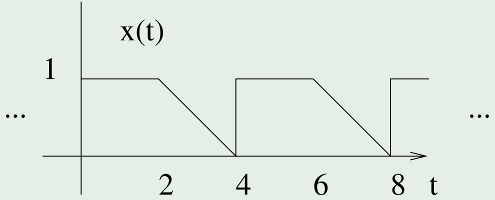
Example

Example

Find the Fourier series representation of the signal $x(t)$ pictured below. Express your results in a real form.



Solution (1)

Method 1

$T_0 = 4 \Longrightarrow \omega_0 = \pi/2.$

- $c_0 = (1/4)[2+1] = 3/4.$
- For $k \neq 0$

$$c_k = 1/4 \int_0^2 1 e^{-jk\omega_0 t}\, dt + 1/4 \int_2^4 (2 - t/2) e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{j2\pi k} + \begin{cases} 0, & k \text{ even} \\ \frac{-1}{k^2\pi^2}, & k \text{ odd}, \end{cases}$$

$$\boxed{x(t) = 3/4 + \sum_{k=1}^{\infty} \frac{1}{\pi k} \sin\left(k \frac{\pi}{2} t\right) + \sum_{k=1 \text{ odd}}^{\infty} \frac{-2}{k^2\pi^2} \cos\left(k \frac{\pi}{2} t\right).}$$

Solution (2)

Method 2

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) == \sum_{n=-\infty}^{\infty} \delta(t-4n) - \frac{1}{2}\sum_{n=-\infty}^{\infty} \mathrm{rect}\left(\frac{t-3-4n}{2}\right)$$

The FS coefficients of $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$ are

$k \neq 0$

$$\begin{aligned} d_k &= \frac{1}{T_0}\int_{T_0} y(t)e^{-jk\omega_0 t}\,dt = \frac{1}{4}\int_0^4 y(t)e^{-jk\omega_0 t}\,dt \\ &= \frac{1}{4}\int_0^4 \delta(t)e^{-jk\omega_0 t}\,dt - \frac{1}{8}\int_2^4 e^{-jk\omega_0 t}\,dt \\ &= \frac{1}{4} - \frac{1}{4} - \frac{1}{-jk\omega_0}e^{-jk\omega_0 t}\Big|_2^4 \\ &= \frac{1}{4} - \frac{1}{8jk\omega_0}\left(e^{-4jk\omega_0} - e^{-2jk\omega_0}\right) \end{aligned}$$

Solution (3)

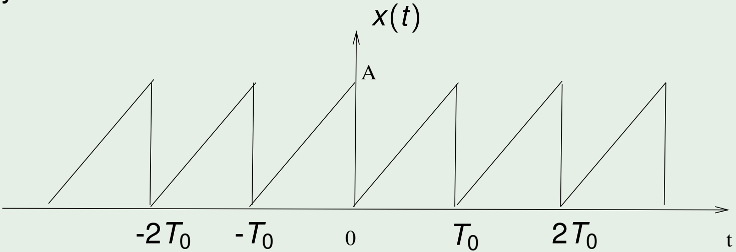
Since $d_k = jk\omega_0 c_k$,

$$
\begin{aligned}
c_k &= \frac{1}{jk\omega_0} d_k = \frac{1}{4jk\omega_0} - \frac{1}{8(jk\omega_0)^2}\left(e^{-4jk\omega_0} - e^{-2jk\omega_0}\right) \\
&= \frac{1}{j2\pi k} + \frac{1}{j2\pi^2 k^2}\left(e^{-j2\pi k} - e^{-j\pi k}\right) \quad (\omega_0 = \pi/2) \\
&= \frac{1}{j2\pi k} + \begin{cases} 0, & k \text{ even} \\ \frac{-1}{k^2\pi^2}, & k \text{ odd}, \end{cases} .
\end{aligned}
$$

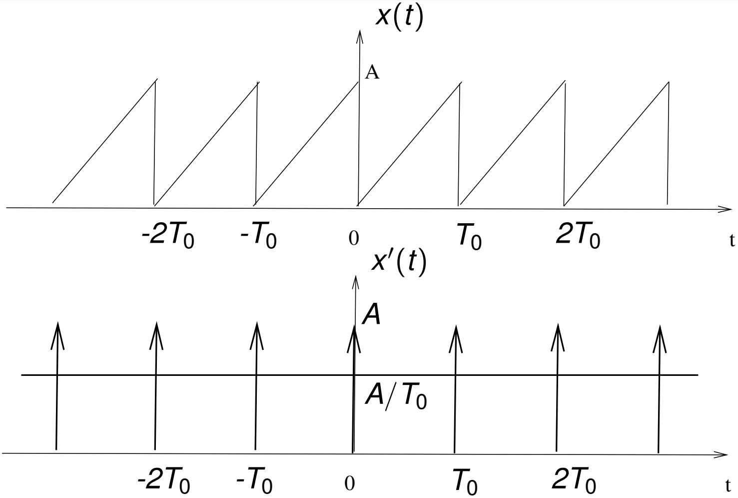
Example

Example

Find the FS of $x(t)$ using the differentiation property. Express your results in a real form.

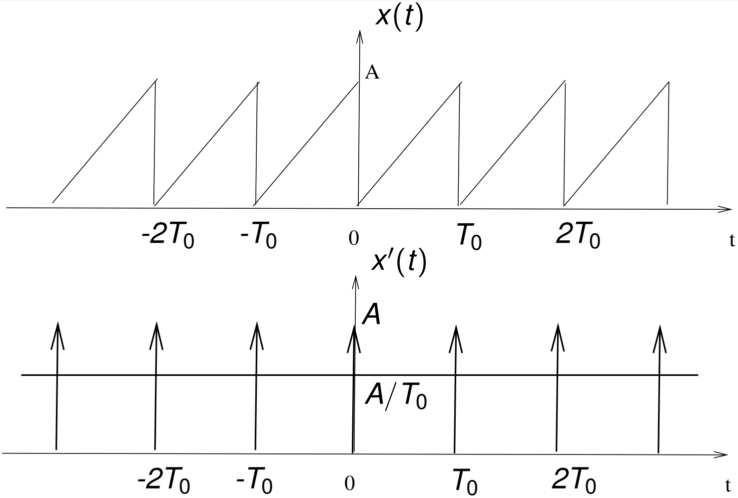


Solution (1)



$$x'(t) = \frac{A}{T_0} + g(t), \quad g(t) = A \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

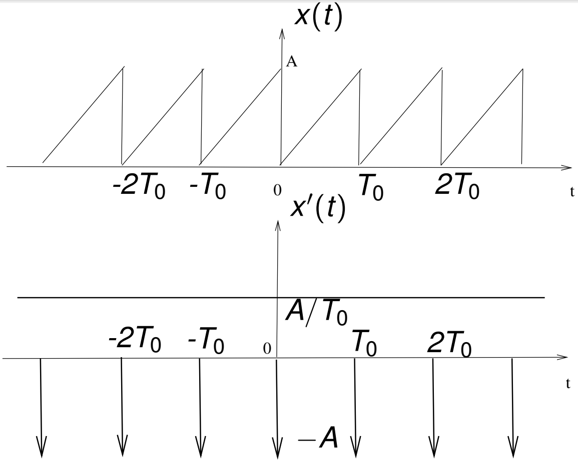
Solution (1)



$$x'(t) = \frac{A}{T_0} + g(t), \quad g(t) = A \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

Wrong!

Solution (2)



$$x'(t) = \frac{A}{T_0} - g(t), \quad g(t) = A \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

Solution (2)

The FS coefficients for $g(t)$ is

$$d'_k = \frac{1}{T_0} \underbrace{\int_{T_0} \delta(t) e^{-jk\omega_0 t}\, dt}_{=1 \text{ sifting property}} = \frac{1}{T_0}, \quad \forall k$$

$$g(t) = A \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = A \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 t}$$

$$x'(t) = \frac{A}{T_0} - A \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}$$

$$d_k = \begin{cases} 0, & k = 0 \\ -A\frac{1}{T_0}, & k \neq 0 \end{cases}$$

Solution (2)

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$$d'_k = \frac{1}{T_0} \underbrace{\int_{T_0} \delta(t) e^{-jk\omega_0 t}\, dt}_{=1 \textit{ sifting property}} = \frac{1}{T_0}, \quad \forall k$$

$$g(t) = A \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = A \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 t}$$

$$x'(t) = \frac{A}{T_0} - A \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}$$

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$$g(t) = A \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = A \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 t}$$

$$x'(t) = \frac{A}{T_0} - A \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}$$

$$d_k = \begin{cases} 0, & k = 0 \\ -A\frac{1}{T_0}, & k \neq 0 \end{cases}$$

Solution (3)

$$c_k = \begin{cases} \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T_0}\int_0^{T_0} \frac{A}{T_0} t\,dt = \frac{A}{2}, & k = 0 \\ \frac{d_k}{jk\omega_0} = -\frac{A}{T_0 jk\omega_0} = j\frac{A}{2\pi k}, & k \neq 0 \end{cases}$$

$$x(t) = \frac{A}{2} + \sum_{k=-\infty, k\neq 0}^{\infty} j\frac{A}{2\pi k} e^{jk\omega_0 t}$$

Solution (3)

$$c_k = \begin{cases} \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T_0}\int_0^{T_0} \frac{A}{T_0}t\,dt = \frac{A}{2}, & k = 0 \\ \frac{d_k}{jk\omega_0} = -\frac{A}{T_0 jk\omega_0} = j\frac{A}{2\pi k}, & k \neq 0 \end{cases}$$

$$x(t) = \frac{A}{2} + \sum_{k=-\infty, k\neq 0}^{\infty} j\frac{A}{2\pi k}e^{jk\omega_0 t}$$

Solution (4)

$$
\begin{aligned}
x(t) &= \frac{A}{2} + \sum_{k=1}^{\infty} j \left[\frac{A}{2\pi k} e^{jk\omega_0 t} - \frac{A}{2\pi k} e^{-jk\omega_0 t} \right] \\
&= \frac{A}{2} - \frac{A}{\pi} \sum_{k=1}^{\infty} \frac{e^{jk\omega_0 t} - e^{-jk\omega_0 t}}{2jk} \\
&= \boxed{\frac{A}{2} - \frac{A}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin(k\omega_0 t)}
\end{aligned}
$$