Ray Structures on Teichmuller space

Huiping Pan
joint work with Michael Wolf

South China University of Technology
华南理工大学

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Main results

Ideas of proof

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Ideas of proof

Teichmüller space

S: closed oriented surface of genus $g \geq 2$
Teichmüller space $\mathcal{T}(S) := \{(X, f)\}/\sim$, where

- X is a Riemann surface (resp. hyperbolic surface) and $f : S \to X$ is an orientation preserving homeomorphism.
- $(X_1, f_1) \sim (X_2, f_2)$ if there is a conformal (resp. isometric) map $X_2 \to X_1$ isotopic to $f_1 \circ f_2^{-1}$.

Theorem

The Teichmüller space $\mathcal{T}(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$.

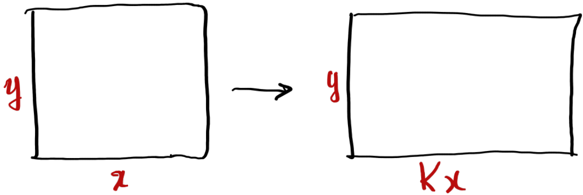
Rays on $\mathcal{T}(S)$: Teichmüller geodesic ray, Thurston geodesic ray, Weil-Peterson geodesic ray, harmonic map rays, earthquake ray, grafting ray, lines of minimal...

Teichmüller map

$x+iy$

$$f : \mathbb{C} \to \mathbb{C}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} Kx \\ y \end{pmatrix}.$$

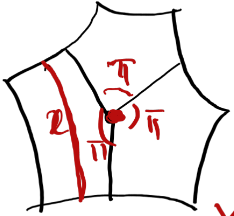


X : Riemann surface:

$\Phi(z)dz^2$: holo. quadratic differential on X

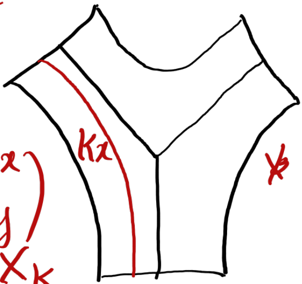
$= dz^2 = (dx + i\,dy)^2$

$z\,dz$



$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} K x \\ y \end{pmatrix}$$

$$X \longrightarrow X_K$$



Teichmüller map and Teichmüller geodesic ray

Teichmüller map: let X be a Riemann surface, Φ a holomorphic quadratic differential on X. Locally $\Phi = dz^2 = (dx + idy)^2$ at regular points. For each $K \geq 1$, define $\Phi_k(z) \triangleq (Kdx + dy)^2$ locally. Let X_K be the Riemann surface underlying Φ_K.

Teichmuller geodesic ray $= \{X_K : K \geq 1\}$.

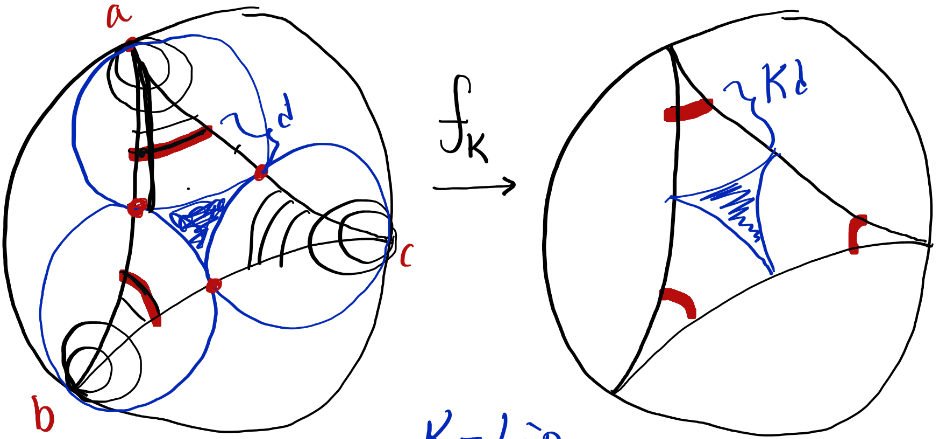
Theorem (Teichmüller 1940s)

For any $X, Y \in \mathcal{T}(S)$, there exists a unique Teichmüller map $f_0 : X \to Y$ realizing $\inf\{K(f) : f \overset{homeo}{\sim} \mathrm{id}\}$, where $K(f)$ is the quasiconformal constant of f.

The Teichmüller metric:

$$d_T(X, Y) \triangleq \frac{1}{2} \log \inf \left\{ K(f) \;\middle|\; \begin{array}{c} \text{homeom. } f : X \to Y \\ \text{isotopic to the identity} \end{array} \right\}$$

Thurston stretch map



K-Lip

expand $\widehat{ab}$, $\widehat{ac}$, $\widehat{bc}$ by K

$(\mathbb{H}^2, d)$

X, λ ($X \setminus \lambda$ = union of ideal triangles)

$\tilde{X} \to X_K$

Theorem (Thurston 1986)

For any closed hyperbolic surface X, for any maximal geodesic lamination λ, there is a new hyperbolic structure $\mathrm{stretch}(X, \lambda, t)$ *depending analytically on $t \geq 0$ such that*

(a) *the identity map $X \to \mathrm{stretch}(X, \lambda, t)$ is e^t-Lipschitz;*

(b) *the identity map expands arc length of λ by the factor e^t.*

Thurston stretch ray $= \{\mathrm{stretch}(X, \lambda, t) : t \geq 0\}$.
The Thurston metric:

$$d_{Th}(X, Y) = \log \inf \left\{ Lip(f) \;\middle|\; \begin{array}{c} \text{homeom. } f : X \to Y \\ \text{isotopic to the identity} \end{array} \right\}$$

Theorem (Thurston 1986)

For any $X, Y \in \mathcal{T}(S)$, there is a geodesic from X to Y which is a concatenation of stretch lines. In particular, $Lip(X, Y)$ is realized by some homeomorphism.

Remark: 1) Stretch lines are rare. 2) d_{Th} is NOT uniquely geodesic.

Harmonic maps between surfaces

$(X, \sigma|dz|^2)$, $(Y, \rho|dw|^2)$: compact Riemannian surfaces.
$w : X \to Y$ differentiable.

- total energy: $E(w; \sigma, \rho) := \int_X \frac{\rho(w(z))}{\sigma(z)}(|w_z|^2 + |w_{\bar{z}}|^2) dz d\bar{z}$
- w is harmonic if it is a critical point of $E(\cdot; \sigma, \rho)$.
- Hopf differential: $w^*(\rho) = \Phi dz^2 + \sigma e(w) dz d\bar{z} + \overline{\Phi} d\bar{z}^2$
- w is harmonic $\Rightarrow \Phi dz^2$ is holomorphic .

Suppose that (Y, ρ) has negative curvature, then

- Existence: Eells-Sampsom, Hamilton
- Uniqueness: Al'ber, Hartman,
- Injectivity: Schoen-Yau, Sampson, Schoen-Jost, Li-Tam, Markovic, Benoist-Hulin

Harmonic map (dual) rays

$Q(X)$: space of holomorphic quadratic differentials on X.

Theorem (Wolf 1989, Hitchin 1987)

The map $\Pi : \mathcal{T}(S) \to Q(X)$ sending $Y \in \mathcal{T}(S)$ to the Hopf differential of the harmonic map $X \to Y$ is a homeomorphism.

Harmonic map ray: $\mathbf{HR}(X, \phi) = \{\Pi^{-1}(t\Phi) : t \geq 0\}$

Theorem (Wolf 1998)

For any hyperbolic surface $Y \in \mathcal{T}(S)$, any measured foliation λ, there exists a unique Riemann surface $X \in \mathcal{T}(S)$ such that the horizontal measured foliation of the Hoff differential of the harmonic map $X \to Y$ is equivalent to λ.

Harmonic map dual ray

$$\mathbf{HDR}(Y, \lambda) \triangleq \left\{ X \in \mathcal{T}(S) \,\middle|\, \begin{array}{l} \text{horizontal measured foliation of} \\ \mathbf{Hopf}(X, Y) \text{ is equivalent to } t\lambda \end{array} \right\}$$

Transition between rays

Theorem (P.-Wolf, 2022)

- *For any hyperbolic surface Y and any measured foliation λ, the family of harmonic map rays $\mathbf{HR}(X_t, Y)$ converges to a unique Thurston geodesic ray locally uniformly, as the domain X_t diverges along the harmonic dual ray $\mathbf{HDR}(Y, \lambda)$.*
- *For any Riemann surface X, and any holomorphic quadratic differential Φ on X, the family of harmonic map dual rays $\mathbf{HDR}(Y_t, t\lambda)$ converges to a unique Teichmüller geodesic locally uniformly, as the target Y_t diverges along the harmonic map ray $\mathbf{HR}(X, \Phi)$.*

Remark: (1) If λ is a maximal measured lamination, then the limiting geodesic is a Thurston stretch ray. (2) For any divergent sequence $X_n \in \mathcal{T}(S)$, the sequence of harmonic map rays $\mathbf{HR}(X_n, R)$ contains a subsequence which converges to some Thurston geodesic. (3) Similar statements for disks also hold.

Transition between rays

Theorem (P.-Wolf, 2022)

- *For any hyperbolic surface Y and any measured foliation λ, the family of harmonic map rays* $\mathbf{HR}(X_t, Y)$ *converges to a unique Thurston geodesic ray locally uniformly, as the domain* X_t *diverges along the harmonic dual ray* $\mathbf{HDR}(Y, \lambda)$.
- *For any Riemann surface X, and any holomorphic quadratic differential Φ on X, the family of harmonic map dual rays* $\mathbf{HDR}(Y_t, t\lambda)$ *converges to a unique Teichmüller geodesic locally uniformly, as the target* Y_t *diverges along the harmonic map ray* $\mathbf{HR}(X, \Phi)$.

Related results: (1) Gerstenhaber-Rauch program 1954 verified by Mese 2004; (2) Bonsante-Mondello-Schlenker 2013: landslide flow to earthquake flow; (3) comparisons between lines of minima and Teichmüller rays by Choi-Rafi-Series 2008, between grafting rays and Teichmüller rays by Choi-Dumas-Rafi 2012 and by Gupta 2014.

Piecewise harmonic stretch lines

Using harmonic maps from punctured surfaces to crowned surfaces, we generalize Thurston's construction of stretch maps from maximal geodesic lamination to non-maximal geodesic laminations.

Let Y be a hyperbolic surface, λ be a geodesic lamination. Let Y^i be the complementary components of $Y \setminus \lambda$.

Deforming crowned surfaces. Let X^i a punctured Riemann surface homeomorphic to Y^i. Let $w^i : X^i \to Y^i$ be a surjective harmonic diffeomorphism with Hopf differential ϕ^i. For any $t > 0$, let Y^i_t be the unique crowned hyperbolic surface such that $w^i_t : X^i \to Y^i_t$ is a surjective harmonic diffeomorphism with Hopf differential $t\phi^i$.

Gluing process. Let Y_t be the hyperbolic surface obtained by replacing Y^i by Y^i_t using the map

$$w^i_t \circ w^i : Y^i \to Y^i_t$$

Let $f_t : Y \to Y_t$ be map obtained by "gluing" $w^i_t \circ w^i$.

Theorem (P.-Wolf, 2022)

Let $Y \in \mathcal{T}(S)$ be any closed hyperbolic surface, and let λ be any geodesic lamination. Then for any surjective harmonic diffeomorphism $f : X \to Y \setminus \lambda$ from some (possibly disconnected) punctured surface X, there is a new hyperbolic surface $Y_t \in \mathcal{T}(S)$ depending analytically on $\{t > 0\}$ such that

(a) *the identity map $f_t : X \to Y_t \setminus \lambda$ is a surjective harmonic map $f_t : X \to Y_t \setminus \lambda$ with Hopf differential $t\mathbf{Hopf}(f)$;*

(b) *for any $0 < s < t$, the identity map $(f_t \circ f_s^{-1})$ is $\sqrt{t/s}$-Lipschitz with (pointwise) Lipschitz constant strictly less than $\sqrt{t/s}$ in $S - \lambda$, and exactly expands arc length of λ by the constant factor $\sqrt{t/s}$.*

Related constructions: Papadopoulos-Yamada (2017), Huang-Papadopoulos (2019), Calderon-Farre (2021), Gueritaud-Kassel (2017), Alessandrini-Disarlo (2019), Daskalopoulos-Uhlenbeck (2020, 2022)

Harmonic stretch lines

A piecewise harmonic stretch lines is called a harmonic stretch line if it is a limit of harmonic map rays.

Theorem (P.-Wolf, 2022)

For any hyperbolic surfaces $X, Y \in \mathcal{T}(S)$, there is a unique harmonic stretch line from X to Y.

Remark: 1) Thurston stretch lines are rare; 2) the Thurston metric is not uniquely geodesic.

Corollary (P.-Wolf, 2022)

For any pair of (homeomorphic) boarded hyperbolic surfaces X, Y, the infimum of Lipschitz constants among all Lipschitz maps $X \to Y$ homotopic to the identity is realized by some homeomorphism.

Visual boundary and exponential maps

Theorem (P.-Wolf, 2022)

For any $Y \in \mathcal{T}(S)$ and any projective measured lamination $[\eta]$, there exists a unique harmonic stretch ray starting at Y, which converges to $[\eta]$ in the Thurston compactification.

Moreover, these rays foliate $\mathcal{T}(S)$ if we fix Y and let $[\eta]$ vary in $\mathcal{PML}(S)$, or if we fix $[\eta]$ and let Y vary in $\mathcal{T}(S)$.

Two versions of geodesic flow of the Thurston metric

Version 1: Using the exponential map to define

$$\psi_t : \mathcal{T}(S) \times \mathcal{PML}(S) \longrightarrow \mathcal{T}(S) \times \mathcal{PML}(S)$$

such that the orbit through $(Y, [\eta]) \in \mathcal{T}(S) \times \mathcal{PML}(S)$ is the harmonic stretch line which passes through Y and converges to $[\eta]$. Moreover, every harmonic stretch line appears as a (forward) orbit.

Version 2: The second version of Thurston geodesic flow is

$$\phi_t : \mathcal{T}(S) \times \mathcal{PML}(S) \to \mathcal{T}(S) \times \mathcal{PML}(S)$$

such that the orbit through $(Y, [\eta]) \in \mathcal{T}(S) \times \mathcal{PML}(S)$ is the stretch line $\mathbf{SR}_{Y,[\eta]}$ obtained as limits of harmonic map rays $\mathbf{HR}_{X_t,Y}$ as X_t degenerates along the harmonic map dual ray $\mathbf{HDR}_{Y,\eta}$

Theorem (P.-Wolf, 2022)

For any hyperbolic surface Y and any measured foliation λ, the family of harmonic map rays $\mathbf{HR}(X_t, Y)$ *converges to a unique Thurston geodesic ray, as the domain X_t diverges along the harmonic dual ray* $\mathbf{HDR}(Y, \lambda)$.

Outline of the proof

- Step 1: $\mathbf{HR}(X_t, Y)$ subconverges to a Thurston geodesic ray
- Step 2: Any limit of $\mathbf{HR}(X_t, Y)$ is a harmonic map ray $\mathbf{HR}(X_\infty, Y \setminus \lambda)$ from some punctured Riemann surface such that the horizontal foliation of $\mathbf{Hopf}(X_\infty, Y \setminus \lambda)$ is $\infty \cdot \lambda$
- Step 3: Uniqueness of such limit ray using minimal graphs valued in $\mathbb{R}$-tree.

Step 1: Subconvergence using high energy estimate

$Y_s \in \mathbf{HR}(X, \Phi)$ with $\|\Phi\| = 1$, $f_s : X \to Y_s$ harmonic, the pullback metric of Y_s via f_s:

$$f_s^* Y_s = 2t(\log \frac{1}{|\mu(z,s)|} + 1)\mathrm{d}x^2 + 2t(\log \frac{1}{|\mu(z,s)|} - 1)\mathrm{d}y^2.$$

Wolf (1989): $\forall z \in X$ with $\Phi(z) \neq 0$, $|\mu(s,z)| \nearrow 1$ as $s \to \infty$.
$\Longrightarrow$: $0 < s < s'$, $f_t \circ (f_{s'})^{-1} : Y_s \to Y_{s'}$ is $\sqrt{s'/s}$-Lipschitz.
$\Longrightarrow$: $d_{Th}(Y_s, Y_{s'}) \leq \log \sqrt{s'/s}$.

Minsky (1992): $E(f_s) = \|s\Phi\| + O(1) = \ell_{Y_s}(\sqrt{s}\lambda) + O(1)$.
$\Longrightarrow$: $\frac{\ell_{Y_{s'}}(\lambda)}{\ell_{Y_s}(\lambda)} = \sqrt{s'/s} + O(\sqrt{1/\|\mathbf{Hopf}(X, Y_s)\|})$
$\Longrightarrow$: $d_{Th}(Y_s, Y_{s'}) \overset{Thurston}{=} \sup_{\alpha \in \mathcal{ML}} \frac{\ell_{Y_s'}(\alpha)}{\ell_{Y_s}(\alpha)} \geq \log \sqrt{s'/s} + o(1)$

In summary: $\mathbf{HR}(X_t, Y)$ subconverges to a Thurston geodesic ray locally uniformly.

Step 2: limits of $\mathbf{HR}(X_t, Y)$ are harmonic map rays from punctured surfaces

Let $X_t \in \mathbf{HDR}_{Y,\lambda}$. Let $h_t : X_t \to Y$ be the harmonic map with Hopf differential Φ_t such that the horizontal foliation of Φ_t is $t\lambda$. Let $Z_t := \{z_t^i, \cdots, z_t^k\}$ be the set of zeros of Φ_t.

Lemma (P.-Wolf)

Any divergent positive sequence $t_n \to \infty$ contains a subsequence, still denoted by t_n for simplicity, such that

- $(X_{t_n}, |\Phi_{t_n}|, Z_{t_n})$ *convergence to a flat surface* $(X_\infty, |\Phi_\infty|, Z_\infty)$ *in the Gromov-Hausdorff sense, where (1)* X_∞ *is homeomorphic to* $Y \setminus \lambda$*; (2)* Φ_∞ *is a meromorphic differential, whose horizontal foliaton is* $\infty \cdot \lambda$*, i.e. the union of half-infinite cylinders or half-planes corresponding to* λ*;*
- $h_{t_n} : X_{t_n} \to Y$ *converges to the harmonic map* $h_\infty : X_\infty \to Y \setminus \lambda$ *with Hopf differential* Φ_∞

Step 3: uniqueness of the generalized Jenkins-Serrin problem

Admissible foliation: a measured foliation η on a crowned surface V (as an open surface) is said to be admissible if it is the horizontal foliation of some meromorphic differential ϕ on a punctured Riemann surface U homeomorphic to V with poles of order ≥ 2 at punctures.

Let $\tilde{\eta}$ be the lift of η to the universal cover $\tilde{V}$ of V. The leaf space of defines an admissible dual tree T_η. The projection map $\tilde{V} \to T_\eta$ along leaves of $\tilde{\eta}$ defines an admissible boundary correspondence $\partial\tilde{V} \to \partial T_\eta$.

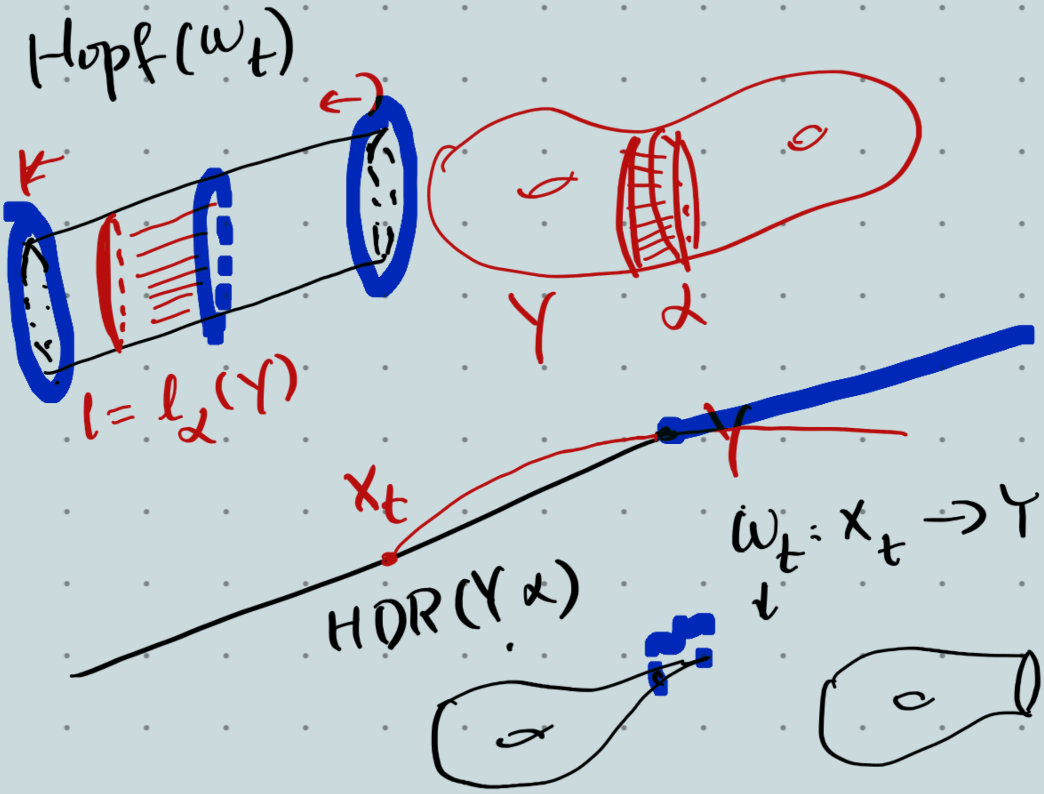
Any limit pair (X_∞, Φ_∞) obtained in Step 2 lifts to an equivariant minimal graph in $\widetilde{(Y \setminus \lambda)} \times T_{\infty \cdot \lambda}$ with an admissible boundary correspondence.

Theorem (P.-Wolf)

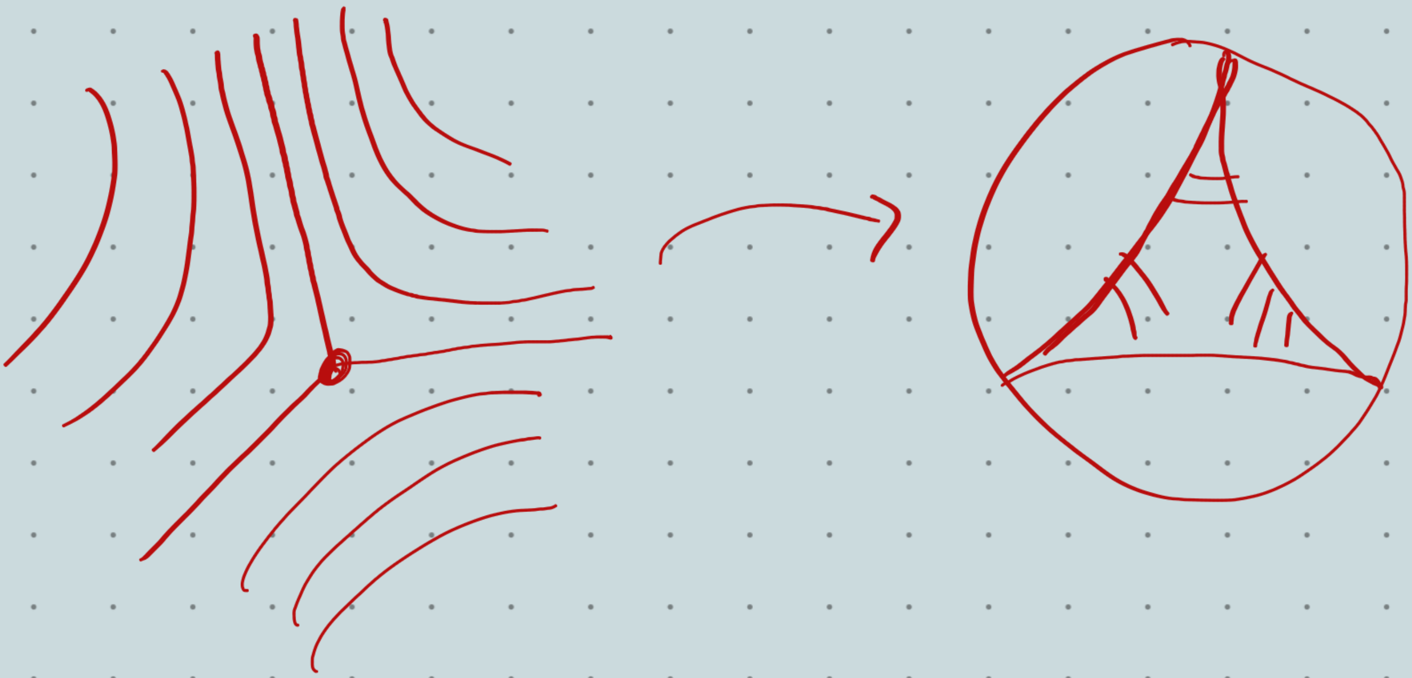
Let V be a crowned hyperbolic surface. Let T be an admissible dual tree. Then there exists a unique $\pi_1(Y)$-equivariant minimal graph in $\widetilde{V} \times T$ with a prescribed admissible boundary correspondence.

Remark: Jenkins-Serrin (1966), Nelli-Rosenberg (2002) minimal graphs over 2n-gons and valued in $\mathbb{R}$ with boundary values alternatively $+\infty$ and $-\infty$.

THANK YOU!!!



Hopf(ω_t)
$l = l_\alpha(Y)$
Y
α
X_t
HDR(Y, α)
$\omega_t : X_t \to Y$
t



$z\,dz^2$