Equivariant Log-concavity and Equivariant Kahler Packages

(or: Shadows of Hodge Theory)

Tao Gui

joint with Rui Xiong(熊锐)

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1. Kähler packages
2. Equivariant log-concavity
3. Equivariant Kähler packages

What is a Kähler package?

- X: mathematical object of "complex dimension" d
- $H^*(X) = \bigoplus_{k=0}^{2d} H^k(X)$ "cohomology" of X
- $P(-,-) : H^k(X) \times H^{2d-k}(X) \to \mathbb{R}, \forall k \leq d$ "Poincaré pairing"
- $L : H^k(X) \to H^{k+2}(X)$ "Lefschetz operator"

We expect that $(H^*, P(-,-), L)$ satisfy:

- Poincaré duality: $P(-,-)$ is non-degenerate.
- Hard Lefschetz ("Théorème de Lefschetz vache"):

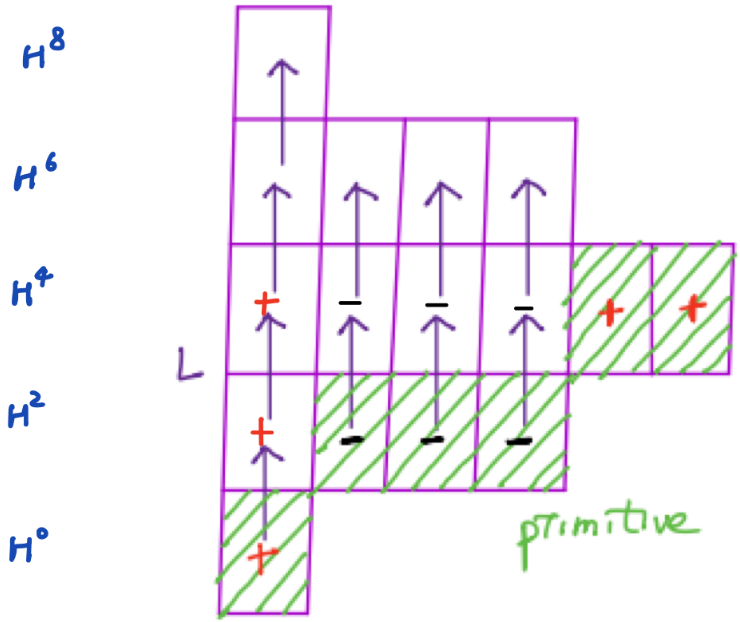
$$\forall k \leq d, \quad L^{d-k} : H^k(X) \xrightarrow{\sim} H^{2d-k}(X).$$

- Hodge–Riemann relations: the bilinear form

$$\langle -,- \rangle_L = P(-, L^{d-k}-) : H^k(X) \times H^k(X) \longrightarrow \mathbb{R}$$

is $(-1)^k$-definite on the kernel of L^{d-k+1}.

A picture to keep in mind



Kähler packages are both ubiquitous and fundamental

- Smooth complex projective variety/ compact Kähler manifold (Poincaré'1895, Lefschetz'24, Hodge'30s, Chern'51, et. al.)
- Smooth projective variety over $\mathbb{F}_q$ (Grothendieck'68) $\Rightarrow$ Weil conjectures'49
- Singular projective variety (Goresky-MacPherson'80, Beilinson-Bernstein-Deligne'82, et. al.) $\Rightarrow$ Decomposition theorem
- Polytope (Stanley'80, Bressler-Lunts'03, Karu'04, et. al.) $\Rightarrow$ Mcmullen g-conjecture'80, Aleksandrov-Fenchel inequality
- Coxeter group (Soergel'92, Elias-Williamson'14, Williamson'16) $\Rightarrow$ Kazhdan-Lusztig conjectures'79, Jantzen conjectures'79
- Matroid (Huh and his collaborators) $\Rightarrow$ Various combinatorial conjectures

According to Geordie Williamson, this is "A mystery for the 21st century!"

Where and how can we find a Kähler package?

Kähler packages have some numerical shadows:

- Poincaré duality: "Betti numbers" are symmetric! (Poincaré)
- Hard Lefschetz: "Betti numbers" are unimodal in parity! (Stanley)
- Hodge–Riemann relations: "intersection numbers" are log-concave! (Huh)
  Why? Essentially because

$$b^2 \geq ac \text{ if and only if } \begin{vmatrix} a & b \\ b & c \end{vmatrix} \leq 0.$$

Miracle: in some cases,
"Betti numbers" = "intersection numbers"!

Log-concavity

- A sequence $b_0, b_1, b_2, \ldots$ of non-negative real numbers is called *log-concave* if, for all i,

$$b_i^2 \geq b_{i-1} b_{i+1}.$$

- A polynomial $f(t) = \sum b_i t^i$ with non-negative coefficients is called log-concave if its sequence of coefficients is log-concave.
- A sequence $b_0, b_1, b_2, \ldots$ or a polynomial $f(t) = \sum b_i t^i$ is *strongly log-concave* if, for all $i \leq j \leq k \leq l$ with $i + l = j + k$, we have

$$b_j b_k \geq b_i b_l.$$

- It can be shown that a sequence or polynomial is strongly log-concave if and only if it is log-concave with *no internal zeros*.

hyperplane arrangement

- V: finite dimensional vector space over $\mathbb{C}$
- $\mathcal{A}$: finite collection of central hyperplanes in V
- Consider the complement

$$U(\mathcal{A}) := V \setminus \bigcup_{H \in \mathcal{A}} H$$

- $b_i(\mathcal{A}) := \dim H^i(U(\mathcal{A}); \mathbb{Q})$ Betti numbers
- $\pi_{\mathcal{A}}(t) := \sum_{i \geq 0} b_i(\mathcal{A}) t^i$ Poincaré polynomial

Example: Ordered configuration space / Coxeter arrangement

Let $\mathcal{A}_n := \{H_{ij}\}$, where $H_{ij} := \{z \in \mathbb{C}^n \mid z_i = z_j\} \subset \mathbb{C}^n$. Then

$$U(\mathcal{A}_n) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, \forall\, i \neq j\} =: \mathrm{Conf}(n, \mathbb{C}),$$

and (exercise) $\pi_{\mathcal{A}_n}(t) = \prod_{k=1}^{n-1} (1 + kt)$.

Thm (Hoggar conj '74 [Huh, 2012])

For any $\mathcal{A}, \pi_{\mathcal{A}}(t)$ is strongly log-concave.

Any arrangement $\mathcal{A}$ determines a matroid M_A.
Given a matroid M, one can define the Orlik-Solomon algebra $\mathrm{OS}^*(M)$, and there is a canonical isomorphism

$$\mathrm{OS}^*\left(M_{\mathcal{A}}\right) \cong H^*(U(\mathcal{A}); \mathbb{Z}).$$

Let

$$\pi_M(t) := \sum_{i \geq 0} t^i \dim \mathrm{OS}^i(M).$$

Thm (Heron-Rota-Welsh conj '70s [AHK, 2018])

For any matroid $M, \pi_M(t)$ is strongly log-concave.

Equivariant log-concavity

- Let G be a finite group, and let

$$V = \bigoplus_{i \geq 0} V^i$$

be a graded representation of G with V^i finite dimensional.

- We say that V is *(weakly) log-concave* if, for all i.

$V^i \otimes V^i$ contains a subrepresentation isomorphic to $V^{i-1} \otimes V^{i+1}$.

- V is *strongly equivariantly log-concave* if, for all $i \leq j \leq k \leq l$ with $i + l = j + k$,

$V^j \otimes V^k$ contains a subrepresentation isomorphic to $V^i \otimes V^l$.

Equivariant log-concavity conjecture

- V: finite dimensional vector space over $\mathbb{C}$,
- $\mathcal{A}$: finite collection of hyperplanes in V.
- G acts linearly on V, preserving $\mathcal{A} \rightsquigarrow$ action of G on $U(\mathcal{A})$.

Example

The symmetric group S_n acts on $\mathbb{C}^n$ preserving $\mathcal{A}_n$. this induces an action on $U(\mathcal{A}_n) = \mathrm{Conf}(n, \mathbb{C})$ by permuting labels of the points.

Conjecture (Matherne-Miyata-Proudfoot-Ramos, 2021)

The cohomology ring $H^*(U(\mathcal{A}); \mathbb{Q})$ is a strongly equivariantly log-concave graded representation of G.
More generally, if M is a matroid with an action of G, the Orlik-Solomon algebra $\mathrm{OS}^*(M)$ is a strongly equivariantly log-concave graded representation of G.

Flag variety

- $\mathcal{F}_n$: (full) flag variety over $\mathbb{C}$, parametrizing all complete flags in a n-dimensional $\mathbb{C}$-vector space V

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$$

- $\mathcal{F}_n \cong \mathrm{GL}(n, \mathbb{C})/B_n \Rightarrow \dim \mathcal{F}_n = n(n-1)/2$

Theorem (Borel, 1935) "Borel's picture"

As a graded ring,

$$H^*(\mathcal{F}_n, \mathbb{Z}) \cong \mathbb{Z}[t_1, \ldots, t_n] / (\sigma_1, \ldots, \sigma_n),$$

where the t_j 's are of degree 2 and the σ_i 's are the elementary symmetric polynomials in the variables t_j 's.

Equivariant log-concavity conjecture on flag variety

As a corollary, $P_{\mathcal{F}_n}(q) := \sum_i b_{2i}q^i = \prod_{k=0}^{n-1}\left(1+q+\cdots+q^k\right)$, which is a symmetric, unimodal, and log-concave polynomial.

Conj (G., Conjecture 1 in arXiv:2205.05408)

For all integer $n \geq 1$, the cohomology ring of the flag variety is equivariantly log-concave as graded representation of S_n.

In terms of Kronecker coefficients, it is equivalent to

Conj (G., Conjecture 5 in arXiv:2205.05408)

For all natural number n and $1 \leq i \leq n(n-1)/2-1$,

$$\sum_{\lambda \vdash n}\sum_{\mu \vdash n} b_{\lambda,i-1}b_{\mu,i+1}g^{\nu}_{\lambda\mu} \leq \sum_{\lambda \vdash n}\sum_{\mu \vdash n} b_{\lambda,i}b_{\mu,i}g^{\nu}_{\lambda\mu}$$

hold for all $\nu \vdash n$, where $g^{\nu}_{\lambda\mu}$ are the Kronecker coefficients, and $b_{\lambda,i}$ are the "fake degrees".

Using the representation stability theory, we have the following

Theorem (G.)

Above conjectures hold for degree ≤ 3 and co-degree ≤ 3, i.e., $\forall n$:

$$\left(H^{2*1}\right)^{\otimes 2} \supseteq H^{2*0} \otimes H^{2*2}, \left(H^{2*(\binom{n}{2}-1)}\right)^{\otimes 2} \supseteq H^{2*\binom{n}{2}} \otimes H^{2*(\binom{n}{2}-2)},$$
$$\left(H^{2*2}\right)^{\otimes 2} \supseteq H^{2*1} \otimes H^{2*3}, \left(H^{2*(\binom{n}{2}-2)}\right)^{\otimes 2} \supseteq H^{2*(\binom{n}{2}-1)} \otimes H^{2*(\binom{n}{2}-3)},$$
$$\left(H^{2*3}\right)^{\otimes 2} \supseteq H^{2*2} \otimes H^{2*4}, \left(H^{2*(\binom{n}{2}-3)}\right)^{\otimes 2} \supseteq H^{2*(\binom{n}{2}-2)} \otimes H^{2*(\binom{n}{2}-4)}.$$

The numerical evidence suggests the stronger conjecture

Conj (G., Conjecture 7 in arXiv:2205.05408)

For all natural number n, $\nu \vdash n$, and $1 \leq i \leq n(n-1)/2 - 1$, the "difference" sequence $d_{\nu,i}$ is symmetric and unimodal, where

$$d_{\nu,i} := \sum_{\lambda \vdash n} \sum_{\mu \vdash n} b_{\lambda,i} b_{\mu,i} g^{\nu}_{\lambda\mu} - \sum_{\lambda \vdash n} \sum_{\mu \vdash n} b_{\lambda,i-1} b_{\mu,i+1} g^{\nu}_{\lambda\mu}.$$

Why equivariant?

- Our feeling: establishing equivariant log-concavity might be simpler than establishing log-concavity!
- Example: a general endomorphism on a vector space is an isomorphism, however establishing a specific one might be very difficult.
- But *symmetry reduces choices*!
- Our idea: *lift the "1-dimensional" sequence to a "2-dimensional" array, and establishing the (equivariant) hard Lefschetz theorem for each "anti-diagonal", which implies the strongly (equivariant) log-concavity of the original sequence.*

Illustration

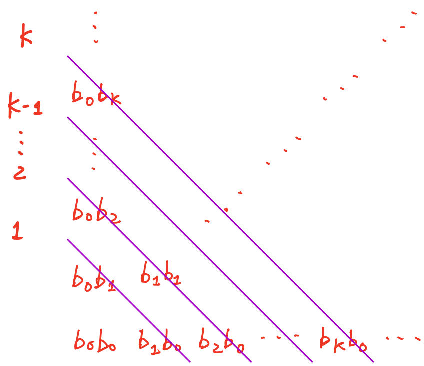
$b_0, b_1, \dots, b_k, \dots$

$\Updownarrow$

$\forall k:$

$b_0 b_k \le b_1 b_{k-1} \le \cdots \ge b_{k-1} b_1 \ge b_k b_0$

$\Updownarrow$



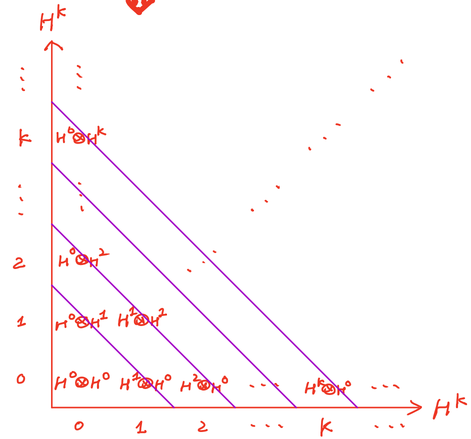
$H^* = \bigoplus_k H^k$

$\Updownarrow$

$\forall k:$

$H^0 \otimes H^k \hookrightarrow H^1 \otimes H^{k-1} \hookrightarrow \cdots \twoheadrightarrow H^{k-1} \otimes H^1 \twoheadrightarrow H^k \otimes H^0$

$\Updownarrow$



Equivariant Kähler packages

In attacking the above conjectures, we discover two new Kähler packages that are *equivariant* and have *no geometric origin*. The equivariant log-concavity hints at our discoveries for these structures.

Polynomial rings

- Consider $S_n \curvearrowright B\left(S^1\right)^n = (\mathbb{CP}^\infty)^n$, the classifying space $\rightsquigarrow S_n \curvearrowright H^*((\mathbb{CP}^\infty)^n, \mathbb{R}) = \mathbb{R}[t_1, \ldots, t_n]$
- Betti numbers $b_{2i} = \binom{n+i-1}{i}$, log-concave for fixed n.
- Fix a pair (n, m), consider the graded $\mathbb{R}$-vector space

$$H_{n,m} = \bigoplus_{i=0}^{m} H_{n,m}^{-m+2i}, \text{ with } H_{n,m}^{-m+2i} := D^i \otimes R^{m-i},$$

  where $D = \mathbb{R}[d_1, \ldots, d_n]$, $R = \mathbb{R}[x_1, \ldots, x_n]$
- Define a pairing $\langle -, - \rangle$ on $H_{n,m}$ by interpreting the d_i as differential operators $\frac{\partial}{\partial x_i}$ "homology-cohomology pairing"
- Define $L : H_{n,m}^i \longrightarrow H_{n,m}^{i+2}$ to be the linear map

$$L := \sum_{i=1}^{n} d_i \otimes \frac{\partial}{\partial x_i},$$

  where d_i acts by multiplication.

Theorem (G.-Xiong)

For any pair of natural numbers (n, m), we have

(a) (PD) The bilinear pairing $\langle -, - \rangle : H_{n,m} \times H_{n,m} \longrightarrow \mathbb{R}$ is S_n-equivariant and graded non-degenerate;

(b) (HL) $L : H^i_{n,m} \longrightarrow H^{i+2}_{n,m}$ is S_n-equivariant and satisfies the hard Lefschetz theorem;

(c) (HR) For all $0 \leq i \leq m/2$, the bilinear form

$$(a, b)_L^{-m+2i} = \left\langle a, L^{m-2i} b \right\rangle : H^{-m+2i}_{n,m} \times H^{-m+2i}_{n,m} \longrightarrow \mathbb{R}$$

is S_n-equivariant and $(-1)^i$-definite on $\ker\left(L^{m-2i+1}\right)$.

Corollary

$H^*((\mathbb{CP}^\infty)^n, \mathbb{R}) = \mathbb{R}[t_1, \ldots, t_n]$ is strongly equivariantly log-concave and the inclusions of S_n-representations are given by the operator $L = \sum_{i=1}^n t_i \otimes \frac{\partial}{\partial t_i}$.

Exterior algebras

- $S_n \curvearrowright (S^1)^n \rightsquigarrow S_n \curvearrowright H^*((S^1)^n, \mathbb{R}) = \Lambda_{\mathbb{R}}[\alpha_1, \ldots, \alpha_n]$
- Betti numbers $b_{2i} = \binom{n}{i}$, log-concave for fixed n
- Fix a pair (n, m) with $m \leq 2n$, consider

$$H'_{n,m} = \bigoplus_{i=0}^{m} (H'_{n,m})^{-m+2i}, \text{ with } (H'_{n,m})^{-m+2i} := \Lambda^i \otimes (\Lambda^*)^{m-i},$$

  where $\Lambda = \Lambda(V) = \Lambda_{\mathbb{R}}[\theta_1, \ldots, \theta_n]$, $\Lambda^* = \Lambda(V^*)$
- Define a pairing $\langle -, - \rangle$ on $H'_{n,m}$ by the natural duality between V and V^* "homology-cohomology pairing"
- Define

$$L : (H'_{n,m})^i \longrightarrow (H'_{n,m})^{i+2}$$

  to be the linear map

$$L := \sum_{k=1}^{n} e_{\theta_k} \otimes i_{\theta_k}.$$

Theorem (G.-Xiong)

For any pair of natural numbers (n, m) with $m \leq 2n$, we have S_n-equivariant PD, HL and HR for $H'_{n,m}$.

Corollary

$H^*((S^1)^n, \mathbb{R}) = \Lambda_{\mathbb{R}}[\alpha_1, \dots, \alpha_n]$ is strongly equivariantly log-concave and the inclusions of S_n-representations are given by the operator

$$L = \sum_{k=1}^{n} e_{\alpha_k} \otimes i_{\alpha_k^*}.$$

where e_{α_k} and $i_{\alpha_k^*}$ are the exterior product and interior product, respectively, and α_k^* is the linear functional on $V = \langle \alpha_1, \dots, \alpha_n \rangle$ satisfying $\alpha_k^*(\alpha_l) = \delta_{kl} = \begin{cases} 0, & \text{if } k \neq l, \\ 1, & \text{if } k = l. \end{cases}$

Some example of "Betti numbers"...

$\mathbb{R}[x_1, x_2]$:

	⋮				
	⋮				
4	4			⋰	
3	3	6	⋰		
2	2	4	6		
1	1	2	3	4	⋯ ⋯
	1	2	3	4	

$\mathbb{R}[x_1, x_2, x_3]$:

	⋮				
	⋮				
10	10			⋰	
6	6	18	⋰		
3	3	9	18		
1	1	3	6	10	⋯ ⋯
	1	3	6	10	

$\Lambda[\alpha_1, \alpha_2]$:

1	1	2	1
2	2	4	2
1	1	2	1
	1	2	1

$\Lambda[\alpha_1, \alpha_2, \alpha_3]$:

1	1	3	3	1
3	3	9	9	3
3	3	9	9	3
1	1	3	3	1
	1	3	3	1

$\Lambda[\alpha_1, \alpha_2, \alpha_3, \alpha_4]$:

1	1	4	6	4	1
4	4	16	24	16	4
6	6	24	36	24	6
4	4	16	24	16	4
1	1	4	6	4	1
	1	4	6	4	1

An "exotic" example

For the exterior algebra, we find that "usual" gradings on tensor products satisfy PD and HL but NOT satisfy HR.

- Fix any n, consider the graded $\mathbb{R}$-vector space

$$H_n = \bigoplus_{i=0}^{2n} H_n^{-n+i}, \text{ with } H_n^{-n+i} := \bigoplus_{j+k=i} \Lambda^j \otimes (\Lambda^*)^k,$$

  i.e., consider the "usual" gradings (up to a degree shift).
- Define a pairing on H_n by the multiplication map

$$H_n^{-n+i} \otimes H_n^{n-i} \to H_n^n \cong \mathbb{R}.$$

- We define

$$L : H_n^i \longrightarrow H_n^{i+2}$$

  to be the linear map

$$L := \sum_{k}^{n} e_{\theta_k} \otimes e_{\xi_k}.$$

Theorem

For any natural numbers n, we have

(a) (PD) The bilinear pairing $\langle -,-\rangle : H_n \times H_n \longrightarrow \mathbb{R}$ is S_n-equivariant and graded non-degenerate;

(b) (HL) [Kim-Rhoades, 2020; G.-Xiong] $L : H_n^i \longrightarrow H_n^{i+2}$ is S_n-equivariant and satisfies the hard Lefschetz theorem;

(c) [G.-Xiong] No HR, which can be seen by the signature!

There is also an S_n-equivariant $\mathfrak{sl}_2$-action on the "usual" gradings of the tensor product of the polynomial ring. But neither Poincaré duality nor hard Lefschetz holds because the "usual" gradings are infinite-dimensional.

Features of constructions

- The constructions are purely algebraic and have no "geometric origin".
- The Poincaré pairing $\langle -, - \rangle$, the Lefschetz operator L, and the Lefschetz form $(a, b)_L^{-m+2i}$ are all S_n-equivariant, which is rare in Lefschetz theory.
- The adjoint ("down") operator f of the Lefschetz operator L can be written explicitly. In other algebraic/combinatorial setting, direct constructions of f seems difficult.
- The proof of hard Lefschetz and Hodge–Riemann relations is simple/natural and it takes advantage of the geometry of the product of projective spaces.

Questions

I would like to finish with some questions:

- Is there any geometric interpretation of these constructions?
- Do these equivariant Kähler packages, especially the Hodge–Riemann relations, have some implications for the diagonal coinvariant ring and the fermionic diagonal coinvariant ring?
- It would be interesting to know whether our approach could shed some new light on other (equivariant) log-concavity questions and conjectures.
- It is interesting to see whether similar (equivariant) Hodge-theoretic structures exist in the settings of other (equivalent) log-concavity conjectures.