Equivariant Log-concavity and Equivariant Kahler Packages

(or: Shadows of Hodge Theory)

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YMSC Topology Seminar

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1. Kähler packages
2. Equivariant log-concavity
3. Equivariant Kähler packages

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- Hodge–Riemann relations: the bilinear form

$$\langle -,- \rangle_L = P(-, L^{d-k}-) : H^k(X) \times H^k(X) \longrightarrow \mathbb{R}$$

is $(-1)^k$-definite on the kernel of L^{d-k+1}.

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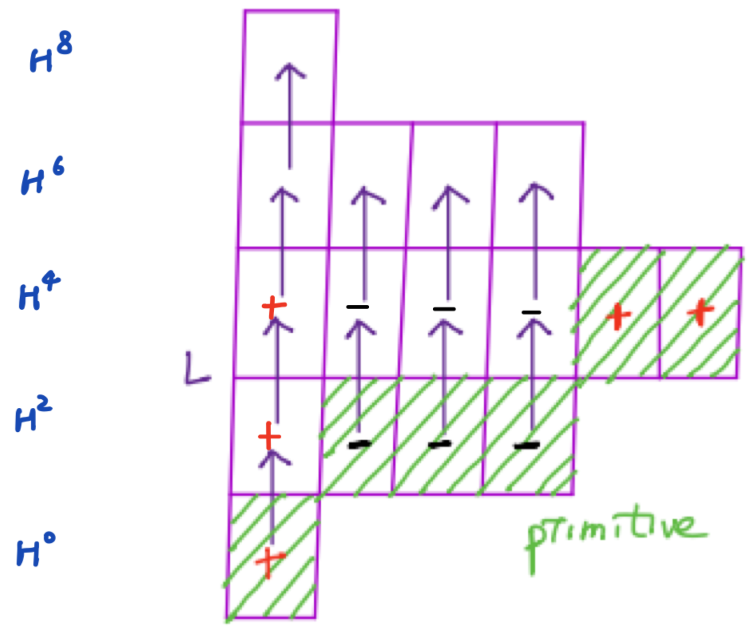

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According to Geordie Williamson, this is "A mystery for the 21st century!"

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Miracle: in some cases,
"Betti numbers" = "intersection numbers"!

Log-concavity

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- A sequence $b_0, b_1, b_2, \ldots$ or a polynomial $f(t) = \sum b_i t^i$ is *strongly log-concave* if, for all $i \leq j \leq k \leq l$ with $i + l = j + k$, we have

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- It can be shown that a sequence or polynomial is strongly log-concave if and only if it is log-concave with *no internal zeros.*

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Example: Ordered configuration space / Coxeter arrangement

Let $\mathcal{A}_n := \{H_{ij}\}$, where $H_{ij} := \{z \in \mathbb{C}^n \mid z_i = z_j\} \subset \mathbb{C}^n$. Then

$$U(\mathcal{A}_n) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, \forall\, i \neq j\} =: \mathrm{Conf}(n, \mathbb{C}),$$

and (exercise) $\pi_{\mathcal{A}_n}(t) = \prod_{k=1}^{n-1}(1 + kt)$.

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For any $\mathcal{A}, \pi_{\mathcal{A}}(t)$ is strongly log-concave.

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Any arrangement $\mathcal{A}$ determines a matroid M_A.
Given a matroid M, one can define the Orlik-Solomon algebra $\mathrm{OS}^*(M)$, and there is a canonical isomorphism

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Conjecture (Matherne-Miyata-Proudfoot-Ramos, 2021)

The cohomology ring $H^*(U(\mathcal{A}); \mathbb{Q})$ is a strongly equivariantly log-concave graded representation of G.
More generally, if M is a matroid with an action of G, the Orlik-Solomon algebra $\mathrm{OS}^*(M)$ is a strongly equivariantly log-concave graded representation of G.

Flag variety

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Theorem (Borel, 1935) "Borel's picture"

As a graded ring,

$$H^*(\mathcal{F}_n, \mathbb{Z}) \cong \mathbb{Z}[t_1, \ldots, t_n]/(\sigma_1, \ldots, \sigma_n),$$

where the t_j 's are of degree 2 and the σ_i 's are the elementary symmetric polynomials in the variables t_j 's.

Equivariant log-concavity conjecture on flag variety

As a corollary, $P_{\mathcal{F}_n}(q) := \sum_i b_{2i} q^i = \prod_{k=0}^{n-1} \left(1 + q + \cdots + q^k\right)$, which is a symmetric, unimodal, and log-concave polynomial.

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For all integer $n \geq 1$, the cohomology ring of the flag variety is equivariantly log-concave as graded representation of S_n.

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In terms of Kronecker coefficients, it is equivalent to

Conj (G., Conjecture 5 in arXiv:2205.05408)

For all natural number n and $1 \leq i \leq n(n-1)/2 - 1$,

$$\sum_{\lambda \vdash n} \sum_{\mu \vdash n} b_{\lambda, i-1} b_{\mu, i+1} g_{\lambda\mu}^{\nu} \leq \sum_{\lambda \vdash n} \sum_{\mu \vdash n} b_{\lambda, i} b_{\mu, i} g_{\lambda\mu}^{\nu}$$

hold for all $\nu \vdash n$, where $g_{\lambda\mu}^{\nu}$ are the Kronecker coefficients, and $b_{\lambda,i}$ are the "fake degrees".

Using the representation stability theory, we have the following

Theorem (G.)

Above conjectures hold for degree ≤ 3 and co-degree ≤ 3, i.e., $\forall n$:

$$\left(H^{2*1}\right)^{\otimes 2} \supseteq H^{2*0} \otimes H^{2*2}, \left(H^{2*(\binom{n}{2}-1)}\right)^{\otimes 2} \supseteq H^{2*\binom{n}{2}} \otimes H^{2*(\binom{n}{2}-2)},$$
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The numerical evidence suggests the stronger conjecture

Conj (G., Conjecture 7 in arXiv:2205.05408)

For all natural number n, $\nu \vdash n$, and $1 \leq i \leq n(n-1)/2 - 1$, the "difference" sequence $d_{\nu,i}$ is symmetric and unimodal, where

$$d_{\nu,i} := \sum_{\lambda \vdash n} \sum_{\mu \vdash n} b_{\lambda,i} b_{\mu,i} g^{\nu}_{\lambda\mu} - \sum_{\lambda \vdash n} \sum_{\mu \vdash n} b_{\lambda,i-1} b_{\mu,i+1} g^{\nu}_{\lambda\mu}.$$

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- Example: a general endomorphism on a vector space is an isomorphism, however establishing a specific one might be very difficult.
- But *symmetry reduces choices*!
- Our idea: *lift the "1-dimensional" sequence to a "2-dimensional" array, and establishing the (equivariant) hard Lefschetz theorem for each "anti-diagonal", which implies the strongly (equivariant) log-concavity of the original sequence.*

An illustration

$b_0, b_1, \cdots, b_k, \cdots$

$\Updownarrow$

$\forall k:$

$b_0 b_k \leq b_1 b_{k-1} \leq \cdots \geq b_{k-1} b_1 \geq b_k b_0$

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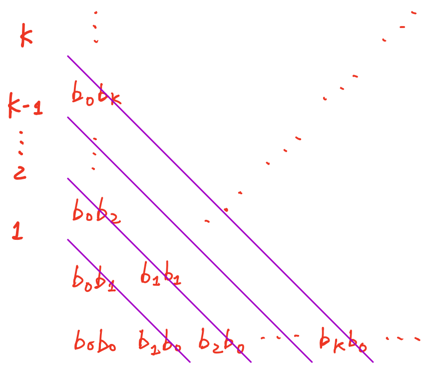

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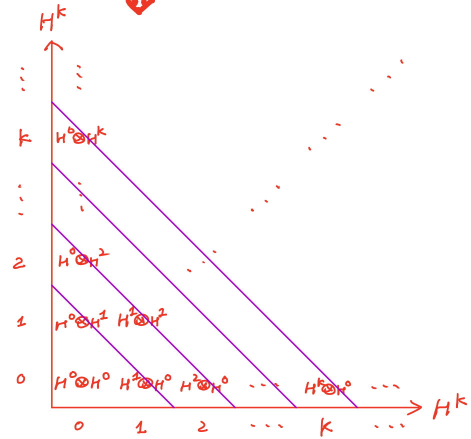

1. Kähler packages
2. Equivariant log-concavity
3. Equivariant Kähler packages

Equivariant Kähler packages

In attacking the above conjectures, we discover two new Kähler packages that are *equivariant* and have *no geometric origin*. The equivariant log-concavity hints at our discoveries for these structures.

Polynomial rings

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- Consider $S_n \curvearrowright B\left(S^1\right)^n = (\mathbb{CP}^\infty)^n$, the classifying space
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$$H_{n,m} = \bigoplus_{i=0}^{m} H_{n,m}^{-m+2i}, \text{ with } H_{n,m}^{-m+2i} := D^i \otimes R^{m-i},$$

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  where d_i acts by multiplication.

Theorem (G.-Xiong)

For any pair of natural numbers (n, m), we have

(a) (PD) The bilinear pairing $\langle -, - \rangle : H_{n,m} \times H_{n,m} \longrightarrow \mathbb{R}$ is S_n-equivariant and graded non-degenerate;

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$H^*((\mathbb{CP}^\infty)^n, \mathbb{R}) = \mathbb{R}[t_1, \ldots, t_n]$ is strongly equivariantly log-concave and the inclusions of S_n-representations are given by the operator $L = \sum_{i=1}^n t_i \otimes \frac{\partial}{\partial t_i}$.

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where e_{α_k} and $i_{\alpha_k^*}$ are the exterior product and interior product, respectively, and α_k^* is the linear functional on $V = \langle \alpha_1, \ldots, \alpha_n \rangle$ satisfying $\alpha_k^*(\alpha_l) = \delta_{kl} = \begin{cases} 0, & \text{if } k \neq l, \\ 1, & \text{if } k = l. \end{cases}$

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$\mathbb{R}[x_1, x_2]$:

	⋮					
	⋮					
4	4			⋰		
3	3	6	⋰			
2	2	4	6			
1	1	2	3	4	⋯	⋯
	1	2	3	4		

$\mathbb{R}[x_1, x_2, x_3]$:

	⋮					
	⋮					
10	10			⋰		
6	6	18	⋰			
3	3	9	18			
1	1	3	6	10	⋯	⋯
	1	3	6	10		

$\Lambda[\alpha_1, \alpha_2]$:

1	1	2	1
2	2	4	2
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4	4	16	24	16	4
6	6	24	36	24	6
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There is also an S_n-equivariant $\mathfrak{sl}_2$-action on the "usual" gradings of the tensor product of the polynomial ring. But neither Poincaré duality nor hard Lefschetz holds because the "usual" gradings are infinite-dimensional.

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- The adjoint ("down") operator f of the Lefschetz operator L can be written explicitly. In other algebraic/combinatorial setting, direct constructions of f seems difficult.
- The proof of hard Lefschetz and Hodge–Riemann relations is simple/natural and it takes advantage of the geometry of the product of projective spaces.

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- It would be interesting to know whether our approach could shed some new light on other (equivariant) log-concavity questions and conjectures.
- It is interesting to see whether similar (equivariant) Hodge-theoretic structures exist in the settings of other (equivalent) log-concavity conjectures.