VE216
Introduction to Signals and Systems

HOMEWORK 2

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1. Since $f(t)$ is the gain of the system, $y\mathrm{d}t = \mathcal{T}x(t-t_0) = f(t)x(t-t_0)$ while $y(t-t_0) = f(t-t_0)x(t-t_0)$. Since there exists t_0, t_1 that $f(t_0) \neq f(t_1)$, there exists input signal $x(t)$ and time shift t_0 that $y\mathrm{d}t \neq y(t-t_0)$, thus the system is time-invariant, i.e. a system with time-variant gain connot be time-invariant. We can find a counterexample that if $f(t) = t$, $x(t) = \delta(t)$, then $y\mathrm{d}t = t \cdot \delta(t-t_0) \neq (t-t_0)\delta(t-t_0) = y(t-t_0)$, so $x(t) \xrightarrow{\mathcal{T}} y(t)$ is not a time-invariant system.

2.

(a)

$$\begin{aligned}
y\mathrm{d}t &= \mathcal{T}x(t-t_0) \\
&= \int_{-\infty}^{t}\left[\int_{-\infty}^{s} x(\tau - t_0 - 5)\mathrm{d}\tau\right]\mathrm{d}s \\
&= \int_{-\infty}^{t}\left[\int_{-\infty}^{s-t_0} x(\tau' - 5)\mathrm{d}\tau'\right]\mathrm{d}s \\
&= \int_{-\infty}^{t-t_0}\left[\int_{-\infty}^{s'} x(\tau' - 5)\mathrm{d}\tau'\right]\mathrm{d}s' \\
y(t-t_0) &= \int_{-\infty}^{t-t_0}\left[\int_{-\infty}^{s} x(\tau - 5)\mathrm{d}\tau\right]\mathrm{d}s
\end{aligned}$$

Since $y\mathrm{d}t = y(t-t_0)$, this system is time-invariant. Since $y(t)$ is linear, $y(t)$ is an LTI system, the impulse response of this system is the output when the input is an impulse signal, i.e.,

$$\boxed{h(t) = \int_{-\infty}^{t}\left[\int_{-\infty}^{s} \delta(\tau - 5)\mathrm{d}\tau\right]\mathrm{d}s = \int_{-\infty}^{t} u(s-5)\mathrm{d}s = (t-5)u(t-5).}$$

(b)

$$y\mathrm{d}t = \int_{-1}^{3} e^{-(t-\tau)^2}x(\tau - t_0)\tau = \int_{-1-t_0}^{3-t_0} e^{-(t-t_0-\tau')^2}x(\tau')\mathrm{d}\tau'$$

$$y(t-t_0) = \int_{-1}^{3} e^{-(t-t_0-\tau)^2}x(\tau)\mathrm{d}\tau$$

Since there exists $x(t)$, t_0 that $y\mathrm{d}t \neq y(t-t_0)$, $\boxed{\text{this system is NOT time-invariant}}$.

(c)

$$\begin{aligned}
y\mathrm{d}t &= \int_{-3}^{3} \tau^2 x(t - t_0 - \tau)\mathrm{d}\tau + \int_{-\infty}^{t+1} (t-\tau+3)^{-2}x(\tau - t_0)\mathrm{d}\tau \\
&= \int_{-3}^{3} \tau^2 x(t - t_0 - \tau)\mathrm{d}\tau + \int_{-\infty}^{t-t_0+1} (t - t_0 - \tau' + 3)^{-2}x(\tau')\mathrm{d}\tau'
\end{aligned}$$

$$y(t-t_0) = \int_{-3}^{3} \tau^2 x(t-t_0-\tau)\mathrm{d}\tau + \int_{-\infty}^{t-t_0+1} (t-t_0-\tau+3)^{-2} x(\tau)\mathrm{d}\tau$$

Since $y\mathrm{d}t = y(t-t_0)$, this system is time-invariant. Since $y(t)$ is linear, $y(t)$ is an LTI system, the impulse response of this system is

$$\begin{aligned}
h(t) &= \int_{-3}^{3} \tau^2 \delta(t-\tau)\mathrm{d}\tau + \int_{-\infty}^{t+1} (t-\tau+3)^{-2} \delta(\tau)\mathrm{d}\tau \\
&= \int_{-3}^{3} t^2 \delta(\tau)\mathrm{d}\tau + \int_{-\infty}^{t+1} (t+3)^{-2} \delta(\tau)\mathrm{d}\tau \\
&= t^2 \cdot \mathrm{rect}(\frac{t}{6}) + (t+3)^{-2} u(t+1) \\
&= \begin{cases} 0 & t < -3 \\ t^2 & -3 \le t < -1 \\ t^2 + (t+3)^{-2} & -1 \le t \le 3 \\ (t+3)^{-2} & t > 3 \end{cases}
\end{aligned}$$

3.

(a) If $y(t) = h(t) * x(t)$, i.e.

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau)\mathrm{d}\tau,$$

then by commutative property and associative property

$$h(t) * x(t-3) = [x(t) * \delta(t-3)] * h(t) = [x(t) * h(t)] * \delta(t-3) = y(t) * \delta(t-3) = y(t-3),$$

Hence, if $y(t) = h(t) * x(t)$ then $\boxed{y(t-3) = h(t) * x(t-3)}$.

(b) For example, if $x(t) = e^{-at}u(t)$ and $h(t) = u(t)$, then when $t > 0$

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t \\ 0, & \text{other} \end{cases}$$

and when $t < 0$, $y(t) = 0$. When $t > 0$,

$$y(t) = \int_0^t e^{-a\tau}\mathrm{d}\tau = -\frac{1}{a} e^{-a\tau}\Big|_0^t = \frac{1}{a}(1 - e^{-at}),$$

so for all of t,

$$y(t) = \frac{1}{a}(1 - e^{-at})u(t),$$

$$y(t-3) = \frac{1}{a}(1 - e^{-a(t-3)})u(t-3).$$

Next, we can see $x(t-3) = e^{-a(t-3)}u(t-3)$, $h(t-3) = u(t-3)$, then when $t > 6$

$$x(\tau-3)h(t-\tau-3) = \begin{cases} e^{-a(\tau-3)}, & 3 < \tau < t-3 \\ 0, & \text{other} \end{cases}$$

and when $t < 6$, $y(t) = 0$. When $t > 6$,

$$h(t-3) * x(t-3) = \int_3^{t-3} e^{-a(\tau-3)}\mathrm{d}\tau = -\frac{1}{a} e^{-a(\tau-3)}\Big|_3^{t-3} = \frac{1}{a}(1 - e^{-a(t-6)}),$$

so for all of t,

$$h(t-3) * x(t-3) = \frac{1}{a}(1 - e^{-a(t-6)})u(t-6).$$

Therefore, $\boxed{y(t-3) \neq h(t-3) * x(t-3)}$.

(c) To repeat (a) and (b) for multiplication, we first give a simple proof of the correct statement. If $y(t) = h(t) \cdot x(t)$, then by time-shifting we can directly derive that

$$y(t-3) = h(t-3) \cdot x(t-3).$$

To give a counterexample for the incorrect statement, we assume that $x(t) = u(t)$, $h(t) = \delta(t)$, then $y(t) = h(t) \cdot x(t) = \delta(t)$, $y(t-3) = \delta(t-3)$,

$$h(t) \cdot x(t-3) = \delta(t) \cdot u(t-3) = 0 \neq y(t-3).$$

Therefore, $y(t) = h(t) \cdot x(t)$ does not imply that $y(t-3) = h(t) \cdot x(t-3)$.

4.

(a) By the definition of convolution integral, the range that $y(t)$ is non-zero must at least start at $a+c$. After conversion, $x_2(t-\tau)$ is non-zero over the range $t-d \leq t \leq t-c$. Considering the product of $x_1(\tau)$ and $x_2(t-\tau)$, when shifting $x_2(t-\tau)$ from $t-c = -\infty$ to a, $x_1(\tau)x_2(t-\tau)$ is zero. Then, the product will become non-zero until $x_2(t-\tau)$ is shifted to $t-d > b$, so the range ends at $t = b+d$. Therefore, the range of values of t for which $y(t)$ is possibly non-zero is $\boxed{a+c \leq t \leq b+d}$.

(b) Using $\text{rect}(t) = u(t+1/2) - u(t-1/2)$, we have

$$\text{rect}((t-2)/2) = u(t-1) - u(t-3)$$

$$\text{rect}((t+3)/4) = u(t+5) - u(t+1)$$

so

$$\begin{aligned}
&\text{rect}((t-2)/2) * \text{rect}((t+3)/4) \\
&= [u(t-1) - u(t-3)] * [u(t+5) - u(t+1)] \\
&= u(t-1) * u(t+5) - u(t-1) * u(t+1) - u(t-3) * u(t+5) + u(t-3) * u(t+1) \\
&\qquad \text{(Distributive property)}
\end{aligned}$$

The convolution integral of unit step function and unit step function is unit ramp function denoted as $R(t)$

$$u(t) * u(t) = \int_{-\infty}^{+\infty} u(\tau)u(t-\tau) = tu(t) = R(t)$$

We can prove the time-shifting property of convolution integral

$$\begin{aligned}
h(t-t_1) * x(t-t_2) &= [x(t) * \delta(t-t_2)] * [h(t) * \delta(t-t_1)] \\
&= [x(t) * h(t)] * [\delta(t-t_1) * \delta(t-t_2)] \\
&= y(t) * \delta(t-t_1-t_2) \\
&= y(t-t_1-t_2)
\end{aligned}$$

Thus,

$$u(t-t_1) * u(t-t_2) = R(t-t_1-t_2).$$

Thus,

$$\begin{aligned}
&\text{rect}((t-2)/2) * \text{rect}((t+3)/4) \\
&= R(t+4) - R(t) - R(t+2) + R(t-2) \\
&= (t+4)u(t+4) - tu(t) - (t+2)u(t+2) + (t-2)u(t-2) \\
&= \begin{cases} 0 & t < -4 \\ t+4 & -4 \leq t < -2 \\ 2 & -2 \leq t < 0 \\ -t+2 & 0 \leq t < 2 \\ 0 & t \geq 2 \end{cases}
\end{aligned}$$

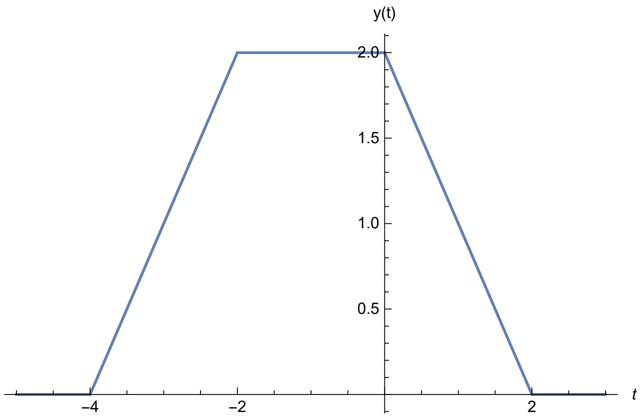

Figure 1. Sketch of $\mathrm{rect}((t-2)/2) * \mathrm{rect}((t+3)/4)$.

We can see that the non-zero range of the convolution integral is $-4 < t < 2$. $\mathrm{rect}((t-2)/2)$ is non-zero over the range $1 < t < 3$ and $\mathrm{rect}((t+3)/4)$ is non-zero over the range $-5 < t < -1$, so $a = 1$, $b = 3$, $c = -5$, $d = -1$, $a + c = -4$, $b + d = 2$, according to (a) the non-zero range is $-4 < t < 2$, which is exactly what we have, so our result in part (a) is true.

5.

(a) The response $y(t)$ is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} e^{-\alpha\tau}u(\tau)e^{-\beta(t-\tau)}u(t-\tau)\mathrm{d}\tau.$$

Since

$$u(\tau) = \begin{cases} 1, & t - \tau \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$u(t-\tau) = \begin{cases} 1, & \tau \leq t \\ 0, & \text{otherwise} \end{cases}$$

we can change the limit of the integral and simply the expression:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} e^{-\alpha\tau}u(\tau)e^{-\beta(t-\tau)}u(t-\tau)\mathrm{d}\tau \\ &= \int_{0}^{\infty} e^{-\alpha\tau}e^{-\beta(t-\tau)}u(t-\tau)\mathrm{d}\tau \\ &= \int_{0}^{t} e^{-\alpha\tau-\beta(t-\tau)}\mathrm{d}\tau \\ &= \int_{0}^{t} e^{-(\alpha-\beta)\tau}e^{-\beta t}\mathrm{d}\tau \end{aligned}$$

When $\alpha \neq \beta$, we calculate the integral as

$$
\begin{aligned}
y(t) &= e^{-\beta t} \int_0^t e^{-(\alpha-\beta)\tau} \mathrm{d}\tau \\
&= e^{-\beta t} \frac{e^{-(\alpha-\beta)\tau}}{-(\alpha-\beta)} \bigg|_0^t \\
&= \frac{e^{-\beta t}}{\beta-\alpha} (e^{(\beta-\alpha)t} - 1) \\
&= \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha}
\end{aligned}
$$

When $\alpha = \beta$,

$$
\begin{aligned}
y(t) &= \int_0^t e^{-(\alpha-\alpha)\tau} e^{-\alpha t} \mathrm{d}\tau \\
&= \int_0^t e^{-\alpha t} \mathrm{d}\tau \\
&= t e^{-\alpha t}
\end{aligned}
$$

Since the convolution exists from 0 to ∞, the response $y(t)$ is

$$
\boxed{y(t) = \begin{cases} \dfrac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha} u(t), & \alpha \neq \beta \\ t e^{-\alpha t} u(t), & \alpha = \beta \end{cases}}
$$

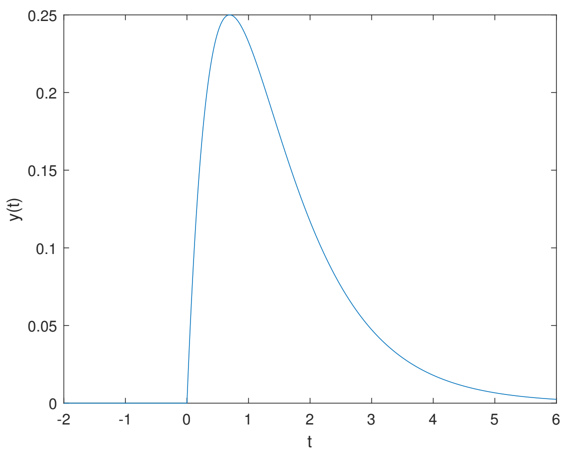

Figure 2. Sketch of $y(t)$ when $\alpha = 1$ and $\beta = 2$.

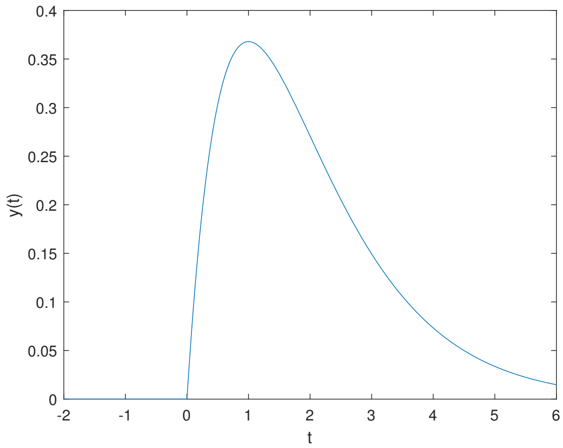

Figure 3. Sketch of $y(t)$ when $\alpha = \beta = 1$.

(b) According to the provided conditions, $x(t) = at + b$ and $h(t) = \frac{4}{3}(u(t) - u(t-1)) - \frac{1}{3}\delta(t-2)$, then

$$x(\tau) = a\tau + b$$

$$h(t-\tau) = \frac{4}{3}(u(t-\tau) - u(t-\tau-1)) - \frac{1}{3}\delta(t-\tau-2)$$

The response

$$\begin{aligned}
y(t) &= x(t) * h(t) \\
&= x(t) * (\frac{4}{3}(u(t) - u(t-1)) - \frac{1}{3}\delta(t-2)) \\
&= \frac{4}{3}x(t) * (u(t) - u(t-1)) - \frac{1}{3}x(t) * \delta(t-2) \\
&= \frac{4}{3}\int_{-\infty}^{\infty}(a\tau + b)(u(t-\tau) - u(t-\tau-1))\mathrm{d}\tau - \frac{1}{3}\int_{-\infty}^{\infty}(a\tau + b)\delta(t-\tau-2)\mathrm{d}\tau \\
&= \frac{4}{3}\int_{t-1}^{t}(a\tau + b)\mathrm{d}\tau - \frac{1}{3}(a(t-2) + b) \\
&= \frac{4}{3}(a\frac{\tau^2}{2} + b\tau)\bigg|_{t-1}^{t} - \frac{1}{3}(a(t-2) + b) \\
&= \frac{4}{3}(\frac{a}{2}(2t-1) + b) - \frac{1}{3}(at - 2a + b) \\
&= \frac{4}{3}at - \frac{2a}{3} + \frac{4b}{3} - \frac{1}{3}at + \frac{2a}{3} - \frac{b}{3} \\
&= at + b
\end{aligned}$$

Therefore, the response is $\boxed{y(t) = at + b}$.

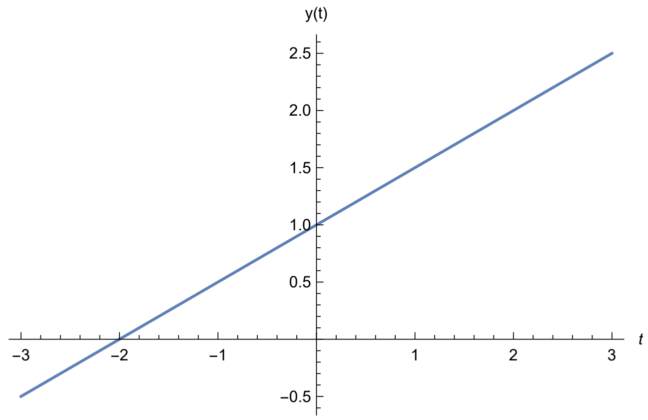

Figure 4. Sketch of $y(t)$ when $a = \frac{1}{2}$ and $b = 1$.

6.

(a)

$$y\mathrm{d}t = \sum_{n=-\infty}^{\infty} x(t-t_0)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x(nT-t_0)$$

$$y(t-t_0) = \sum_{n=-\infty}^{\infty} x(t-t_0)\delta(t-t_0-nT) = \sum_{n=-\infty}^{\infty} x(nT)$$

Since there exists $x(t)$ and t_0 that $y\mathrm{d}t \neq y(t-t_0)$, this system is NOT time-invariant.

(b) To sketch $y(t)$ we would like to deal with $x(t)$ first.

$$\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} \delta(t-nT) * h(t) \\
&= \int_{-\infty}^{\infty} (\sum_{k=-\infty}^{\infty} \delta(t-nT-\tau))h(\tau)\mathrm{d}\tau \\
&= \sum_{k=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \delta(t-nT-\tau)h(\tau)\mathrm{d}\tau\right] \\
&= \sum_{k=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \delta(nT+\tau-t)h(\tau)\mathrm{d}\tau\right] \\
&= \sum_{k=-\infty}^{\infty} h(t-nT)
\end{aligned}$$

Here we use the time shift property of impulse signal

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)\mathrm{d}\tau = x(t)\int_{-\infty}^{\infty} \delta(t-\tau)\mathrm{d}\tau = x(t).$$

Then, we would like to sketch $y(t)$ for different values of T respectively.

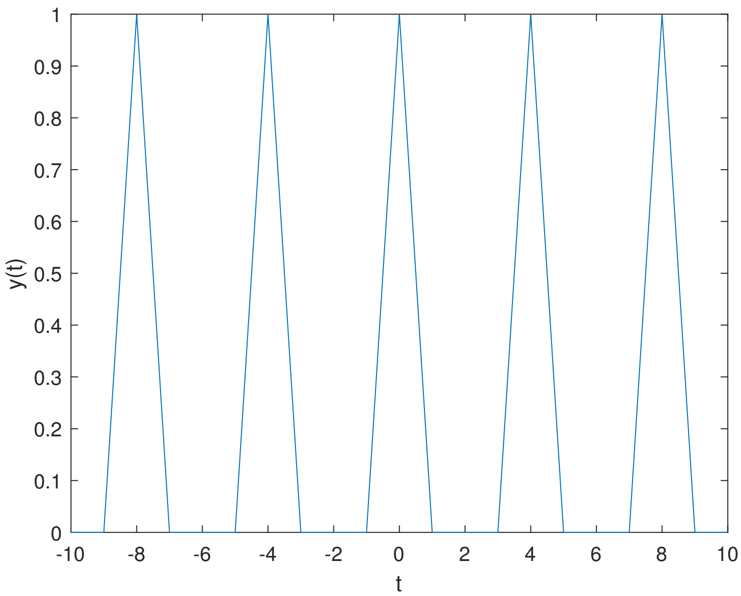

Figure 5. Sketch of $y(t) = \sum_{k=-\infty}^{\infty} h(t - 4T)$ for $T = 4$.

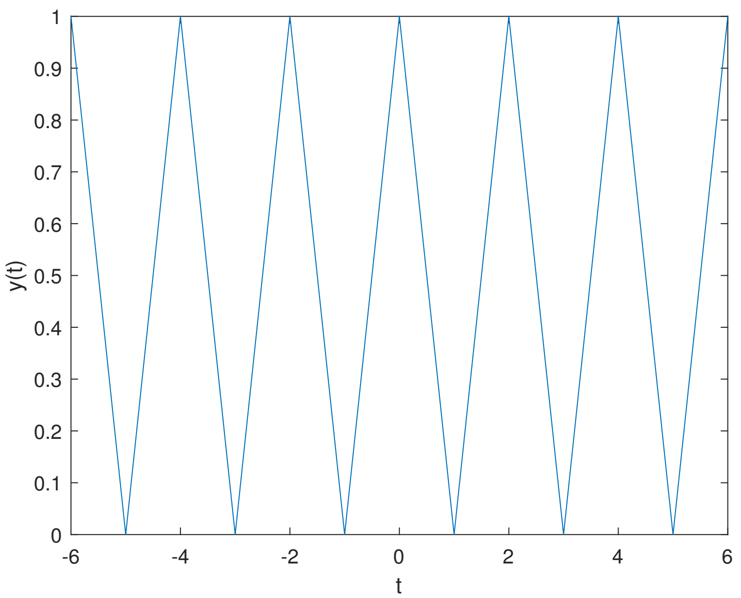

Figure 6. Sketch of $y(t) = \sum_{k=-\infty}^{\infty} h(t - 2T)$ for $T = 2$.

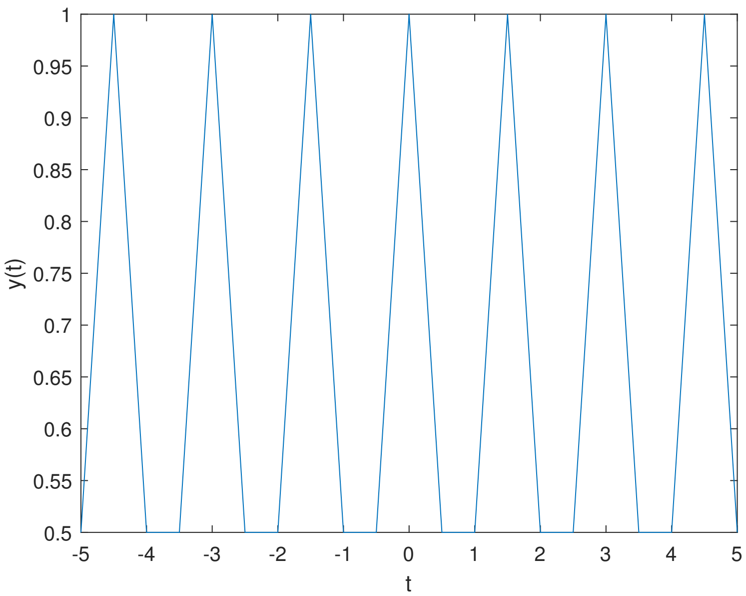

Figure 7. Sketch of $y(t) = \sum_{k=-\infty}^{\infty} h(t - 1.5T)$ for $T = 1.5$.

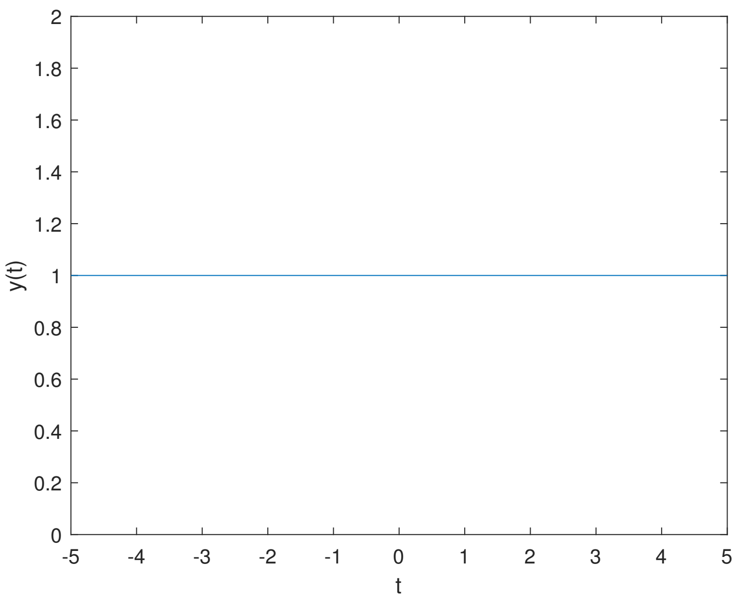

Figure 8. Sketch of $y(t) = \sum_{k=-\infty}^{\infty} h(t - T)$ for $T = 1$.

7.

$$y(t) = (x * h)(t) \Leftrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau \tag{1}$$

(a) By the definition of convolution integral and Fubini's theorem,

$$\begin{aligned}\int_{-\infty}^{\infty} y(t)\mathrm{d}t &= \int_{-\infty}^{\infty} (x * h)(t)\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau\right]\mathrm{d}t \\ &= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)\mathrm{d}t\right]\mathrm{d}\tau\end{aligned}$$

Substitute $x(t)$ for $x(\tau)$ and $h(t)$ for $h(t-\tau)$ using $\mathrm{d}t = \mathrm{d}(t-\tau)$,

$$\begin{aligned}\int_{-\infty}^{\infty} y(t)\mathrm{d}t &= \int_{-\infty}^{\infty} x(t)\left[\int_{-\infty}^{\infty} h(t)\mathrm{d}t\right]\mathrm{d}t \\ &= \left[\int_{-\infty}^{\infty} x(t)\mathrm{d}t\right]\left[\int_{-\infty}^{\infty} h(t)\mathrm{d}t\right]\end{aligned}$$

(b) Differentiate both sides of Eq. (1) with respect to t,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}y(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{-\infty}^{\infty} x(t-\tau)h(\tau)\mathrm{d}\tau\right] \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}[x(t-\tau)h(\tau)]\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} h(\tau)\left[\frac{\mathrm{d}}{\mathrm{d}t}x(t-\tau)\right]\mathrm{d}\tau \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}t}x(t)\right] * h(t)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}y(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{-\infty}^{\infty} h(t-\tau)x(\tau)\mathrm{d}\tau\right] \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}[h(t-\tau)x(\tau)]\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} x(\tau)\left[\frac{\mathrm{d}}{\mathrm{d}t}h(t-\tau)\right]\mathrm{d}\tau \\ &= x(t) * \left[\frac{\mathrm{d}}{\mathrm{d}t}h(t)\right]\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = \left[\frac{\mathrm{d}}{\mathrm{d}t}x(t)\right] * h(t) = x(t) * \left[\frac{\mathrm{d}}{\mathrm{d}t}h(t)\right] \tag{2}$$

8.

(a) Using results from Problem 5(a),

$$\begin{aligned}u(t) * u(t) &= \int_{-\infty}^{\infty} u(\tau)u(t-\tau)\mathrm{d}\tau \\ &= \int_{0}^{t} 1 \cdot 1\mathrm{d}\tau \\ &= t.\end{aligned}$$

Since the convolution exists from 0 to ∞, $u(t)$ is multiplied to the terms

$$\boxed{u(t) * u(t) = tu(t).}$$

(b)

$$\begin{aligned}
u(t) * t^2u(t) &= \int_{-\infty}^{\infty} \tau^2 u(\tau)u(t-\tau)\mathrm{d}\tau \\
&= \int_0^t t^2 \cdot 1\mathrm{d}\tau \\
&= \frac{1}{3}t^3.
\end{aligned}$$

Since the convolution exists from 0 to ∞,

$$\boxed{u(t) * t^2u(t) = \frac{1}{3}t^3u(t).}$$

9. We would like to use the result of Problem 7(b) that $\frac{\mathrm{d}}{\mathrm{d}t}y(t) = [\frac{\mathrm{d}}{\mathrm{d}t}x(t)] * h(t)$. Since $x(t) = u(t)$ and $y(t) = (3-t)\mathrm{rect}(\frac{t-1}{2}) = (3-t)(u(t) - u(t-2))$, the derivatives of $x(t)$ and $y(t)$ are respectively

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \delta(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = (3-t)(\delta(t) - \delta(t-2)) - u(t) + u(t-2)$$

Applying Eq. (2),

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = \delta(t) * h(t) = \int_{-\infty}^{\infty} \delta(t-\tau)h(\tau)\mathrm{d}\tau = h(t)$$

Therefore, the impulse response of the system is

$$\boxed{h(t) = (3-t)(\delta(t) - \delta(t-2)) - u(t) + u(t-2))}$$

10.
(a) For any signals $x_1(t)$, $x_2(t)$ and any constants a_1 and a_2,

$$\mathcal{T}[x_1(t)] = x_1(\sin(t))$$

$$\mathcal{T}[x_2(t)] = x_2(\sin(t))$$

$$\alpha\mathcal{T}[x_1(t)] + \beta\mathcal{T}[x_2(t)] = \alpha x_1(\sin(t)) + \beta x_2(\sin(t))$$

$$\mathcal{T}[\alpha x_1(t) + \beta x_2(t)] = \alpha x_1(\sin(t)) + \beta x_2(\sin(t))$$

$$\mathcal{T}[\alpha x_1(t) + \beta x_2(t)] = \alpha\mathcal{T}[x_1(t)] + \beta\mathcal{T}[x_2(t)]$$

Therefore, the system is $\boxed{\text{linear}}$.

Since $\forall t$ ran $\sin(t) = [-1, 1] \subset$ ran t, if $\exists M_x$ s.t. $|x(t)| \leq M_x < \infty\ \forall t$, then there must exist an M_y s.t.

$$|y(t)| \leq M_y < \infty\ \forall t.$$

Therefore, the system is $\boxed{\text{stable}}$.

Since the output $y(t)$ at any time t may depend on previous and future values of the input signal, the system is $\boxed{\text{NOT causal}}$ and $\boxed{\text{NOT memoryless}}$.

$$y\mathrm{d}t = x(\sin(t) - t_0) \neq x(\sin(t - t_0)) = y(t - t_0)$$

Therefore, the system is $\boxed{\text{NOT time-invariant}}$.

(b) For any signals $x_1(t)$, $x_2(t)$ and any constants a_1 and a_2,

$$\mathcal{T}[x_1(t)] = \frac{\mathrm{d}}{\mathrm{d}t}(e^{-t}x_1(t))$$

$$\mathcal{T}[x_2(t)] = \frac{\mathrm{d}}{\mathrm{d}t}(e^{-t}x_2(t))$$

$$\alpha\mathcal{T}[x_1(t)] + \beta\mathcal{T}[x_2(t)] = \alpha\frac{\mathrm{d}}{\mathrm{d}t}(e^{-t}x_1(t)) + \beta\frac{\mathrm{d}}{\mathrm{d}t}(e^{-t}x_2(t))$$

$$\mathcal{T}[\alpha x_1(t) + \beta x_2(t)] = \frac{\mathrm{d}}{\mathrm{d}t}(e^{-t}(\alpha x_1(t) + \beta x_2(t)) = \alpha\frac{\mathrm{d}}{\mathrm{d}t}(e^{-t}x_1(t)) + \beta\frac{\mathrm{d}}{\mathrm{d}t}(e^{-t}x_2(t))$$

$$\mathcal{T}[\alpha x_1(t) + \beta x_2(t)] = \alpha\mathcal{T}[x_1(t)] + \beta\mathcal{T}[x_2(t)]$$

Therefore, the system is $\boxed{\text{linear}}$.

If we take a bounded input signal $x(t) = \sin(\frac{1}{t})$ so that $-1 \le x(t) \le 1$, then

$$y(t) = \frac{\mathrm{d}}{\mathrm{d}t}[e^{-t}x(t)] = -e^{-t}(\frac{\cos\frac{1}{t}}{t^2} + \sin\frac{1}{t})$$

Since $\lim_{t\to-\infty} y(t) = +\infty$, $y(t)$ is unbounded, i.e. there does not exist an M_y s.t. $|y(t)| \le M_y < \infty\ \forall t$, so the system is $\boxed{\text{NOT stable}}$.

Since we can express $y(t)$ as

$$y(t) = \left(\frac{\mathrm{d}x(t)}{\mathrm{d}t} - x(t)\right)e^{-t}$$

we see that its causality and memory are the same as those of the system in Homework 1 Problem 10(e), again according to the definition of derivative, the output $y(t)$ at any time t may depend on previous and present values of the input signal, the system is $\boxed{\text{causal}}$ and $\boxed{\text{NOT memoryless}}$.

$$y\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}t}[e^{-t}x(t-t_0)] \ne \frac{\mathrm{d}}{\mathrm{d}t}[e^{-(t-t_0)}x(t-t_0)] = y(t-t_0)$$

Therefore, the system is $\boxed{\text{NOT time-invariant}}$.

11.

(a) The input signal $x_2(t)$ is

$$x_2(t) = x_1(t) - x_1(t-2),$$

so the response of the system can be determined by

$$y_2(t) = y_1(t) - y_1(t-2) = 2(1-|t-1|)(u(t) - u(t-2)) - 2(1-|t-3|)(u(t-2) - u(t-4)).$$

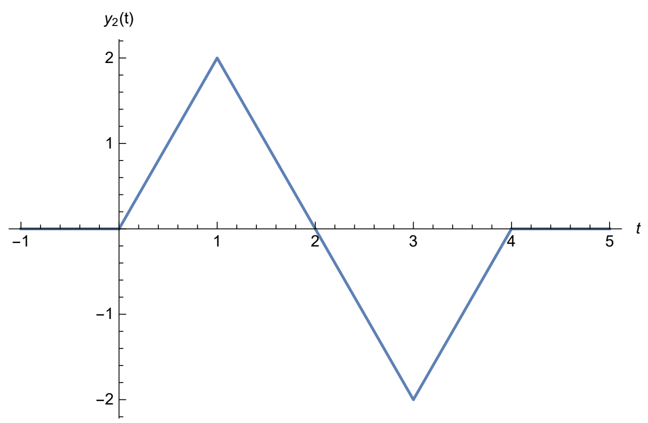

Figure 9. Sketch of the response of the system to the input $x_2(t)$.

(b) The input signal $x_3(t)$ is

$$x_3(t) = x_1(t) + x_1(t+1),$$

so the response of the system can be determined by

$$y_3(t) = y_1(t) + y_1(t+1) = 2(1-|t-1|)(u(t)-u(t-2)) + 2(1-|t|)(u(t+1)-u(t-1)).$$

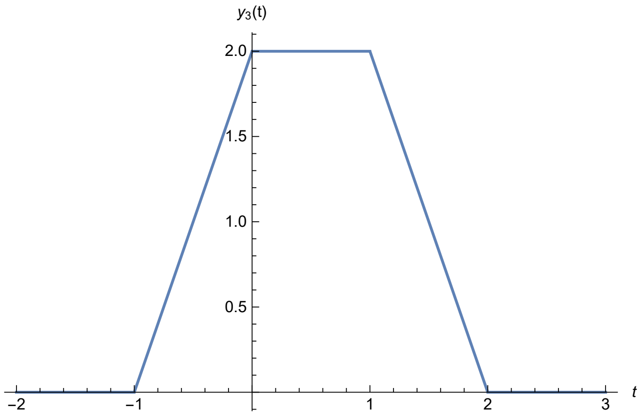

Figure 10. Sketch of the response of the system to the input $x_3(t)$.

12. Since the two signals are complex interpreted by the unit step functions, we would like to compute the convolution by graphs. Denote $x_1(t) = \mathrm{tri}(\frac{t}{2})$ and $x_2(t) = \mathrm{rect}(\frac{t-1}{2})$.

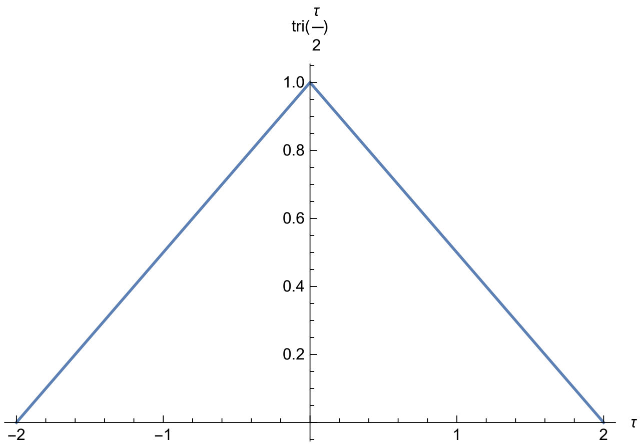

Figure 11. $x_1(\tau) = \mathrm{tri}(\frac{\tau}{2})$.

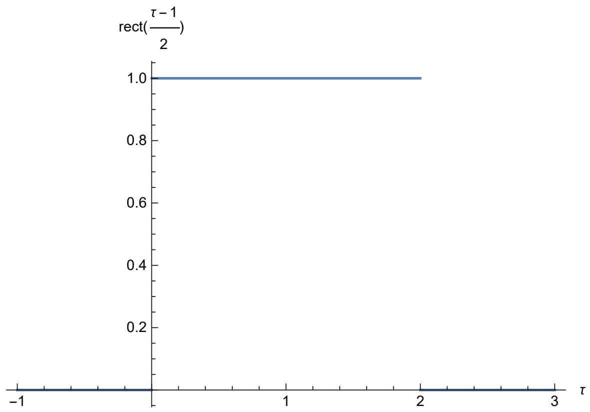

Figure 12. $x_2(\tau) = \text{rect}(\frac{\tau-1}{2})$.

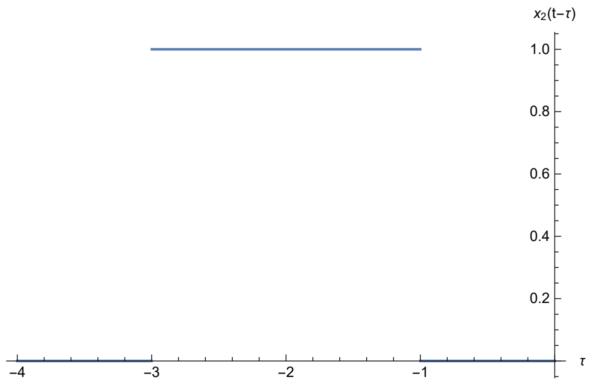

Figure 13. $x_2(t-\tau)$ when $t=-1$.

When $t \leq -2$, $x_1(t) * x_2(t) = 0$.

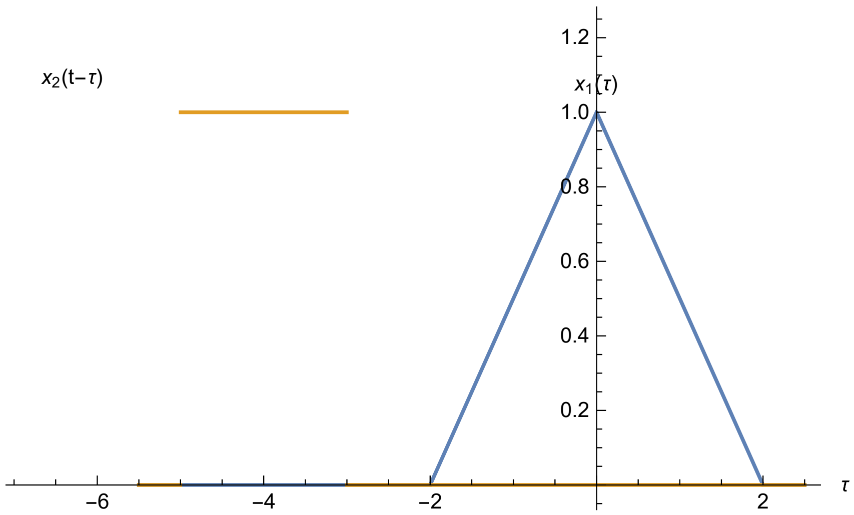

Figure 14. $x_1(\tau)$ and $x_2(t-\tau)$ when $t=-3$.

When $-2 \leq t \leq 0$,

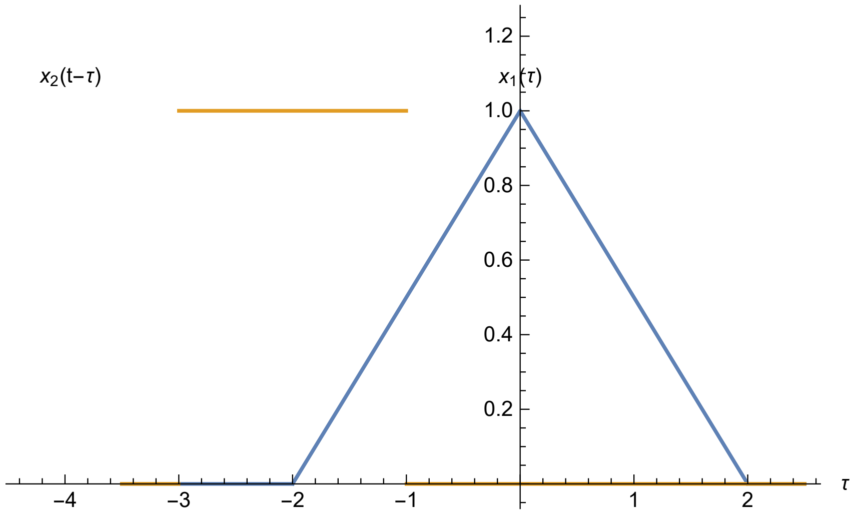

Figure 15. $x_1(\tau)$ and $x_2(t-\tau)$ when $t=-1$.

$$x_1(t) * x_2(t) = \int_{-2}^{t} \left(\frac{1}{2}\tau + 1\right) \cdot 1 \mathrm{d}\tau = \left(\frac{\tau^2}{4} + \tau\right)\Big|_{-2}^{t} = \frac{1}{4}(t+2)^2.$$

When $0 \leq t \leq 2$,

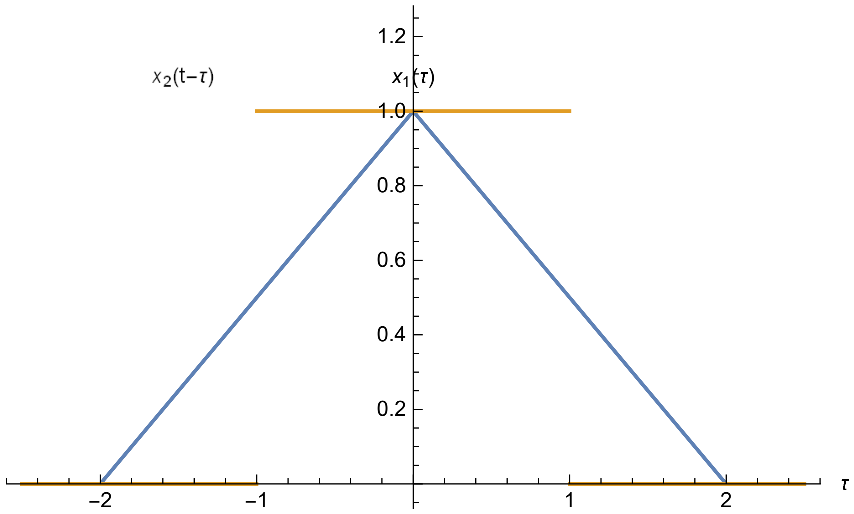

Figure 16. $x_1(\tau)$ and $x_2(t-\tau)$ when $t=1$.

$$x_1(t) * x_2(t) = \int_{t-2}^{t} \left(-\frac{1}{2}|\tau| + 1\right) \cdot 1 \mathrm{d}\tau = -\left(\frac{1}{4}|\tau|^2 - |\tau|\right)\Big|_{t-2}^{t} = -\frac{1}{2}t^2 + t + 1.$$

When $2 \le t \le 4$,

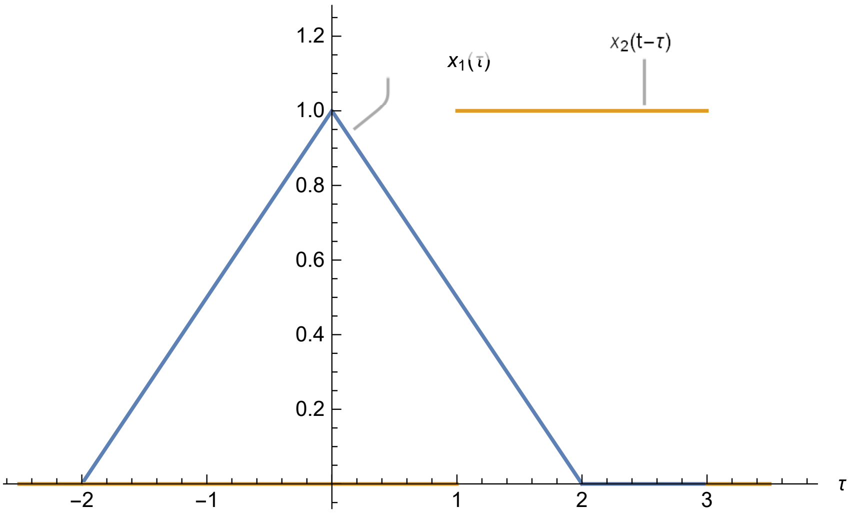

Figure 17. $x_1(\tau)$ and $x_2(t-\tau)$ when $t = 3$.

$$x_1(t) * x_2(t) = \int_{t-2}^{2} \left(-\frac{1}{2}\tau + 1 \right) \cdot 1 \mathrm{d}\tau = -\left(\frac{1}{4}\tau^2 - \tau \right) \Big|_{t-2}^{2} = \frac{1}{4}(t-4)^2.$$

When $t \ge 4$,

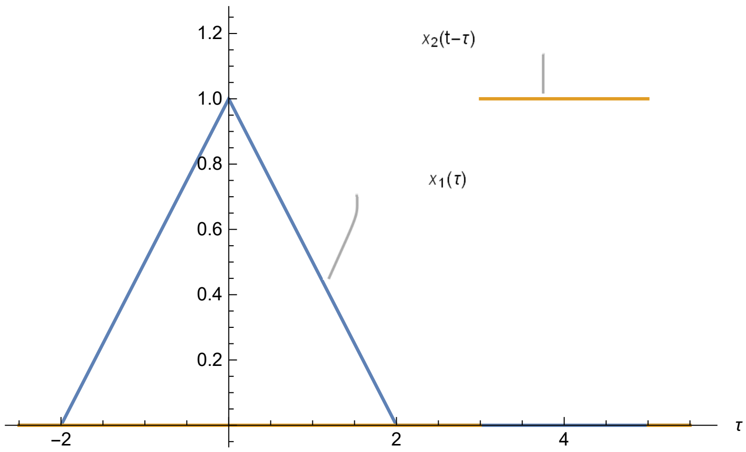

Figure 18. $x_1(\tau)$ and $x_2(t-\tau)$ when $t = 5$.

$$x_1(t) * x_2(t) = 0.$$

In fact, we can rewrite the expression as

$$x(t) = \mathrm{tri}(\frac{\tau}{2}) * \mathrm{rect}(\frac{\tau-1}{2}) = \frac{1}{4}((t+2)^2 u(t+2) - 3t^2 u(t) + 3(t-2)^2 u(t-2) - 4(t-4)^2 u(t-4)).$$

To sum up,

$$y(t)=\begin{cases}0, & t<-2\\ \frac{1}{4}(t+2)^2, & -2\le t<0\\ -\frac{1}{2}t^2+t+1, & 0\le t<2\\ \frac{1}{4}(t-4)^2, & 2\le t<4\\ 0, & t\ge 4\end{cases}$$

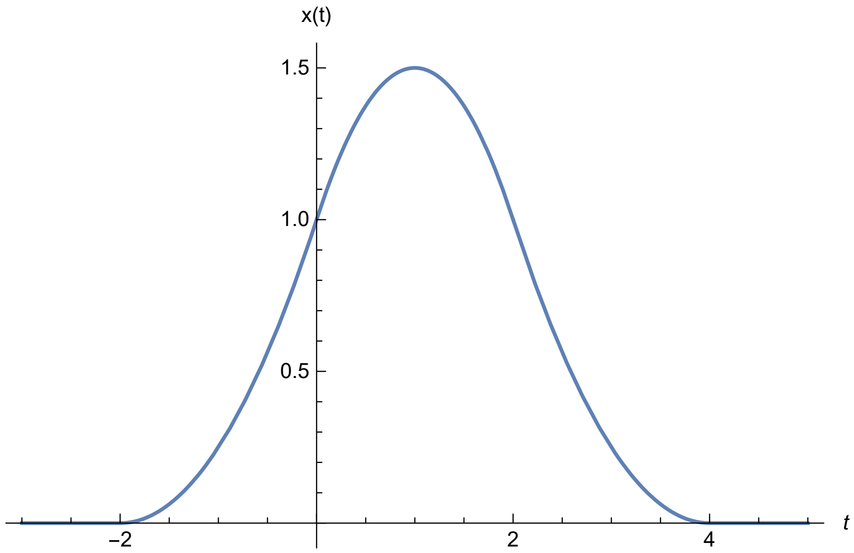

Figure 19. Sketch of $x(t)=\mathrm{tri}(\frac{\tau}{2})*\mathrm{rect}(\frac{\tau-1}{2})$.

13.
(a) The impulse response is

$$h(t)=\int_{-\infty}^{t}(t-\tau)e^{-(t-\tau)}\delta(\tau)\mathrm{d}\tau=\int_{-\infty}^{t}te^{-t}\delta(\tau)\mathrm{d}\tau=te^{-t}u(t).$$

An LTI system is causal iff its impulse response $h(t)=0$ for all $t<0$. Since $h(t)=0$ for all $t<0$, the system is $\boxed{\text{causal}}$.

An LTI system is BIBO stable iff its impulse response is absolutely integrable.

$$\int_{-\infty}^{\infty}|h(t)|\mathrm{d}t=\int_{0}^{\infty}te^{-t}\mathrm{d}t=-(t+1)e^{-t}\Big|_{0}^{\infty}=1.$$

i.e. its impulse response is absolutely integrable, $\int_{-\infty}^{\infty}|h(t)|\mathrm{d}t<\infty$. Therefore, the system is $\boxed{\text{stable}}$.

An LTI system is memoryless or stable iff its impulse response is $h(t)=a\delta(t)$. Since $h(t)$ cannot be $a\delta(t)$ for any a, the system is $\boxed{\text{NOT static}}$.

(b) The impulse response is

$$h(t)=\int_{t-1}^{t+1}e^{-2(t-\tau)}\delta(\tau)\mathrm{d}\tau=\int_{t-1}^{t+1}e^{-2t}\delta(\tau)\mathrm{d}\tau=\mathrm{rect}\left(\frac{t}{2}\right)e^{-2t}.$$

Since $\exists t\ h(t)\neq 0$ for $t<0$, the system is $\boxed{\text{NOT causal}}$.

$$\int_{-\infty}^{\infty}|h(t)|\mathrm{d}t=\int_{-1}^{1}e^{-2t}\mathrm{d}t=-\frac{1}{2}e^{-2t}\Big|_{-1}^{1}=e^{2}-e^{-2}<\infty.$$

Since its impulse response is absolutely integrable, the system is $\boxed{\text{stable}}$.

Since $h(t)$ cannot be $a\delta(t)$ for any a, the system is $\boxed{\text{NOT static}}$.

14. We would first differentiate $x(t)$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}x(t) &= \frac{\mathrm{d}}{\mathrm{d}t}[2e^{-3t}u(t-1)] \\ &= 2e^{-3t}\delta(t-1) + 2u(t-1)(-3)e^{-3t} \\ &= 2e^{-3t}\delta(t-1) - 6e^{-3t}u(t-1) \\ &= 2e^{-3}\delta(t-1) - 3x(t)\end{aligned}$$

Then, $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = 2e^{-3}\delta(t-1) - 3x(t) \to 2e^{-3}h(t-1) - 3y(t)$. Besides, $\frac{\mathrm{d}x(t)}{\mathrm{d}t} \to e^{-2t}u(t) - 3y(t)$, we have

$$2e^{-3}h(t-1) = e^{-2t}u(t)$$

Let $s = t-1$, then

$$h(s) = \frac{e^3}{2}e^{-2(s+1)}u(s+1) = \frac{1}{2}e^{-2s+1}u(s+1)$$

Therefore, the impulse response $h(t)$ of S is

$$\boxed{h(t) = \frac{1}{2}e^{-2t+1}u(t+1)}$$

15.

(a) We have enough information to determine the output $y(t)$.

$$\boxed{y(t) = x(t) * h(t) = 2x_0(t) * h_0(t) = 2(x_0(t) * h_0(t)) = 2y_0(t)}$$

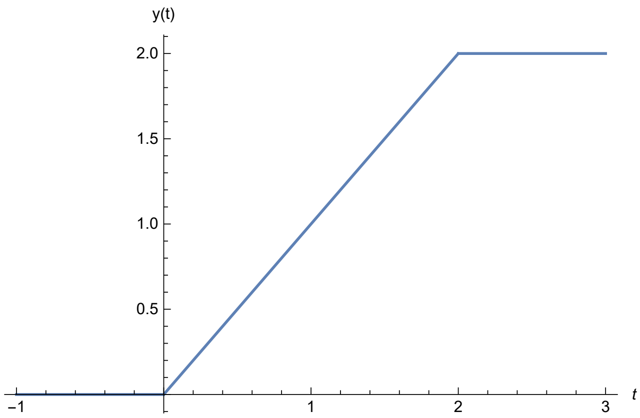

Figure 20. 15(a).

(b) We have enough information to determine the output $y(t)$.

$$\begin{aligned}y(t) &= x(t) * h(t) \\ &= (x_0(t) - x_0(t-2)) * h_0(t+1) \\ &= (x_0(t) * h_0(t+1)) - (x_0(t-2) * h_0(t+1)) \\ &= \boxed{y_0(t+1) - y_0(t-1)}\end{aligned}$$

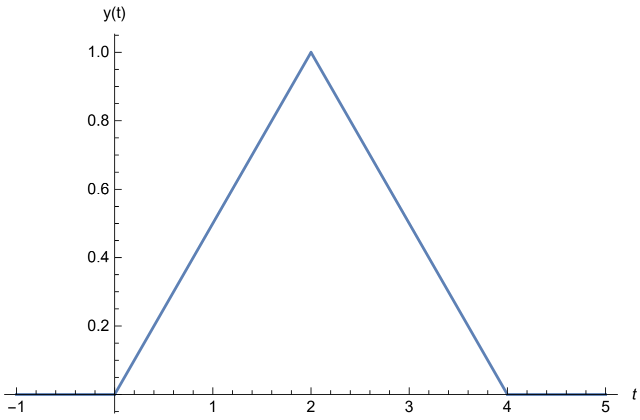

Figure 21. 15(b).

(c) Try express $y(t)$ in terms of $x_0(t)$ and $h_0(t)$

$$y(t) = x(t) * h(t) = x_0(-t) * h_0(t)$$

Since the convolution does not exist, we do not have enough information to determine the output $y(t)$.

(d) We have enough information to determine the output $y(t)$.

$$\boxed{y(t) = x(t) * h(t) = x_0(-t) * h_0(-t) = y_0(-t)}$$

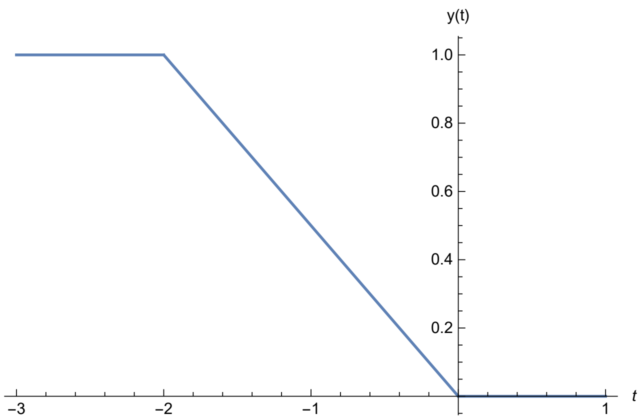

Figure 22. 15(e).

(e) We have enough information to determine the output $y(t)$.

$$\boxed{y(t) = x(t) * h(t) = x_0'(t) * h_0(-t) = y_0'(t)}$$

(f) We have enough information to determine the output $y(t)$. Using the result from Problem 7(b),

$$y(t) = x(t) * h(t) = x_0'(t) * h_0'(t) = y_0''(t)$$

We first find the expression of $y_0(t)$

$$y_0(t) = \frac{1}{2}(tu(t) - (t-2)u(t-2))$$

Then we can calculate its derivatives of $y_0(t)$

$$y_0'(t) = \frac{1}{2}(u(t) - u(t-2))$$

$$y_0''(t) = \frac{1}{2}(\delta(t) - \delta(t-2))$$

Hence,

$$\boxed{y(t) = \frac{1}{2}(\delta(t) - \delta(t-2))}$$

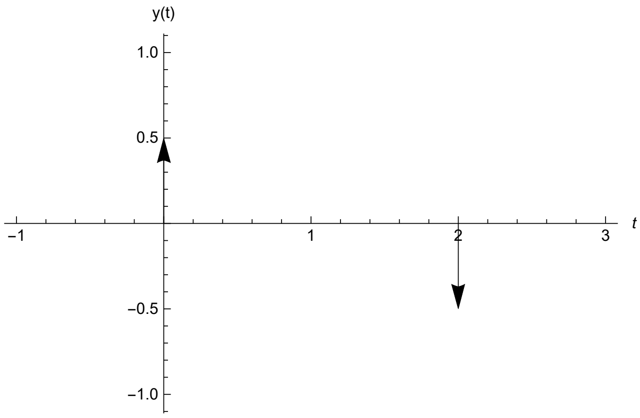

Figure 23. 15(f).

16. Find natural response $y_h(t) = Ce^(st)$.

$$\frac{\mathrm{d}t}{\mathrm{d}t} y_h(t) + 10y_h(t) = 0$$

$$\implies 10Ce^{st} + sCe^{st} = 0$$

$$\implies s = -10$$

$$\implies y_h(t) = Ce^{-10t}$$

We analyze a unit-step input signal

$$x(t) = u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Since $x(t) = 0$ for $t \leq 0$, the condition of initial rest implies the auxiliary condition $y(t) = 0$ for $t \leq 0$.

Since $x(t) = 1$, $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = 0$ for $t \geq 0$, the forced response is

$$y_p(t) = P \text{ for } t \geq 0$$

Thus

$$y(t) = y_h(t) + y_p(t) = Ce^{-10t} + P \text{ for } t \geq 0$$

Plugging into the diffeq yields:

$$10Ce^{-10t} + 10P - 10Ce^{-10}t = 2 \text{ for } t \geq 0$$

$$\implies P = \frac{1}{5}$$

Furthermore, using the initial condition $y(0) = 1$,

$$\begin{aligned} Ce^0 + P &= 1 \\ \implies C + \frac{1}{5} &= 1 \\ \implies C &= \frac{4}{5} \end{aligned}$$

Therefore, the expression of response of the CT system is

$$\boxed{y(t) = \begin{cases} \frac{4}{5}e^{-10t} + \frac{1}{5}, & t \geq 0 \\ 0, & t < 0 \end{cases} = \left(\frac{4}{5}e^{-10t} + \frac{1}{5}\right)u(t)}$$