《Normal generation vs small translation length》

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S closed orientable surface of genus $g \geq 2$

$\mathrm{Mod}(S)$ = mapping class group

$= \mathrm{Homeo}^+(S) / \text{homotopy}$

$\mathrm{Mod}(S)$ action $\curvearrowright X$

where X is either $C(S)$ curve graph

Vertices = free homotopy classes of simple closed curves on S

edge = disjoint representatives

or $T(S)$ Teichmüller space

$= \{ (f, X) : \underset{\text{homeo}}{f : S} \to \underset{\text{hyperbolic surface}}{X} \} / \sim$

$$\begin{tikzcd} & X \arrow[d, "\exists h \text{ isometry}"] \\ S \arrow[ur, "f"] \arrow[r, "g"'] & Y \end{tikzcd} \quad \circlearrowleft \text{ up to homotopy} \qquad (f, X) \sim (g, Y).$$

$g \in G \curvearrowright (X, d)$

$\tau_X(g) = \inf_{x \in X} d(x, g \cdot x)$ ← translation length

n^{th} iterate.

$\ell_X(g) = \lim_{n \to \infty} \frac{d(x, g^n x)}{n}$ ← stable translation length.

$\tau_X(g) \geq \ell_X(g)$

$\ell_X(g^n) = n \cdot \ell_X(g)$

Conj Let $f \in \mathrm{Mod}(S)$ be pseudo-Anosov.

If $\ell_X(f)$ is small enough,

then $\langle\langle f \rangle\rangle = \mathrm{Mod}(S)$.

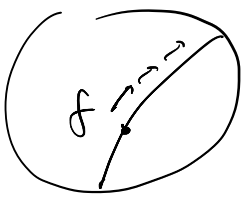
normal closure

Thm (Lanier–Margalit) $S = S_g$, $g \geq 3$

$f \in \mathrm{Mod}(S_g)$ pseudo-Anosov.

If $\ell_T(f) \leq \log \sqrt{2}$,

then $\langle\langle f \rangle\rangle = \mathrm{Mod}(S)$.



L-M criterion

$f \in \mathrm{Mod}(S_g)$

s.c.c.

$N_f(S) :=$ graph of non-separating curves on S

s/t $\alpha - \beta$ iff $\exists\, h \in \mathrm{Mod}(S)$ s/t

$hfh^{-1}(\alpha) = \beta$.

Thm (L-M). If $N_f(S)$ is connected,

then $\langle\langle f \rangle\rangle = \mathrm{Mod}(S_g)$.

Step 1 Let c, d be non-separating curves in S_g

with $i(c, d) = 1$.

Th of Lickorish

$\exists$ non-sep. curves $c_1, \dots, c_{3g-3}$ in S_g

s.t. $i(c_i, c_j) \le 1$,

$T_{c_1}, \dots, T_{c_{3g-3}}$ generate $\mathrm{Mod}(S_g)$.

$\Downarrow$

$[\mathrm{Mod}(S_g), \mathrm{Mod}(S_g)]$ is normally gen'd by

various $[T_{c_i}, T_{c_j}]$.

Claim $[\mathrm{Mod}(S_g), \mathrm{Mod}(S_g)] \le \langle\langle T_c T_d^{-1} \rangle\rangle$

obs $\exists$ homeo h s.t.

$$h \quad c_i \mapsto c$$
$$c_j \mapsto d$$

$$T_c = T_{h(c_i)} = h T_{c_i} h^{-1}$$

$$T_d = T_{h(c_j)} = h T_{c_j} h^{-1}$$

$$[T_{c_i}, T_{c_j}] = \dots = h^{-1} [T_c, T_d] h$$

$\Rightarrow [\mathrm{Mod}(S_j), \mathrm{Mod}(S_j)] \leq \langle\langle [T_c, T_a] \rangle\rangle$.

$\leq \langle\langle T_c T_a^{-1} \rangle\rangle$

$\mathrm{Mod}(S_j) \longrightarrow \mathrm{Mod}(S_j) / \langle\langle T_c T_a^{-1} \rangle\rangle$

Step 2. Suppose $\exists$ non-sep. curve c

so that $i(c, f(c)) = 1$.

Claim $[\mathrm{Mod}(S_j), \mathrm{Mod}(S_j)] \leq \langle\langle f \rangle\rangle$.

$[\mathrm{Mod}(S), \mathrm{Mod}(S)] \leq \underbrace{\langle\langle T_c T_{f(c)}^{-1} \rangle\rangle}_{\leq \langle\langle f \rangle\rangle}$.

$T_{f(c)} = f T_c f^{-1}$

$T_c T_{f(c)}^{-1} = T_c f T_c^{-1} f^{-1} \in \langle\langle f \rangle\rangle$.

Step 3 $N_f(S)$

Let c, d any non-sep. with $i(c, d) = 1$.

$c_0 = c - c_1 - c_2 - \cdots - c_n = d$.

For each $0 \le i \le n-1$, $\exists$ a conjugate f_i of f

s.t. $\underline{f_i(c_i) = c_{i+1}}$.

$$\underline{d} = \underbrace{f_{n-1} \cdot \cdots \cdot f_0}\,(\underline{c})$$

$$[\mathrm{Mod}(S), \mathrm{Mod}(S)] \le \underbrace{\langle\langle f_{n-1} \cdots f_0 \rangle\rangle} \le \langle\langle f \rangle\rangle$$

$\square$

Lemma (Lei Chen) $g \ge 0$ $f \in \mathrm{Mod}(S_g)$

Suppose $\exists$ non sep. curves c & d s.t.

$$i(\underline{c}, d) = 1 \quad \& \quad i(f(\underline{c}), d) = 0.$$

Then $[\mathrm{Mod}(S), \mathrm{Mod}(S)] \le \langle\langle f \rangle\rangle$.

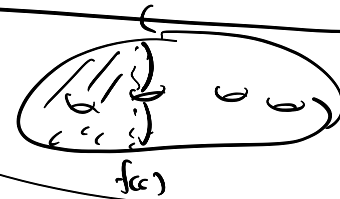
$$[T_c, f] = T_c \underbrace{f \, T_c^{-1} f^{-1}} \in \langle\langle f \rangle\rangle$$

$$= \underline{T_c}\, \underline{T_{f(c)}^{-1}}$$

$$i(d, T_c \cancel{T_{f(c)}^{-1}}(d)) = i(d, \underline{T_c(d)}) = 1$$

Step 2 $\Rightarrow$ $[\mathrm{Mod}(S), \mathrm{Mod}(S)] \le \langle\langle [T_c, f] \rangle\rangle \le \langle\langle f \rangle\rangle$

Lem $g \geq 3$ $f \in \mathrm{Mod}(S_g)$



Suppose $\exists$ non-sep c s/t

$i(c, f(c)) = 0$ & $c, f(c)$ do $\boxed{\text{not}}$ bound a subsurface

Then $[\mathrm{Mod}(S), \mathrm{Mod}(S)] \leq \langle\langle f \rangle\rangle$.

Fact $g \geq 3$.

$$[\mathrm{Mod}(S_g), \mathrm{Mod}(S_g)] = \mathrm{Mod}(S_g)$$

Two directions from Th of L-M.

p-A small transl. length of $\mathcal{T}(S)$

$\Rightarrow$ normal gen.

Q1. Is "it" true for $\mathcal{C}(S)$? (Margalit)

Q2. Can we generalize it to certain reducible elts?

$H \leq \mathrm{Mod}(S_g)$

$\boxed{L_X(H)} = \inf \{ \ell_X(f) \mid f \in H,\ f \text{ is p-A} \}$

<u>L-M</u> $\quad L_{\mathcal{T}}(H) \geq \log \sqrt{2}$ for any $H \trianglelefteq_{\neq} \mathrm{Mod}(S_g)$.

<u>Known results</u>

$L_{\mathcal{T}}(k, g) \asymp \frac{k+1}{g}$ Agol-Leininger-Margalit

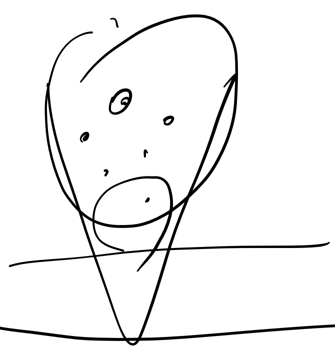
$L_{\mathcal{T}}(\mathrm{Mod}(S_g)) \asymp \frac{1}{g}$ (Penner)

$L_{\mathcal{T}}(\mathcal{I}_g) \asymp 1$ (Farb-Leininger-Margalit)

$L_C(\mathrm{Mod}(S_g)) \asymp \frac{1}{g^2}$ (Kin-Shin)

$L_C(\mathcal{I}_g) \gtrsim \frac{1}{g}$ (B-Shin)

conj $\frac{k+1}{g^2}$



Q2 **Def** $f \in \mathrm{Mod}(S_g)$ is called partly pseudo-Anosov

if $\exists$ embedded subsurface A of S

s.t. A is inv. under f & $f|_A$ is p-A.

Thm (B-Kim-Wu) $S = S_g$, $g \geq 3$

$f \in \mathrm{Mod}(S_g)$ partly p-A s.t. inv. subsurface A

has genus ≥ 3

If $l_T(f) \leq \log \sqrt{2}$, then $\langle\langle f \rangle\rangle = \mathrm{Mod}(S)$.

Idea Minsky's Product region thm '96

Γ multi-curve $\sim S$

$$T(S) \longrightarrow T(S \setminus \Gamma) \times \prod_{\gamma \in \Gamma} \mathbb{H}^2$$

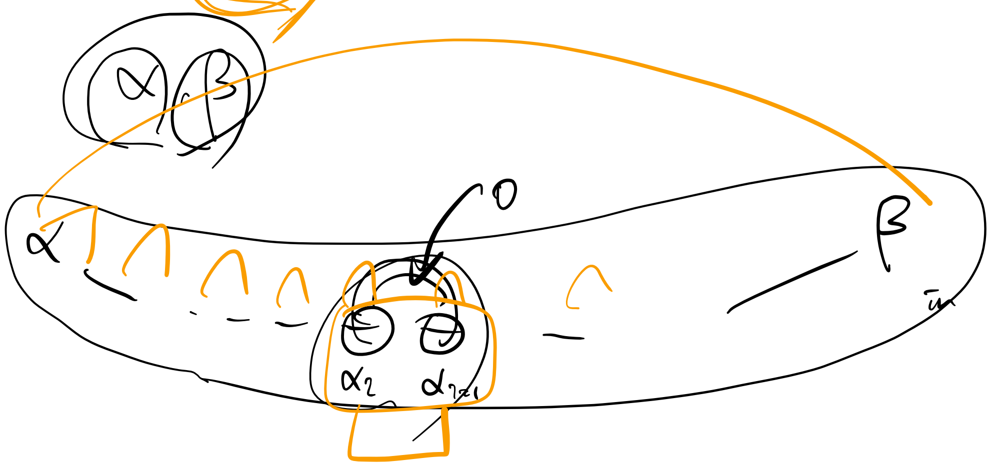
$$l_{T(A \setminus \partial A)}(f|_A) \leq \log \sqrt{2}$$

$\langle\langle f|_A \rangle\rangle_{\mathrm{Mod}(A)}$ contains $[\mathrm{Mod}(A), \mathrm{Mod}(A)]$

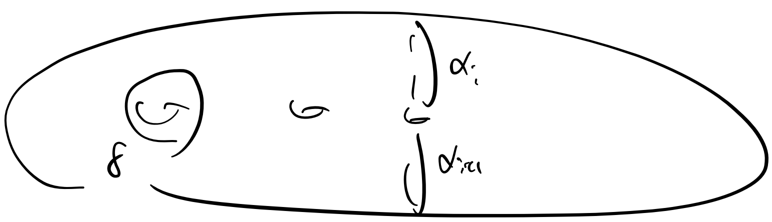
Want to promote

$\langle\langle f \rangle\rangle_{Mod(S)} \quad \cdot\cdot \quad [Mod(S), Mod(S)]$

We show $N_f(S)$ is connected.



in $C(S)$



$\alpha_i - \alpha_{i+1} \qquad \alpha_i - \delta - \alpha_{i+1}$

PMod(A) is perfect (by Powell's thm)