Cyclic Higgs bundles and minimal surfaces in pseudo-hyperbolic spaces

(Joint Seminar on Teich and Related Topics 2022-9-19)

Xin Nie (Yau Center of Southeast University)

Pseudo-hyperbolic spaces

The pseudo-hyperbolic space $\mathbb{H}^{p,q}$ (hyperboloid model):

$$\mathbb{H}^{p,q} = \{x \in \mathbb{R}^{p,q+1} \mid \langle x, x \rangle = -1\},$$

Pseudo-hyperbolic spaces

The pseudo-hyperbolic space $\mathbb{H}^{p,q}$ (hyperboloid model):

$$\mathbb{H}^{p,q} = \{x \in \mathbb{R}^{p,q+1} \mid \langle x, x \rangle = -1\},$$

with the metric restricted from $\mathbb{R}^{p,q+1}$.

Pseudo-hyperbolic spaces

The pseudo-hyperbolic space $\mathbb{H}^{p,q}$ (hyperboloid model):

$$\mathbb{H}^{p,q} = \{x \in \mathbb{R}^{p,q+1} \mid \langle x, x \rangle = -1\},$$

with the metric restricted from $\mathbb{R}^{p,q+1}$.

- $\mathbb{H}^{p,q}$ is a pseudo-Riemannian manifold of constant sectional curvature -1;

Pseudo-hyperbolic spaces

The pseudo-hyperbolic space $\mathbb{H}^{p,q}$ (hyperboloid model):

$$\mathbb{H}^{p,q} = \{x \in \mathbb{R}^{p,q+1} \mid \langle x, x \rangle = -1\},$$

with the metric restricted from $\mathbb{R}^{p,q+1}$.

- $\mathbb{H}^{p,q}$ is a pseudo-Riemannian manifold of constant sectional curvature -1;
- diffeomorphic to $\mathbb{B}^p \times \mathbb{S}^q$;

Pseudo-hyperbolic spaces

The pseudo-hyperbolic space $\mathbb{H}^{p,q}$ (hyperboloid model):

$$\mathbb{H}^{p,q} = \{x \in \mathbb{R}^{p,q+1} \mid \langle x, x \rangle = -1\},$$

with the metric restricted from $\mathbb{R}^{p,q+1}$.

- $\mathbb{H}^{p,q}$ is a pseudo-Riemannian manifold of constant sectional curvature -1;
- diffeomorphic to $\mathbb{B}^p \times \mathbb{S}^q$;
- $\mathrm{Isom}(\mathbb{H}^{p,q}) = \mathrm{O}(p, q+1)$;

Pseudo-hyperbolic spaces

The pseudo-hyperbolic space $\mathbb{H}^{p,q}$ (hyperboloid model):

$$\mathbb{H}^{p,q} = \{x \in \mathbb{R}^{p,q+1} \mid \langle x, x \rangle = -1\},$$

with the metric restricted from $\mathbb{R}^{p,q+1}$.

- $\mathbb{H}^{p,q}$ is a pseudo-Riemannian manifold of constant sectional curvature -1;
- diffeomorphic to $\mathbb{B}^p \times \mathbb{S}^q$;
- $\mathrm{Isom}(\mathbb{H}^{p,q}) = \mathrm{O}(p, q+1)$; not connected, subgroups:

$$\underbrace{\mathrm{SO}(p, q+1)}_{\text{orientation-preserving isom.}} \quad \overset{1:2}{\subset} \mathrm{O}(p, q+1)$$

Pseudo-hyperbolic spaces

The pseudo-hyperbolic space $\mathbb{H}^{p,q}$ (hyperboloid model):

$$\mathbb{H}^{p,q} = \{x \in \mathbb{R}^{p,q+1} \mid \langle x, x \rangle = -1\},$$

with the metric restricted from $\mathbb{R}^{p,q+1}$.

- $\mathbb{H}^{p,q}$ is a pseudo-Riemannian manifold of constant sectional curvature -1;
- diffeomorphic to $\mathbb{B}^p \times \mathbb{S}^q$;
- $\mathrm{Isom}(\mathbb{H}^{p,q}) = \mathrm{O}(p, q+1)$; not connected, subgroups:

$$\underbrace{\mathrm{SO}_0(p, q+1)}_{\substack{\text{orient.-preserving isom.} \\ \text{acting trivially on homology}}} \overset{1:2}{\subset} \underbrace{\mathrm{SO}(p, q+1)}_{\text{orientation-preserving isom.}} \overset{1:2}{\subset} \mathrm{O}(p, q+1)$$

Pseudo-hyperbolic spaces

The pseudo-hyperbolic space $\mathbb{H}^{p,q}$ (hyperboloid model):

$$\mathbb{H}^{p,q} = \{x \in \mathbb{R}^{p,q+1} \mid \langle x, x\rangle = -1\},$$

with the metric restricted from $\mathbb{R}^{p,q+1}$.

- $\mathbb{H}^{p,q}$ is a pseudo-Riemannian manifold of constant sectional curvature -1;
- diffeomorphic to $\mathbb{B}^p \times \mathbb{S}^q$;
- $\mathrm{Isom}(\mathbb{H}^{p,q}) = \mathrm{O}(p, q+1)$; not connected, subgroups:

$$\underbrace{\mathrm{SO}_0(p, q+1)}_{\substack{\text{orient.-preserving isom.} \\ \text{acting trivially on homology}}} \overset{1:2}{\subset} \underbrace{\mathrm{SO}(p, q+1)}_{\text{orientation-preserving isom.}} \overset{1:2}{\subset} \mathrm{O}(p, q+1)$$

- $\mathbf{S}_{p,q} :=$ {timelike totally geodesic q-spheres}

Pseudo-hyperbolic spaces

The pseudo-hyperbolic space $\mathbb{H}^{p,q}$ (hyperboloid model):

$$\mathbb{H}^{p,q} = \{x \in \mathbb{R}^{p,q+1} \mid \langle x, x\rangle = -1\},$$

with the metric restricted from $\mathbb{R}^{p,q+1}$.

- $\mathbb{H}^{p,q}$ is a pseudo-Riemannian manifold of constant sectional curvature -1;
- diffeomorphic to $\mathbb{B}^p \times \mathbb{S}^q$;
- $\mathrm{Isom}(\mathbb{H}^{p,q}) = \mathrm{O}(p, q+1)$; not connected, subgroups:

$$\underbrace{\mathrm{SO}_0(p, q+1)}_{\substack{\text{orient.-preserving isom.}\\ \text{acting trivially on homology}}} \overset{1:2}{\subset} \underbrace{\mathrm{SO}(p, q+1)}_{\text{orientation-preserving isom.}} \overset{1:2}{\subset} \mathrm{O}(p, q+1)$$

- $\mathbf{S}_{p,q} :=$ {timelike totally geodesic q-spheres}
  $= \mathrm{SO}_0(p, q+1)/(\mathrm{SO}(p) \times \mathrm{SO}(q+1)) \cong \mathbf{S}_{q+1,p-1}$
  (Riemannian symmetric space of nonpositive curvature)

Pseudo-hyperbolic spaces

Example. $\mathbb{H}^{2,1}$ = anti-de Sitter (adS) space.

Pseudo-hyperbolic spaces

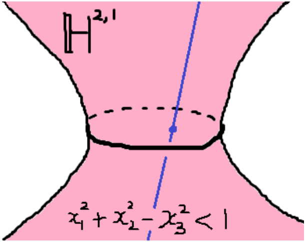
Example. $\mathbb{H}^{2,1}$ = anti-de Sitter (adS) space.

Projectivized picture in the affine plane $\mathbb{R}^{2,1} \cong \{x_4 = 1\} \subset \mathbb{R}^{2,2}$:

Pseudo-hyperbolic spaces

Example. $\mathbb{H}^{2,1}$ = anti-de Sitter (adS) space.

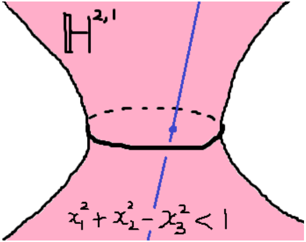
Projectivized picture in the affine plane $\mathbb{R}^{2,1} \cong \{x_4 = 1\} \subset \mathbb{R}^{2,2}$:

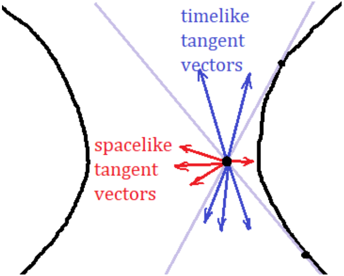


Pseudo-hyperbolic spaces

Example. $\mathbb{H}^{2,1}$ = anti-de Sitter (adS) space.

Projectivized picture in the affine plane $\mathbb{R}^{2,1} \cong \{x_4 = 1\} \subset \mathbb{R}^{2,2}$:

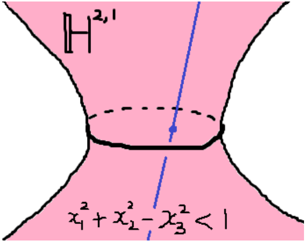


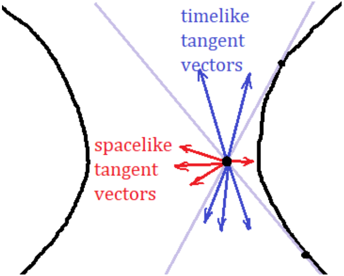


Pseudo-hyperbolic spaces

Example. $\mathbb{H}^{2,1}$ = anti-de Sitter (adS) space.

Projectivized picture in the affine plane $\mathbb{R}^{2,1} \cong \{x_4 = 1\} \subset \mathbb{R}^{2,2}$:



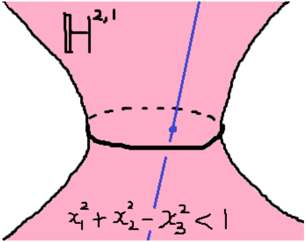


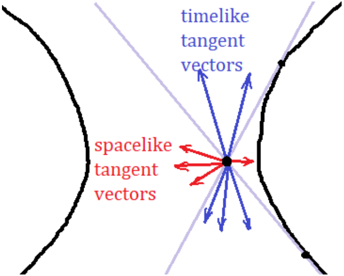
- $\mathrm{SO}_0(2,2) \cong \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$

Pseudo-hyperbolic spaces

Example. $\mathbb{H}^{2,1}$ = anti-de Sitter (adS) space.

Projectivized picture in the affine plane $\mathbb{R}^{2,1} \cong \{x_4 = 1\} \subset \mathbb{R}^{2,2}$:

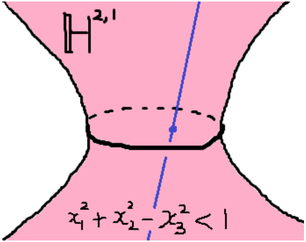


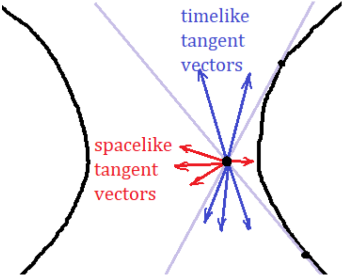


- $\mathrm{SO}_0(2,2) \cong \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$
- $\mathbf{S}_{2,1} = \{\text{timelike closed geodesics}\}$
  $= \mathrm{SO}_0(2,2)/(\mathrm{SO}(2) \times \mathrm{SO}(2)) = \mathbb{H}^2 \times \mathbb{H}^2$.

Pseudo-hyperbolic spaces

Example. $\mathbb{H}^{2,1}$ = anti-de Sitter (adS) space.
Projectivized picture in the affine plane $\mathbb{R}^{2,1} \cong \{x_4 = 1\} \subset \mathbb{R}^{2,2}$:





- $\mathrm{SO}_0(2,2) \cong \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$
- $\mathbf{S}_{2,1} = \{\text{timelike closed geodesics}\}$
  $= \mathrm{SO}_0(2,2)/(\mathrm{SO}(2) \times \mathrm{SO}(2)) = \mathbb{H}^2 \times \mathbb{H}^2$.

All the other $\mathrm{SO}_0(p, q+1)$'s $(p, q \geq 2)$ are simple Lie groups.

Maximal surfaces in $\mathbb{H}^{2,q}$

Minimal submanifold = submanifold Σ whose 2nd f.f.
$\boldsymbol{I\!I} \in C^\infty(\mathrm{Sym}^2 \mathsf{T}^*\Sigma \otimes \mathsf{N})$ has zero trace (w.r.t. 1st f.f.)

Maximal surfaces in $\mathbb{H}^{2,q}$

Minimal submanifold = submanifold Σ whose 2nd f.f. $\boldsymbol{I\!I} \in C^\infty(\mathrm{Sym}^2 \mathsf{T}^*\Sigma \otimes \mathsf{N})$ has zero trace (w.r.t. 1st f.f.)
= submanifold whose volume is stationary w.r.t. compactly supported variations

Maximal surfaces in $\mathbb{H}^{2,q}$

Minimal submanifold = submanifold Σ whose 2nd f.f. $\boldsymbol{I\!I} \in C^\infty(\mathrm{Sym}^2\mathsf{T}^*\Sigma \otimes \mathsf{N})$ has zero trace (w.r.t. 1st f.f.)
= submanifold whose volume is stationary w.r.t. compactly supported variations (critical point for the volume functional).

Maximal surfaces in $\mathbb{H}^{2,q}$

Minimal submanifold = submanifold Σ whose 2nd f.f. $\boldsymbol{I\!I} \in C^\infty(\mathrm{Sym}^2\mathsf{T}^*\Sigma \otimes \mathsf{N})$ has zero trace (w.r.t. 1st f.f.) = submanifold whose volume is stationary w.r.t. compactly supported variations (critical point for the volume functional).

Maximality Theorem

Let Σ be a spacelike minimal submanifold with timelike normal bundle in a pseudo-Riemannian manifold M of negative sectional curvature.

Maximal surfaces in $\mathbb{H}^{2,q}$

Minimal submanifold = submanifold Σ whose 2nd f.f. $\boldsymbol{I\!I} \in C^\infty(\mathrm{Sym}^2\mathsf{T}^*\Sigma \otimes \mathsf{N})$ has zero trace (w.r.t. 1st f.f.) = submanifold whose volume is stationary w.r.t. compactly supported variations (critical point for the volume functional).

Maximality Theorem

Let Σ be a spacelike minimal submanifold with timelike normal bundle in a pseudo-Riemannian manifold M of negative sectional curvature. Then any nonzero, compactly supported, normal deformation of Σ locally decreases the volume.

Maximal surfaces in $\mathbb{H}^{2,q}$

Minimal submanifold = submanifold Σ whose 2nd f.f. $\mathit{II} \in C^\infty(\mathrm{Sym}^2 \mathsf{T}^*\Sigma \otimes \mathsf{N})$ has zero trace (w.r.t. 1st f.f.)
= submanifold whose volume is stationary w.r.t. compactly supported variations (critical point for the volume functional).

Maximality Theorem

Let Σ be a spacelike minimal submanifold with timelike normal bundle in a pseudo-Riemannian manifold M of negative sectional curvature. Then any nonzero, compactly supported, normal deformation of Σ locally decreases the volume.

$\Rightarrow$ Spacelike minimal surfaces in $\mathbb{H}^{2,q}$ are called maximal surfaces.

Proof. 2nd variation formula of volume for minimal submanifolds

Proof. 2nd variation formula of volume for minimal submanifolds ($\xi \in C^\infty(\Sigma, \mathsf{N})$ variational vector field):

$$\frac{\mathsf{d}^2}{\mathsf{d}t^2}\bigg|_{t=0} vol(\Sigma_t) = \int_\Sigma \mathrm{Tr}_g \left(\langle \nabla^{\mathsf{N}}_{\bullet} \xi, \nabla^{\mathsf{N}}_{\bullet} \xi \rangle - \langle A_\xi(\bullet), A_\xi(\bullet) \rangle - \langle R_M(\bullet, \xi)\xi, \bullet \rangle \right) \mathsf{d}vol_g$$

$\bullet$ = slot for tangent vector,
g = 1st fundamental form,
∇^{N} = normal connection,
A_ξ = shape operator associated to ξ,
R_M = curvature tensor of M.

Proof. 2nd variation formula of volume for minimal submanifolds ($\xi \in C^\infty(\Sigma, \mathsf{N})$ variational vector field):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} vol(\Sigma_t) = \int_\Sigma \mathrm{Tr}_g \left(\underbrace{\langle \overbrace{\nabla^{\mathsf{N}}_{\bullet}\xi}^{\text{normal vector}}, \nabla^{\mathsf{N}}_{\bullet}\xi \rangle}_{\leq 0} - \langle A_\xi(\bullet), A_\xi(\bullet)\rangle - \langle R_M(\bullet, \xi)\xi, \bullet\rangle \right) \mathrm{d}vol_g$$

$\bullet$ = slot for tangent vector,
g = 1st fundamental form,
∇^{N} = normal connection,
A_ξ = shape operator associated to ξ,
R_M = curvature tensor of M.

Proof. 2nd variation formula of volume for minimal submanifolds ($\xi \in C^\infty(\Sigma, \mathsf{N})$ variational vector field):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} vol(\Sigma_t) = \int_\Sigma \mathrm{Tr}_g \Big(\underbrace{\langle \overbrace{\nabla^{\mathsf{N}}_{\bullet}\xi}^{\text{normal vector}}, \nabla^{\mathsf{N}}_{\bullet}\xi \rangle}_{\leq 0} - \underbrace{\langle \overbrace{A_\xi(\bullet)}^{\text{tangent vector}}, A_\xi(\bullet) \rangle}_{\geq 0} - \langle R_M(\bullet, \xi)\xi, \bullet \rangle \Big)\, \mathrm{d}vol_g$$

- $\bullet$ = slot for tangent vector,
- g = 1st fundamental form,
- ∇^{N} = normal connection,
- A_ξ = shape operator associated to ξ,
- R_M = curvature tensor of M.

Proof. 2nd variation formula of volume for minimal submanifolds
($\xi \in C^\infty(\Sigma, \mathsf{N})$ variational vector field):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} vol(\Sigma_t) = \int_\Sigma \mathrm{Tr}_g \Big(\underbrace{\langle \overbrace{\nabla^{\mathsf{N}}_{\bullet}}^{\text{normal vector}} \xi, \nabla^{\mathsf{N}}_{\bullet}\xi \rangle}_{\leq 0} - \underbrace{\langle \overbrace{A_\xi(\bullet)}^{\text{tangent vector}}, A_\xi(\bullet) \rangle}_{\geq 0} - \underbrace{\langle R_M(\bullet, \xi)\xi, \bullet \rangle}_{\geq 0\text{, strict on } \{\xi \neq 0\}} \Big)\, \mathrm{d}vol_g$$

$\bullet$ = slot for tangent vector,
g = 1st fundamental form,
∇^{N} = normal connection,
A_ξ = shape operator associated to ξ,
R_M = curvature tensor of M.

Proof. 2nd variation formula of volume for minimal submanifolds ($\xi \in C^\infty(\Sigma, \mathsf{N})$ variational vector field):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} vol(\Sigma_t) = \int_\Sigma \mathrm{Tr}_g \Big(\underbrace{\langle \overbrace{\nabla^{\mathsf{N}}_{\bullet}}^{\text{normal vector}} \xi, \nabla^{\mathsf{N}}_{\bullet}\xi\rangle}_{\leq 0} - \underbrace{\langle \overbrace{A_\xi(\bullet)}^{\text{tangent vector}}, A_\xi(\bullet)\rangle}_{\geq 0} - \underbrace{\langle R_M(\bullet, \xi)\xi, \bullet\rangle}_{\geq 0\text{, strict on } \{\xi \neq 0\}} \Big)\, \mathrm{d}vol_g$$

because $\underbrace{K(v \wedge \xi)}_{-} = \dfrac{\langle R_M(v,\xi)\xi, v\rangle}{\underbrace{\langle v, v\rangle}_{+}\underbrace{\langle \xi, \xi\rangle}_{-} - \cancel{\langle v, \xi\rangle^2}}$

- $\bullet$ = slot for tangent vector,

g = 1st fundamental form,
∇^{N} = normal connection,
A_ξ = shape operator associated to ξ,
R_M = curvature tensor of M.

Maximal surfaces in $\mathbb{H}^{2,q}$

Proof. 2nd variation formula of volume for minimal submanifolds ($\xi \in C^\infty(\Sigma, \mathsf{N})$ variational vector field):

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} vol(\Sigma_t) = \int_\Sigma \mathrm{Tr}_g\Big(\underbrace{\langle \overbrace{\nabla^{\mathsf{N}}_{\bullet}\xi}^{\text{normal vector}}, \nabla^{\mathsf{N}}_{\bullet}\xi\rangle}_{\leq 0} - \underbrace{\langle \overbrace{A_\xi(\bullet)}^{\text{tangent vector}}, A_\xi(\bullet)\rangle}_{\geq 0} - \underbrace{\langle R_M(\bullet,\xi)\xi,\bullet\rangle}_{\geq 0,\ \text{strict on } \{\xi \neq 0\}}\Big)\, \mathrm{d}vol_g$$

$$< 0$$

because $\underbrace{K(v \wedge \xi)}_{-} = \dfrac{\langle R_M(v,\xi)\xi, v\rangle}{\underbrace{\langle v,v\rangle}_{+}\underbrace{\langle \xi,\xi\rangle}_{-} - \cancel{\langle v,\xi\rangle^2}}$

- $\bullet$ = slot for tangent vector,
- g = 1st fundamental form,
- ∇^{N} = normal connection,
- A_ξ = shape operator associated to ξ,
- R_M = curvature tensor of M.

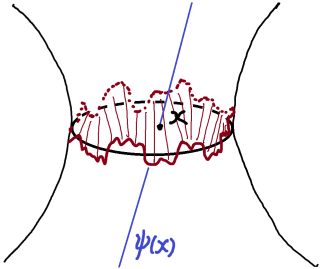
$\square$

Maximal surfaces in $\mathbb{H}^{2,q}$

Spacelike surface $\Sigma \subset \mathbb{H}^{2,q} \rightsquigarrow$ Gauss map $\psi : \Sigma \to \mathbf{S}_{2,q}$

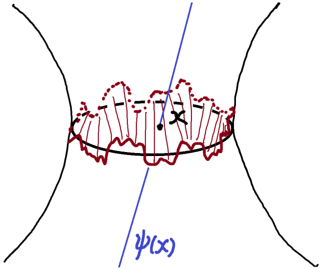
Maximal surfaces in $\mathbb{H}^{2,q}$

Spacelike surface $\Sigma \subset \mathbb{H}^{2,q} \rightsquigarrow$ Gauss map $\psi : \Sigma \to \mathbf{S}_{2,q}$



Maximal surfaces in $\mathbb{H}^{2,q}$

Spacelike surface $\Sigma \subset \mathbb{H}^{2,q} \rightsquigarrow$ Gauss map $\psi : \Sigma \to \mathbf{S}_{2,q}$

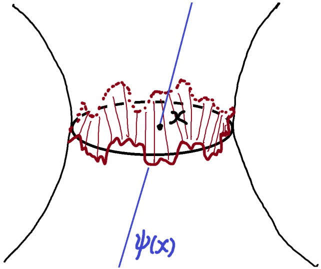


Theorem (analogue for Gauss map of minimal surfaces in $\mathbb{R}^3$)

If Σ is maximal, then ψ is a conformal minimal immersion.

Maximal surfaces in $\mathbb{H}^{2,q}$

Spacelike surface $\Sigma \subset \mathbb{H}^{2,q} \rightsquigarrow$ Gauss map $\psi : \Sigma \to \mathbf{S}_{2,q}$



Theorem (analogue for Gauss map of minimal surfaces in $\mathbb{R}^3$)

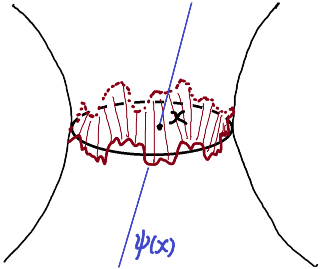
If Σ is maximal, then ψ is a conformal minimal immersion.

Example. Maximal surface in $\mathbb{H}^{2,1}$

$\rightsquigarrow$ conformal minimal immersion $\psi : \Sigma \to \mathbf{S}_{2,1} = \mathbb{H}^2 \times \mathbb{H}^2$

Maximal surfaces in $\mathbb{H}^{2,q}$

Spacelike surface $\Sigma \subset \mathbb{H}^{2,q} \rightsquigarrow$ Gauss map $\psi : \Sigma \to \mathbf{S}_{2,q}$



Theorem (analogue for Gauss map of minimal surfaces in $\mathbb{R}^3$)

If Σ is maximal, then ψ is a conformal minimal immersion.

Example. Maximal surface in $\mathbb{H}^{2,1}$

$\rightsquigarrow$ conformal minimal immersion $\psi : \Sigma \to \mathbf{S}_{2,1} = \mathbb{H}^2 \times \mathbb{H}^2$

$\rightsquigarrow$ pair of harmonic maps $\psi_+, \psi_- : \Sigma \to \mathbb{H}^2$ with opposite Hopf differentials.

Applications in (higher) Teichmüler theory

- (Mess 2007, Bonsante-Schlenker 2010) Closed maximal surfaces in some partial quotients of $\mathbb{H}^{2,1}$ ("globally hyperbolic spacetimes", analogue of quasi-Fuchsian convex cores) are useful for studying pairs of Fuchsian representations $\pi_1(S_g) \to \mathrm{PSL}(2,\mathbb{R})$ (e.g. new proof and extensions of Thurston's Earthquake Theorem).

Applications in (higher) Teichmüler theory

- (Mess 2007, Bonsante-Schlenker 2010) Closed maximal surfaces in some partial quotients of $\mathbb{H}^{2,1}$ ("globally hyperbolic spacetimes", analogue of quasi-Fuchsian convex cores) are useful for studying pairs of Fuchsian representations $\pi_1(S_g) \to \mathrm{PSL}(2,\mathbb{R})$ (e.g. new proof and extensions of Thurston's Earthquake Theorem).
- (Collier-Tholozan-Toulisse 2019) Closed maximal surfaces in some partial quotients of $\mathbb{H}^{2,q}$ are useful for studying maximal representations $\pi_1(S_g) \to \mathrm{SO}_0(2,q+1)$.

Applications in (higher) Teichmüler theory

- (Mess 2007, Bonsante-Schlenker 2010) Closed maximal surfaces in some partial quotients of $\mathbb{H}^{2,1}$ ("globally hyperbolic spacetimes", analogue of quasi-Fuchsian convex cores) are useful for studying pairs of Fuchsian representations $\pi_1(S_g) \to \mathrm{PSL}(2, \mathbb{R})$ (e.g. new proof and extensions of Thurston's Earthquake Theorem).
- (Collier-Tholozan-Toulisse 2019) Closed maximal surfaces in some partial quotients of $\mathbb{H}^{2,q}$ are useful for studying maximal representations $\pi_1(S_g) \to \mathrm{SO}_0(2, q+1)$.

What about minimal surfaces in $\mathbb{H}^{p,q}$, $p > 2$?

Applications in (higher) Teichmüler theory

- (Mess 2007, Bonsante-Schlenker 2010) Closed maximal surfaces in some partial quotients of $\mathbb{H}^{2,1}$ ("globally hyperbolic spacetimes", analogue of quasi-Fuchsian convex cores) are useful for studying pairs of Fuchsian representations $\pi_1(S_g) \to \mathrm{PSL}(2, \mathbb{R})$ (e.g. new proof and extensions of Thurston's Earthquake Theorem).
- (Collier-Tholozan-Toulisse 2019) Closed maximal surfaces in some partial quotients of $\mathbb{H}^{2,q}$ are useful for studying maximal representations $\pi_1(S_g) \to \mathrm{SO}_0(2, q+1)$.

What about minimal surfaces in $\mathbb{H}^{p,q}$, $p > 2$?

- Not maximal (critical point of area functional, but not local maximum/minimum);

Applications in (higher) Teichmüler theory

- (Mess 2007, Bonsante-Schlenker 2010) Closed maximal surfaces in some partial quotients of $\mathbb{H}^{2,1}$ (“globally hyperbolic spacetimes”, analogue of quasi-Fuchsian convex cores) are useful for studying pairs of Fuchsian representations $\pi_1(S_g) \to \mathrm{PSL}(2, \mathbb{R})$ (e.g. new proof and extensions of Thurston’s Earthquake Theorem).
- (Collier-Tholozan-Toulisse 2019) Closed maximal surfaces in some partial quotients of $\mathbb{H}^{2,q}$ are useful for studying maximal representations $\pi_1(S_g) \to \mathrm{SO}_0(2, q+1)$.

What about minimal surfaces in $\mathbb{H}^{p,q}$, $p > 2$?

- Not maximal (critical point of area functional, but not local maximum/minimum);
- No obvious notion of Gauss map to $\mathbf{S}_{p,q}$.

Summary of results

- Introduce a special type of spacelike minimal surfaces Σ (“*A-surfaces*”) in $\mathbb{H}^{n,n}$ with n even or $\mathbb{H}^{n+1,n-1}$ with n odd, related to cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles.

Summary of results

- Introduce a special type of spacelike minimal surfaces Σ (“*A-surfaces*”) in $\mathbb{H}^{n,n}$ with n even or $\mathbb{H}^{n+1,n-1}$ with n odd, related to cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles.
- Have a good notion of Gauss map $\Sigma \to \mathbf{S}_{n,n}$ or $\mathbf{S}_{n+1,n-1}$ (recovers the minimal mapping from the Higgs bundle).

Summary of results

- Introduce a special type of spacelike minimal surfaces Σ (*"A-surfaces"*) in $\mathbb{H}^{n,n}$ with n even or $\mathbb{H}^{n+1,n-1}$ with n odd, related to cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles.
- Have a good notion of Gauss map $\Sigma \to \mathbf{S}_{n,n}$ or $\mathbf{S}_{n+1,n-1}$ (recovers the minimal mapping from the Higgs bundle).
- *Key Theorem.* Normal bundle of Σ has orthogonal splitting $\mathtt{N} = \mathtt{N}^+ \oplus \mathtt{N}^-$ (spacelike/timelike parts). Under some conditions, any compactly supported deformation of Σ along $\mathtt{N}^+$ (resp. $\mathtt{N}^-$) increases (resp. decreases) the area.

Summary of results

- Introduce a special type of spacelike minimal surfaces Σ (*"A-surfaces"*) in $\mathbb{H}^{n,n}$ with n even or $\mathbb{H}^{n+1,n-1}$ with n odd, related to cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles.
- Have a good notion of Gauss map $\Sigma \to \mathbf{S}_{n,n}$ or $\mathbf{S}_{n+1,n-1}$ (recovers the minimal mapping from the Higgs bundle).
- *Key Theorem.* Normal bundle of Σ has orthogonal splitting $\mathsf{N} = \mathsf{N}^+ \oplus \mathsf{N}^-$ (spacelike/timelike parts). Under some conditions, any compactly supported deformation of Σ along N^+ (resp. N^-) increases (resp. decreases) the area.
- *Corollary.* Infinitesimally rigid.

Summary of results

- Introduce a special type of spacelike minimal surfaces Σ (*"A-surfaces"*) in $\mathbb{H}^{n,n}$ with n even or $\mathbb{H}^{n+1,n-1}$ with n odd, related to cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles.
- Have a good notion of Gauss map $\Sigma \to \mathbf{S}_{n,n}$ or $\mathbf{S}_{n+1,n-1}$ (recovers the minimal mapping from the Higgs bundle).
- *Key Theorem.* Normal bundle of Σ has orthogonal splitting $\mathtt{N} = \mathtt{N}^+ \oplus \mathtt{N}^-$ (spacelike/timelike parts). Under some conditions, any compactly supported deformation of Σ along $\mathtt{N}^+$ (resp. $\mathtt{N}^-$) increases (resp. decreases) the area.
- *Corollary.* Infinitesimally rigid.
- New proof of Labourie Immersion Theorem ($\Rightarrow$ Labourie Conjecture for rank 2 groups) for $\mathrm{SO}_0(n, n+1)$ & generalization to Collier's components.

A(lternating)-surfaces

Frenet frame: Given curve $t \mapsto \gamma(t) \in \mathbb{E}^n$, if $\gamma'(t), \gamma''(t), \cdots, \gamma^{(n)}(t)$ are linearly independent for all t,

A(lternating)-surfaces

Frenet frame: Given curve $t \mapsto \gamma(t) \in \mathbb{E}^n$, if $\gamma'(t), \gamma''(t), \cdots, \gamma^{(n)}(t)$ are linearly independent for all t, then $\exists!$ orthonormal moving frame $(e_1(t), \cdots, e_n(t))$ such that

- $e_1 = \gamma'(t)/\|\gamma'(t)\|$ is the unit tangent;

A(lternating)-surfaces

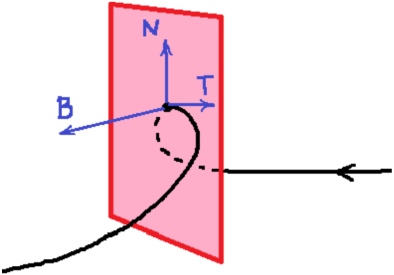
Frenet frame: Given curve $t \mapsto \gamma(t) \in \mathbb{E}^n$, if $\gamma'(t), \gamma''(t), \cdots, \gamma^{(n)}(t)$ are linearly independent for all t, then $\exists!$ orthonormal moving frame $(e_1(t), \cdots, e_n(t))$ such that

- $e_1 = \gamma'(t)/\|\gamma'(t)\|$ is the unit tangent;
- $(e_1'(t), \cdots, e_n'(t)) = (e_1(t), \cdots, e_n(t)) \begin{pmatrix} 0 & -\kappa_2 & & & \\ \kappa_2 & 0 & -\kappa_3 & & \\ & \kappa_3 & 0 & \ddots & \\ & & \ddots & \ddots & -\kappa_n \\ & & & \kappa_n & 0 \end{pmatrix}$.

A(lternating)-surfaces

Frenet frame: Given curve $t \mapsto \gamma(t) \in \mathbb{E}^n$, if $\gamma'(t), \gamma''(t), \cdots, \gamma^{(n)}(t)$ are linearly independent for all t, then $\exists!$ orthonormal moving frame $(e_1(t), \cdots, e_n(t))$ such that

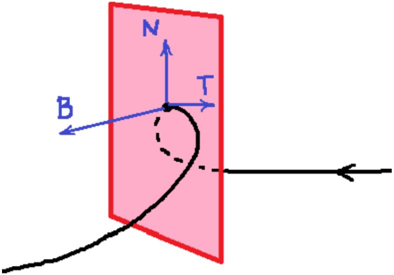
- $e_1 = \gamma'(t)/\|\gamma'(t)\|$ is the unit tangent;
- $(e_1'(t), \cdots, e_n'(t)) = (e_1(t), \cdots, e_n(t)) \begin{pmatrix} 0 & -\kappa_2 & & & \\ \kappa_2 & 0 & -\kappa_3 & & \\ & \kappa_3 & 0 & \ddots & \\ & & \ddots & \ddots & -\kappa_n \\ & & & \kappa_n & 0 \end{pmatrix}.$



A(lternating)-surfaces

Frenet frame: Given curve $t \mapsto \gamma(t) \in \mathbb{E}^n$, if $\gamma'(t), \gamma''(t), \cdots, \gamma^{(n)}(t)$ are linearly independent for all t, then $\exists!$ orthonormal moving frame $(e_1(t), \cdots, e_n(t))$ such that

- $e_1 = \gamma'(t)/\|\gamma'(t)\|$ is the unit tangent;
- $(e_1'(t), \cdots, e_n'(t)) = (e_1(t), \cdots, e_n(t)) \begin{pmatrix} 0 & -\kappa_2 & & & \\ \kappa_2 & 0 & -\kappa_3 & & \\ & \kappa_3 & 0 & \ddots & \\ & & \ddots & \ddots & -\kappa_n \\ & & & \kappa_n & 0 \end{pmatrix}.$

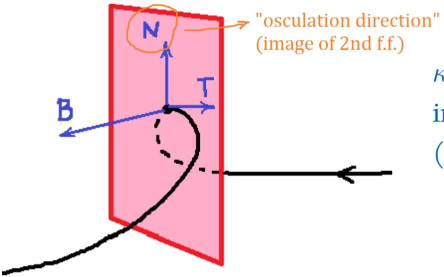


$\kappa_2, \cdots, \kappa_n$ are complete intrinsic invariants of the curve ($n = 3$: *curvature* & *torsion*)

A(lternating)-surfaces

Frenet frame: Given curve $t \mapsto \gamma(t) \in \mathbb{E}^n$, if $\gamma'(t), \gamma''(t), \cdots, \gamma^{(n)}(t)$ are linearly independent for all t, then $\exists!$ orthonormal moving frame $(e_1(t), \cdots, e_n(t))$ such that

- $e_1 = \gamma'(t)/\|\gamma'(t)\|$ is the unit tangent;
- $$(e_1'(t), \cdots, e_n'(t)) = (e_1(t), \cdots, e_n(t)) \begin{pmatrix} 0 & -\kappa_2 & & & \\ \kappa_2 & 0 & -\kappa_3 & & \\ & \kappa_3 & 0 & \ddots & \\ & & \ddots & \ddots & -\kappa_n \\ & & & \kappa_n & 0 \end{pmatrix}.$$



$\kappa_2, \cdots, \kappa_n$ are complete intrinsic invariants of the curve ($n = 3$: *curvature* & *torsion*)

A(lternating)-surfaces

Theorem (Chern 1970)

A(lternating)-surfaces

Theorem (Chern 1970)

For any minimal 2-sphere $S \subset \mathbb{S}^{2n}$ not in any $\mathbb{S}^{2n-1}$,

A(lternating)-surfaces

Theorem (Chern 1970)

For any minimal 2-sphere $S \subset \mathbb{S}^{2n}$ not in any $\mathbb{S}^{2n-1}$, $\exists!$ orthogonal splitting $\mathsf{T}\mathbb{S}^{2n}|_S = L_1 \oplus \cdots \oplus L_n$ s.t.

(a) every L_i has rank 2; $L_1 = \mathsf{T}S$;

A(lternating)-surfaces

Theorem (Chern 1970)

For any minimal 2-sphere $S \subset \mathbb{S}^{2n}$ not in any $\mathbb{S}^{2n-1}$, $\exists!$ orthogonal splitting $\mathsf{T}\mathbb{S}^{2n}|_S = L_1 \oplus \cdots \oplus L_n$ s.t.

(a) every L_i has rank 2; $L_1 = \mathsf{T}S$;

(b) the Levi-Civita connection ∇ on $\mathsf{T}\mathbb{S}^{2n}|_S$ decomposes as

$$\nabla = \begin{pmatrix} \nabla_1 & -\alpha_2^* & & & \\ \alpha_2 & \nabla_2 & -\alpha_3^* & & \\ & \alpha_3 & \nabla_3 & \ddots & \\ & & \ddots & \ddots & -\alpha_n^* \\ & & & \alpha_n & \nabla_n \end{pmatrix};$$

A(lternating)-surfaces

Theorem (Chern 1970)

For any minimal 2-sphere $S \subset \mathbb{S}^{2n}$ not in any $\mathbb{S}^{2n-1}$, $\exists!$ orthogonal splitting $\mathsf{T}\mathbb{S}^{2n}|_S = L_1 \oplus \cdots \oplus L_n$ s.t.

(a) every L_i has rank 2; $L_1 = \mathsf{T}S$;

(b) the Levi-Civita connection ∇ on $\mathsf{T}\mathbb{S}^{2n}|_S$ decomposes as

$$\nabla = \begin{pmatrix} \nabla_1 & -\alpha_2^* & & & \\ \alpha_2 & \nabla_2 & -\alpha_3^* & & \\ & \alpha_3 & \nabla_3 & \ddots & \\ & & \ddots & \ddots & -\alpha_n^* \\ & & & \alpha_n & \nabla_n \end{pmatrix};$$

(c) each $\alpha_i \in \Omega^1(S, \mathrm{Hom}(L_{i-1}, L_i))$ is conformal (i.e. any $\alpha_i(x)v : L_{i-1}|_x \to L_i|_x$ is a conformal linear map of Euclidean planes).

A(lternating)-surfaces

Idea of proof. Find L_2 as the image of the 2nd fundamental form, use the fact that any holomorphic k-differential on $\mathbb{CP}^1$ vanishes to show that α_2 is conformal (a famous idea of Hopf);

Idea of proof. Find L_2 as the image of the 2nd fundamental form, use the fact that any holomorphic k-differential on $\mathbb{CP}^1$ vanishes to show that α_2 is conformal (a famous idea of Hopf); find $L_3, L_4, \cdots$ and show the conformality of $\alpha_3, \alpha_4, \cdots$ inductively in the same way. $\square$

Idea of proof. Find L_2 as the image of the 2nd fundamental form, use the fact that any holomorphic k-differential on $\mathbb{CP}^1$ vanishes to show that α_2 is conformal (a famous idea of Hopf); find $L_3, L_4, \cdots$ and show the conformality of $\alpha_3, \alpha_4, \cdots$ inductively in the same way. $\square$

For surfaces not homeomorphic to $\mathbb{S}^2$:

Idea of proof. Find L_2 as the image of the 2nd fundamental form, use the fact that any holomorphic k-differential on $\mathbb{CP}^1$ vanishes to show that α_2 is conformal (a famous idea of Hopf); find $L_3, L_4, \cdots$ and show the conformality of $\alpha_3, \alpha_4, \cdots$ inductively in the same way. $\square$

For surfaces not homeomorphic to $\mathbb{S}^2$:

Definition

A superminimal surface in a Riemannian manifold M^{2n} is a minimal surface Σ with an orthogonal splitting $\mathsf{T}M|_\Sigma = L_1 \oplus \cdots \oplus L_n$ satisfying conditions (a), (b), (c).

Idea of proof. Find L_2 as the image of the 2nd fundamental form, use the fact that any holomorphic k-differential on $\mathbb{CP}^1$ vanishes to show that α_2 is conformal (a famous idea of Hopf); find $L_3, L_4, \cdots$ and show the conformality of $\alpha_3, \alpha_4, \cdots$ inductively in the same way. $\square$

For surfaces not homeomorphic to $\mathbb{S}^2$:

Definition

A superminimal surface in a Riemannian manifold M^{2n} is a minimal surface Σ with an orthogonal splitting $\mathsf{T}M|_\Sigma = L_1 \oplus \cdots \oplus L_n$ satisfying conditions (a), (b), (c).

Cond.(c) relaxed s.t. only $\alpha_2, \cdots, \alpha_{n-1}$ are assumed conformal (no assumption on α_n) $\Rightarrow$ quasi-superminimal surface.

Idea of proof. Find L_2 as the image of the 2nd fundamental form, use the fact that any holomorphic k-differential on $\mathbb{CP}^1$ vanishes to show that α_2 is conformal (a famous idea of Hopf); find $L_3, L_4, \cdots$ and show the conformality of $\alpha_3, \alpha_4, \cdots$ inductively in the same way. $\square$

For surfaces not homeomorphic to $\mathbb{S}^2$:

Definition

A superminimal surface in a Riemannian manifold M^{2n} is a minimal surface Σ with an orthogonal splitting $\mathsf{T}M|_\Sigma = L_1 \oplus \cdots \oplus L_n$ satisfying conditions (a), (b), (c).
Cond.(c) relaxed s.t. only $\alpha_2, \cdots, \alpha_{n-1}$ are assumed conformal (no assumption on α_n) $\Rightarrow$ quasi-superminimal surface.

Still makes sense for *pseudo-Riemannian* M if every L_i is (positive or negative) definite.

A(lternating)-surfaces

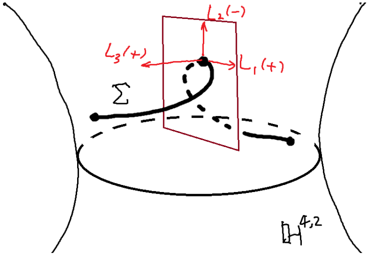
Definition

When M is pseudo-Riemannian, Σ is called an A-surface if it is quasi-superminimal and L_i is positive (resp. negative) definite for all i odd (resp. even) (*alternating* space-/time-likeness).

A(lternating)-surfaces

Definition

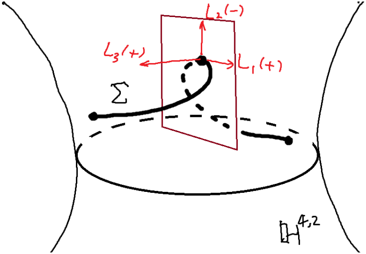
When M is pseudo-Riemannian, Σ is called an A-surface if it is quasi-superminimal and L_i is positive (resp. negative) definite for all i odd (resp. even) (*alternating* space-/time-likeness).



A(lternating)-surfaces

Definition

When M is pseudo-Riemannian, Σ is called an A-surface if it is quasi-superminimal and L_i is positive (resp. negative) definite for all i odd (resp. even) (*alternating* space-/time-likeness).

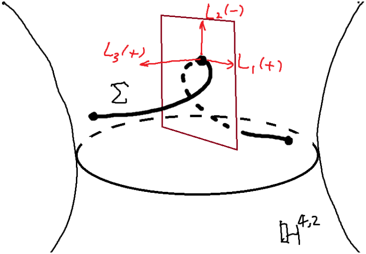


$\Rightarrow (p, q) = (n, n)$ (resp. $(n+1, n-1)$) when n is even (resp. odd).

A(lternating)-surfaces

Definition

When M is pseudo-Riemannian, Σ is called an A-surface if it is quasi-superminimal and L_i is positive (resp. negative) definite for all i odd (resp. even) (*alternating* space-/time-likeness).



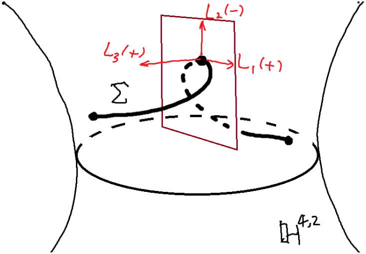
$\Rightarrow (p, q) = (n, n)$ (resp. $(n+1, n-1)$) when n is even (resp. odd).

$$\underset{+}{L_1} \oplus \underset{-}{L_2} \oplus \underset{+}{L_3} \oplus \underset{-}{L_4} \qquad \underset{+}{L_1} \oplus \underset{-}{L_2} \oplus \underset{+}{L_3} \oplus \underset{-}{L_4} \oplus \underset{+}{L_5}$$

A(lternating)-surfaces

Definition

When M is pseudo-Riemannian, Σ is called an A-surface if it is quasi-superminimal and L_i is positive (resp. negative) definite for all i odd (resp. even) (*alternating* space-/time-likeness).



$\Rightarrow (p,q) = (n,n)$ (resp. $(n+1, n-1)$) when n is even (resp. odd).

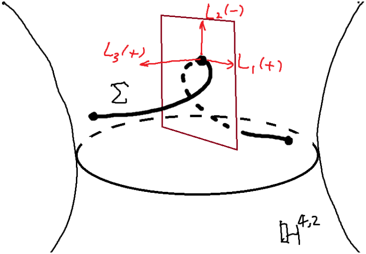
$$\underset{+}{L_1} \oplus \underset{-}{L_2} \oplus \underset{+}{L_3} \oplus \underset{-}{L_4} \qquad \underset{+}{L_1} \oplus \underset{-}{L_2} \oplus \underset{+}{L_3} \oplus \underset{-}{L_4} \oplus \underset{+}{L_5}$$

Gauss map $\psi : \Sigma \to \mathbf{S}_{p,q}$ of an A-surface $\Sigma \subset \mathbb{H}^{p,q}$:

A(lternating)-surfaces

Definition

When M is pseudo-Riemannian, Σ is called an A-surface if it is quasi-superminimal and L_i is positive (resp. negative) definite for all i odd (resp. even) (*alternating* space-/time-likeness).



$\Rightarrow (p,q) = (n,n)$ (resp. $(n+1, n-1)$) when n is even (resp. odd).

$$\underset{+}{L_1} \oplus \underset{-}{L_2} \oplus \underset{+}{L_3} \oplus \underset{-}{L_4} \qquad \underset{+}{L_1} \oplus \underset{-}{L_2} \oplus \underset{+}{L_3} \oplus \underset{-}{L_4} \oplus \underset{+}{L_5}$$

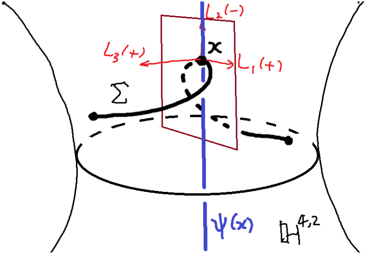
Gauss map $\psi : \Sigma \to \mathbf{S}_{p,q}$ of an A-surface $\Sigma \subset \mathbb{H}^{p,q}$:

$\psi(x) :=$ totally geodesic q-sphere generated by $(L_2 \oplus L_4 \oplus \cdots)|_x$.

A(lternating)-surfaces

Definition

When M is pseudo-Riemannian, Σ is called an A-surface if it is quasi-superminimal and L_i is positive (resp. negative) definite for all i odd (resp. even) (*alternating* space-/time-likeness).



$\Rightarrow (p,q) = (n,n)$ (resp. $(n+1, n-1)$) when n is even (resp. odd).

$$\underset{+}{L_1} \oplus \underset{-}{L_2} \oplus \underset{+}{L_3} \oplus \underset{-}{L_4} \qquad \underset{+}{L_1} \oplus \underset{-}{L_2} \oplus \underset{+}{L_3} \oplus \underset{-}{L_4} \oplus \underset{+}{L_5}$$

Gauss map $\psi : \Sigma \to \mathbf{S}_{p,q}$ of an A-surface $\Sigma \subset \mathbb{H}^{p,q}$:

$\psi(x) :=$ totally geodesic q-sphere generated by $(L_2 \oplus L_4 \oplus \cdots)|_x$.

Infinitesimal rigidity

Generalization of Maximality Theorem:

Min-Max Theorem (N. 2022)

Given A-surface $\Sigma \subset \mathbb{H}^{p,q}$ with $\mathsf{T}\mathbb{H}^{p,q}|_\Sigma = L_1 \oplus \cdots \oplus L_n$ $(L_1 = \mathsf{T}\Sigma)$.

Infinitesimal rigidity

Generalization of Maximality Theorem:

Min-Max Theorem (N. 2022)

Given A-surface $\Sigma \subset \mathbb{H}^{p,q}$ with $\mathsf{T}\mathbb{H}^{p,q}|_\Sigma = L_1 \oplus \cdots \oplus L_n$ ($L_1 = \mathsf{T}\Sigma$). Suppose that each of $L_2, \cdots, L_n$, viewed as a hermitian holomorphic line bundle, has *semi*-negative Chern form (no assumption on L_1).

Infinitesimal rigidity

Generalization of Maximality Theorem:

Min-Max Theorem (N. 2022)

Given A-surface $\Sigma \subset \mathbb{H}^{p,q}$ with $\mathsf{T}\mathbb{H}^{p,q}|_\Sigma = L_1 \oplus \cdots \oplus L_n$ $(L_1 = \mathsf{T}\Sigma)$. Suppose that each of $L_2, \cdots, L_n$, viewed as a hermitian holomorphic line bundle, has *semi*-negative Chern form (no assumption on L_1). Then any nonzero, compactly supported deformation of Σ along $\mathsf{N}^+ := L_3 \oplus L_5 \oplus \cdots$ (resp. $\mathsf{N}^- := L_2 \oplus L_4 \oplus \cdots$) locally increases (resp. decreases) the area.

Infinitesimal rigidity

Generalization of Maximality Theorem:

Min-Max Theorem (N. 2022)

Given A-surface $\Sigma \subset \mathbb{H}^{p,q}$ with $\mathsf{T}\mathbb{H}^{p,q}|_\Sigma = L_1 \oplus \cdots \oplus L_n$ $(L_1 = \mathsf{T}\Sigma)$. Suppose that each of $L_2, \cdots, L_n$, viewed as a hermitian holomorphic line bundle, has *semi*-negative Chern form (no assumption on L_1). Then any nonzero, compactly supported deformation of Σ along $\mathsf{N}^+ := L_3 \oplus L_5 \oplus \cdots$ (resp. $\mathsf{N}^- := L_2 \oplus L_4 \oplus \cdots$) locally increases (resp. decreases) the area.

Remark. Cannot expect such property from minimal surfaces in $\mathbb{H}^p \subset \mathbb{H}^{p,q}$ (unless Σ is totally geodesic or close to t.g.).

Infinitesimal rigidity

Generalization of Maximality Theorem:

Min-Max Theorem (N. 2022)

Given A-surface $\Sigma \subset \mathbb{H}^{p,q}$ with $\mathsf{T}\mathbb{H}^{p,q}|_\Sigma = L_1 \oplus \cdots \oplus L_n$ ($L_1 = \mathsf{T}\Sigma$). Suppose that each of $L_2, \cdots, L_n$, viewed as a hermitian holomorphic line bundle, has *semi*-negative Chern form (no assumption on L_1). Then any nonzero, compactly supported deformation of Σ along $\mathsf{N}^+ := L_3 \oplus L_5 \oplus \cdots$ (resp. $\mathsf{N}^- := L_2 \oplus L_4 \oplus \cdots$) locally increases (resp. decreases) the area.

Remark. Cannot expect such property from minimal surfaces in $\mathbb{H}^p \subset \mathbb{H}^{p,q}$ (unless Σ is totally geodesic or close to t.g.).
What is different for A-surfaces: the 2nd fundamental form $\mathrm{Sym}^2\mathsf{T}\Sigma = \mathrm{Sym}^2 L_1 \to \mathsf{N}$ takes values in L_2, which is timelike (A-surfaces have "timelike osculation").

Infinitesimal rigidity

Recall: Any minimal submanifold Σ has a Jacobi operator

$L_\Sigma : C^\infty(\Sigma, \mathsf{N}) \to C^\infty(\Sigma, \mathsf{N})$ ($L_\Sigma = \Delta^{\mathsf{N}} + 0\text{th order part}$)

Infinitesimal rigidity

Recall: Any minimal submanifold Σ has a Jacobi operator
$L_\Sigma : C^\infty(\Sigma, \mathsf{N}) \to C^\infty(\Sigma, \mathsf{N})$ ($L_\Sigma = \Delta^{\mathsf{N}} + 0$th order part)
s.t. $\forall$ deformation (Σ_t) with variational vector field $\xi \in C^\infty(\Sigma, \mathsf{N})$,

Infinitesimal rigidity

Recall: Any minimal submanifold Σ has a Jacobi operator
$L_\Sigma : C^\infty(\Sigma, \mathsf{N}) \to C^\infty(\Sigma, \mathsf{N})$ ($L_\Sigma = \Delta^{\mathsf{N}} + 0$th order part)
s.t. $\forall$ deformation (Σ_t) with variational vector field $\xi \in C^\infty(\Sigma, \mathsf{N})$,

- $\frac{\mathsf{d}}{\mathsf{d}t^2}\big|_{t=0} vol(\Sigma_t) = -\int_\Sigma \langle L_\Sigma \xi, \xi \rangle$,

Infinitesimal rigidity

Recall: Any minimal submanifold Σ has a Jacobi operator

$L_\Sigma : C^\infty(\Sigma, \mathsf{N}) \to C^\infty(\Sigma, \mathsf{N})$ ($L_\Sigma = \Delta^{\mathsf{N}} + 0$th order part)

s.t. $\forall$ deformation (Σ_t) with variational vector field $\xi \in C^\infty(\Sigma, \mathsf{N})$,

- $\frac{\mathrm{d}}{\mathrm{d}t^2}\big|_{t=0} vol(\Sigma_t) = -\int_\Sigma \langle L_\Sigma \xi, \xi \rangle$,
- if very Σ_t is minimal, then $L_\Sigma \xi = 0$.

Infinitesimal rigidity

Recall: Any minimal submanifold Σ has a Jacobi operator $L_\Sigma : C^\infty(\Sigma, \mathsf{N}) \to C^\infty(\Sigma, \mathsf{N})$ ($L_\Sigma = \Delta^{\mathsf{N}} +$ 0th order part) s.t. $\forall$ deformation (Σ_t) with variational vector field $\xi \in C^\infty(\Sigma, \mathsf{N})$,

- $\frac{\mathrm{d}}{\mathrm{d}t^2}\Big|_{t=0} vol(\Sigma_t) = -\int_\Sigma \langle L_\Sigma \xi, \xi \rangle$,
- if very Σ_t is minimal, then $L_\Sigma \xi = 0$.

So those ξ with $L_\Sigma \xi = 0$ are call Jacobi fields and can be viewed as the infinitesimal deformations of Σ.

Infinitesimal rigidity

Recall: Any minimal submanifold Σ has a Jacobi operator
$L_\Sigma : C^\infty(\Sigma, \mathsf{N}) \to C^\infty(\Sigma, \mathsf{N})$ ($L_\Sigma = \Delta^{\mathsf{N}} + 0$th order part)
s.t. $\forall$ deformation (Σ_t) with variational vector field $\xi \in C^\infty(\Sigma, \mathsf{N})$,

- $\frac{\mathsf{d}}{\mathsf{d}t^2}\Big|_{t=0} vol(\Sigma_t) = -\int_\Sigma \langle L_\Sigma \xi, \xi\rangle$,
- if very Σ_t is minimal, then $L_\Sigma \xi = 0$.

So those ξ with $L_\Sigma \xi = 0$ are call Jacobi fields and can be viewed as the infinitesimal deformations of Σ.

Corollary of Min-Max Theorem (Infinitesimal Rigidity)

Under the same assumption, Σ does not have any nonzero compactly supported Jacobi field.

Infinitesimal rigidity

Recall: Any minimal submanifold Σ has a Jacobi operator
$L_\Sigma : C^\infty(\Sigma, \mathsf{N}) \to C^\infty(\Sigma, \mathsf{N})$ ($L_\Sigma = \Delta^{\mathsf{N}} + 0$th order part)
s.t. $\forall$ deformation (Σ_t) with variational vector field $\xi \in C^\infty(\Sigma, \mathsf{N})$,

- $\frac{\mathsf{d}}{\mathsf{d}t^2}\Big|_{t=0} vol(\Sigma_t) = -\int_\Sigma \langle L_\Sigma \xi, \xi \rangle$,
- if very Σ_t is minimal, then $L_\Sigma \xi = 0$.

So those ξ with $L_\Sigma \xi = 0$ are call Jacobi fields and can be viewed as the infinitesimal deformations of Σ.

Corollary of Min-Max Theorem (Infinitesimal Rigidity)

Under the same assumption, Σ does not have any nonzero compactly supported Jacobi field.

Proof. A-surface Σ NOT inf. rigid $\Rightarrow \exists \xi = \xi_- + \xi_- \not\equiv 0$ with $L_\Sigma \xi = 0$.

Infinitesimal rigidity

Recall: Any minimal submanifold Σ has a Jacobi operator
$L_\Sigma : C^\infty(\Sigma, \mathsf{N}) \to C^\infty(\Sigma, \mathsf{N})$ ($L_\Sigma = \Delta^{\mathsf{N}} + 0$th order part)
s.t. $\forall$ deformation (Σ_t) with variational vector field $\xi \in C^\infty(\Sigma, \mathsf{N})$,

- $\frac{\mathrm{d}}{\mathrm{d}t^2}\Big|_{t=0} vol(\Sigma_t) = -\int_\Sigma \langle L_\Sigma \xi, \xi \rangle$,
- if very Σ_t is minimal, then $L_\Sigma \xi = 0$.

So those ξ with $L_\Sigma \xi = 0$ are call Jacobi fields and can be viewed as the infinitesimal deformations of Σ.

Corollary of Min-Max Theorem (Infinitesimal Rigidity)

Under the same assumption, Σ does not have any nonzero compactly supported Jacobi field.

Proof. A-surface Σ NOT inf. rigid $\Rightarrow \exists \xi = \xi_- + \xi_- \not\equiv 0$ with $L_\Sigma \xi = 0$.
$\Rightarrow \langle L_\Sigma \xi, \xi_+ \rangle = 0$, $\langle L_\Sigma \xi, \xi_- \rangle = 0$

Infinitesimal rigidity

Recall: Any minimal submanifold Σ has a Jacobi operator
$L_\Sigma : C^\infty(\Sigma, \mathsf{N}) \to C^\infty(\Sigma, \mathsf{N})$ ($L_\Sigma = \Delta^{\mathsf{N}} + 0$th order part)
s.t. $\forall$ deformation (Σ_t) with variational vector field $\xi \in C^\infty(\Sigma, \mathsf{N})$,

- $\frac{\mathrm{d}}{\mathrm{d}t^2}\Big|_{t=0} vol(\Sigma_t) = -\int_\Sigma \langle L_\Sigma \xi, \xi \rangle$,
- if very Σ_t is minimal, then $L_\Sigma \xi = 0$.

So those ξ with $L_\Sigma \xi = 0$ are call Jacobi fields and can be viewed as the infinitesimal deformations of Σ.

Corollary of Min-Max Theorem (Infinitesimal Rigidity)

Under the same assumption, Σ does not have any nonzero compactly supported Jacobi field.

Proof. A-surface Σ NOT inf. rigid $\Rightarrow \exists \xi = \xi_- + \xi_- \not\equiv 0$ with $L_\Sigma \xi = 0$.
$\Rightarrow \langle L_\Sigma \xi, \xi_+ \rangle = 0$, $\langle L_\Sigma \xi, \xi_- \rangle = 0$
$\Rightarrow \langle L_\Sigma \xi_+, \xi_+ \rangle = -\langle L_\Sigma \xi_-, \xi_+ \rangle$, $\langle L_\Sigma \xi_-, \xi_- \rangle = -\langle L_\Sigma \xi_+, \xi_- \rangle$

Infinitesimal rigidity

Recall: Any minimal submanifold Σ has a Jacobi operator

$L_\Sigma : C^\infty(\Sigma, \mathsf{N}) \to C^\infty(\Sigma, \mathsf{N})$ ($L_\Sigma = \Delta^{\mathsf{N}}$ + 0th order part)

s.t. $\forall$ deformation (Σ_t) with variational vector field $\xi \in C^\infty(\Sigma, \mathsf{N})$,

- $\frac{\mathsf{d}}{\mathsf{d}t^2}\Big|_{t=0} vol(\Sigma_t) = -\int_\Sigma \langle L_\Sigma \xi, \xi \rangle$,
- if very Σ_t is minimal, then $L_\Sigma \xi = 0$.

So those ξ with $L_\Sigma \xi = 0$ are call Jacobi fields and can be viewed as the infinitesimal deformations of Σ.

Corollary of Min-Max Theorem (Infinitesimal Rigidity)

Under the same assumption, Σ does not have any nonzero compactly supported Jacobi field.

Proof. A-surface Σ NOT inf. rigid $\Rightarrow \exists \xi = \xi_- + \xi_- \not\equiv 0$ with $L_\Sigma \xi = 0$.

$\Rightarrow \langle L_\Sigma \xi, \xi_+ \rangle = 0$, $\langle L_\Sigma \xi, \xi_- \rangle = 0$

$\Rightarrow \langle L_\Sigma \xi_+, \xi_+ \rangle = -\langle L_\Sigma \xi_-, \xi_+ \rangle, \langle L_\Sigma \xi_-, \xi_- \rangle = -\langle L_\Sigma \xi_+, \xi_- \rangle$

Min-Max Thm. $\Rightarrow$ integral <0, >0, resp.

Infinitesimal rigidity

Recall: Any minimal submanifold Σ has a Jacobi operator

$L_\Sigma : C^\infty(\Sigma, \mathsf{N}) \to C^\infty(\Sigma, \mathsf{N})$ ($L_\Sigma = \Delta^{\mathsf{N}}$ + 0th order part)

s.t. $\forall$ deformation (Σ_t) with variational vector field $\xi \in C^\infty(\Sigma, \mathsf{N})$,

- $\frac{\mathrm{d}}{\mathrm{d}t^2}\Big|_{t=0} vol(\Sigma_t) = -\int_\Sigma \langle L_\Sigma \xi, \xi\rangle$,
- if very Σ_t is minimal, then $L_\Sigma \xi = 0$.

So those ξ with $L_\Sigma \xi = 0$ are call Jacobi fields and can be viewed as the infinitesimal deformations of Σ.

Corollary of Min-Max Theorem (Infinitesimal Rigidity)

Under the same assumption, Σ does not have any nonzero compactly supported Jacobi field.

Proof. A-surface Σ NOT inf. rigid $\Rightarrow \exists \xi = \xi_- + \xi_- \not\equiv 0$ with $L_\Sigma \xi = 0$.

$\Rightarrow \langle L_\Sigma \xi, \xi_+\rangle = 0$, $\langle L_\Sigma \xi, \xi_-\rangle = 0$

$\Rightarrow \langle L_\Sigma \xi_+, \xi_+\rangle = -\langle L_\Sigma \xi_-, \xi_+\rangle$, $\langle L_\Sigma \xi_-, \xi_-\rangle = -\langle L_\Sigma \xi_+, \xi_-\rangle$

Min-Max Thm. $\Rightarrow$ integral <0, >0, resp.

integrals are equal because L_Σ is self-adjoint

Infinitesimal rigidity

Recall: Any minimal submanifold Σ has a Jacobi operator
$L_\Sigma : C^\infty(\Sigma, \mathsf{N}) \to C^\infty(\Sigma, \mathsf{N})$ ($L_\Sigma = \Delta^{\mathsf{N}}$ + 0th order part)
s.t. $\forall$ deformation (Σ_t) with variational vector field $\xi \in C^\infty(\Sigma, \mathsf{N})$,

- $\frac{\mathrm{d}}{\mathrm{d}t^2}\big|_{t=0} vol(\Sigma_t) = -\int_\Sigma \langle L_\Sigma \xi, \xi \rangle$,
- if very Σ_t is minimal, then $L_\Sigma \xi = 0$.

So those ξ with $L_\Sigma \xi = 0$ are call Jacobi fields and can be viewed as the infinitesimal deformations of Σ.

Corollary of Min-Max Theorem (Infinitesimal Rigidity)

Under the same assumption, Σ does not have any nonzero compactly supported Jacobi field.

Proof. A-surface Σ NOT inf. rigid $\Rightarrow \exists \xi = \xi_- + \xi_- \not\equiv 0$ with $L_\Sigma \xi = 0$.
$\Rightarrow \langle L_\Sigma \xi, \xi_+ \rangle = 0$, $\langle L_\Sigma \xi, \xi_- \rangle = 0$
$\Rightarrow \langle L_\Sigma \xi_+, \xi_+ \rangle = -\langle L_\Sigma \xi_-, \xi_+ \rangle$, $\langle L_\Sigma \xi_-, \xi_- \rangle = -\langle L_\Sigma \xi_+, \xi_- \rangle$

contradiction!

Min-Max Thm. $\Rightarrow$ integral <0, >0, resp.

integrals are equal because L_Σ is self-adjoint

$\square$

Remark. This is an infinite-dimensional generalization of the linear-algebraic fact "if $A = \begin{pmatrix} A_+ & * \\ * & A_- \end{pmatrix}$ is symmetric and the blocks A_+ and A_- are positive/negative definite resp., then $\det(A) \neq 0$"

Remark. This is an infinite-dimensional generalization of the linear-algebraic fact "if $A = \begin{pmatrix} A_+ & * \\ * & A_- \end{pmatrix}$ is symmetric and the blocks A_+ and A_- are positive/negative definite resp., then $\det(A) \neq 0$" (saddle-type critical points are non-degenerate).

Higgs bundles

A Higgs bundle on a closed Riemann surface Σ is a pair (E, Φ) where

- E is a holomorphic vector bundle on Σ;
- $\Phi \in H^0(\Sigma, K_\Sigma \otimes \mathrm{End}(E))$ ($\mathrm{End}(E)$-valued holomorphic 1-form).

Higgs bundles

A Higgs bundle on a closed Riemann surface Σ is a pair (E, Φ) where

- E is a holomorphic vector bundle on Σ;
- $\Phi \in H^0(\Sigma, K_\Sigma \otimes \mathrm{End}(E))$ ($\mathrm{End}(E)$-valued holomorphic 1-form).

Given a complex reductive group $G_\mathbb{C}$ (e.g. $\mathrm{SL}(n, \mathbb{C})$),
replace E by a holomorphic principal $G_\mathbb{C}$-bundle
$\rightsquigarrow$ $G_\mathbb{C}$-Higgs bundle.

Higgs bundles

A Higgs bundle on a closed Riemann surface Σ is a pair (E, Φ) where

- E is a holomorphic vector bundle on Σ;
- $\Phi \in H^0(\Sigma, K_\Sigma \otimes \mathrm{End}(E))$ ($\mathrm{End}(E)$-valued holomorphic 1-form).

Given a complex reductive group $G_\mathbb{C}$ (e.g. $\mathrm{SL}(n, \mathbb{C})$),
replace E by a holomorphic principal $G_\mathbb{C}$-bundle
$\rightsquigarrow$ $G_\mathbb{C}$-Higgs bundle.

Given a real semisimple Lie group G (e.g. $\mathrm{SL}(n, \mathbb{R})$),
$G_\mathbb{C}$-Higgs bundle ($G_\mathbb{C}$ = complexificationof G) + some extra structure
$\rightsquigarrow$ G-Higgs bundle.

Higgs bundles

(Part of) nonabelian Hodge correspondence:

$\deg(E) = 0$ and (E, Φ) is stable

$\Rightarrow \exists!$ hermitian metric h on E satisfying $F_{\nabla^h} + [\Phi, \Phi^{*_h}] = 0$;

Higgs bundles

(Part of) nonabelian Hodge correspondence:

$\deg(E) = 0$ and (E, Φ) is stable

$\Rightarrow \exists!$ hermitian metric h on E satisfying $F_{\nabla^h} + [\Phi, \Phi^{*_h}] = 0$;

the connection $D = \nabla^h + \Phi + \Phi^{*_h}$ is flat.

Higgs bundles

(*Part of*) *nonabelian Hodge correspondence:*

$\deg(E) = 0$ and (E, Φ) is stable

$\Rightarrow \exists!$ hermitian metric h on E satisfying $F_{\nabla^h} + [\Phi, \Phi^{*_h}] = 0$;

the connection $D = \nabla^h + \Phi + \Phi^{*_h}$ is flat.

(harmonic metric, Hitchin equation, Hitchin connection)

Higgs bundles

(*Part of*) *nonabelian Hodge correspondence:*

$\deg(E) = 0$ and (E, Φ) is stable

$\Rightarrow \exists!$ hermitian metric h on E satisfying $F_{\nabla^h} + [\Phi, \Phi^{*_h}] = 0$;

the connection $D = \nabla^h + \Phi + \Phi^{*_h}$ is flat.

(harmonic metric, Hitchin equation, Hitchin connection)

$\Rightarrow$ Give representation $\rho : \pi_1(\Sigma) \to \mathrm{GL}(n, \mathbb{C})$ and

ρ-equivariant harmonic map $f : \widetilde{\Sigma} \to \mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n)$.

Higgs bundles

(*Part of*) *nonabelian Hodge correspondence:*

$\deg(E) = 0$ and (E, Φ) is stable

$\Rightarrow \exists!$ hermitian metric h on E satisfying $F_{\nabla^h} + [\Phi, \Phi^{*_h}] = 0$;

the connection $D = \nabla^h + \Phi + \Phi^{*_h}$ is flat.

(harmonic metric, Hitchin equation, Hitchin connection)

$\Rightarrow$ Give representation $\rho : \pi_1(\Sigma) \to \mathrm{GL}(n, \mathbb{C})$ and

ρ-equivariant harmonic map $f : \widetilde{\Sigma} \to \mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n)$.

If (E, Φ) is G-Higgs bundle for real semisimple G

Higgs bundles

(*Part of*) *nonabelian Hodge correspondence:*

$\deg(E) = 0$ and (E, Φ) is stable

$\Rightarrow \exists!$ hermitian metric h on E satisfying $F_{\nabla^h} + [\Phi, \Phi^{*_h}] = 0$;

the connection $D = \nabla^h + \Phi + \Phi^{*_h}$ is flat.

(harmonic metric, Hitchin equation, Hitchin connection)

$\Rightarrow$ Give representation $\rho : \pi_1(\Sigma) \to \mathrm{GL}(n, \mathbb{C})$ and

ρ-equivariant harmonic map $f : \widetilde{\Sigma} \to \mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n)$.

If (E, Φ) is G-Higgs bundle for real semisimple G

$\Rightarrow \rho$ has image in G; f has image in G/K.

Cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles

Special case for today.

$$E = L_n^{-1} \oplus \cdots \oplus L_2^{-1} \oplus L_1^{-1} \oplus \mathcal{O} \oplus L_1 \oplus L_2 \oplus \cdots \oplus L_n$$

$$\Phi = \left(\begin{array}{ccccc|c|ccccc}
0 & & & & & & & & & \boldsymbol{\beta} & \\
\boldsymbol{\alpha}_n & 0 & & & & & & & & & \boldsymbol{\beta} \\
& \boldsymbol{\alpha}_{n-1} & \ddots & & & & & & & & \\
& & \ddots & \ddots & & & & & & & \\
& & & \boldsymbol{\alpha}_2 & 0 & & & & & & \\
\hline
& & & & \boldsymbol{\alpha}_1 & 0 & & & & & \\
\hline
& & & & & \boldsymbol{\alpha}_1 & 0 & & & & \\
& & & & & & \boldsymbol{\alpha}_2 & \ddots & & & \\
& & & & & & & \ddots & \ddots & & \\
& & & & & & & & \boldsymbol{\alpha}_{n-1} & 0 & \\
& & & & & & & & & \boldsymbol{\alpha}_n & 0
\end{array}\right)$$

is a cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundle

Cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles

Special case for today.

$$E = L_n^{-1} \oplus \cdots \oplus L_2^{-1} \oplus L_1^{-1} \oplus \mathcal{O} \oplus L_1 \oplus L_2 \oplus \cdots \oplus L_n$$

$$\Phi = \left(\begin{array}{ccccc|c|ccccc}
0 & & & & & & & & & \boldsymbol{\beta} & \\
\boldsymbol{\alpha}_n & 0 & & & & & & & & & \boldsymbol{\beta} \\
 & \boldsymbol{\alpha}_{n-1} & \ddots & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & \\
 & & & \boldsymbol{\alpha}_2 & 0 & & & & & & \\
\hline
 & & & & \boldsymbol{\alpha}_1 & 0 & & & & & \\
\hline
 & & & & & \boldsymbol{\alpha}_1 & 0 & & & & \\
 & & & & & & \boldsymbol{\alpha}_2 & \ddots & & & \\
 & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & \boldsymbol{\alpha}_{n-1} & 0 & \\
 & & & & & & & & & \boldsymbol{\alpha}_n & 0
\end{array}\right)$$

is a cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundle

$L_1, \cdots, L_n$:
holo. line bundle

$\alpha_1 \in H^0(\Sigma, K_\Sigma L_1)$

$\alpha_i \in H^0(\Sigma, K_\Sigma L_{i-1}^{-1} L_i)$
$i = 2, \cdots, n$

$\beta \in H^0(\Sigma, K_\Sigma L_{n-1}^{-1} L_n^{-1})$

Cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles

Special case for today.

$$E = L_n^{-1} \oplus \cdots \oplus L_2^{-1} \oplus L_1^{-1} \oplus \mathcal{O} \oplus L_1 \oplus L_2 \oplus \cdots \oplus L_n$$

$$\Phi = \left(\begin{array}{ccccc|c|ccccc}
0 & & & & & & & & & \boldsymbol{\beta} & \\
\boldsymbol{\alpha}_n & 0 & & & & & & & & & \boldsymbol{\beta} \\
& \boldsymbol{\alpha}_{n-1} & \ddots & & & & & & & & \\
& & \ddots & \ddots & & & & & & & \\
& & & \boldsymbol{\alpha}_2 & 0 & & & & & & \\
\hline
& & & & \boldsymbol{\alpha}_1 & 0 & & & & & \\
\hline
& & & & & \boldsymbol{\alpha}_1 & 0 & & & & \\
& & & & & & \boldsymbol{\alpha}_2 & \ddots & & & \\
& & & & & & & \ddots & \ddots & & \\
& & & & & & & & \boldsymbol{\alpha}_{n-1} & 0 & \\
& & & & & & & & & \boldsymbol{\alpha}_n & 0
\end{array}\right)$$

$L_1, \cdots, L_n$: holo. line bundle

$\alpha_1 \in H^0(\Sigma, K_\Sigma L_1)$

$\alpha_i \in H^0(\Sigma, K_\Sigma L_{i-1}^{-1} L_i)$

$i = 2, \cdots, n$

$\beta \in H^0(\Sigma, K_\Sigma L_{n-1}^{-1} L_n^{-1})$

is a cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundle

(= "sub-cyclic" $\mathrm{SL}(2n+1, \mathbb{R})$-Higgs bundle)

Cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles

Properties.

- (Baraglia, Collier-Li) The harmonic metric h on $E = L_n^{-1} \oplus \cdots \oplus L_1^{-1} \oplus \mathcal{O} \oplus L_1 \oplus \cdots \oplus L_n$ has the form $h = h_n^{-1} \oplus \cdots \oplus h_1^{-1} \oplus 1 \oplus h_1 \oplus \cdots \oplus h_n$ for hermitian metric h_i on L_i.

Cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles

Properties.

- (Baraglia, Collier-Li) The harmonic metric h on $E = L_n^{-1} \oplus \cdots \oplus L_1^{-1} \oplus \mathcal{O} \oplus L_1 \oplus \cdots \oplus L_n$ has the form $h = h_n^{-1} \oplus \cdots \oplus h_1^{-1} \oplus 1 \oplus h_1 \oplus \cdots \oplus h_n$ for hermitian metric h_i on L_i.
- The harmonic map $f : \widetilde{\Sigma} \to \mathbf{S}_{n,n} = \mathbf{S}_{n+1,n-1}$ is a conformal minimal immersion.

Cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles

Sub-cases.

- If $L_1 = K_\Sigma^{-1}$, $\boldsymbol{\alpha}_1 = 1 \in H^0(\Sigma, K_\Sigma L_1) = H^0(\Sigma, \mathcal{O})$, $L_2 = K_\Sigma^{-2}$, $\boldsymbol{\alpha}_2 = 1 \in H^0(\Sigma, K_\Sigma L_1^{-1} L_2) = H^0(\Sigma, \mathcal{O})$, $\cdots, L_n = K_\Sigma^{-n}$, $\boldsymbol{\alpha}_n = 1$

Cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles

Sub-cases.

- If $L_1 = K_\Sigma^{-1}$, $\boldsymbol{\alpha}_1 = 1 \in H^0(\Sigma, K_\Sigma L_1) = H^0(\Sigma, \mathcal{O})$, $L_2 = K_\Sigma^{-2}$, $\boldsymbol{\alpha}_2 = 1 \in H^0(\Sigma, K_\Sigma L_1^{-1} L_2) = H^0(\Sigma, \mathcal{O})$, $\cdots$, $L_n = K_\Sigma^{-n}$, $\boldsymbol{\alpha}_n = 1 \Rightarrow \rho$ is in the Hitchin component $\mathbf{Hit}(S_g, \mathrm{SO}_0(n, n+1)) \subset \mathbf{Hit}(S_g, \mathrm{SL}(2n+1, \mathbb{R}))$.

Cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles

Sub-cases.

- If $L_1 = K_\Sigma^{-1}$, $\boldsymbol{\alpha}_1 = 1 \in H^0(\Sigma, K_\Sigma L_1) = H^0(\Sigma, \mathcal{O})$,
  $L_2 = K_\Sigma^{-2}$, $\boldsymbol{\alpha}_2 = 1 \in H^0(\Sigma, K_\Sigma L_1^{-1} L_2) = H^0(\Sigma, \mathcal{O})$,
  $\cdots$, $L_n = K_\Sigma^{-n}$, $\boldsymbol{\alpha}_n = 1 \Rightarrow \rho$ is in the Hitchin component
  $\mathbf{Hit}(S_g, \mathrm{SO}_0(n, n+1)) \subset \mathbf{Hit}(S_g, \mathrm{SL}(2n+1, \mathbb{R}))$.
- If moreover $n = 3 \Rightarrow \rho$ is in $\mathbf{Hit}(S_g, G_2') \subset \mathbf{Hit}(S_g, \mathrm{SO}_0(3, 4))$.

Cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles

Sub-cases.

- If $L_1 = K_\Sigma^{-1}$, $\boldsymbol{\alpha}_1 = 1 \in H^0(\Sigma, K_\Sigma L_1) = H^0(\Sigma, \mathcal{O})$, $L_2 = K_\Sigma^{-2}$, $\boldsymbol{\alpha}_2 = 1 \in H^0(\Sigma, K_\Sigma L_1^{-1} L_2) = H^0(\Sigma, \mathcal{O})$, $\cdots, L_n = K_\Sigma^{-n}$, $\boldsymbol{\alpha}_n = 1 \Rightarrow \rho$ is in the Hitchin component $\mathbf{Hit}(S_g, \mathrm{SO}_0(n, n+1)) \subset \mathbf{Hit}(S_g, \mathrm{SL}(2n+1, \mathbb{R}))$.
- If moreover $n = 3 \Rightarrow \rho$ is in $\mathbf{Hit}(S_g, G_2') \subset \mathbf{Hit}(S_g, \mathrm{SO}_0(3,4))$.
- More generally, if $L_k = K_\Sigma^{-k}$ and $\alpha_k = 1$ for $k = 1, \cdots, n-1$

Cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles

Sub-cases.

- If $L_1 = K_\Sigma^{-1}$, $\boldsymbol{\alpha}_1 = 1 \in H^0(\Sigma, K_\Sigma L_1) = H^0(\Sigma, \mathcal{O})$, $L_2 = K_\Sigma^{-2}$, $\boldsymbol{\alpha}_2 = 1 \in H^0(\Sigma, K_\Sigma L_1^{-1} L_2) = H^0(\Sigma, \mathcal{O})$, $\cdots$, $L_n = K_\Sigma^{-n}$, $\boldsymbol{\alpha}_n = 1 \Rightarrow \rho$ is in the Hitchin component $\mathbf{Hit}(S_g, \mathrm{SO}_0(n, n+1)) \subset \mathbf{Hit}(S_g, \mathrm{SL}(2n+1, \mathbb{R}))$.
- If moreover $n = 3 \Rightarrow \rho$ is in $\mathbf{Hit}(S_g, G_2') \subset \mathbf{Hit}(S_g, \mathrm{SO}_0(3,4))$.
- More generally, if $L_k = K_\Sigma^{-k}$ and $\alpha_k = 1$ for $k = 1, \cdots, n-1$ $\Rightarrow \rho$ is in a Collier component $\mathbf{Col}_d(S_g, \mathrm{SO}_0(n, n+1))$, $0 \leq d \leq (2g-2)n$

Cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles

Sub-cases.

- If $L_1 = K_\Sigma^{-1}$, $\boldsymbol{\alpha}_1 = 1 \in H^0(\Sigma, K_\Sigma L_1) = H^0(\Sigma, \mathcal{O})$, $L_2 = K_\Sigma^{-2}$, $\boldsymbol{\alpha}_2 = 1 \in H^0(\Sigma, K_\Sigma L_1^{-1} L_2) = H^0(\Sigma, \mathcal{O})$, $\cdots$, $L_n = K_\Sigma^{-n}$, $\boldsymbol{\alpha}_n = 1 \Rightarrow \rho$ is in the Hitchin component $\mathbf{Hit}(S_g, \mathrm{SO}_0(n, n+1)) \subset \mathbf{Hit}(S_g, \mathrm{SL}(2n+1, \mathbb{R}))$.
- If moreover $n = 3 \Rightarrow \rho$ is in $\mathbf{Hit}(S_g, G_2') \subset \mathbf{Hit}(S_g, \mathrm{SO}_0(3, 4))$.
- More generally, if $L_k = K_\Sigma^{-k}$ and $\alpha_k = 1$ for $k = 1, \cdots, n-1$ $\Rightarrow \rho$ is in a Collier component $\mathbf{Col}_d(S_g, \mathrm{SO}_0(n, n+1))$, $0 \leq d \leq (2g-2)n$ ($\mathbf{Col}_d$ consists of Θ-*positive Anosov* $\mathrm{SO}_0(n, n+1)$-*representations;*

Cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles

Sub-cases.

- If $L_1 = K_\Sigma^{-1}$, $\boldsymbol{\alpha}_1 = 1 \in H^0(\Sigma, K_\Sigma L_1) = H^0(\Sigma, \mathcal{O})$, $L_2 = K_\Sigma^{-2}$, $\boldsymbol{\alpha}_2 = 1 \in H^0(\Sigma, K_\Sigma L_1^{-1} L_2) = H^0(\Sigma, \mathcal{O})$, $\cdots$, $L_n = K_\Sigma^{-n}$, $\boldsymbol{\alpha}_n = 1 \Rightarrow \rho$ is in the Hitchin component $\mathbf{Hit}(S_g, \mathrm{SO}_0(n, n+1)) \subset \mathbf{Hit}(S_g, \mathrm{SL}(2n+1, \mathbb{R}))$.
- If moreover $n = 3 \Rightarrow \rho$ is in $\mathbf{Hit}(S_g, G_2') \subset \mathbf{Hit}(S_g, \mathrm{SO}_0(3,4))$.
- More generally, if $L_k = K_\Sigma^{-k}$ and $\alpha_k = 1$ for $k = 1, \cdots, n-1$ $\Rightarrow \rho$ is in a Collier component $\mathbf{Col}_d(S_g, \mathrm{SO}_0(n, n+1))$, $0 \leq d \leq (2g-2)n$ ($\mathbf{Col}_d$ consists of Θ-*positive Anosov* $\mathrm{SO}_0(n, n+1)$-*representations*; $\mathbf{Col}_{(2g-2)n} = \mathbf{Hit}$;

Cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles

Sub-cases.

- If $L_1 = K_\Sigma^{-1}$, $\boldsymbol{\alpha}_1 = 1 \in H^0(\Sigma, K_\Sigma L_1) = H^0(\Sigma, \mathcal{O})$, $L_2 = K_\Sigma^{-2}$, $\boldsymbol{\alpha}_2 = 1 \in H^0(\Sigma, K_\Sigma L_1^{-1} L_2) = H^0(\Sigma, \mathcal{O})$, $\cdots$, $L_n = K_\Sigma^{-n}$, $\boldsymbol{\alpha}_n = 1 \Rightarrow \rho$ is in the Hitchin component $\mathbf{Hit}(S_g, \mathrm{SO}_0(n, n+1)) \subset \mathbf{Hit}(S_g, \mathrm{SL}(2n+1, \mathbb{R}))$.
- If moreover $n = 3 \Rightarrow \rho$ is in $\mathbf{Hit}(S_g, G_2') \subset \mathbf{Hit}(S_g, \mathrm{SO}_0(3,4))$.
- More generally, if $L_k = K_\Sigma^{-k}$ and $\alpha_k = 1$ for $k = 1, \cdots, n-1$ $\Rightarrow \rho$ is in a Collier component $\mathbf{Col}_d(S_g, \mathrm{SO}_0(n, n+1))$, $0 \leq d \leq (2g-2)n$ ($\mathbf{Col}_d$ consists of Θ-*positive Anosov* $\mathrm{SO}_0(n, n+1)$-*representations*; $\mathbf{Col}_{(2g-2)n} = \mathbf{Hit}$; $n = 1 \rightsquigarrow$ the familiar components for $\mathrm{SO}_0(2,1) = \mathrm{PSL}(2, \mathbb{R})$).

Cyclic $\mathrm{SO}_0(n, n+1)$-Higgs bundles

Sub-cases.

- If $L_1 = K_\Sigma^{-1}$, $\boldsymbol{\alpha}_1 = 1 \in H^0(\Sigma, K_\Sigma L_1) = H^0(\Sigma, \mathcal{O})$, $L_2 = K_\Sigma^{-2}$, $\boldsymbol{\alpha}_2 = 1 \in H^0(\Sigma, K_\Sigma L_1^{-1} L_2) = H^0(\Sigma, \mathcal{O})$, $\cdots$, $L_n = K_\Sigma^{-n}$, $\boldsymbol{\alpha}_n = 1 \Rightarrow \rho$ is in the Hitchin component $\mathbf{Hit}(S_g, \mathrm{SO}_0(n, n+1)) \subset \mathbf{Hit}(S_g, \mathrm{SL}(2n+1, \mathbb{R}))$.
- If moreover $n = 3 \Rightarrow \rho$ is in $\mathbf{Hit}(S_g, G_2') \subset \mathbf{Hit}(S_g, \mathrm{SO}_0(3,4))$.
- More generally, if $L_k = K_\Sigma^{-k}$ and $\alpha_k = 1$ for $k = 1, \cdots, n-1$ $\Rightarrow \rho$ is in a Collier component $\mathbf{Col}_d(S_g, \mathrm{SO}_0(n, n+1))$, $0 \leq d \leq (2g-2)n$ ($\mathbf{Col}_d$ consists of Θ*-positive Anosov* $\mathrm{SO}_0(n, n+1)$*-representations*; $\mathbf{Col}_{(2g-2)n} = \mathbf{Hit}$; $n = 1 \rightsquigarrow$ the familiar components for $\mathrm{SO}_0(2,1) = \mathrm{PSL}(2, \mathbb{R})$).

Theorem (N. 2022)

If $L_1 = K_\Sigma^{-1}$ and $\boldsymbol{\alpha}_1 = 1$, Then $\exists!$ A-surface immersion $F : \widetilde{\Sigma} \to \mathbb{H}^{n,n}$ (if n is even) or $\mathbb{H}^{n+1,n-1}$ (if n is odd) such that f is the composition of F with the Gauss map of F.

Labourie Immersion Theorem & Labourie Conjecture

Labourie Immersion Theorem (2017)

Given any split real semisimple Lie group G of rank ≥ 2 (e.g. $\mathrm{SL}(n,\mathbb{R})$, $n \geq 3$), the holonomy map

$$\Psi : \left\{ \begin{array}{l} \text{genus } g \text{ Riemann surfaces w.} \\ \text{conformal } G\text{-Higgs bundles} \\ \text{in Hitchin section} \end{array} \right\} / \sim \; \to \mathbf{Hit}(S_g, G)$$

$$(E, \Phi) \mapsto [\rho]$$

Labourie Immersion Theorem & Labourie Conjecture

Labourie Immersion Theorem (2017)

Given any split real semisimple Lie group G of rank ≥ 2 (e.g. $\mathrm{SL}(n, \mathbb{R}), n \geq 3$), the holonomy map

$$\Psi : \left\{ \begin{array}{l} \text{genus } g \text{ Riemann surfaces w.} \\ \text{conformal } G\text{-Higgs bundles} \\ \text{in Hitchin section} \end{array} \right\} / \sim \;\to \mathbf{Hit}(S_g, G)$$

$$(E, \Phi) \mapsto [\rho]$$

Labourie Conjecture: Ψ is injective

Labourie Immersion Theorem & Labourie Conjecture

Labourie Immersion Theorem (2017)

Given any split real semisimple Lie group G of rank ≥ 2 (e.g. $\mathrm{SL}(n, \mathbb{R})$, $n \geq 3$), the holonomy map

$$\Psi : \left\{ \begin{array}{l} \text{genus } g \text{ Riemann surfaces w.} \\ \text{conformal } G\text{-Higgs bundles} \\ \text{in Hitchin section} \end{array} \right\} / \sim \; \to \mathbf{Hit}(S_g, G)$$

$$(E, \Phi) \mapsto [\rho]$$

Labourie Conjecture: Ψ is injective (already known to be surjective).

Labourie Immersion Theorem & Labourie Conjecture

Labourie Immersion Theorem (2017)

Given any split real semisimple Lie group G of rank ≥ 2 (e.g. $\mathrm{SL}(n,\mathbb{R}), n \geq 3$), the holonomy map

$$\Psi : \left\{ \begin{array}{l} \text{genus } g \text{ Riemann surfaces w.} \\ \text{conformal } G\text{-Higgs bundles} \\ \text{in Hitchin section} \end{array} \right\} / \sim \;\to \mathbf{Hit}(S_g, G)$$

$$(E, \Phi) \mapsto [\rho]$$

restricts to an immersion on the cyclic locus

Labourie Conjecture: Ψ is injective (already known to be surjective).

Labourie Immersion Theorem & Labourie Conjecture

Labourie Immersion Theorem (2017)

Given any split real semisimple Lie group G of rank ≥ 2 (e.g. $\mathrm{SL}(n,\mathbb{R}), n \geq 3$), the holonomy map

$$\Psi : \left\{\begin{array}{l}\text{genus } g \text{ Riemann surfaces w.} \\ \text{conformal } \underline{G\text{-Higgs bundles}} \\ \underline{\text{in Hitchin section}}\ {}^{\text{constructed from}}_{\text{holo. differentials}}\end{array}\right\} / \sim \;\to \mathbf{Hit}(S_g, G)$$

$$(E, \Phi) \mapsto [\rho]$$

restricts to an immersion on the cyclic locus

Labourie Conjecture: Ψ is injective (already known to be surjective).

Labourie Immersion Theorem & Labourie Conjecture

Labourie Immersion Theorem (2017)

Given any split real semisimple Lie group G of rank ≥ 2 (e.g. $\mathrm{SL}(n,\mathbb{R}), n \geq 3$), the holonomy map

$$\Psi : \left\{ \begin{array}{l} \text{genus } g \text{ Riemann surfaces w.} \\ \text{conformal } \underline{G\text{-Higgs bundles}} \\ \underline{\text{in Hitchin section}} \end{array} \right\} / \sim \;\to \mathbf{Hit}(S_g, G)$$

constructed from holo. differentials

holo. quadratic diff. part =0

$$(E, \Phi) \mapsto [\rho]$$

restricts to an immersion on the cyclic locus

Labourie Conjecture: Ψ is injective (already known to be surjective).

Labourie Immersion Theorem & Labourie Conjecture

Labourie Immersion Theorem (2017)

Given any split real semisimple Lie group G of rank ≥ 2 (e.g. $\mathrm{SL}(n,\mathbb{R}), n \geq 3$), the holonomy map

not fixed

$$\Psi : \left\{ \begin{array}{l} \text{genus } g \text{ Riemann surfaces w.} \\ \text{conformal } G\text{-Higgs bundles} \\ \text{in Hitchin section} \end{array} \right\} / \sim \;\to \mathbf{Hit}(S_g, G)$$

constructed from holo. differentials

holo. quadratic diff. part =0

$$(E, \Phi) \mapsto [\rho]$$

restricts to an immersion on the cyclic locus

Labourie Conjecture: Ψ is injective (already known to be surjective).

Labourie Immersion Theorem & Labourie Conjecture

Labourie Immersion Theorem (2017)

Given any split real semisimple Lie group G of rank ≥ 2 (e.g. $\mathrm{SL}(n,\mathbb{R}), n \geq 3$), the holonomy map

not fixed

$$\Psi : \left\{ \begin{array}{l} \text{genus } g \text{ Riemann surfaces w.} \\ \text{conformal } G\text{-Higgs bundles} \\ \text{in Hitchin section} \end{array} \right\} / \sim \;\to \mathbf{Hit}(S_g, G)$$

constructed from holo. differentials

holo. quadratic diff. part =0

$$(E, \Phi) \mapsto [\rho]$$

holo. vector bundle on $\mathcal{T}_g$

restricts to an immersion on the cyclic locus

Labourie Conjecture: Ψ is injective (already known to be surjective).

Labourie Immersion Theorem & Labourie Conjecture

Labourie Immersion Theorem (2017)

Given any split real semisimple Lie group G of rank ≥ 2 (e.g. $\mathrm{SL}(n, \mathbb{R}), n \geq 3$), the holonomy map

not fixed

MCG-equivariant

$$\Psi : \left\{ \begin{array}{l} \text{genus } g \text{ Riemann surfaces w.} \\ \text{conformal } G\text{-Higgs bundles} \\ \text{in Hitchin section} \end{array} \right\} / \sim \; \to \mathbf{Hit}(S_g, G)$$

constructed from holo. differentials

holo. quadratic diff. part $=0$

$$(E, \Phi) \mapsto [\rho]$$

holo. vector bundle on $\mathcal{T}_g$

restricts to an immersion on the cyclic locus

Labourie Conjecture: Ψ is injective (already known to be surjective).

Labourie Immersion Theorem & Labourie Conjecture

Labourie Immersion Theorem (2017)

Given any split real semisimple Lie group G of rank ≥ 2 (e.g. $\mathrm{SL}(n,\mathbb{R}), n \geq 3$), the holonomy map

not fixed

MCG-equivariant

$$\Psi : \left\{ \begin{array}{l} \text{genus } g \text{ Riemann surfaces w.} \\ \text{conformal } G\text{-Higgs bundles} \\ \text{in Hitchin section} \end{array} \right\} / \sim \; \to \mathbf{Hit}(S_g, G)$$

constructed from holo. differentials

holo. quadratic diff. part =0

$$(E, \Phi) \mapsto [\rho]$$

holo. vector bundle on $\mathcal{T}_g$

restricts to an immersion on the cyclic locus (i.e. locus where only the highest holomorphic differential is nonzero).

Labourie Conjecture: Ψ is injective (already known to be surjective).

Labourie Immersion Theorem & Labourie Conjecture

Labourie Immersion Theorem (2017)

Given any split real semisimple Lie group G of rank ≥ 2 (e.g. $\mathrm{SL}(n,\mathbb{R})$, $n \geq 3$), the holonomy map

not fixed

MCG-equivariant

$$\Psi : \left\{ \begin{matrix} \text{genus } g \text{ Riemann surfaces w.} \\ \text{conformal } G\text{-Higgs bundles} \\ \text{in Hitchin section} \end{matrix} \right\} / \sim \;\to\; \mathbf{Hit}(S_g, G)$$

constructed from holo. differentials

holo. quadratic diff. part =0

$$(E, \Phi) \mapsto [\rho]$$

holo. vector bundle on $\mathcal{T}_g$

restricts to an immersion on the cyclic locus (i.e. locus where only the highest holomorphic differential is nonzero).

Labourie Conjecture: Ψ is injective (already known to be surjective).

Corollary (proof of Labourie Conjecture in rank 2)

If G has rank 2 (the whole domain of Ψ is cyclic), then Ψ is injective.

Labourie Immersion Theorem & Labourie Conjecture

Labourie Immersion Theorem (2017)

Given any split real semisimple Lie group G of rank ≥ 2 (e.g. $\mathrm{SL}(n,\mathbb{R})$, $n \geq 3$), the holonomy map

not fixed

MCG-equivariant

$$\Psi : \left\{ \begin{array}{l} \text{genus } g \text{ Riemann surfaces w.} \\ \text{conformal } G\text{-Higgs bundles} \\ \text{in Hitchin section} \end{array} \right\} / \sim \; \to \mathbf{Hit}(S_g, G)$$

constructed from holo. differentials

holo. quadratic diff. part $=0$

$$(E, \Phi) \mapsto [\rho]$$

holo. vector bundle on $\mathcal{T}_g$

restricts to an immersion on the cyclic locus (i.e. locus where only the highest holomorphic differential is nonzero).

Labourie Conjecture: Ψ is injective (already known to be surjective).

Corollary (proof of Labourie Conjecture in rank 2)

If G has rank 2 (the whole domain of Ψ is cyclic), then Ψ is injective.

Markovic, Markovic-Sagman-Smillie, Sagman-Smillie: disproved the conjecture for rank ≥ 3.

Strategy for LIT through infinitesimal rigidity

- Given (E, Φ) in the domain of Ψ, consider the conformal minimal immersion $f : \widetilde{\Sigma} \to G/K$.

Strategy for LIT through infinitesimal rigidity

- Given (E, Φ) in the domain of Ψ, consider the conformal minimal immersion $f : \widetilde{\Sigma} \to G/K$.

Lemma

If the quotient closed minimal surface in $\rho(\pi_1(S_g))\backslash(G/K)$ given by f is infinitesimally rigid, then $\mathrm{d}\Psi$ is nondegenerate at (E, Φ).

Strategy for LIT through infinitesimal rigidity

- Given (E, Φ) in the domain of Ψ, consider the conformal minimal immersion $f : \widetilde{\Sigma} \to G/K$.

Lemma

If the quotient closed minimal surface in $\rho(\pi_1(S_g))\backslash(G/K)$ given by f is infinitesimally rigid, then $\mathrm{d}\Psi$ is nondegenerate at (E, Φ).

Proof sketch. If degenerate, then $\exists$ path (E_t, Φ_t) s.t. (ρ_t) is stationary.

Strategy for LIT through infinitesimal rigidity

- Given (E, Φ) in the domain of Ψ, consider the conformal minimal immersion $f : \widetilde{\Sigma} \to G/K$.

Lemma

If the quotient closed minimal surface in $\rho(\pi_1(S_g))\backslash(G/K)$ given by f is infinitesimally rigid, then $\mathrm{d}\Psi$ is nondegenerate at (E, Φ).

Proof sketch. If degenerate, then $\exists$ path (E_t, Φ_t) s.t. (ρ_t) is stationary.

Strategy for LIT through infinitesimal rigidity

- Given (E, Φ) in the domain of Ψ, consider the conformal minimal immersion $f : \widetilde{\Sigma} \to G/K$.

Lemma

If the quotient closed minimal surface in $\rho(\pi_1(S_g))\backslash(G/K)$ given by f is infinitesimally rigid, then $\mathrm{d}\Psi$ is nondegenerate at (E, Φ).

Proof sketch. If degenerate, then $\exists$ path (E_t, Φ_t) s.t. (ρ_t) is stationary. Equivariance:
$f_t(\gamma.x) = \rho_t(\gamma) f_t(x)$

Strategy for LIT through infinitesimal rigidity

- Given (E, Φ) in the domain of Ψ, consider the conformal minimal immersion $f : \widetilde{\Sigma} \to G/K$.

Lemma

If the quotient closed minimal surface in $\rho(\pi_1(S_g))\backslash(G/K)$ given by f is infinitesimally rigid, then $\mathrm{d}\Psi$ is nondegenerate at (E, Φ).

Proof sketch. If degenerate, then $\exists$ path (E_t, Φ_t) s.t. (ρ_t) is stationary. Equivariance:

$$f_t(\gamma.x) = \rho_t(\gamma) f_t(x) \;\Rightarrow\; \dot{f}_0(\gamma.x) = \dot{\rho}_0(\gamma) f_0(x) + \rho_0(\gamma)\dot{f}_0(x)$$

Strategy for LIT through infinitesimal rigidity

- Given (E, Φ) in the domain of Ψ, consider the conformal minimal immersion $f : \widetilde{\Sigma} \to G/K$.

Lemma

If the quotient closed minimal surface in $\rho(\pi_1(S_g))\backslash(G/K)$ given by f is infinitesimally rigid, then $\mathrm{d}\Psi$ is nondegenerate at (E, Φ).

Proof sketch. If degenerate, then $\exists$ path (E_t, Φ_t) s.t. (ρ_t) is stationary. Equivariance:

$$f_t(\gamma.x) = \rho_t(\gamma) f_t(x) \;\Rightarrow\; \dot{f}_0(\gamma.x) = \cancel{\dot{\rho}_0(\gamma)} f_0(x) + \rho_0(\gamma)\dot{f}_0(x)$$

Strategy for LIT through infinitesimal rigidity

- Given (E, Φ) in the domain of Ψ, consider the conformal minimal immersion $f : \widetilde{\Sigma} \to G/K$.

Lemma

If the quotient closed minimal surface in $\rho(\pi_1(S_g))\backslash(G/K)$ given by f is infinitesimally rigid, then $\mathrm{d}\Psi$ is nondegenerate at (E, Φ).

Proof sketch. If degenerate, then $\exists$ path (E_t, Φ_t) s.t. (ρ_t) is stationary. Equivariance:

$$f_t(\gamma.x) = \rho_t(\gamma)f_t(x) \;\Rightarrow\; \dot{f}_0(\gamma.x) = \cancel{\dot{\rho}_0(\gamma)}f_0(x) + \rho_0(\gamma)\dot{f}_0(x)$$

This means $\dot{f}_0$ is a Jacobi field on the quotient minimal surface, contradicting rigidity. $\square$

Strategy for LIT through infinitesimal rigidity

- Given (E, Φ) in the domain of Ψ, consider the conformal minimal immersion $f : \widetilde{\Sigma} \to G/K$.

Lemma

If the quotient closed minimal surface in $\rho(\pi_1(S_g))\backslash(G/K)$ given by f is infinitesimally rigid, then $d\Psi$ is nondegenerate at (E, Φ).

Proof sketch. If degenerate, then $\exists$ path (E_t, Φ_t) s.t. (ρ_t) is stationary. Equivariance:

$$f_t(\gamma.x) = \rho_t(\gamma) f_t(x) \Rightarrow \dot{f}_0(\gamma.x) = \cancel{\dot{\rho}_0(\gamma)} f_0(x) + \rho_0(\gamma)\dot{f}_0(x)$$

This means $\dot{f}_0$ is a Jacobi field on the quotient minimal surface, contradicting rigidity. □

- However, nobody knows how to show such a rigidity result...

Strategy for LIT through infinitesimal rigidity

- ... nevertheless, one can replace f by some other surfaces with nice rigidity property and apply the same strategy:

Strategy for LIT through infinitesimal rigidity

- ... nevertheless, one can replace f by some other surfaces with nice rigidity property and apply the same strategy:
  - (Labourie 2007, Loftin 2004) hyperbolic affine spheres in $\mathbb{R}^3$ $\rightsquigarrow$ LIT for $\mathrm{SL}(3,\mathbb{R})$;

Strategy for LIT through infinitesimal rigidity

- … nevertheless, one can replace f by some other surfaces with nice rigidity property and apply the same strategy:
  - (Labourie 2007, Loftin 2004) hyperbolic affine spheres in $\mathbb{R}^3$ $\rightsquigarrow$ LIT for $\mathrm{SL}(3, \mathbb{R})$;
  - (Labourie 2017) cyclic surfaces on some homogeneous space of $G_{\mathbb{C}}$ $\rightsquigarrow$ LIT for any rank 2 split G;

Strategy for LIT through infinitesimal rigidity

- ... nevertheless, one can replace f by some other surfaces with nice rigidity property and apply the same strategy:
  - (Labourie 2007, Loftin 2004) hyperbolic affine spheres in $\mathbb{R}^3$ $\rightsquigarrow$ LIT for $\mathrm{SL}(3,\mathbb{R})$;
  - (Labourie 2017) cyclic surfaces on some homogeneous space of $G_{\mathbb{C}}$ $\rightsquigarrow$ LIT for any rank 2 split G;
  - (Collier-Tholozan-Toulisse 2019) maximal surfaces in $\mathbb{H}^{2,n}$ $\rightsquigarrow$ generalization of Labourie Conjecture to maximal representations in $\mathrm{SO}_0(2,n+1)$

Strategy for LIT through infinitesimal rigidity

- ... nevertheless, one can replace f by some other surfaces with nice rigidity property and apply the same strategy:
  - (Labourie 2007, Loftin 2004) hyperbolic affine spheres in $\mathbb{R}^3$ $\rightsquigarrow$ LIT for $\mathrm{SL}(3,\mathbb{R})$;
  - (Labourie 2017) cyclic surfaces on some homogeneous space of $G_{\mathbb{C}}$ $\rightsquigarrow$ LIT for any rank 2 split G;
  - (Collier-Tholozan-Toulisse 2019) maximal surfaces in $\mathbb{H}^{2,n}$ $\rightsquigarrow$ generalization of Labourie Conjecture to maximal representations in $\mathrm{SO}_0(2,n+1)$
  - (N. 2022) A-surfaces $\rightsquigarrow$ LIT for Collier's component $\mathbf{Col}_d(S_g,\mathrm{SO}_0(n,n+1))$ with $d \neq 0$

Strategy for LIT through infinitesimal rigidity

- ... nevertheless, one can replace f by some other surfaces with nice rigidity property and apply the same strategy:
  - (Labourie 2007, Loftin 2004) hyperbolic affine spheres in $\mathbb{R}^3$ $\rightsquigarrow$ LIT for $\mathrm{SL}(3,\mathbb{R})$;
  - (Labourie 2017) cyclic surfaces on some homogeneous space of $G_{\mathbb{C}}$ $\rightsquigarrow$ LIT for any rank 2 split G;
  - (Collier-Tholozan-Toulisse 2019) maximal surfaces in $\mathbb{H}^{2,n}$ $\rightsquigarrow$ generalization of Labourie Conjecture to maximal representations in $\mathrm{SO}_0(2,n+1)$
  - (N. 2022) A-surfaces $\rightsquigarrow$ LIT for Collier's component $\mathbf{Col}_d(S_g,\mathrm{SO}_0(n,n+1))$ with $d \neq 0$ ($\Rightarrow$ new proof of Labourie Conjecture for G_2').

Strategy for LIT through infinitesimal rigidity

- ... nevertheless, one can replace f by some other surfaces with nice rigidity property and apply the same strategy:
  - (Labourie 2007, Loftin 2004) hyperbolic affine spheres in $\mathbb{R}^3$ $\rightsquigarrow$ LIT for $\mathrm{SL}(3,\mathbb{R})$;
  - (Labourie 2017) cyclic surfaces on some homogeneous space of $G_{\mathbb{C}}$ $\rightsquigarrow$ LIT for any rank 2 split G;
  - (Collier-Tholozan-Toulisse 2019) maximal surfaces in $\mathbb{H}^{2,n}$ $\rightsquigarrow$ generalization of Labourie Conjecture to maximal representations in $\mathrm{SO}_0(2,n+1)$
  - (N. 2022) A-surfaces $\rightsquigarrow$ LIT for Collier's component $\mathbf{Col}_d(S_g,\mathrm{SO}_0(n,n+1))$ with $d \neq 0$ ($\Rightarrow$ new proof of Labourie Conjecture for G_2').

Strategy for LIT through infinitesimal rigidity

- ... nevertheless, one can replace f by some other surfaces with nice rigidity property and apply the same strategy:
  - (Labourie 2007, Loftin 2004) hyperbolic affine spheres in $\mathbb{R}^3$ $\rightsquigarrow$ LIT for $\mathrm{SL}(3,\mathbb{R})$;
  - (Labourie 2017) cyclic surfaces on some homogeneous space of $G_{\mathbb{C}}$ $\rightsquigarrow$ LIT for any rank 2 split G;
  - (Collier-Tholozan-Toulisse 2019) maximal surfaces in $\mathbb{H}^{2,n}$ $\rightsquigarrow$ generalization of Labourie Conjecture to maximal representations in $\mathrm{SO}_0(2,n+1)$
  - (N. 2022) A-surfaces $\rightsquigarrow$ LIT for Collier's component $\mathbf{Col}_d(S_g,\mathrm{SO}_0(n,n+1))$ with $d\neq 0$ ($\Rightarrow$ new proof of Labourie Conjecture for G_2').

Remark. The Min-Max Theorem requires a negativity assumption on some Chern forms.

Strategy for LIT through infinitesimal rigidity

- ... nevertheless, one can replace f by some other surfaces with nice rigidity property and apply the same strategy:
  - (Labourie 2007, Loftin 2004) hyperbolic affine spheres in $\mathbb{R}^3$ $\rightsquigarrow$ LIT for $\mathrm{SL}(3,\mathbb{R})$;
  - (Labourie 2017) cyclic surfaces on some homogeneous space of $G_{\mathbb{C}}$ $\rightsquigarrow$ LIT for any rank 2 split G;
  - (Collier-Tholozan-Toulisse 2019) maximal surfaces in $\mathbb{H}^{2,n}$ $\rightsquigarrow$ generalization of Labourie Conjecture to maximal representations in $\mathrm{SO}_0(2,n+1)$
  - (N. 2022) A-surfaces $\rightsquigarrow$ LIT for Collier's component $\mathbf{Col}_d(S_g,\mathrm{SO}_0(n,n+1))$ with $d\neq 0$ ($\Rightarrow$ new proof of Labourie Conjecture for G_2').

Remark. The Min-Max Theorem requires a negativity assumption on some Chern forms. In the above application, this is guaranteed by:

Strategy for Labourie Immersion Theorem

Lemma (Dai-Li, N.)

Let $h_1, \cdots, h_n$ be hermitian metrics on holo. line bundles $L_1, \cdots, L_n$ over a closed Riemann surface solving the Hitchin eqn. for

$$E = L_n^{-1} \oplus \cdots \oplus L_2^{-1} \oplus L_1^{-1} \oplus \mathcal{O} \oplus L_1 \oplus L_2 \oplus \cdots \oplus L_n,$$

$$\Phi = \left(\begin{array}{ccccc|c|ccccc}
0 & & & & & & & & & \boldsymbol{\beta} & \\
\boldsymbol{\alpha}_n & 0 & & & & & & & & & \boldsymbol{\beta} \\
 & \boldsymbol{\alpha}_{n-1} & \ddots & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & \\
 & & & \boldsymbol{\alpha}_2 & 0 & & & & & & \\
\hline
 & & & & \boldsymbol{\alpha}_1 & 0 & & & & & \\
\hline
 & & & & & \boldsymbol{\alpha}_1 & 0 & & & & \\
 & & & & & & \boldsymbol{\alpha}_2 & \ddots & & & \\
 & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & \boldsymbol{\alpha}_{n-1} & 0 & \\
 & & & & & & & & & \boldsymbol{\alpha}_n & 0
\end{array}\right).$$

Suppose $(\boldsymbol{\alpha}_1) \prec (\boldsymbol{\alpha}_2) \prec \cdots \prec (\boldsymbol{\alpha}_{n-1}) \prec \min\{(\boldsymbol{\alpha}_n), (\boldsymbol{\beta})\}$. Then $\partial^2_{z\bar{z}} h_i \leq 0$ for $i = 2, \cdots, n$, with strict inequality except at the obvious zeros of $\partial^2_{z\bar{z}} h_i$.

Strategy for Labourie Immersion Theorem

Lemma (Dai-Li, N.)

Let $h_1, \cdots, h_n$ be hermitian metrics on holo. line bundles $L_1, \cdots, L_n$ over a closed Riemann surface solving the Hitchin eqn. for

$$E = L_n^{-1} \oplus \cdots \oplus L_2^{-1} \oplus L_1^{-1} \oplus \mathcal{O} \oplus L_1 \oplus L_2 \oplus \cdots \oplus L_n,$$

$$\Phi = \left(\begin{array}{ccccc|c|ccccc}
0 & & & & & & & & & \boldsymbol{\beta} & \\
\boldsymbol{\alpha}_n & 0 & & & & & & & & & \boldsymbol{\beta} \\
 & \boldsymbol{\alpha}_{n-1} & \ddots & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & \\
 & & & \boldsymbol{\alpha}_2 & 0 & & & & & & \\
\hline
 & & & & \boldsymbol{\alpha}_1 & 0 & & & & & \\
\hline
 & & & & & \boldsymbol{\alpha}_1 & 0 & & & & \\
 & & & & & & \boldsymbol{\alpha}_2 & \ddots & & & \\
 & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & \boldsymbol{\alpha}_{n-1} & 0 & \\
 & & & & & & & & & \boldsymbol{\alpha}_n & 0
\end{array}\right).$$

Suppose $\overbrace{(\boldsymbol{\alpha}_1)}^{\text{divisor of } \alpha_1} \prec (\boldsymbol{\alpha}_2) \prec \cdots \prec (\boldsymbol{\alpha}_{n-1}) \prec \min\{(\boldsymbol{\alpha}_n), (\boldsymbol{\beta})\}$. Then $\partial^2_{z\bar{z}} h_i \leq 0$ for $i = 2, \cdots, n$, with strict inequality except at the obvious zeros of $\partial^2_{z\bar{z}} h_i$.

Strategy for Labourie Immersion Theorem

Lemma (Dai-Li, N.)

Let $h_1, \cdots, h_n$ be hermitian metrics on holo. line bundles $L_1, \cdots, L_n$ over a closed Riemann surface solving the Hitchin eqn. for

$$E = L_n^{-1} \oplus \cdots \oplus L_2^{-1} \oplus L_1^{-1} \oplus \mathcal{O} \oplus L_1 \oplus L_2 \oplus \cdots \oplus L_n,$$

$$\Phi = \left(\begin{array}{ccccc|c|ccccc}
0 & & & & & & & & & \boldsymbol{\beta} & \\
\boldsymbol{\alpha}_n & 0 & & & & & & & & & \boldsymbol{\beta} \\
 & \boldsymbol{\alpha}_{n-1} & \ddots & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & \\
 & & & \boldsymbol{\alpha}_2 & 0 & & & & & & \\
\hline
 & & & & \boldsymbol{\alpha}_1 & 0 & & & & & \\
\hline
 & & & & & \boldsymbol{\alpha}_1 & 0 & & & & \\
 & & & & & & \boldsymbol{\alpha}_2 & \ddots & & & \\
 & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & \boldsymbol{\alpha}_{n-1} & 0 & \\
 & & & & & & & & & \boldsymbol{\alpha}_n & 0
\end{array}\right).$$

$<$ or both empty

Suppose $\overbrace{(\boldsymbol{\alpha}_1)}^{\text{divisor of } \alpha_1} \prec (\boldsymbol{\alpha}_2) \prec \cdots \prec (\boldsymbol{\alpha}_{n-1}) \prec \min\{(\boldsymbol{\alpha}_n), (\boldsymbol{\beta})\}$. Then $\partial^2_{z\bar{z}} h_i \leq 0$ for $i = 2, \cdots, n$, with strict inequality except at the obvious zeros of $\partial^2_{z\bar{z}} h_i$.

Strategy for Labourie Immersion Theorem

Lemma (Dai-Li, N.)

Let $h_1, \cdots, h_n$ be hermitian metrics on holo. line bundles $L_1, \cdots, L_n$ over a closed Riemann surface solving the Hitchin eqn. for

$$E = L_n^{-1} \oplus \cdots \oplus L_2^{-1} \oplus L_1^{-1} \oplus \mathcal{O} \oplus L_1 \oplus L_2 \oplus \cdots \oplus L_n,$$

holds in par. when $\alpha_1 = \cdots = \alpha_{n-1} = 1$ (Collier components)

$$\Phi = \left(\begin{array}{ccccc|c|ccccc}
0 &&&&&&&&& \boldsymbol{\beta} & \\
\boldsymbol{\alpha}_n & 0 &&&&&&&&& \boldsymbol{\beta} \\
& \boldsymbol{\alpha}_{n-1} & \ddots &&&&&&&& \\
&& \ddots & \ddots &&&&&&& \\
&&& \boldsymbol{\alpha}_2 & 0 &&&&&& \\
\hline
&&&& \boldsymbol{\alpha}_1 & 0 &&&&& \\
\hline
&&&&& \boldsymbol{\alpha}_1 & 0 &&&& \\
&&&&&& \boldsymbol{\alpha}_2 & \ddots &&& \\
&&&&&&& \ddots & \ddots && \\
&&&&&&&& \boldsymbol{\alpha}_{n-1} & 0 & \\
&&&&&&&&& \boldsymbol{\alpha}_n & 0
\end{array}\right).$$

$<$ or both empty

divisor of α_1

Suppose $\boxed{(\boldsymbol{\alpha}_1) \prec (\boldsymbol{\alpha}_2) \prec \cdots \prec (\boldsymbol{\alpha}_{n-1}) \prec \min\{(\boldsymbol{\alpha}_n), (\boldsymbol{\beta})\}.}$ Then $\partial^2_{z\bar{z}} h_i \le 0$ for $i = 2, \cdots, n$, with strict inequality except at the obvious zeros of $\partial^2_{z\bar{z}} h_i$.

Thank you for your attention!