Vv285 Honors Mathematics III
Functions of a Multiple Variables

Summer 2019 Term Project

Date Due: 11:59 PM, Thursday, the 1st of August 2019

General Information

The goal of this term project is to help you apply your knowledge of mathematics in extended tasks that are beyond the scope of ordinary assignments. **It is strongly recommended that you do not leave the entire project to the last minute** but rather commence work on individual parts as soon as you are able to do so.

Group Work

You will be divided into groups of *4–5 students* each.

> Each group member must be familar with and have contributed to each part of the project report. **You may not divide up the work in such a way that only certain members are involved with certain parts.** In the event of an Honor Code violation (plagiarism or other), all members of the group will be held equally responsible for the violation. Exceptions may only be made, at my discretion, in exceptional situations.

It is therefore all group members' duty to ensure that all collaborators' contributions are plausibly their own and to check on all collaborators' work progress and verify their contributions within reason.

Project Report

The term project will be submitted **electronically only** as a typed report. Handwritten submission will not be accepted! It is recommended that you use a professional type-setting program (such as LaTeX) for your report. Unless you are able to ensure a unified font size and style for formulas and text in Microsoft Word, use of Word is *not recommended.*

Your report should have the appearance, style and contents of a professional report. It should be comprehensible without reference to this document and should be comprehensible by any other student in this course. It is strongly suggested that all members of the project team proof-read the report before submission. **The report should not look like the solution to an assignment.** Do not structure the section titles as "Answer to Question i)" or similarly.

The following are a list of projects, one of which should be completed during this term. The goal of the projects is to allow you to perform some long-term in-depth research on an interesting subject, in a much more intensive way than is possible in standard course work. The projects are intentionally open-ended, i.e., after an initial part that will get you started, you determine yourselves in what direction and how far you want to take the project. In particular, you can perform work on a computer, using a computer algebra system to visualize certain aspects, or a programming language (Pascal, C++, java) to write a program that will perform numerical calculations, or even create a web page for your project! This is not specified in the projects, but can be developed as you see fit. If you perform experiments or carry out research, you can also make a video of your activities and include it in your project report. It is up to you how far you develop these projects.

Guidelines

The following guidelines apply:

- You will be randomly assigned into Project Teams of 4-5 students each Each group will then choose a single project, and submit this project as a group. No individual work will be credited, and all members of a group will receive the same grade, except in special cases.
- Each group will create a joint Project Report, typed on a word processor such as Word or LaTeX. The project report must be submitted in a PDF file format and include:
  - formulation of your aims for this project,
  - a detailed report of your research/experiments/other activities,
  - your results and calculations,
  - a conclusion - were the aims of the project achieved? Why or why not?,
  - a bibliography,
  - supplementary material
- Needless to say, your work must be original. Naturally, many groups will choose the same project, but each group should use its own methods and interpretations to deal with the project. If you find that you are running out of ideas to pursue, contact me. Please also read the UM-SJTU JI Honor Code carefully. **Any** information from third parties (books, web sites, even conversations) that you use in your project must be accounted for in the bibliography, with a reference in the text.

Grading

This term project accounts for 10% of the course grade; it will be scored based on

- **Form (4 points):** Does the report contain essential elements, such as a cover page (with title, date, list of authors), a synopsis (abstract giving the main conclusions of the project), table of contents, clear section headings, introduction, clear division into sections and appendices with informative titles and bibliography (if applicable)? Are the pages numbered? Are the text and formulas composed in a unified font? Are all figures (graphs and images) clearly labeled with identifiable source?
- **Language (4 points):** Is the style of english appropriate for a technical report? Do not treat the project as an assignment and simply number your results like part-exercises. Your text should be a single, coherent whole. The text should be a pleasant read for anyone wanting to find out about the subject matter.

  Errors in grammar and orthography (use a spell-checker!) will be penalized. Make sure that the report is interesting to read. Avoid simply repeating sentences by cut-and-paste.
- **Content (12 points):** Are the mathematical and statistical methods and deductions clearly exhibited and easy to follow? Are the conclusions well-supported by the mathematical analysis? It is important to not just copy calculations from elsewhere, but to fully make them your own, adding details and comments where necessary.

All group members will generally receive the same grade for the term project. Exceptions are possible in certain circumstances, such as a group member not contributing to the project.

On Plagiarism

Study JI's Honor Code carefully. **Any** information from third parties (books, web sites, even conversations) that you use in your project must be accounted for in the bibliography, with a reference in the text. Follow the rules regarding the correct attribution of sources that you have learned in your English course (e.g., Vy100, Vy200). All members of a group are jointly responsible for the correct attribution of all sources in all parts of the project essay, i.e., any plagiarism will be considered a violation of the Honor Code by all group members. Every group member has a duty to confirm the origin of any part of the text.

The following list includes some specific examples of plagiarism:

- Use of any passage of three words or longer from another source without proper attribution. Use of any phrase of three words or more must be enclosed in quotation marks (“example, example, example”). This excludes set phrases (e.g., “and so on”, “it follows that”) and very precise technical terminology (e.g., “without loss of generality”, “reject the null hypothesis”) that cannot be paraphrased,
- Use of material from an uncredited source, making very minor changes (like word order or verb tense) to avoid the three-word rule.
- Inclusion of facts, data, ideas or theories originally thought of by someone else, without giving that person (organization, etc.) credit.
- Paraphrasing of ideas or theories without crediting the original thinker.
- Use of images, computer code and other tools and media without appropriate credit to their creator and in accordance with relevant copyright laws.

Project 1: A Perfect Pendulum

Pendula[1] have been used for time-keeping and in physical and engineering applicastions for millenia. Their periodic motion and simple construction made them well-suited for the construction of clockworks. However, they poseess a basic drawback when used for these purposes: their periods depend on the initial displacement. The present project investigates this effect in detail.

i) Give a definition of a "mathematical pendulum" of length l and mass m. What is the difference to a "physical pendulum"? Show that the energy of a mathematical pendulum is given by

$$E(\theta, \dot{\theta}) = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1 - \cos\theta),$$

where $\theta = \theta(t)$ is the angle of displacement and $\dot{\theta}$ is the time-derivative of θ. Here g is the acceleration due to gravity, assumed constant. Derive the *pendulum equation*

$$\frac{d^2\theta(t)}{dt^2} + \frac{g}{l}\sin(\theta(t)) = 0.$$

ii) Suppose that a pendulum is held at an angle $\theta(0) = \theta_0$ at time $t = 0$ and then let go. Show that

$$|\dot{\theta}| = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}.$$

For a single period, the map $t \mapsto \theta$ is bijective and hence invertible. Use the Inverse Function Theorem of Vv186 to derive the equation

$$T = 4\sqrt{\frac{l}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

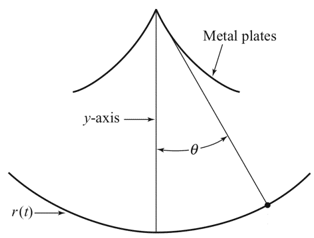
Using an appropriate substitution, show that

$$T = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \sin^2(\theta_0/2)\sin^2\phi}}$$

Note that this is a complete elliptic integral of the first kind (see Assignment 12 of Vv186). Give a formula relating the period of the pendulum to the arithmetic-geometric mean. Using an appropriate series expansion, show that the period is approximately given by

$$T \approx 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\theta_0^2}{16}\right) \approx 2\pi\sqrt{\frac{l}{g}}.$$

The mass of a simple pendulum moves along a circular path, held there by its string. Suppose the shape of this path could be modified (e.g. by varying the effective length of the string in some way, e.g., using plates). The famous mathematical *tautochrone problem* (sometimes also called the *isochrone problem*) asks whether there exists a path along which the period of such a pendulum would not depend on θ_0. This problem was solved by Christiaan Huygens in 1659 and the main part of this project is concerned with its solution.



Pendulum constrained by plates. Figure taken from [2].

iii) Consider the one-dimensional motion of a point mass on the real axis, travelling along a trajectory $\gamma\colon \mathbb{R} \to \mathbb{R}$ according to Newton's law $F \circ \gamma(t) = m\gamma''(t)$. Suppose that there exists a function $U\colon \mathbb{R} \to \mathbb{R}$ such that $F(x) = U'(x)$ for all $x \in \mathbb{R}$ (called the *potential energy*), $U(0) = 0$ and U is monotonically increasing on $[0, \infty)$.

$$U(\gamma(t)) + \frac{m}{2}\gamma'(t)^2 = \text{const.}$$

Given a potential U, Newton's law together with $\gamma(0) = x_1$ for some $x_1 > 0$, $\gamma'(0) = 0$ can be used to determine a trajectory such that $\gamma(T) = 0$. In general, $T = T(x_1)$. We seek to determine U such that T that is actually independent of x_1. First, show that

$$T(x_1) = \sqrt{\frac{m}{2}} \int_0^{x_1} \frac{1}{\sqrt{U(x_1) - U(x)}}\, dx$$

[1]Either *pendulums* or *pendula* is accepted as the plural of pendulum.

Substitute in the integral as follows:

$$y^2 = \frac{U(x)}{U(x_1)}$$

and then show that U must satisfy the equation $U'(x) = c \cdot \sqrt{U(x)}$ for some $c > 0$. Assuming that $U(0) = 0$ and that U is strictly increasing for $x \geq 0$, this equation has a unique solution on $[0, \infty)$.

iv) Explain why the tautochone must be a path along which

$$\frac{d^2 s}{dt^2} = -ks, \tag{1}$$

where t is time and s is the path length. Does a simple pendulum (as discussed in parts i) and ii) above) satisfy this relation?

The following questions are based on the section "Huygens Discovers the Isochrone" of [2], which is freely available for download at http://sofia.nmsu.edu/~history/.

v) Follow Exercise 3.2 of [2] to deduce that a curve satisfying (1) is a cycloid and give the parametric equations (parametrization) of this cycloid. Note that a proof that a cycloid curve satisfies (1) can be commonly found, e.g., in [3]. However, such a proof depends on already knowing the solution curve. You are asked here to deduce the cycloid from the equation alone.

vi) Follow the ideas discussed in Exercises 3.4 and 3.5 of [2] to describe the construction of a tautochronous pendulum.

References

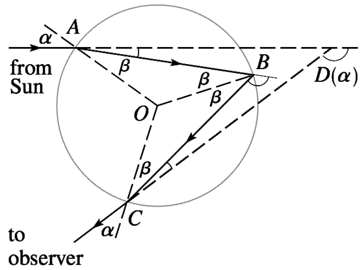
[1] A. Belyaev. Plane and space curves. Curvature. Curvature-based features. www.math.utah.edu/~palais/dnamath/04gm_curves.pdf, 2006. Web. Accessed July 10^{th}, 2016.

[2] R. Knoebel, A. Laubenbacher, R. Lodder, and D. Pengelley. *Mathematical Masterpieces: Further Chronicles by the Explorers*. Undergraduate Texts in Mathematics. Springer, 2007. Several sections can be downloaded here.

[3] E. W. Weisstein. Tautochrone problem. From MathWorld–A Wolfram Web Resource. Web. Accessed April 11^{th}, 2012.

Project Option 2: Rainbows

This project, including the description and the figures below, is adapted from "The calculus of rainbows" at the end of Section 4.1 of [2]. The goal of this project is to analyze the physics underlying the formation of rainbows. The physical principles involved are based on the scattering of light in water droplets and provide an illustration of focal points in geometrical optics.

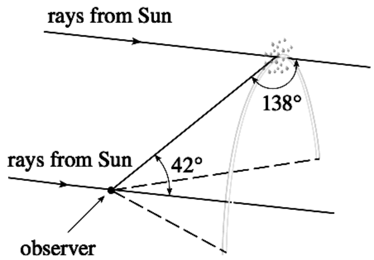


Formation of the primary rainbow. Figure taken from [2].

Your basic tasks for this project are the following:

i) Consider a single ray of light entering a spherical water droplet and reflected within it exactly once before exiting. This is illustrated in the figure at right, which has been adapted from Stewart's book. Show that the angle of deviation $D(\alpha)$ satisfies

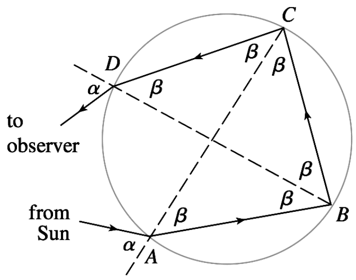
$$D(\alpha) = \pi + 2\alpha - 4\beta.$$



The rainbow angle. Figure taken from [2].

ii) It can be shown that of all the deflection angles $\delta = D(\alpha)$, only the light rays reflected to $\delta_0 = D(\alpha_0)$ are visible, where $D'(\alpha_0) = 0$. (The reason for this is investigated further below.) Show that $\alpha_0 \approx 59.4°$ and $\delta_0 = D(\alpha_0) \approx 138°$. The angle $180° - 138° = 42°$ is called the *rainbow angle.* Explain also why a rainbow appears as an arc.

iii) Explain what the seven "primary collors are" and why the rainbow has these colors in their positions.

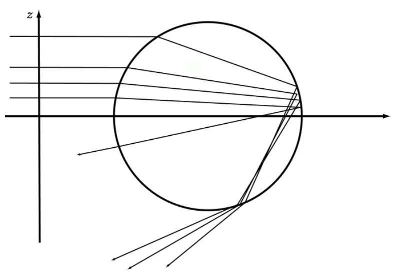


Formation of the secondary rainbow. Figure taken from [2].

iv) A secondary rainbow is formed when the light rays are reflected twice within a water droplet. Find the *secondary rainbow angle.* Why is the intensity of the secondary rainbow less than that of the primary rainbow? Is the order of the colors the same? Why or why not? Is the width of the secondary rainbow smaller or greater than that of the primary rainbow?

v) Analogously, a tertiary rainbow is formed when a light ray is thrice reflected within a water droplet. Investigate the tertiary rainbow in the same way as above.

After you have completed these tasks, extend your project by considering one or more of the following:



Scatteirng of rallel rays.

i) The light rays emanating from the sun can be considered to be parallel ue to the large distance of the sun from the earth. In the two-dimensional model we used above, the sun light enters a water droplet as a family of parallel rays, which we can parametrize by $z \in \mathbb{R}$, the *impact parameter.* (There is of course a geometric relation between z and α. We will call α he *impact angle* and wrk with α instead of z.) Consider a fixed "scattering angle" δ_0. Then all light rays wuch that $D(\alpha) = \delta_0$ are said to be scattered in the direction δ_0. Fix δ_0 and consider the interval $I_\varepsilon := [\delta_0 - \varepsilon, \delta_0 + \varepsilon]$ for some small $\varepsilon > 0$. Suppose that there exists an interval $J_{\varepsilon'} := [\alpha_0 - \varepsilon', \alpha_0 + \varepsilon']$ such that $D\colon J_{\varepsilon'} \to I_\varepsilon$ is surjective.

Then the density of the light rays scattered in directions in the interval $[\delta_0 - \varepsilon, \delta_0 + \varepsilon]$

$$\varrho(J_{\varepsilon'}) := \frac{(\alpha_0 + \varepsilon') - (\alpha_0 + \varepsilon')}{(\delta_0 + \varepsilon) - (\delta_0 - \varepsilon)} = \frac{2\varepsilon'}{D(\alpha_0 + \varepsilon') - D(\alpha_0 - \varepsilon')}$$

Show that

$$\varrho(\alpha_0) := \lim_{\varepsilon \to 0} \varrho(J_{\varepsilon'}) = \frac{1}{D'(\alpha_0)}$$

When $D'(\alpha_0) = 0$ there is no interval $J_{\varepsilon'}$ such that $D\colon J_{\varepsilon'} \to I_\varepsilon$ is surjective and the density of the light ray, in the geometrical-optical theory, becomes infinite. The scattering angle δ_0 is called a *focal point at infinity.*

ii) Investigate whether the region under a rainbow is lighter or darker than that ouside the rainbow. What about the region between the primary and the secondary rainbow?

iii) Investigate rainbows of order 4 and greater. Find and prove a formula for $D(\alpha)$ for a rainbow of kth order. Find the order of the seven color bands in a rainbow of order 12!

iv) It is also possible that a light ray is not relected at all within a water droplet, but exits on its first contact of the water-air interface. It is hence merely refracted twice and gives rise to a rainbow of "order zero". This is known as *glory scattering*. Investigate this. Are there also visible color bands in the glory? Why or why not?

References

[1] L. Cowley. Atmospheric optics. http://www.atoptics.co.uk, 2015. Web. Accessed June 26$^{\text{th}}$, 2015.

[2] J. Stewart. *Calculus*. Higher Education Press, 5 edition, 2003.

Project Option 3: $\pi = \mathbf{243F6A8885A308D31}\ldots$

The number π has fascinated mathematicians for millenia. From ancient approximations through Archimedes's circle approximations by polygons through techniques based on calculus, advances in mathematics have been accompanied by new methods for finding ever more decimal digits of π. We have already discovered and proven some fascinating identites, such as

- The Leibniz series $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$
- The Euler series $\frac{\pi^2}{6} = \sum_{n=0}^{\infty} \frac{1}{n^2}$
- The Wallis formula $\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9} \cdots$

More sophisticated formulas have been used to calculate millions of digits of π, starting from the first digit and continuing from there. But all of these formulas have required finding all n digits in order to calculate the $(n+1)st$ digit.

Recently (by mathematical standards) a new formula was discovered that is not only of a simple and elegant form, but also allows the calculation of arbitrary *hexadecimal* digits of π:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right). \tag{2}$$

This is known as the *Borwein-Bailey-Plouffe (BBP) formula.*

i) Give an elementary proof of (2), as described, e.g., in [1].

ii) The BBP formula was not directly discovered by a human, but rather turned up through the use of the PSLQ computer algorithm. Give a general introduction to what the PSLQ algorithm does: what is an integer relation of real numbers? What is the basic idea that PSLQ uses to obtain such a relation? What is the norm of a relation? You don't have to go into the technical details (e.g., Hermite reduction, LQ-decomposition), but if you do, make sure you explain them properly.

iii) The BBP formula can be used to calculate an arbitrary hexadecimal digit of π, *without calculating the previous digits.* Explain what a hexadecimal digit is and explain further how exactly (2) can be adapted to find auch a digit. You should introduce necessary concepts such as modulo calculations and exponentiations modulo bases. The wikipedia article [3] about the algorithm states "There is a possibility that a particular computation will be akin to failing to add a small number (e.g. 1) to the number 999999999999999, and that the error will propagate to the most significant digit." which seems to indicate that the algorithm may not be entirely reliable. Explain!

iv) Implement the algorithm of in a programming language or a computer algebra system, e.g., Mathematica and test it by calculating hexadecimal digits and comparing with some known table of hexadecimal digits.

v) *Optional:* Include a biography of the Borweins, of Bailey, and of Plouffe as well as a discussion of Plouffe's statement [2] in your project report. Or don't. That's up to you. But learn from that story.

References

[1] D. H. Bailey, S. M. Plouffe, P. B. Borwein, and J. M. Borwein. The quest for pi. *The Mathematical Intelligencer*, 19(1):50–56, Dec 1997.

[2] S. M. Plouffe. The story behind a formula for pi. https://groups.google.com/forum/!topic/sci.math.symbolic/a3kVKVYJhgc, 2003. Web. Accessed July 20^{th}, 2017.

[3] Wikipedia. Bailey–Borwein–Plouffe formula — wikipedia, the free encyclopedia, 2017. Web. Accessed July 20^{th}, 2017.