# UM-SJTU Joint Institute, Physics Laboratory I
# Measurement Uncertainty Analysis Worksheet*
# Exercise 5

## WS-1 Natural Angular Frequency

The uncertainty for ten periods is found first. Then the result for the natural frequency is given along with its uncertainty.

The type-B uncertainty for $T_{10}$ is $\Delta_{T_{10},B} = 0.001$ s. To find the type-A uncertainty, we first find the standard deviation

$$s_{T_{10}} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(T_{10,i} - \overline{T}_{10})^2} = \underline{\hspace{3cm}}\ [\underline{\hspace{1cm}}].$$

We have $n =$ ________, so the type-A uncertainty $\Delta_{T_{10},A}$ is calculated as

$$\Delta_{T_{10},A} = \frac{t_{0.95}}{\sqrt{n}} s_{T_{10}} = \underline{\hspace{1.5cm}} \times \underline{\hspace{1.5cm}} = \underline{\hspace{1.5cm}}\ [\underline{\hspace{1cm}}].$$

Hence the uncertainty for $T_{10}$ is given by

$$u_{T_{10}} = \sqrt{\Delta^2_{T_{10},A} + \Delta^2_{T_{10},B}} = \underline{\hspace{3cm}}\ [\underline{\hspace{1cm}}].$$

The period is found indirectly by measuring the ten periods. Therefore, its uncertainty $u_T$ of a single period is found by applying the uncertainty propagation formula

$$\boxed{u_T} = \sqrt{\left(\frac{\partial T}{\partial T_{10}} u_{T_{10}}\right)^2} = \frac{u_{T_{10}}}{10} = \boxed{\underline{\hspace{1.5cm}}\ [\underline{\hspace{1cm}}]},$$

Hence the period is given by

$$\boxed{T = \underline{\hspace{1.5cm}} \pm \underline{\hspace{1.5cm}}\ [\underline{\hspace{1cm}}]},$$

*Created by Peng Wenhao, edited by Fan Yixing, Mateusz Krzyzosiak [rev. 1.2] with relative uncertainty

$$\boxed{u_{rT}} = \frac{u_T}{T} \times 100\% = \boxed{\underline{\hspace{3cm}}}\%$$

The natural angular frequency $\omega_0$ is found from the formula $\omega_0 = 2\pi/T$, so by the uncertainty propagation formula and the fact that

$$\frac{\partial\omega_0}{\partial T} = -\frac{2\pi}{T^2},$$

we obtain

$$\boxed{u_{\omega_0}} = \left|\frac{\partial\omega_0}{\partial T} u_T\right| = \boxed{\underline{\hspace{6cm}}\ [\underline{\hspace{1.5cm}}]}$$

with the relative uncertainty

$$\boxed{u_{\mathrm{r},\omega_0}} = \frac{u_{\omega_0}}{\omega_0} \times 100\% = \boxed{\underline{\hspace{3cm}}\%}$$

## WS-2 Damping Coefficient

The damping coefficient is found indirectly form measurements of the period $T$ and the amplitude $\theta$ as $\beta = \frac{1}{5T}\ln(\theta_i/\theta_{i+5})$.

The uncertainty each single measurement of the amplitude is $u_\theta =$ ________°, so the uncertainty of the logarithm of the quotient of them $q_i = \ln(\theta_i/\theta_{i+5})$ is found from the uncertainty propagation formula

$$\Delta_{q_i,B} = \sqrt{\left(\frac{\partial(\ln(\theta_i/\theta_{i+5}))}{\partial\theta_i}\right)^2 u_\theta^2 + \left(\frac{\partial(\ln(\theta_i/\theta_{i+5}))}{\partial\theta_{i+5}}\right)^2 u_\theta^2} = \sqrt{\left(\frac{u_\theta}{\theta_{i+5}}\right)^2 + \left(\frac{u_\theta}{\theta_i}\right)^2}$$

For example, for $i = 1$,

$$\Delta_{q_1,B} = \sqrt{\underline{\hspace{8cm}}} = \underline{\hspace{4cm}}.$$

The results for all five sequences of measurements are given in Table WS-1.

| $i$ | $\Delta_{q_i,B}$ |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |

Table WS-1: Type-B uncertainties for $q_i$.

The overall type-B uncertainty for the quotient can be estimated as the maximum of uncertainties listed in in Table WS-1

$$\Delta_{q,B} = \_\_\_\_\_\_\_\_.$$

To estimate the type-A uncertainty of $q$, the standard deviation of $q$ is calculated as

$$s_q = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(q_i - \overline{q})^2} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Hence the type-A uncertainty for $n = 5$ is calculated as

$$\Delta_{q,A} = \frac{t_{0.95}}{\sqrt{n}} s_q = \_\_\_\_\_\_\_\_ \times \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_,$$

and the combined uncertainty

$$u_q = \sqrt{\Delta_{q,B}^2 + \Delta_{q,A}^2} = \sqrt{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} = \_\_\_\_\_\_\_\_.$$

A single measurement for ten periods is recorded as $T_{10}$ = ________ ± ________ [_____]. Hence $T$ = ________ ± ________ [_____].

Then the uncertainty propagation equation is used to calculate the uncertainty for the damping coefficient $\beta = \frac{1}{5T}q$ as

$$\begin{aligned}\boxed{u_\beta} &= \sqrt{\left(\frac{\partial \beta}{\partial T}\right)^2 u_T^2 + \left(\frac{\partial \beta}{\partial q}\right)^2 u_q^2} = \sqrt{\left(-\frac{q}{5T^2}\right) u_T^2 + \left(\frac{1}{5T}\right)^2 u_q^2} \\ &= \sqrt{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} = \boxed{\_\_\_\_\_\_\_\_ \ [\_\_\_\_\_]}\end{aligned}$$

with relative uncertainty

$$\boxed{u_{\mathrm{r},\beta}} = \frac{u_\beta}{\beta} \times 100\% = \boxed{\_\_\_\_\_\_\_\_\%}$$

## WS-3 The $\theta_{st}$-$\omega$ and $\varphi$-$\omega$ Characteristics of Forced Oscillations

On the graphs included in the report, the uncertainty is shown in the form of error bars.[1] In both the $\varphi$ vs. $(\omega/\omega_0)$ graph and the $\theta_{\mathrm{st}}$ vs. $(\omega/\omega_0)$ graph, the

[1] Please follow this part to find the uncertainties and mark them on the graphs of the phase shift $\varphi$ vs. $(\omega/\omega_0)$ graph and the amplitude of steady-state oscillations $\theta_{\mathrm{st}}$ vs. $(\omega/\omega_0)$.

measurements of $\varphi$ and $\theta_{\mathrm{st}}$ are single measurements with uncertainty _________$^\circ$, determined by the resolution of our equipment. However, to find the uncertainty of $(\omega/\omega_0)$ we need to derive it from the uncertainty propagation formula. Let us introduce symbols $Q = \frac{\omega}{\omega_0}$, $T_{10,\mathrm{natural}} = N$ and $T_{10,\mathrm{driven}} = D$, where the uncertainty of $D$ is again the minimum scale (resolution) of the equipment used. Since these are single measurements, we have

$$Q = \frac{\omega}{\omega_0} = \frac{T_{10,\mathrm{natural}}}{T_{10,\mathrm{driven}}} = \frac{N}{D}$$

and the uncertainty of the ratio $Q$, found from the uncertainty propagation formula, is

$$u_Q = \sqrt{\left(\frac{\partial Q}{\partial N} u_N\right)^2 + \left(\frac{\partial Q}{\partial D} u_D\right)^2} = \sqrt{\left(\frac{u_N}{D}\right)^2 + \left(\frac{N u_D}{D^2}\right)^2}$$

In particular, with $N =$ _________ [_____], $u_N =$ _________ [_____], and $u_D =$ _________ [_____], so with every set of $N$ and $D$ a unique uncertainty is generated. For instance,[2] for $D =$ _________ [_____], we can calculate $Q$ as

$$Q = \frac{N}{D} = \underline{\hspace{4cm}} = \underline{\hspace{3cm}}$$

with uncertainty $u_Q$ calculated as

$$\boxed{u_Q} = \sqrt{\underline{\hspace{10cm}}} = \boxed{\underline{\hspace{5cm}}},$$

and

$$\boxed{u_\varphi = 1^\circ = 0.017\ \mathrm{rad}} \qquad \boxed{u_{\theta_{\mathrm{st}}} = 1^\circ = 0.017\ \mathrm{rad}}.$$

[2]Here, based on your measurement data, give one sample calculation for a chosen value of $\omega/\omega_0$. All values of the calculated uncertainties $u_Q$ that you have used to plot error bars, should be given in the *Results* section, where tables with the data for the plots $\varphi$ vs. $(\omega/\omega_0)$ and $\theta_{\mathrm{st}}$ vs. $(\omega/\omega_0)$ is included.