A_\beta^{-1} = (16, 9, 4/3)$, which is feasible for the dual of (P_b) . We have

$$f(b) = \bar{y}'b = 16b_1 + 9b_2 + (4/3)b_3,$$

for $b \in \mathcal{B}$ where $\mathcal{B} \subset \mathbb{R}^m$ is the solution set of m linear inequalities (more details below). We have

$$\frac{\partial f}{\partial b_i} = \bar{y}_i, \quad i = 1, \dots, m.$$

The \mathcal{B} represents that the change to a single right-hand side element b_i should keep $\bar{b}_i > 0$ for $i = 1, \dots, m$. Let the right hand change be $b + \Delta_i e^i$. Then $A_\beta^{-1}(b + \Delta_i e^i) \geq \mathbf{0}$. Let $h^i := A_\beta^{-1}e^i$. This means that Δ_i must be in the interval $[L_i, U_i]$, where

$$L_i := \max_{k: h_k^i > 0} -\bar{b}_k / h_k^i$$

, and

$$U_i := \min_{k: h_k^i < 0} -\bar{b}_k / h_k^i$$

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We can confirm through AMPL that $\Delta_1 \in [-2, +\infty)$, $\Delta_2 \in [-2, 4]$, and $\Delta_3 \in [-12, +\infty)$ (consistent with Table 1).

Next, we define a function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ via

$$(P^c) \quad \begin{aligned} g(c) &:= \min && c'x \\ &&& Ax = b; \\ &&& x \geq \mathbf{0}, \end{aligned}$$

with the same parameters. Consider a fixed basis $\beta = \{1, 4, 5\}$ for (P^c) . Similarly, we have $g(c) = c'_\beta \bar{x}_\beta = 2c_1 + c_4 + 4c_5$. If only changing a single c_i and the optimal basis keeps, the interval of c_i are: $c_1 \in [-23/3, 18]$, $c_2 \in [-64, +\infty)$, $c_3 \in [18, +\infty)$, $c_4 \in [-25.5, 12]$, and $c_5 \in [3, 75]$.

2. EXERCISE 6.2 ILLUSTRATE GLOBAL SENSITIVITY ANALYSIS

Problem: Using AMPL, make an original example, with at least three constraints, graphing the objective value of (P), as a single $b[i]$ is varied from $-\infty$ to $+\infty$. As you work on this, bear in mind Theorem 6.3.

Solution: Consider the standard-form problem (P) with $m = 3$ and $n = 5$

$$(P) \quad \begin{aligned} \min \quad & c'x \\ Ax \quad &= b; \\ x \quad &\geq \mathbf{0}, \end{aligned}$$

where

$$A := \begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & -4 & 2 & 2 & 0 \\ 0 & -9 & 0 & 6 & 3 \end{pmatrix}, \quad b := (1, 2, 18)', \quad c := (16, 7, 20, 10, 4)'.$$

Using AMPL, at each time we vary only one b_i from b and check the range of b_i in which the optimal basis for (P) keeps the same. We show the optimal basis as follows:

b_1	$(-\infty, -13)$	$(-13, -1)$	$(-1, +\infty)$
optimal basis β	$\{2, 3, 4\}$	$\{2, 4, 5\}$	$\{1, 4, 5\}$
$c'_\beta A_\beta^{-1}$	$(-12.8 \ 10 \ -3.8)$	$(-23/3 \ -17/6 \ 4/3)$	$(16 \ 9 \ 4/3)$
objective value	$-12.8b_1 - 48.4$	$(-23/3)b_1 + 55/3$	$16b_1 + 42$

b_2	$(-\infty, 0)$	$(0, 6)$	$(6, +\infty)$
optimal basis β	$\{1, 2, 5\}$	$\{1, 4, 5\}$	$\{1, 3, 4\}$
$c'_\beta A_\beta^{-1}$	$(16 \ -8.75 \ 4/3)$	$(16 \ 9 \ 4/3)$	$(16 \ 10 \ 18)$
objective value	$-8.75b_2 + 40$	$9b_2 + 40$	$10b_2 + 34$

b_3	$(-\infty, 0)$	$(0, 6)$	$(6, +\infty)$
optimal basis β	$\{1, 2, 3\}$	$\{1, 3, 4\}$	$\{1, 4, 5\}$
$c'_\beta A_\beta^{-1}$	$(16 \ 10 \ -7)$	$(16 \ 10 \ 1)$	$(16 \ 9 \ 4/3)$
objective value	$-7b_3 + 36$	$b_3 + 36$	$(4/3)b_3 + 34$

TABLE 1. Changing single b_i from $-\infty$ to $+\infty$

As we get the above data from AMPL, we graph the objective value of (P) as a single b_i is varied from $-\infty$ to $+\infty$ in Figure 1.

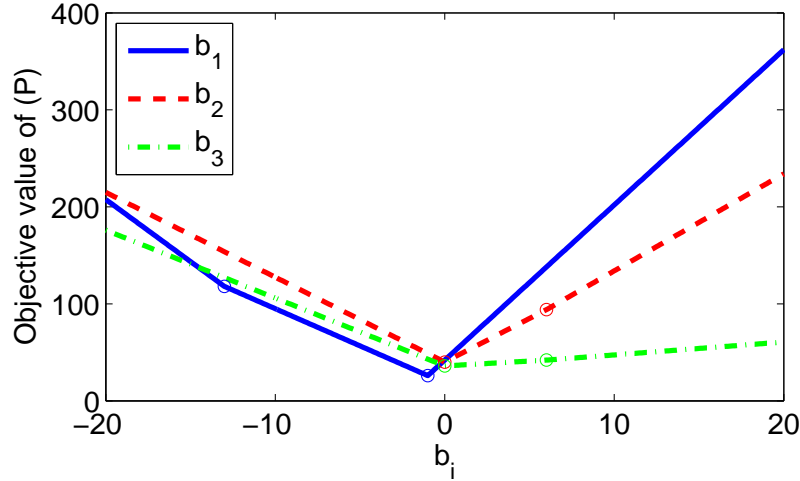


FIGURE 1. Objective value of (P) as a single b_i is varied from $-\infty$ to $+\infty$

3. EXERCISE 6.3 “I feel that I know the change that is needed.” – MAHATMA GANDHI

Problem: We are given $2m$ numbers satisfying $L_i \leq 0 \leq U_i$, $i = 1, 2, \dots, m$. Let β be an optimal basis for all of the m problems

$$(3.1) \quad \begin{aligned} \min \quad & c'x \\ \text{s.t.} \quad & Ax = b + \Delta_i e^i; \\ & x \geq 0. \end{aligned}$$

for all Δ_i satisfying $L_i \leq \Delta_i \leq U_i$. Lets be clear on what this means: For each i individually, the basis β is optimal when the i th right-hand side component is changed from b_i to $b_i + \Delta_i$, as long as Δ_i is in the interval $[L_i, U_i]$.

The point of this problem is to be able to say something about *simultaneously* changing all of the b_i . Prove that we can simultaneously change b_i to

$$\tilde{b}_i := b_i + \lambda_i \left\{ \begin{array}{c} L_i \\ U_i \end{array} \right\}$$

where $\lambda_i \geq 0$, when $\sum_{i=1}^m \lambda_i \leq 1$. [Note that in the formula above, for each i we can $i = 1$ pick either L_i (a decrease) or U_i (an increase)].

Proof:

Let $D_i = \left\{ \begin{array}{c} L_i \\ U_i \end{array} \right\}$, so $\max_{k: h_k^i > 0} -\frac{\bar{b}_k}{h_k^i} \leq D_i \leq \min_{k: h_k^i < 0} -\frac{\bar{b}_k}{h_k^i}$, where $\bar{b} = A_\beta^{-1}b \geq 0$, $h^i = A_\beta^{-1}e^i$, and $k = 1 \dots m$.

Thus, what we need to prove is that the basis β is optimal when we simultaneously change b_i to $\tilde{b}_i := b_i + \lambda_i D_i$, where $\sum_{i=1}^m \lambda_i = 1$. In the other word, we need to prove that the basis β is optimal for (3.2), where $\sum_{i=1}^m \lambda_i = 1$.

$$(3.2) \quad \begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax = b + \sum_{i=1}^m \lambda_i D_i e^i; \\ & x \geq 0. \end{array}$$

Consider a fixed and optimal basis β for (3.3). For β to be optimal for (3.2), we need $A_\beta^{-1}(b + \sum_{i=1}^m \lambda_i D_i e^i) = \bar{b} + \sum_{i=1}^m \lambda_i D_i h^i \geq 0$.

$$(3.3) \quad \begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax = b; \\ & x \geq 0. \end{array}$$

For $k = 1 \dots m$,

$$\begin{aligned} & \bar{b}_k + \sum_{i=1}^m \lambda_i D_i h_k^i \\ &= \bar{b}_k + \sum_{i=1, h_k^i > 0}^m \lambda_i D_i h_k^i + \sum_{i=1, h_k^i < 0}^m \lambda_i D_i h_k^i \\ &\geq \bar{b}_k + \sum_{i=1, h_k^i > 0}^m \lambda_i (\max_{k': h_{k'}^i > 0} -\frac{\bar{b}_{k'}}{h_{k'}^i}) h_k^i + \sum_{i=1, h_k^i < 0}^m \lambda_i (\min_{k': h_{k'}^i < 0} -\frac{\bar{b}_{k'}}{h_{k'}^i}) h_k^i \\ &\geq \bar{b}_k + \sum_{i=1, h_k^i > 0}^m \lambda_i (-\frac{\bar{b}_k}{h_k^i}) h_k^i + \sum_{i=1, h_k^i < 0}^m \lambda_i (-\frac{\bar{b}_k}{h_k^i}) h_k^i \\ &= \bar{b}_k + \sum_{i=1, h_k^i > 0}^m \lambda_i (-\bar{b}_k) + \sum_{i=1, h_k^i < 0}^m \lambda_i (-\bar{b}_k) \\ &= \bar{b}_k + \sum_{i=1}^m \lambda_i (-\bar{b}_k) \\ &= \bar{b}_k + (-\bar{b}_k) \sum_{i=1}^m \lambda_i \\ &\geq \bar{b}_k + (-\bar{b}_k) \\ &= 0 \end{aligned}$$

Thus, $\bar{b} + \sum_{i=1}^m \lambda_i D_i h^i \geq 0$, and β is feasible and hence still optimal for (3.2). ■

4. EXERCISE 6.4 DOMAIN FOR OBJECTIVE VARIATIONS

Problem: Prove Theorem 6.4: The domain of g is a convex set.

Proof:

Suppose that c^j is in the domain of g , for $j = 1, 2$. Therefore, there exist x^j that are feasible for (4.1), for $j = 1, 2$.

$$(4.1) \quad \begin{array}{ll} \min & c^j x \\ \text{s.t.} & Ax = b; \\ & x \geq 0. \end{array}$$

For any $0 < \lambda < 1$, consider $\hat{c} := \lambda c^1 + (1 - \lambda)c^2$, to prove the domain of g is a convex set, we need to prove that (4.2) is feasible and not unbounded.

$$(4.2) \quad \begin{array}{ll} \min & \hat{c}'x \\ \text{s.t.} & Ax = b; \\ & x \geq 0. \end{array}$$

Consider $\hat{x} := \lambda x^1 + (1 - \lambda)x^2$,

$\therefore x^j$ that are feasible for (4.1), for $j = 1, 2$

$\therefore x^j \geq 0$, and $Ax^j = b$, for $j = 1, 2$

$$\therefore A\hat{x} = A(\lambda x^1 + (1 - \lambda)x^2) = \lambda Ax^1 + (1 - \lambda)Ax^2 = \lambda b + (1 - \lambda)b = b$$

$$\therefore \lambda > 0$$

$$\therefore \hat{x} = \lambda x^1 + (1 - \lambda)x^2 \geq 0$$

$$\therefore \hat{x} \text{ is a feasible solution to (4.2)}$$

$$\therefore (4.2) \text{ is feasible}$$

Consider the objective function $\tilde{c}'x$ of (4.2),

$$\therefore c^j \text{ is in the domain of } g, \text{ for } j = 1, 2$$

$$\therefore (4.1) \text{ is not unbounded for } j = 1, 2$$

$$\therefore \exists k^j \in \mathbb{R}, \text{ for any } x, \text{ such that } c^{j'}x \geq k^j, \text{ for } j = 1, 2$$

$$\therefore \text{for a feasible solution } \bar{x} \text{ of (4.2), } \tilde{c}'\bar{x} = (\lambda c^1 + (1 - \lambda)c^2)'\bar{x} = \lambda c^{1'}\bar{x} + (1 - \lambda)c^{2'}\bar{x} \geq (\lambda k^1 + (1 - \lambda)k^2)$$

$$\therefore k^1, k^2 \in \mathbb{R}, 0 < \lambda < 1$$

$$\therefore (4.2) \text{ is feasible and not unbounded}$$

$$\therefore \hat{c} \text{ is in the domain } g$$

$$\therefore \text{the domain of } g \text{ is a convex set}$$

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5. EXERCISE 6.5 CONCAVE PIECEWISE-LINEAR FUNCTION

Problem: Prove Theorem 6.5: g is a concave piecewise-linear function on its domain.

Proof:

The function g is

$$(5.1) \quad \begin{aligned} g(c) &:= \min && c'x \\ \text{s.t.} &&& Ax = b; \\ &&& x \geq 0. \end{aligned}$$

So a basis β is feasible or not for (5.1), independent for c_β . So g can be written as

$$g(c) = \min \left\{ c'_\beta (A_\beta^{-1}b) : \beta \text{ is a feasible basis for (5.1)} \right\}$$

$$\therefore c'_\beta (A_\beta^{-1}b) \text{ are affine functions}$$

$$\therefore g \text{ is the pointwise minimum of a finite number of affine functions}$$

$$\therefore g \text{ is a concave piecewise-linear function}$$

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