# VE 216 Spring 2020
## Lab 1: LTI Systems
## Part I: Intro & Pre-lab Assignment

### Note: Pre-lab Borrowed from UMich EECS 216

## 1    Introduction

The first several weeks of VE 216: Signals and Systems have been dedicated to introducing the notion of a *linear time-invariant system* (LTI) and characterizing the input/output relationships for such a system. In particular, the properties of *linearity* and *time-invariance* have been described and it has been shown that the general solution of an LTI system can be computed by convolving the input with the systems impulse function, $h(t)$. Furthermore, even though systems often have multiple inputs and multiple outputs, we restrict ourselves in VE 216 to LTI systems that have a single input and a single output.

In this set of laboratory experiments, you will apply these basic concepts of LTI systems to a simple RC circuit. You will measure the response of the circuit under a variety of conditions and compare your results with the corresponding theoretical predications.

## 2    Goals

- To become familiar with the laboratory equipment: power supply, signal generator, digital oscilloscope, computer data acquisition system (Scope Connect).

- To review basic concepts of linear time-invariant systems.

- To illustrate several possible ways to determine the impulse response of a physical system from measured data.

- To use linearity, time-invariance and impulse response to compute the output of an LTI system when the input is a step, a pulse, or a more complicated signal. You will compare these calculations with actual measurements.

- To measure the frequency response of an LTI system and compare against theory.

## 3    Technical Background Information

An RC circuit is used so that the computations are easy and physically meaningful. The same procedures can be applied to much more complicated systems.

### 3.1    RC circuit

The RC circuit shown below is an example of a simple LTI system. Of course, there are many other LTI systems that do not involve circuits at all.
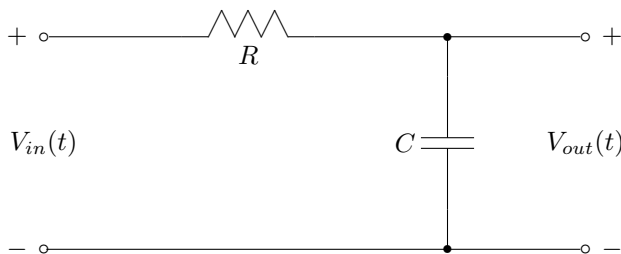
Figure 3.1.1: RC Circuit

We will take the system input to be the voltage $V_{in}(t)$, while the system output is the voltage, $V_{out}(t)$, dropped across the capacitor. Notice that these voltages, in general, will be functions of time, $t$.

## 3.2 When is a linear circuit a linear system?

Using Kirchoffs current and voltage laws, one can easily derive a differential equation model of the RC-circuit in Figure 3.1.1, namely

$$RC\frac{dV_{out}(t)}{dt} + V_{out}(t) = V_{in}(t). \tag{3.2.1}$$

Appealing to basic knowledge of ODEs from a sophomore level math course, the *total solution* is seen to be

$$V_{out}(t) = V_0 e^{-t/RC} + \int_0^t \frac{1}{RC} e^{-(t-\tau)/RC} V_{in}(\tau)d\tau, \quad t \geq 0, \tag{3.2.2}$$

where the *initial condition* at time zero is $V_{out}(0) = V_0$. It is very easy to verify that $V_{out}(t)$ is a linear function $V_{in}(t)$ if, and only if, $V_{out}(0) = V_0 = 0$, that is, the initial voltage on the capacitor has to be zero. This point is emphasized because you will have to assure this in the laboratory by either waiting for the charge to decay on the capacitor or by shorting the capacitor with a wire.

**Reassuring remark:** We will learn how to deal comfortably with nonzero initial conditions when we study the Laplace transform. For the time being, it is important to realize that we are assuming zero initial conditions when our models arise from a differential equation.

## 3.3 Impulse Response

The impulse response, $h(t)$, of an LTI system is, by definition, the output response when the input of the system is a delta function, $\delta(t)$. Of course, the delta function is a mathematical idealization. In practice, $h(t)$ can be well approximated by the response of the system when the input is a pulse of very short duration (compared with the response time of the system) and unit area, such as $p_\Delta(t) = \frac{1}{\Delta}(u(t) - u(t - \Delta))$ for $\Delta > 0$ sufficiently small.

Note that in order to keep the area of the pulse equal to unity, the amplitude has to increase as the pulse duration gets shorter. Often, this is a problem in a practical system as a large voltage pulse may fry an amplifier, for example. One way to get around this is to use linearity and realize that if the input is scaled by "b", then the output will be scaled by "b" as well. Consequently, if the measured response is divided by "b", an approximation of the impulse response is obtained.

## 3.4 Step Response

For any LTI system the output can be expressed as

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} x(t - \tau)h(\tau)d\tau \tag{3.4.1}$$

where $x(t)$ denotes the input and $*$ denotes the convolution operation. The output resulting when the input is a unit step function, $x(t) = u(t)$, is called the unit step response. Simple manipulation leads to

$$y_{step}(t) = u(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)u(t-\tau)d\tau + \int_{t}^{\infty} h(\tau)u(t-\tau)d\tau \qquad (3.4.2)$$

which, because the unit step function is equal to 1 for $t - \tau > 0$ and equal to 0 for $t - \tau < 0$ step response simplifies to

$$y_{step}(t) = \int_{-\infty}^{t} h(\tau)d\tau. \qquad (3.4.3)$$

By taking the derivative of $y_{step}(t)$ with respect to $t$, we obtain, by the fundamental theorem of calculus,

$$\frac{dy_{step}(t)}{dt} = \frac{d}{dt} \int_{-\infty}^{t} h(\tau)d\tau = h(t). \qquad (3.4.4)$$

Thus the impulse response can be computed from the unit step response by calculating the derivative of the step response with respect to time. This is a useful observation because it is sometimes easier to apply a step input to a physical system than it is to apply (an approximation of) an impulse.

We now have two ways of determining the impulse response from data. A third way will be hinted at a little later.

# 4    Pre-Lab Homework Assignment

In the following exercises, use Matlab when you are asked to plot results. Your data and plots should be saved for later comparison with the experimental results that you will obtain in the lab. **You need to bring duplicate copies of your prelab homework solutions to the lab so that you can immediately compare (qualitatively), while in the laboratory, your measured results against theory. One set of your solutions will be collected at the beginning of the lab by the TAs. Each solution should be labeled by the corresponding problem number (e.g., 4-3).**

All of these exercises refer to the RC circuit shown in Fig. 3.1.1, where $V_{in}(t)$ is the input and $V_{out}(t)$ is the output. If we are doing calculations from some finite initial time, we always assume the corresponding initial condition to be zero.

## 4.1

(a) Using the differential equation that relates $V_{in}(t)$ to $V_{out}(t)$, verify by direct substitution into the differential equation that

$$y_{step}(t) = (1 - e^{-t/RC})u(t) = \begin{cases} (1 - e^{-t/RC}) & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad (4.1.1)$$

is indeed the step response, that is, the solution corresponding to $V_{in}(t) = u(t)$.

**Hint:**   When taking derivatives, evaluate them separately for $t > 0$ and $t < 0$, and ignore any problems at $t = 0$!

(b) For $R = 1\,\mathrm{k\Omega}$ and $C = 1\,\mu\mathrm{F}$, use Matlab to plot the step response. Use the Matlab axis command `axis([0 0.01 0 1])` for your plot. The axis command is important so that everyone will have the right scales for their plots.

## 4.2

Using the unit step response, find the impulse response, $h(t)$, by differentiation.

**4.3**

(a) Using linearity and time-invariance together with the step response, compute the output when the input is a pulse of amplitude $b/\Delta$ and duration $\Delta$, that is,

$$V_{in}(t) = \frac{b}{\Delta}(u(t) - u(t - \Delta)).$$

In the following, $y_{b,\Delta}(t)$ is used to denote $V_{out}(t)$ when $V_{in}(t) = \frac{b}{\Delta}(u(t) - u(t - \Delta))$ as this allows the dependence on parameters "$\Delta$" and "$b$" to be shown.

**Optional [i.e. not required]:** Verify that you obtain the same answer analytically by convolving $h(t)$ with $V_{in}(t)$. Set up the convolution "graphically" so that you can find the proper limits of integration.

(b) With $b = 1$, compute $\lim_{\Delta \to 0} y_{b,\Delta}(t)$, where $y_{b,\Delta}(t)$ is the output computed in part (a). How does your result compare with $h(t)$ computed in problem 4.2?

**Hint:** You can use l'Hôpital's rule to evaluate $\lim_{\Delta \to 0} \frac{e^{\Delta/RC} - 1}{\Delta}$ or you can use the relation $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(c) For $R = 1\,\text{k}\Omega$ and $C = 1\,\mu\text{F}$, use Matlab to plot (on a single plot) $y_{b,\Delta}(t)$ for pulse inputs with the following parameters: $b = 10^{-4}\,\text{V s}$ and $\Delta = 10^{-3}\,\text{s}$, $0.5 \times 10^{-3}\,\text{s}$ and $0.1 \times 10^{-3}\,\text{s}$ which correspond respectively to pulses of

  (i) duration 1 ms and amplitude 100 mV

  (ii) duration 0.5 ms and amplitude 200 mV

  (iii) duration 0.1 ms and amplitude 1 V

Also on the same plot, give $10^{-4}h(t)$, where $h(t)$ is the impulse response. For your plots use the Matlab axis command `axis([0 0.005 0 0.1])`.

Note: Since $V_{in}(t)$ approaches a delta function when $b = 1$ and $\Delta \to 0$, and since the (i)-(iii) above all correspond to $b = 0.0001$, the plots for cases (i)-(iii) should approach the plot of $bh(t)$ as $\Delta$ decreases. Do you observe this to be the case?

**Hint:** In order for you to verify that your answers for parts (i)-(iii) above are correct, you are provided with the following partial results (i.e., a few data points):

  (i) $V_{out}(0.5\,\text{ms}) = 0.0393\,\text{V}$, $V_{out}(1\,\text{ms}) = 0.0632\,\text{V}$, $V_{out}(1.5\,\text{ms}) = 0.0383\,\text{V}$

  (ii) $V_{out}(0.5\,\text{ms}) = 0.0787\,\text{V}$, $V_{out}(1\,\text{ms}) = 0.0477\,\text{V}$, $V_{out}(1.5\,\text{ms}) = 0.0289\,\text{V}$

  (iii) $V_{out}(0.5\,\text{ms}) = 0.0638\,\text{V}$, $V_{out}(1\,\text{ms}) = 0.0387\,\text{V}$, $V_{out}(1.5\,\text{ms}) = 0.0235\,\text{V}$

Note that V denotes volts!

**4.4**

For $R = 1\,\text{k}\Omega$ and $C = 1\,\mu\text{F}$, determine the output when the input is

$$V_{in}(t) = [u(t) - u(t - 0.01)] + 200(t - 0.011)[u(t - 0.011) - u(t - 0.016)].$$

**Hint:** In order to verify the correctness of your answer, you are provided the following partial results (i.e., a few data points): $V_{out}(2\,\text{ms}) = 0.87\,\text{V}$, $V_{out}(12\,\text{ms}) = 0.21\,\text{V}$, $V_{out}(18\,\text{ms}) = 0.11\,\text{V}$, where V denotes volts.

## 4.5

Use the phasor circuit analysis you learned in VE 215 to compute the frequency transfer function, $H(j\omega)$.

**Remark:** Recall how this quantity is used in circuit analysis: if the input to the circuit is

$$V_{in}(t) = A\cos(\omega t),$$

then the output will be

$$V_{out}(t) = A|H(j\omega)|\cos(\omega t + \angle H(j\omega))$$

Consequently, the frequency response function can be directly measured. We will see in class that the notion of a frequency response function makes sense for any LTI system, and not just for circuits! As an **optional** exercise, you may wish to compute

$$\Gamma(j\omega) \equiv \int_{-\infty}^{\infty} h(t)\exp(-j\omega t)dt$$

and see that is exactly equal to $H(j\omega)$, ? showing that the frequency response function can be computed directly from the impulse response. A little later in class, when we study Fourier transforms, we will see that it goes the other way Too: if you know the frequency response function, you can determine the impulse response function. Hence, being able to measure the frequency response function provides an alternative manner to determine the impulse response of the system.

**Remark:** You may evaluate the integral for $\Gamma(j\omega)$ by breaking it into its real and imaginary parts via Euler's formula, *i.e.*

$$\Gamma(j\omega) \equiv \int_{-\infty}^{\infty} h(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} h(t)\sin(\omega t)dt$$

or by using the fact that

$$\int \exp(-at)dt = -\frac{1}{a}\exp(-at)$$

even when $a$ is complex-valued.

## 4.6

Recall that $\omega$ is angular frequency in units of radians/s, and this quantity can be expressed in terms of the frequency, $f$, in units of cycles/s (or Hz) using the relationship $\omega = 2\pi f$. Construct a table of $|H(j2\pi f_c)|$ and $\angle|H(j2\pi f_c)|$ (degrees) for the following values of $f_c$: 50 Hz, 200 Hz, 500 Hz, 1kHz and 5kHz. Convert the phase delay, $\angle|H(j2\pi f_c)|$, to a corresponding time-delay, $\tau_d$, where $2\pi f_c\tau_d = -\angle H(j2\pi f_c)$ and $H(j2\pi f_c)$ is expressed in units of radians. Also include the time delay values in your table.

**Hint:** In order for you to verify that your answers are correct, you are given the following partial results: $|H(j2\pi(200\,\text{Hz}))| = 0.6227$, $\angle H(j2\pi(200\,\text{Hz})) = -51.4881\,\text{deg}$, and $\tau_d(j2\pi(200\,\text{Hz})) = 0.7151\,\text{ms}$.

**Remark:** The reason for doing these computations is related to the remark in the previous problem, namely, you can measure $|H(j2\pi f_c)|$ and $\angle H(j2\pi f_c)$ by applying $\cos(2\pi f_c t)$ at the input of the circuit and observing the output, and once you have measured the frequency response function , you can compute the impulse response if you need it.