Hitchin represenations and mimimal surfaces

Nathaniel Sagman (joint work with P.Smillie and V.Marković-P.Smillie)

University of Luxembourg

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Outline

- Hitchin representations and the Hitchin base

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- Minimal surfaces and Labourie's Conjecture

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- Proofs

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$$\{\text{A component of discrete faithful representations}\} \subset \mathrm{Rep}(\mathrm{PSL}(2,\mathbb{R})).$$

- Basic facts: $\mathbf{T}_g \simeq_{C^\infty} \mathbb{R}^{6g-6}$, $\mathbf{T}_g$ has a complex structure, the mapping class group $MCG(\Sigma_g)$ acts properly discontinuously.

- When unspecified, G is a semisimple real Lie group with no compact factors with finite center, i.e., $G = G_1 \times \cdots \times G_m$, with each G_i simple, non-compact type, and finite center.

Definition (Wienhard)

A higher Teichmüller space is a union of connected components of $\mathrm{Rep}(G)$ consisting entirely of discrete and faithful representations.

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- $\mathrm{Hit}(G)$ is homeomorphic to $\mathbb{R}^{(2g-2)\dim(G)}$ (Hitchin, 1990), the Hitchin component is a higher Teichmüller space (Labourie, Fock-Goncharov, 2006), and MCG acts properly discontinuously (Labourie, 2006).

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- From now on, $\mathrm{Hit}(n) = \mathrm{Hit}(\mathrm{PSL}(n, \mathbb{R}))$.

Definition

The Hitchin base for $\mathrm{PSL}(n, \mathbb{C})$ is $B(S) = \oplus_{i=2}^{n} H^0(S, \mathcal{K}^i) \simeq_{C^\infty} \mathbb{C}^{\dim G(g-1)}$.

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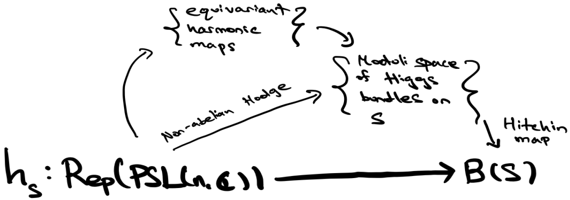
- A section of $\mathcal{K}^i$ is a tensor q locally of the form $q = q(z)dz^i$, $q(z)$ holomorphic.
- There is a surjective map



Hitchin section $b_S : B(S) \to \mathrm{Rep}(\mathrm{PSL}(n, \mathbb{C}))$ such that $b_S(B(S)) = \mathrm{Hit}(n)$.

More on the Hitchin base for $\mathrm{PSL}(n,\mathbb{C})$

Let

$$\mathfrak{t}^{\mathbb{C}} = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} : a_i \in \mathbb{C}, \sum a_i = 0 \right\}, \quad \text{(Cartan subalgebra)}.$$

A point in $B(S)$ is equivalent to

- a finite degree branched covering $S_B \to S$ (the cameral cover) whose deck group has a faithful representation $\eta : \mathrm{Deck}(S_B) \to S_n$ (the Weyl group).
- On S_B, a η-equivariant holomorphic 1-form with values in the trivial $\mathfrak{t}^{\mathbb{C}}$-bundle (satisfying a condition relative to the branch points).

The action of $\alpha \in S_n$ on the $\mathfrak{t}^{\mathbb{C}}$-bundle is

$$\alpha \cdot \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = \begin{pmatrix} a_{\alpha(1)} & & \\ & \ddots & \\ & & a_{\alpha(n)} \end{pmatrix}.$$

That is, we have a matrix of 1-forms on a cover S_B, whose entries are permuted by the Deck group action.

In one direction.

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- Let $\sigma_2, \dots, \sigma_n$ be the elementary symmetric polynomials, acting on ϕ by
$$\sigma_2(\phi) = \sum_{j<k} \phi_j \phi_k, \sigma_3(\phi) = \sum_{j<k<\ell} \phi_j \phi_k \phi_\ell, \dots, \sigma_n(\phi) = \prod_{j=1}^{n} \phi_j.$$

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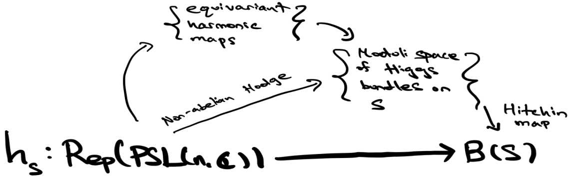
- $\sigma_i(\phi)$ defines a holomorphic i-differential on S_B.
- For any $\alpha \in S_n$, $\sigma_i(\alpha \cdot \phi) = \sigma_i(\phi)$. By equivariance, each σ_i descends to a holomorphic i-differential q_i on S.



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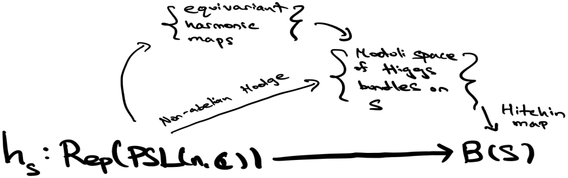

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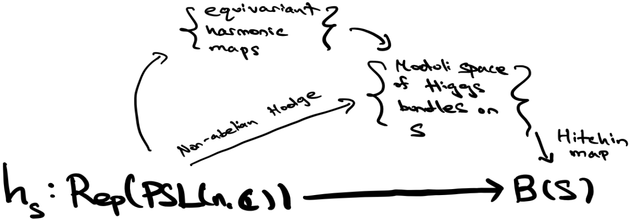
- The Hitchin section $b_S : \mathrm{B}(S) \to \mathrm{Hit}(n)$ associates the data of the cover S_B and the (abelian) section ϕ to a unique Hitchin representation.

Hitchin's parametrization for $G = \mathrm{PSL}(n, \mathbb{R})$



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Two issues with Hitchin's parametrization:

- No $\mathrm{MCG}(\Sigma_g)$ equivariance.
- In the induced complex structure, the inclusion $\mathbf{T}_g \to \mathrm{Hit}(n)$ is not holomorphic.

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- Given $[\rho] \in \mathrm{Rep}(G)$, we consider C^2 maps $h : \tilde{\Sigma_g} \to G/K$ that are ρ-equivariant: for all $\gamma \in \pi_1(\Sigma_g)$, $h \circ \gamma^{-1} = \rho(\gamma)h$.

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- Given $[\rho] \in \mathrm{Rep}(G)$, we consider C^2 maps $h : \tilde{\Sigma_g} \to G/K$ that are ρ-equivariant: for all $\gamma \in \pi_1(\Sigma_g)$, $h \circ \gamma^{-1} = \rho(\gamma)h$.
- Since ρ is acting by isometries on $(G/K, \nu)$, the pullback $h^*\mathrm{dvol}_\nu$ is $\pi_1(\Sigma_g)$-invariant, and hence descends to a 2-form on Σ_g.

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- For general admissible G, let K be a maximal compact subgroup. Then G/K admits a G-invariant metric ν, so that $(G/K, \nu)$ is a symmetric space of non-compact type.
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- Given $[\rho] \in \mathrm{Rep}(G)$, we consider C^2 maps $h : \tilde{\Sigma_g} \to G/K$ that are ρ-equivariant: for all $\gamma \in \pi_1(\Sigma_g)$, $h \circ \gamma^{-1} = \rho(\gamma)h$.
- Since ρ is acting by isometries on $(G/K, \nu)$, the pullback $h^*\mathrm{dvol}_\nu$ is $\pi_1(\Sigma_g)$-invariant, and hence descends to a 2-form on Σ_g.
- The area of the map is

$$\mathrm{Area}(h) = A(h) = \int_{\Sigma_g} h^*\mathrm{dvol}_\nu.$$

Definition

h is minimal if it is a critical point of A, i.e., for all smooth variations h_t through C^2 ρ-equivariant maps,

$$\frac{d}{dt}|_{t=0} A(h_t) = 0.$$

Theorem (Labourie)

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- **Fact:** for $G = \mathrm{PSL}(n, \mathbb{R})$, there exists a weakly conformal minimal map $h : \tilde{S} \to (G/K, \nu)$ if and only if $h_S(\rho) = (0, q_3, \ldots, q_n) \in \oplus_{i=2}^n H^0(S, \mathcal{K}^i)$, i.e., the q_2 term vanishes (or $\sigma_2(\phi) = 0$).

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Labourie Conjecture

Given $\rho : \pi_1(\Sigma_g) \to \mathrm{PSL}(n, \mathbb{R})$ Hitchin, there exists a unique ρ-equivariant minimal surface in the symmetric space.

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- At the time, known for $n = 2, 3$ (Labourie, Loftin).
- A positive resolution of the conjecture means that L provides a $\mathrm{MCG}(\Sigma_g)$-equivariant parametrization of $\mathrm{Hit}(n)$. Note also the inclusion $\mathbf{T}_g \to \mathbf{M}$ is holomorphic.

In general, recall G/K is a symmetric space, and let ν be a G-invariant metric.

Generalized Labourie Conjecture

Given $\rho : \pi_1(\Sigma_g) \to G$ in any higher Teichmüller space, there exists a unique minimal surface in $(G/K, \nu)$.

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The rank of G is the dimension of a maximal flat subspace of G/K. For example: $\mathrm{rank}(\mathrm{PSL}(n, \mathbb{R})) = n - 1$.

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Theorem (Labourie, Collier-Toulisse-Tholozan, Schoen, Wan)

The generalized Labourie conjecture holds for all Hitchin components and maximal components for Lie groups of rank 2.

Hitchin: $\mathrm{PSL}(3, \mathbb{R})$, $\mathrm{PSp}(4, \mathbb{R})$, G_2' (Labourie), Maximal: $\mathrm{PSL}(2, \mathbb{R})^2$ (Schoen, Wan), and Hermitian Lie groups like $\mathrm{PSO}_0(2, n)$ (Collier-Tholozan-Toulisse). See also work of Nie on G_2'.

One simpler example of a higher Teichmüller space is $\mathbf{T}_g^n \subset \mathrm{Rep}(\mathrm{PSL}(2,\mathbb{R})^n)$.

Theorem (Marković, 2021)

For all genus g large enough and $n \geq 3$, there exists product of n discrete faithful representations $\rho = (\rho_1, \ldots, \rho_n) : \pi_1(\Sigma_g) \to \mathrm{PSL}(2,\mathbb{R})^n$ with multiple minimal surfaces.

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Marković-S.-Smillie, 2022

New geometric proof of the above result, works for all genus $g \geq 2$.

Results

Theorem (S.-Smillie, 2022)

For all $g \geq 3$ and $n \geq 4$, there exists a Hitchin representation $\rho : \pi_1(\Sigma_g) \to \mathrm{PSL}(n, \mathbb{R})$ with multiple minimal surfaces. That is, Labourie's conjecture fails for $n \geq 4$.

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More generally, let G be simple split real or a product of simple split real Lie groups. We say that $\rho : \pi_1(\Sigma_g) \to G$ is Hitchin if each factor is Hitchin. Let ν be any G-invariant metric on G/K.

Theorem (S.-Smillie, 2022)

For all $g \geq 3$ and G as above with rank ≥ 3, there exists a Hitchin representation $\rho : \pi_1(\Sigma_g) \to G$ with multiple minimal surfaces in $(G/K, \nu)$.

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- The results hold for every known higher Teichmüller space such that the restriction of the Hitchin map is surjective.
- We can prove the same result as well for $g = 2$. Writing up.

Corollary

For $n \geq 4$, $L : \mathbf{M} \to \mathrm{Hit}(n)$ has no continuous section.

Corollary

For all G as above with rank at least 3, there exists a Hitchin representation with multiple area minimizing minimal surfaces.

Questions

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What are the fibers of $L : \mathbf{M} \to \mathrm{Hit}(n)$?

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Is there a "good" parametrization of $\mathrm{Hit}(n)$?

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- We produce a Hitchin representation together with an unstable minimal surface. Since it must have a stable minimal surface as well, it must have at least two minimal surfaces.

Heuristic: minimal maps to buildings

Fix a Riemann surface S.

- Given $[\rho] \in \mathrm{Rep}(\mathrm{PSL}(n, \mathbb{C}))$ with $h_S([\rho]) = (0, q_3, \ldots, q_n)$, there exists a Hitchin ray $(\rho_R)_{R \in [1,\infty)}$: a path of representations starting at ρ with $h_S(\rho_R) = (0, R^3 q_3, \ldots, R^n q_n)$ and minimal maps h_R.

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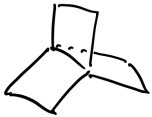
Rank 1 (M.Wolf):

$\mathbb{R}$-trees

Rank 2:



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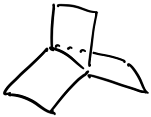

- Heuristic: minimal surfaces in rank k buildings look like minimal surfaces in $\mathbb{R}^k$. For $k \geq 3$, minimal surfaces in $\mathbb{R}^k$ are often unstable.

Minimal maps from the cameral cover

- Starting with $(0, q_3, \ldots, q_n)$ take the branched cover $\tau : S_B \to S$ and the holomorphic $\mathfrak{t}^{\mathbb{C}}$-valued 1-form

$$\phi = \begin{pmatrix} \phi_1 & & \\ & \ddots & \\ & & \phi_n \end{pmatrix}$$

with $\sigma_1(\phi) := \sum_j \phi_j = 0$ and $q_2 = \sigma_2(\phi) = 0$, $q_i = \sigma_i(\phi)$ for $i > 2$.

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- Identifying the hyperplane with $\mathbb{R}^{n-1}$, we write $f : \tilde{S}_B \to \mathbb{R}^{n-1}$.

Minimal maps from the cameral cover

- Taking stock, we have a ray of representations $R \mapsto \rho_R : \pi_1(S) \to G = \mathrm{PSL}(n, \mathbb{C})$ with minimal maps h_R and an equivariant minimal map $f : \tilde{S}_B \to \mathbb{R}^{n-1}$.

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- Let $B \subset S_B$ be the branch locus of $\tau : S_B \to S$. Lift h_R to $\tilde{h}_R : \tilde{S}_B \to (G/K, \nu)$. The following is a consequence of an estimate of T. Mochizuki.

Loosely stated proposition

For "generic" Hitchin rays, the intrinsic data of $\tilde{h}_R$, rescaled by $1/R^2$, converges locally uniformly on $\tilde{S}_B - B$ to the intrinsic data of f. The convergence at $z \in \tilde{S}_B$ is $O(e^{-cd(z,B)R})$.

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- That is, away from B, the intrinsic data of the limiting minimal map to the building should look like f.
- **Remark:** for general G, we have a branched covering $S_B \to S$ with $\mathrm{Deck}(S_B/S)$ inside the Weyl group of the Lie algebra of G, and a minimal map to $\mathbb{R}^{\mathrm{rank} G}$.

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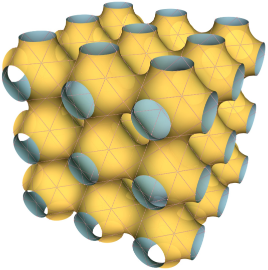

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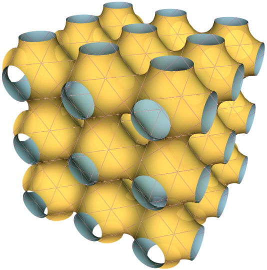

- **Remark:** in genus 2, every equivariant minimal surface is contained in a 2-plane, and hence stable!

The index of minimal maps

Set $G = \mathrm{PSL}(n, \mathbb{C})$, $K = \mathrm{PSU}(n, \mathbb{C})$.

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Theorem (S.-Smillie 2022)

$\liminf_{R \to \infty} \mathrm{Ind}(h_R) \geq \mathrm{Ind}(f)$.

- For $n \geq 4$ and $g \geq 3$, we can choose an unstable minimal map $f : \tilde{S} \to \mathbb{R}^{n-1}$.

Disproof of the Labourie conjecture

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- For R sufficiently large, h_R is unstable, and hence ρ_R admits at least two minimal surfaces.

A Question

- For $\mathrm{PSL}(2,\mathbb{R})^n$, we can show $\mathrm{Ind}(h_R)$ is non-decreasing with R and $\lim_{R\to\infty}\mathrm{Ind}(h_R)=\mathrm{Ind}(f)$ (Marković-S.-Smillie).

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- For more general G, we have Hitchin rays such that $\mathrm{Ind}(h_R) > \mathrm{Ind}(f)$, although we have no example in the Hitchin component.

Question

What more can we say about $\mathrm{Ind}(h_R)$ in terms of $\mathrm{Ind}(f)$?

Thanks for listening