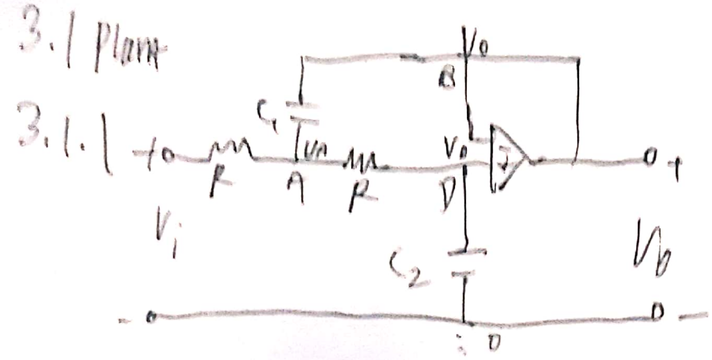
VE311 Prelab 3 Liu Yihan [illegible] 518021910998

3.1 Plant

3.1.1



At node A

$$\begin{cases} \dfrac{V_i - V_A}{R} + \dfrac{v_0 - V_A}{\frac{1}{j\omega C_1}} = \dfrac{V_A - V_0}{R} \\ \dfrac{V_A}{R + \frac{1}{j\omega C_2}} = \dfrac{V_0}{\frac{1}{j\omega C_2}} \end{cases}$$

$$V_A = \left(R + \frac{1}{j\omega C_2}\right) j\omega C_2 V_0 = (j\omega C_2 R + 1) V_0$$

$$V_A - V_0 = j\omega C_2 R V_0$$

$$\frac{V_i}{R} - \frac{(j\omega C_2 R + 1)V_0}{R} - j\omega C_1 \, j\omega C_2 R V_0 = j\omega C_2$$

$$\frac{V_i}{R} = \left(2j\omega C_2 + \frac{1}{R} - \omega^2 C_1 C_2 R\right) V_0$$

$$\frac{V_0}{V_i} = \frac{1}{-\omega^2 C_1 C_2 R^2 + 2j\omega C_2 R + 1}$$

Let $s = j\omega$ $$\frac{V_0(s)}{V_i(s)} = \frac{1}{C_1C_2R^2 s^2 + 2C_2 R s + 1} = \frac{\frac{1}{C_1C_2R^2}}{s^2 + \frac{2}{C_1R}s + \frac{1}{C_1C_2R^2}} = \frac{a_1}{s^2 + a_2 s + a_3}$$

$$\boxed{a_1 = \frac{1}{C_1C_2R^2}} \quad \boxed{a_2 = \frac{2}{C_1R}} \quad \boxed{a_3 = \frac{1}{C_1C_2R^2}}$$

3.1.2 poles: $$\frac{-2C_2R \pm \sqrt{4C_2^2R^2 - 4C_1C_2R^2}}{2C_1C_2R^2} = \frac{-C_2 \pm \sqrt{C_2^2 - C_1C_2}}{C_1C_2R}$$

with $C_1 = 10^{-4}F$, $R = 10^4 \Omega$

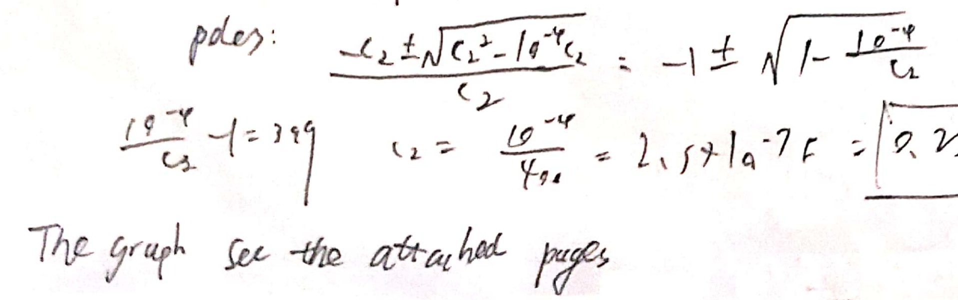
poles: $$\frac{-C_2 \pm \sqrt{C_2^2 - 10^{-4}C_2}}{C_2} = -1 \pm \sqrt{1 - \frac{10^{-4}}{C_2}} = -1 \pm j\sqrt{399}$$

$$\frac{10^{-4}}{C_2} - 1 = 399 \qquad C_2 = \frac{10^{-4}}{400} = 2.5 \times 10^{-7} F = \boxed{0.25\ \mu F}$$

3.1.3 The graph see the attached pages

3.3 PD controller with 2-degrees of Freedom

3.3.1



$$\begin{cases} Y(s) = D(s)P(s) \\ D(s) = E(s)G(s) - G_2(s)Y(s) \\ E(s) = X(s) - Y(s) \end{cases}$$

$$\Rightarrow Y(s) = \left(X(s)G(s) - Y(s)G(s) - Y(s)G_2(s)\right)P(s)$$

$$Y(s)\left(1 + (G(s) + G_2(s))P(s)\right) = X(s)G(s)$$

closed-loop response $G_{cl}(s) = \dfrac{Y(s)}{X(s)} = \dfrac{G(s)P(s)}{1 + G(s)P(s) + G_2(s)P(s)}$

3.3.2 closed-loop response

$$G_{cl}(s) = \frac{K_p \cdot P(s)}{1 + K_p P(s) + K_D s P(s)}$$

3.3.3 $P(s) = \dfrac{1}{GL_aR^2s^2 + 2L_aRs + 1} = \dfrac{1}{2.5\times10^{-3}s^2 + 5\times10^{-3}s + 1}$

closed-loop response $G_{cl}(s) = \dfrac{\dfrac{K_p}{2.5\times10^{-3}s^2 + 5\times10^{-3}s + 1}}{1 + \dfrac{K_p + K_D s}{2.5\times10^{-3}s^2 + 5\times10^{-3}s + 1}} = \dfrac{K_p}{2.5\times10^{-3}s^2 + (5\times10^{-3} + K_D)s + K_p + 1} = \dfrac{400K_p}{s^2 + (400K_D + 2)s + 400(K_p + 1)}$

3.4 Unit Step Response of PD Controller with 2-Degrees of Freedom

3.4.1 poles

$$s_1 = \frac{-K_D - 5\times10^{-3} + \sqrt{(5\times10^{-3} + K_D)^2 - 10^{-2}(K_p + 1)}}{5\times10^{-3}} = -200K_D - 1 + \sqrt{(200K_D + 1)^2 - 400(K_p + 1)}$$

$$s_2 = \frac{-K_D - 5\times10^{-3} - \sqrt{(5\times10^{-3} + K_D)^2 - 10^{-2}(K_p + 1)}}{5\times10^{-3}} = -200K_D - 1 - \sqrt{(200K_D + 1)^2 - 400(K_p + 1)}$$

(a) $(200K_D + 1)^2 > 400(K_p + 1)$
$200K_D + 1 > 20\sqrt{K_p + 1}$ distinct and real-valued

(b) $200K_D + 1 = 20\sqrt{K_p + 1}$ identical and real valued

(c) $200K_D + 1 < 20\sqrt{K_p + 1}$ a pair of complex conjugate numbers

3.4.2 $G_{cl}(s) = \dfrac{400K_p}{(s - s_1)(s - s_2)} = \dfrac{400K_p}{s_1 - s_2}\left(\dfrac{1}{s - s_1} - \dfrac{1}{s - s_2}\right)$

To find unit step response $y(t) = g_{cl}(t) * u(t)$

$$Y(s) = G_{cl}(s)\cdot\frac{1}{s} = \frac{400K_p}{s(s - s_1)(s - s_2)} = 400K_p\left(\frac{1}{(s_1 - s_2)s_1(s - s_1)} + \frac{1}{s_1 s_2 \cdot s} - \frac{1}{(s_1 - s_2)s_2(s - s_2)}\right)$$

Doing inverse Laplace transform

$$y(t) = \left(\frac{400K_p}{s_1 s_2} + \frac{400K_p}{(s_1 - s_2)s_1}e^{s_1 t} + \frac{400K_p}{(s_2 - s_1)s_2}e^{s_2 t}\right)u(t)$$

(continued)

$$\boxed{A_1 = \frac{400k_p}{s_1 s_2}} \quad \boxed{A_2 = \frac{400k_p}{(s_1-s_2)s_1}} \quad \boxed{A_3 = \frac{400k_p}{(s_2-s_1)s_2}}$$

$$\frac{dy}{dt} = \frac{400k_p}{s_1-s_2}(e^{s_1t}-e^{s_2t})$$

$s_1 > s_2 \quad s_1 - s_2 > 0$

$e^{s_1t} > e^{s_2t} \quad e^{s_1t} - e^{s_2t} > 0$

Thus, $\frac{dy}{dt} > 0$ for all t, $y(t)$ is monotonically increasing

$y(t) \geq y(-1) = 0 \quad \Rightarrow y(t) \geq 0$

$y(t) < \lim_{t\to\infty} y(t)$

$$\lim_{t\to\infty} y(t) = A_1 = \frac{400k_p}{s_1 s_2} = \frac{400k_p}{400(k_p+1)} = \frac{k_p}{k_p+1}$$

$\lim_{t\to\infty} y(t) < 1 \Rightarrow y(t) < 1$

thus $0 \leq y(t) < 1$

3.4.3 For critically-damped system, $G_{cl}(s) = \frac{400k_p}{(s-s_1)^2}$

$$Y(s) = \frac{400k_p}{s(s-s_1)^2} = 400k_p\left(\frac{1}{s_1^2 \cdot s} + \frac{-1}{s_1^2(s-s_1)} + \frac{1}{(s-s_1)^2 s_1}\right)$$

$$y(t) = \left(\frac{400k_p}{s_1^2} + \frac{-400k_p}{s_1^2}e^{s_1t} + \frac{400k_p}{s_1}te^{s_1t}\right)u(t)$$

$$\boxed{A_1 = \frac{400k_p}{s_1^2}} \quad \boxed{A_2 = -\frac{400k_p}{s_1^2}} \quad \boxed{A_3 = \frac{400k_p}{s_1}}$$

$$\frac{dy}{dt} = 400k_p\left[\left(-\frac{1}{s_1}e^{s_1t} + \frac{1+s_1t}{s_1}e^{s_1t}\right)u(t) + \left(\left(\frac{t}{s_1} - \frac{1}{s_1^2}\right)e^{s_1t} + \frac{1}{s_1^2}\right)u(t)\right]$$

$= 400k_p\left(s_1te^{s_1t}u(t) - \frac{1}{s_1^2} + \frac{1}{s_1^2}\right)$

$= 400k_p s_1 t e^{s_1t} u(t) > 0$ for all t

thus $y(t)$ is a monotonically increasing function of t

$y(t) \geq y(-1) = 0 \Rightarrow y(t) \geq 0$

$y(t) < \lim_{t\to\infty} y(t)$

$$\lim_{t\to\infty} y(t) = A_1 = \frac{400k_p}{s_1^2} = \frac{400k_p}{(\sqrt{400k_p+1})^2} = \frac{400k_p}{400(k_p+1)} = \frac{k_p}{k_p+1}$$

$\lim_{t\to\infty} y(t) < 1 \Rightarrow y(t) < 1$

thus $0 \leq y(t) < 1$

3.4.4 $a = \mathrm{Re}\{s_1\} = -200k_p - 1$ $\qquad s_1 s_2 = a^2+b^2 \quad s_1 = a+jb$

$b = \mathrm{Im}\{s_1\} = \sqrt{400(k_p+1) - (200k_p+1)^2}$ $\qquad s_1+s_2 = 2a \quad s_2 = a-jb$

$s_1 - s_2 = 2jb$

$$Y(s) = \frac{A_1}{s} + B_1 \frac{(s-a)\cos\theta - b\sin\theta}{b^2 + (s-a)^2}$$

$$y(t) = \left(\frac{400k_p}{s_1 s_2} + \frac{400k_p}{(s_1-s_2)s_1} e^{s_1 t} - \frac{400k_p}{(s_1-s_2)s_2} e^{s_2 t}\right)u(t)$$

$$= \frac{400k_p}{a^2+b^2} + 400k_p e^{at}\left(\frac{e^{jbt}}{2jb(a+jb)} + \frac{e^{-jbt}}{2jb(a-jb)}\right)u(t)$$

$$\frac{e^{jbt}}{2jb(a+jb)} = \frac{e^{jbt}(-2jab - 2b^2)}{4a^2b^2+4b^4} = \frac{\sqrt{4a^2b^2+4b^4}}{4a^2b^2+4b^4} e^{\theta j} e^{bjt} \qquad \theta = \arccos\frac{b}{\sqrt{a^2+b^2}} + \pi$$

$$\frac{e^{-jbt}}{2jb(jb-a)} = \frac{e^{-jbt}(2jab+2b^2)}{4a^2b^2+4b^4} = \frac{\sqrt{4a^2b^2+4b^4}}{4a^2b^2+4b^4} e^{\theta j} e^{-bjt}$$

$$y(t) = \left(\frac{400k_p}{a^2+b^2} + \frac{400k_p e^{at}}{\sqrt{4a^2b^2+4b^4}}\left(e^{jbt+j\theta} + e^{-jbt-j\theta}\right)\right)u(t)$$

$$= \left(\frac{400k_p}{a^2+b^2} + \frac{400k_p}{b\sqrt{a^2+b^2}}\cos(bt+\theta)e^{at}\right)u(t)$$

$$\boxed{A_1 = \frac{400k_p}{a^2+b^2}} \quad \boxed{B_1 = \frac{400k_p}{b\sqrt{a^2+b^2}}} \quad \boxed{\theta = \arccos\frac{b}{\sqrt{a^2+b^2}} + \pi}$$

$$\lim_{t\to\infty} y(t) = A_1 = \frac{400k_p}{a^2+b^2} = \frac{400k_p}{s_1 s_2} = \frac{400k_p}{400(k_p+1)} = \frac{k_p}{k_p+1}$$

Period $T = \frac{2\pi}{b}$ Maximum value will occur at $t = \frac{\pi}{b}$ in first period ($t>0$)

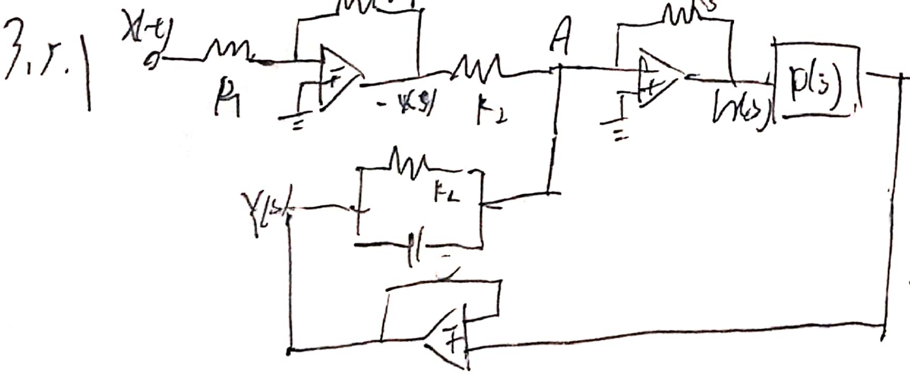
Since $a<0$, e^{at} is decreasing

$$\max(y(t)) = \frac{400k_p}{a^2+b^2} + \frac{400k_p}{b\sqrt{a^2+b^2}}\frac{b}{\sqrt{a^2+b^2}} e^{\frac{\pi a}{b}} = \frac{400k_p}{400(k_p+1)} + \frac{400k_p}{400(k_p+1)} e^{\pi a/b}$$

$$= \frac{k_p}{k_p+1}\left[1 + e^{\pi a/b}\right]$$

3.5 Op-Amp Realization of PD Controller with 2-Degrees of Freedom

3.5.1



At node A

$$\rightarrow y(t) \quad \frac{X(s)}{R_2} = \frac{W(s)}{R_3} + \frac{Y(s)}{R_2} + CsY(s)$$

$$W(s) = \frac{R_3}{R_2}(X(s) - Y(s)) - R_3 C s Y(s)$$

$$\boxed{K_p = \frac{R_3}{R_2}} \qquad \boxed{K_D = R_3 C}$$