Hyperbolic Geometry and Quantum Invariants

Tian Yang
Texas A&M University

Qinghua University, December 21, 2021

Part I: Overview

Two very different approaches to 3-manifolds:

Two very different approaches to 3-manifolds:

- Hyperbolic geometry

Two very different approaches to 3-manifolds:

- Hyperbolic geometry

  Thurston, Perelman, Agol ...

Two very different approaches to 3-manifolds:

- Hyperbolic geometry

  Thurston, Perelman, Agol ...

- Quantum invariants

Two very different approaches to 3-manifolds:

- Hyperbolic geometry

  Thurston, Perelman, Agol ...

- Quantum invariants

  Jones, Witten, Reshetikhin, Turaev ...

K = knot in S^3

K = knot in S^3

- Hyperbolic geometry

K = knot in S^3

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on $S^3 \smallsetminus K$.

K = knot in S^3

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on $S^3 \setminus K$.
  - Invariants such as the volume,

K = knot in S^3

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on $S^3 \setminus K$.
  - Invariants such as the volume, the Chern-Simons invariant

K = knot in S^3

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on $S^3 \setminus K$.
  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion

K = knot in S^3

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on $S^3 \smallsetminus K$.
  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.

K = knot in S^3

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on $S^3 \setminus K$.
  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.
- Quantum invariants

K = knot in S^3

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on $S^3 \setminus K$.
  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.
- Quantum invariants
  - Constructed using representation theory of quantum groups.

K = knot in S^3

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on $S^3 \setminus K$.
  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.
- Quantum invariants
  - Constructed using representation theory of quantum groups.
  - Colored Jones polynomials

K = knot in S^3

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on $S^3 \setminus K$.
  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.
- Quantum invariants
  - Constructed using representation theory of quantum groups.
  - Colored Jones polynomials $\mathrm{J}_n(K, t) \in \mathbb{Z}[t, t^{-1}]$,

K = knot in S^3

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on $S^3 \setminus K$.
  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.
- Quantum invariants
  - Constructed using representation theory of quantum groups.
  - Colored Jones polynomials $\mathrm{J}_n(K,t) \in \mathbb{Z}[t,t^{-1}]$, $n \in \mathbb{N}$.

K = knot in S^3

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on $S^3 \smallsetminus K$.
  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.
- Quantum invariants
  - Constructed using representation theory of quantum groups.
  - Colored Jones polynomials $\mathrm{J}_n(K, t) \in \mathbb{Z}[t, t^{-1}]$, $n \in \mathbb{N}$.

$\mathrm{J}_n($  $, t)$

$$= \sum_{j=0}^{n-1} \prod_{k=1}^{j} \left(t^{\frac{n-k}{2}} - t^{-\frac{n-k}{2}}\right)\left(t^{\frac{n+k}{2}} - t^{-\frac{n+k}{2}}\right)$$

- Volume Conjecture (Kashaev '97)

- Volume Conjecture (Kashaev '97) K hyperbolic knot,

- Volume Conjecture (Kashaev '97) K hyperbolic knot, then

$$\lim_{n\to\infty} \frac{2\pi}{n} \ln \left| \mathrm{J}_n(K, e^{\frac{2\pi i}{n}}) \right| = \mathrm{Vol}(S^3 \setminus K),$$

where $\mathrm{Vol}(S^3 \setminus K)$ is the volume of the hyperbolic metric on the knot complement.

- Volume Conjecture (Kashaev '97) K hyperbolic knot, then

$$\lim_{n\to\infty} \frac{2\pi}{n} \ln \left| \mathrm{J}_n(K, e^{\frac{2\pi i}{n}}) \right| = \mathrm{Vol}(S^3 \setminus K),$$

where $\mathrm{Vol}(S^3 \setminus K)$ is the volume of the hyperbolic metric on the knot complement. Namely,

$$\left| \mathrm{J}_n(K, e^{\frac{2\pi i}{n}}) \right| \asymp e^{\frac{n}{2\pi} \mathrm{Vol}(S^3 \setminus K)} \quad \text{as} \quad n \to \infty.$$

- Volume Conjecture (Kashaev '97) K hyperbolic knot, then

$$\lim_{n\to\infty} \frac{2\pi}{n} \ln \left| \mathrm{J}_n(K, e^{\frac{2\pi i}{n}}) \right| = \mathrm{Vol}(S^3 \setminus K),$$

where $\mathrm{Vol}(S^3 \setminus K)$ is the volume of the hyperbolic metric on the knot complement. Namely,

$$\left| \mathrm{J}_n(K, e^{\frac{2\pi i}{n}}) \right| \asymp e^{\frac{n}{2\pi} \mathrm{Vol}(S^3 \setminus K)} \quad \text{as} \quad n \to \infty.$$

- There are knots with same volume, but different colored Jones polynomials.

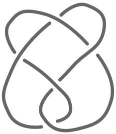
- Volume Conjecture (Kashaev '97) K hyperbolic knot, then

$$\lim_{n\to\infty} \frac{2\pi}{n} \ln \left| \mathrm{J}_n(K, e^{\frac{2\pi i}{n}}) \right| = \mathrm{Vol}(S^3 \setminus K),$$

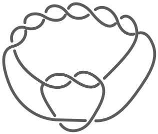
  where $\mathrm{Vol}(S^3 \setminus K)$ is the volume of the hyperbolic metric on the knot complement. Namely,

$$\left| \mathrm{J}_n(K, e^{\frac{2\pi i}{n}}) \right| \asymp e^{\frac{n}{2\pi} \mathrm{Vol}(S^3 \setminus K)} \quad \text{as} \quad n \to \infty.$$

- There are knots with same volume, but different colored Jones polynomials.



5_2 knot



$(-2, 3, 7)$-pretzel knot

- **Volume Conjecture** (Kashaev '97) K hyperbolic knot, then

$$\lim_{n\to\infty} \frac{2\pi}{n} \ln \left| \mathrm{J}_n(K, e^{\frac{2\pi i}{n}}) \right| = \mathrm{Vol}(S^3 \setminus K),$$

where $\mathrm{Vol}(S^3 \setminus K)$ is the volume of the hyperbolic metric on the knot complement. Namely,

$$\left| \mathrm{J}_n(K, e^{\frac{2\pi i}{n}}) \right| \asymp e^{\frac{n}{2\pi}\mathrm{Vol}(S^3 \setminus K)} \quad \text{as} \quad n \to \infty.$$

- **Asymptotic Expansion Conjecture** (Witten '89, Gukov '05)

$$\mathrm{J}_n(K, e^{\frac{2\pi i}{n}}) = \frac{n^{\frac{3}{2}}}{\sqrt{\mathrm{Tor}(S^3 \setminus K)}} e^{\frac{n}{2\pi}(\mathrm{Vol}(S^3 \setminus K) + i\mathrm{CS}(S^3 \setminus K))} \left(1 + O\left(\frac{1}{n}\right)\right)$$

as $n \to \infty$,

- Volume Conjecture (Kashaev '97) K hyperbolic knot, then

$$\lim_{n\to\infty} \frac{2\pi}{n} \ln \left| \mathrm{J}_n(K, e^{\frac{2\pi i}{n}}) \right| = \mathrm{Vol}(S^3 \setminus K),$$

where $\mathrm{Vol}(S^3 \setminus K)$ is the volume of the hyperbolic metric on the knot complement. Namely,

$$\left| \mathrm{J}_n(K, e^{\frac{2\pi i}{n}}) \right| \asymp e^{\frac{n}{2\pi}\mathrm{Vol}(S^3 \setminus K)} \quad \text{as} \quad n \to \infty.$$

- Asymptotic Expansion Conjecture (Witten '89, Gukov '05)

$$\mathrm{J}_n(K, e^{\frac{2\pi i}{n}}) = \frac{n^{\frac{3}{2}}}{\sqrt{\mathrm{Tor}(S^3 \setminus K)}} e^{\frac{n}{2\pi}(\mathrm{Vol}(S^3 \setminus K) + i\mathrm{CS}(S^3 \setminus K))} \left(1 + O\left(\frac{1}{n}\right)\right)$$

as $n \to \infty$, where $\mathrm{CS}(S^3 \setminus K)$ is the Chern-Simons invariant and $\mathrm{Tor}(S^3 \setminus K)$ is the adjoint twisted Reidemeister torsion of the hyperbolic metric on the knot complement.

M = 3-manifold

M = 3-manifold (closed, oriented ...)

M = 3-manifold (closed, oriented ...)

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on M.
  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.

M = 3-manifold (closed, oriented ...)

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on M.
  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.
- Quantum invariants
  - Constructed using representation theory of quantum groups.

M = 3-manifold (closed, oriented ...)

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on M.
  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.
- Quantum invariants
  - Constructed using representation theory of quantum groups.
  - Reshetikhin-Turaev invariants

M = 3-manifold (closed, oriented ...)

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on M.
  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.
- Quantum invariants
  - Constructed using representation theory of quantum groups.
  - Reshetikhin-Turaev invariants $\mathrm{RT}_r(M, q) \in \mathbb{C}$,

M = 3-manifold (closed, oriented ...)

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on M.
  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.
- Quantum invariants
  - Constructed using representation theory of quantum groups.
  - Reshetikhin-Turaev invariants $\mathrm{RT}_r(M, q) \in \mathbb{C}$, $r \in \mathbb{N}$.

M = 3-manifold (closed, oriented ...)

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on M.
  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.

- Quantum invariants
  - Constructed using representation theory of quantum groups.
  - Reshetikhin-Turaev invariants $\mathrm{RT}_r(M, q) \in \mathbb{C}$, $r \in \mathbb{N}$.
  - Mathematical realization of Witten's invariants

M = 3-manifold (closed, oriented ...)

- Hyperbolic geometry
  - Complete metric of constant curvature -1 on M.
  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.
- Quantum invariants
  - Constructed using representation theory of quantum groups.
  - Reshetikhin-Turaev invariants $\mathrm{RT}_r(M, q) \in \mathbb{C}$, $r \in \mathbb{N}$.
  - Mathematical realization of Witten's invariants (at $q = e^{\frac{\pi i}{r}}$).

- Conjecture (Chen-Y. '15)

- Conjecture (Chen-Y. '15) M closed oriented hyperbolic,

- Conjecture (Chen-Y. '15) M closed oriented hyperbolic, then

$$\lim_{\substack{r\to\infty \\ odd}} \frac{4\pi}{r}\ln\left|\mathrm{RT}_r\left(M, e^{\frac{2\pi i}{r}}\right)\right| = \mathrm{Vol}(M),$$

where $\mathrm{Vol}(M)$ is the volume of the hyperbolic metric on M.

- Conjecture (Chen-Y. '15) M closed oriented hyperbolic, then

$$\lim_{\substack{r \to \infty \\ odd}} \frac{4\pi}{r} \ln \left| \mathrm{RT}_r\left(M, e^{\frac{2\pi i}{r}}\right) \right| = \mathrm{Vol}(M),$$

where $\mathrm{Vol}(M)$ is the volume of the hyperbolic metric on M. Namely,

$$\left| \mathrm{RT}_r\left(M, e^{\frac{2\pi i}{r}}\right) \right| \asymp e^{\frac{r}{4\pi}\mathrm{Vol}(M)} \quad \text{as} \quad r \to \infty, \ \text{odd}.$$

- Conjecture (Chen-Y. '15) M closed oriented hyperbolic, then

$$\lim_{\substack{r\to\infty\\ odd}} \frac{4\pi}{r}\ln\left|\mathrm{RT}_r\left(M, e^{\frac{2\pi i}{r}}\right)\right| = \mathrm{Vol}(M),$$

where $\mathrm{Vol}(M)$ is the volume of the hyperbolic metric on M. Namely,

$$\left|\mathrm{RT}_r\left(M, e^{\frac{2\pi i}{r}}\right)\right| \asymp e^{\frac{r}{4\pi}\mathrm{Vol}(M)} \quad \text{as} \quad r\to\infty, \ \text{odd}.$$

- Conjecture (Ohtsuki '16, Gang-Romo-Yamazaki '16)

$$\mathrm{RT}_r\left(M, e^{\frac{2\pi i}{r}}\right) = \frac{1}{\sqrt{\mathrm{Tor}(M)}} e^{\frac{r}{4\pi}(\mathrm{Vol}(M)+i\mathrm{CS}(M))}\left(1+O\left(\frac{1}{r}\right)\right)$$

as $r\to\infty$, odd, where $\mathrm{CS}(M)$ is the Chern-Simons invariant and $\mathrm{Tor}(M)$ is the adjoint twisted Reidemeister torsion of the hyperbolic metric on M.

- Volume Conjecture (Kashaev '97) K hyperbolic knot, then

$$\lim_{n\to\infty} \frac{2\pi}{n} \ln \left| \mathrm{J}_n(K, e^{\frac{2\pi i}{n}}) \right| = \mathrm{Vol}(S^3 \setminus K),$$

where $\mathrm{Vol}(S^3 \setminus K)$ is the volume of the hyperbolic metric on the knot complement. Namely,

$$\left| \mathrm{J}_n(K, e^{\frac{2\pi i}{n}}) \right| \asymp e^{\frac{n}{2\pi} \mathrm{Vol}(S^3 \setminus K)} \quad \text{as} \quad n \to \infty.$$

- Asymptotic Expansion Conjecture (Witten '89, Gukov '05)

$$\mathrm{J}_n(K, e^{\frac{2\pi i}{n}}) = \frac{n^{\frac{3}{2}}}{\sqrt{\mathrm{Tor}(S^3 \setminus K)}} e^{\frac{n}{2\pi}(\mathrm{Vol}(S^3 \setminus K) + i\mathrm{CS}(S^3 \setminus K))} \Big(1 + O\big(\frac{1}{n}\big)\Big)$$

as $n \to \infty$, where $\mathrm{CS}(S^3 \setminus K)$ is the Chern-Simons invariant and $\mathrm{Tor}(S^3 \setminus K)$ is the adjoint twisted Reidemeister torsion of the hyperbolic metric on the knot complement.

Part II: Asymptotics of relative Reshetikhin-Turaev invariants

Part II: Asymptotics of relative Reshetikhin-Turaev invariants

Joint work with Ka Ho Wong

$(M, L) = (\text{3-manifold, link in } M)$

$(M, L) = (\text{3-manifold, link in } M)$

- Hyperbolic geometry
  - Hyperbolic cone metric

(M, L) = (3-manifold, link in M)

- Hyperbolic geometry
  - Hyperbolic cone metric (metric on M with constant curvature -1 on $M \setminus L$ and singular locus L).

(M, L) = (3-manifold, link in M)

- Hyperbolic geometry
  - Hyperbolic cone metric (metric on M with constant curvature -1 on $M \setminus L$ and singular locus L).
  - cone angles $= 0 \iff$ complete hyperbolic metric on $M \setminus L$.

(M, L) = (3-manifold, link in M)

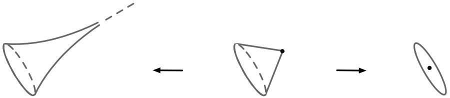
- Hyperbolic geometry
  - Hyperbolic cone metric (metric on M with constant curvature -1 on $M \setminus L$ and singular locus L).
  - cone angles $= 0 \iff$ complete hyperbolic metric on $M \setminus L$.

    cone angles $= 2\pi \iff$ complete hyperbolic metric on M.

$(M, L) = $ (3-manifold, link in M)

- Hyperbolic geometry
  - Hyperbolic cone metric (metric on M with constant curvature -1 on $M \setminus L$ and singular locus L).
  - cone angles $= 0 \iff$ complete hyperbolic metric on $M \setminus L$.

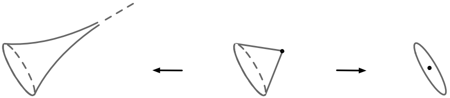
    cone angles $= 2\pi \iff$ complete hyperbolic metric on M.



$(M, L) = (\text{3-manifold, link in } M)$

- Hyperbolic geometry
  - Hyperbolic cone metric (metric on M with constant curvature -1 on $M \setminus L$ and singular locus L).
  - cone angles $= 0 \iff$ complete hyperbolic metric on $M \setminus L$.

    cone angles $= 2\pi \iff$ complete hyperbolic metric on M.

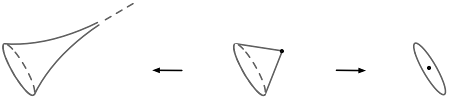


  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.

$(M, L) = $ (3-manifold, link in M)

- Hyperbolic geometry
  - Hyperbolic cone metric (metric on M with constant curvature -1 on $M \setminus L$ and singular locus L).
  - cone angles $= 0 \iff$ complete hyperbolic metric on $M \setminus L$.

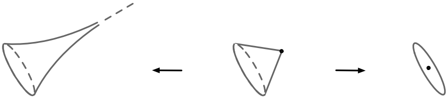
    cone angles $= 2\pi \iff$ complete hyperbolic metric on M.



  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.
- Quantum invariants
  - Relative Reshetikhin-Turaev invariants

$(M, L) = (\text{3-manifold, link in } M)$

- Hyperbolic geometry
  - Hyperbolic cone metric (metric on M with constant curvature -1 on $M \setminus L$ and singular locus L).
  - cone angles $= 0 \iff$ complete hyperbolic metric on $M \setminus L$.
    cone angles $= 2\pi \iff$ complete hyperbolic metric on M.

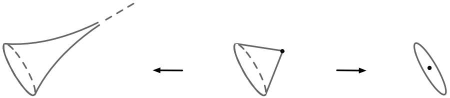


  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.
- Quantum invariants
  - Relative Reshetikhin-Turaev invariants $\mathrm{RT}_r(M, L, \mathrm{n}) \in \mathbb{C}$,

(M, L) = (3-manifold, link in M)

- Hyperbolic geometry
  - Hyperbolic cone metric (metric on M with constant curvature -1 on $M \setminus L$ and singular locus L).
  - cone angles $= 0 \iff$ complete hyperbolic metric on $M \setminus L$.

    cone angles $= 2\pi \iff$ complete hyperbolic metric on M.

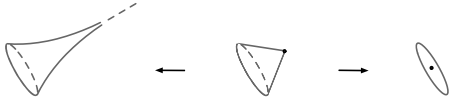


  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.
- Quantum invariants
  - Relative Reshetikhin-Turaev invariants $\mathrm{RT}_r(M, L, \mathrm{n}) \in \mathbb{C}$,

    $r \in \mathbb{N}$,

$(M, L) = (\text{3-manifold, link in } M)$

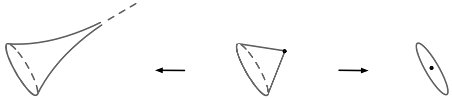
- Hyperbolic geometry
  - Hyperbolic cone metric (metric on M with constant curvature -1 on $M \setminus L$ and singular locus L).
  - cone angles $= 0 \iff$ complete hyperbolic metric on $M \setminus L$.
    cone angles $= 2\pi \iff$ complete hyperbolic metric on M.



  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.
- Quantum invariants
  - Relative Reshetikhin-Turaev invariants $\mathrm{RT}_r(M, L, \mathrm{n}) \in \mathbb{C}$,
    $r \in \mathbb{N},\ \mathrm{n} = (n_1, \ldots, n_{|L|})$,

$(M, L) = $ (3-manifold, link in M)

- Hyperbolic geometry
  - Hyperbolic cone metric (metric on M with constant curvature -1 on $M \setminus L$ and singular locus L).
  - cone angles $= 0 \iff$ complete hyperbolic metric on $M \setminus L$.
    cone angles $= 2\pi \iff$ complete hyperbolic metric on M.

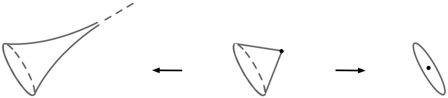


  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.
- Quantum invariants
  - Relative Reshetikhin-Turaev invariants $\mathrm{RT}_r(M, L, \mathrm{n}) \in \mathbb{C}$,
    $r \in \mathbb{N}$, $\mathrm{n} = (n_1, \ldots, n_{|L|})$, $n_k \in \{0, \ldots, r-2\}$.

(M, L) = (3-manifold, link in M)

- Hyperbolic geometry
  - Hyperbolic cone metric (metric on M with constant curvature -1 on $M \setminus L$ and singular locus L).
  - cone angles $= 0 \iff$ complete hyperbolic metric on $M \setminus L$.

    cone angles $= 2\pi \iff$ complete hyperbolic metric on M.



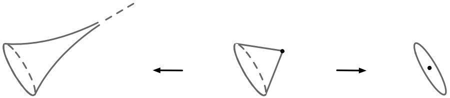
  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.
- Quantum invariants
  - Relative Reshetikhin-Turaev invariants $\mathrm{RT}_r(M, L, \mathrm{n}) \in \mathbb{C}$,

    $r \in \mathbb{N}$, $\mathrm{n} = (n_1, \ldots, n_{|L|})$, $n_k \in \{0, \ldots, r-2\}$.
  - $M = S^3 \implies \mathrm{RT}_r(M, L, n) = \mathrm{J}_n(L, q^2)$.

(M, L) = (3-manifold, link in M)

- Hyperbolic geometry
  - Hyperbolic cone metric (metric on M with constant curvature -1 on $M \setminus L$ and singular locus L).
  - cone angles $= 0 \iff$ complete hyperbolic metric on $M \setminus L$.

    cone angles $= 2\pi \iff$ complete hyperbolic metric on M.



  - Invariants such as the volume, the Chern-Simons invariant and the twisted Reidemeister torsion of this metric.

- Quantum invariants
  - Relative Reshetikhin-Turaev invariants $\mathrm{RT}_r(M, L, \mathrm{n}) \in \mathbb{C}$, $r \in \mathbb{N}$, $\mathrm{n} = (n_1, \ldots, n_{|L|})$, $n_k \in \{0, \ldots, r-2\}$.
  - $M = S^3 \implies \mathrm{RT}_r(M, L, n) = \mathrm{J}_n(L, q^2)$.

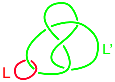
    $\mathrm{n} = 0 \implies \mathrm{RT}_r(M, L, \mathrm{n}) = \mathrm{RT}_r(M)$.

- M = obtained from surgery along $L' \subset S^3$,

- M = obtained from surgery along $L' \subset S^3$, $L \subset S^3 \setminus L' \subset M$.

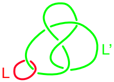
- M = obtained from surgery along $L' \subset S^3$, $L \subset S^3 \setminus L' \subset M$.

$$(M, L) =$$



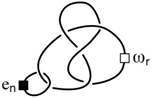
- M = obtained from surgery along $L' \subset S^3$, $L \subset S^3 \setminus L' \subset M$.

$$(M, L) =$$



- Definition (Reshetikhin-Turaev '91, Lickorish '93)

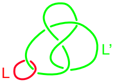
$$\mathrm{RT}_r(M, L, \mathrm{n}) = c \cdot \Big\langle$$



$$\Big\rangle,$$

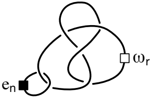
- M = obtained from surgery along $L' \subset S^3$, $L \subset S^3 \setminus L' \subset M$.

$$(M, L) =$$



- Definition (Reshetikhin-Turaev '91, Lickorish '93)

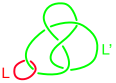
$$\mathrm{RT}_r(M, L, \mathrm{n}) = c \cdot \Big\langle \quad \Big\rangle,$$



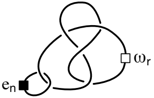
where

- ω_r = Kirby coloring, e_n = n-th Jones-Wenzl projector.

- M = obtained from surgery along $L' \subset S^3$, $L \subset S^3 \setminus L' \subset M$.

$(M, L) =$ 

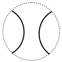
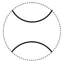
- Definition (Reshetikhin-Turaev '91, Lickorish '93)

$\mathrm{RT}_r(M, L, \mathrm{n}) = c \cdot \Big\langle$ 

$\Big\rangle$,

where

  - ω_r = Kirby coloring, e_n = n-th Jones-Wenzl projector.
  - $\langle\ \rangle$ = Kauffman bracket given by

(1) $\Big\langle$  $\Big\rangle = q^{\frac{1}{2}} \Big\langle$  $\Big\rangle + q^{-\frac{1}{2}} \Big\langle$  $\Big\rangle$,

(2) $\Big\langle$  $\sqcup D \Big\rangle = (-q - q^{-1})\langle D \rangle$.

- Conjecture 1 (Wong-Y. '20)

- Conjecture 1 (Wong-Y. '20) (M, L).

- Conjecture 1 (Wong-Y. '20) (M, L).
  $\mathrm{n}^{(r)} = (n_1^{(r)}, \dots, n_{|L|}^{(r)}), \quad n_k^{(r)} \in \{0, ..., r-2\},$

- Conjecture 1 (Wong-Y. '20) (M, L).
  $\mathrm{n}^{(r)} = (n_1^{(r)}, \dots, n_{|L|}^{(r)}), \quad n_k^{(r)} \in \{0, ..., r-2\}$, such that

- Conjecture 1 (Wong-Y. '20) (M, L).

  $\mathrm{n}^{(r)} = (n_1^{(r)}, \dots, n_{|L|}^{(r)}), \quad n_k^{(r)} \in \{0, ..., r-2\}$, such that

  (1) $\theta = (\theta_1, \dots, \theta_{|L|}), \quad \theta_k = \lim_{r\to\infty} \left| \frac{4\pi n_k^{(r)}}{r} - 2\pi \right|$ exists,

- **Conjecture 1** (Wong-Y. '20) (M, L).
  $\mathrm{n}^{(r)} = (n_1^{(r)}, \dots, n_{|L|}^{(r)}), \quad n_k^{(r)} \in \{0, ..., r-2\}$, such that
  (1) $\theta = (\theta_1, \dots, \theta_{|L|}), \quad \theta_k = \lim_{r\to\infty} \left| \frac{4\pi n_k^{(r)}}{r} - 2\pi \right|$ exists,
  (2) $M_{L_\theta} = M$ with a hyperbolic cone metric with cone angles θ along L exists.

- **Conjecture 1** (Wong-Y. '20) (M, L).

  $\mathrm{n}^{(r)} = (n_1^{(r)}, \dots, n_{|L|}^{(r)}), \quad n_k^{(r)} \in \{0, ..., r-2\}$, such that

  (1) $\theta = (\theta_1, \dots, \theta_{|L|}), \quad \theta_k = \lim_{r\to\infty} \left| \frac{4\pi n_k^{(r)}}{r} - 2\pi \right|$ exists,

  (2) $M_{L_\theta} = M$ with a hyperbolic cone metric with cone angles θ along L exists.

  Then

$$\lim_{\substack{r\to\infty \\ odd}} \frac{4\pi}{r} \ln \left| \mathrm{RT}_r\big(M, L, \mathrm{n}^{(r)}\big) \right| = \mathrm{Vol}(M_{L_\theta}).$$

- Conjecture 1 (Wong-Y. '20) (M, L).

  $\mathrm{n}^{(r)} = (n_1^{(r)}, \dots, n_{|L|}^{(r)})$, $\quad n_k^{(r)} \in \{0, ..., r-2\}$, such that

  (1) $\theta = (\theta_1, \dots, \theta_{|L|})$, $\quad \theta_k = \lim_{r\to\infty} \left| \frac{4\pi n_k^{(r)}}{r} - 2\pi \right|$ exists,

  (2) $M_{L_\theta} = M$ with a hyperbolic cone metric with cone angles θ along L exists.

  Then

$$\lim_{\substack{r \to \infty \\ odd}} \frac{4\pi}{r} \ln \left| \mathrm{RT}_r\big(M, L, \mathrm{n}^{(r)}\big) \right| = \mathrm{Vol}(M_{L_\theta}).$$

- Conjecture 2 (Wong-Y. '21)

$$\mathrm{RT}_r\big(M, L, \mathrm{n}^{(r)}\big) = \frac{e^{\frac{1}{2}\sum \mathrm{l}_k}}{\sqrt{\mathrm{Tor}(M_{L_\theta})}} e^{\frac{r}{4\pi}(\mathrm{Vol}(M_{L_\theta}) + i\mathrm{CS}(M_{L_\theta}))} \Big(1 + O\big(\frac{1}{r}\big)\Big)$$

  as $r \to \infty$, odd,

- Conjecture 1 (Wong-Y. '20) (M, L).

  $\mathrm{n}^{(r)} = (n_1^{(r)}, \dots, n_{|L|}^{(r)})$, $\quad n_k^{(r)} \in \{0, ..., r-2\}$, such that

  (1) $\theta = (\theta_1, \dots, \theta_{|L|})$, $\quad \theta_k = \lim_{r\to\infty} \left| \frac{4\pi n_k^{(r)}}{r} - 2\pi \right|$ exists,

  (2) $M_{L_\theta} = M$ with a hyperbolic cone metric with cone angles θ along L exists.

  Then

$$\lim_{\substack{r\to\infty \\ odd}} \frac{4\pi}{r} \ln \left| \mathrm{RT}_r\left(M, L, \mathrm{n}^{(r)}\right) \right| = \mathrm{Vol}(M_{L_\theta}).$$

- Conjecture 2 (Wong-Y. '21)

$$\mathrm{RT}_r\left(M, L, \mathrm{n}^{(r)}\right) = \frac{e^{\frac{1}{2}\sum \mathrm{l}_k}}{\sqrt{\mathrm{Tor}(M_{L_\theta})}} e^{\frac{r}{4\pi}(\mathrm{Vol}(M_{L_\theta}) + i\mathrm{CS}(M_{L_\theta}))} \left(1 + O\left(\frac{1}{r}\right)\right)$$

  as $r \to \infty$, odd, where l_k is the length of the k-th component of L in M_{L_θ}.

- Theorem 1 (Wong-Y. '20,'21) Conjecture 1 & 2 are true for pairs (M, L) such that

- **Theorem 1** (Wong-Y. '20,'21) Conjecture 1 & 2 are true for pairs (M, L) such that
  (1) $M \setminus L \cong$ a fundamental shadow link complement,

- **Theorem 1** (Wong-Y. '20,'21) Conjecture 1 & 2 are true for pairs (M, L) such that
  (1) $M \setminus L \cong$ a fundamental shadow link complement,
  (2) M is obtained from the fslc by an integral surgery,

- Theorem 1 (Wong-Y. '20,'21) Conjecture 1 & 2 are true for pairs (M, L) such that
  (1) $M \diagdown L \cong$ a fundamental shadow link complement,
  (2) M is obtained from the fslc by an integral surgery,
  (3) cone angles θ are sufficiently small.

- Theorem 1 (Wong-Y. '20,'21) Conjecture 1 & 2 are true for pairs (M, L) such that
  (1) $M \setminus L \cong$ a fundamental shadow link complement,
  (2) M is obtained from the fslc by an integral surgery,
  (3) cone angles θ are sufficiently small.
- (Costantino-Thurston '08) Every closed oriented 3-manifold is obtained from some fundamental shadow link complement by an integral surgery.

- Theorem 1 (Wong-Y. '20,'21) Conjecture 1 & 2 are true for pairs (M, L) such that
  (1) $M \setminus L \cong$ a fundamental shadow link complement,
  (2) M is obtained from the fslc by an integral surgery,
  (3) cone angles θ are sufficiently small.
- (Costantino-Thurston '08) Every closed oriented 3-manifold is obtained from some fundamental shadow link complement by an integral surgery.
- If can push θ from small to 2π, then solve the Volume Conjecture for all closed oriented hyperbolic 3-manifolds!

- Theorem 1 (Wong-Y. '20,'21) Conjecture 1 & 2 are true for pairs (M, L) such that
  (1) $M \setminus L \cong$ a fundamental shadow link complement,
  (2) M is obtained from the fslc by an integral surgery,
  (3) cone angles θ are sufficiently small.

- (Costantino-Thurston '08) Every closed oriented 3-manifold is obtained from some fundamental shadow link complement by an integral surgery.

- If can push θ from small to 2π, then solve the Volume Conjecture for all closed oriented hyperbolic 3-manifolds!

- Theorem 2 (Wong-Y. '21) Conjectures 1 & 2 are true for pairs (M, L) such that

$$M \setminus L \cong S^3 \setminus$$ 

  and M is obtained by integral surgery,

- Theorem 1 (Wong-Y. '20,'21) Conjecture 1 & 2 are true for pairs (M, L) such that
  1. $M \setminus L \cong$ a fundamental shadow link complement,
  2. M is obtained from the fslc by an integral surgery,
  3. cone angles θ are sufficiently small.
- (Costantino-Thurston '08) Every closed oriented 3-manifold is obtained from some fundamental shadow link complement by an integral surgery.
- If can push θ from small to 2π, then solve the Volume Conjecture for all closed oriented hyperbolic 3-manifolds!
- Theorem 2 (Wong-Y. '21) Conjectures 1 & 2 are true for pairs (M, L) such that

$$M \setminus L \cong S^3 \setminus$$ 

  and M is obtained by integral surgery, for all $\theta \in [0, 2\pi]$.

Part III: An explicit formula of adjoint twisted Reidemeister torsion

Joint work with Ka Ho Wong

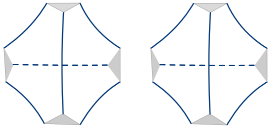
Fundamental shadow link complement

Fundamental shadow link complement

- Take truncated tetrahedra $\Delta_1, \ldots, \Delta_n$.

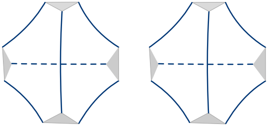
Fundamental shadow link complement

- Take truncated tetrahedra $\Delta_1, \ldots, \Delta_n$.



Fundamental shadow link complement

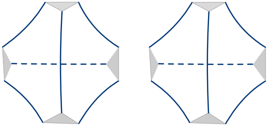
- Take truncated tetrahedra $\Delta_1, \ldots, \Delta_n$.



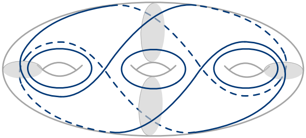
- Glue along triangles to get a handlebody.

Fundamental shadow link complement

- Take truncated tetrahedra $\Delta_1, \ldots, \Delta_n$.

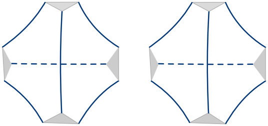


- Glue along triangles to get a handlebody.

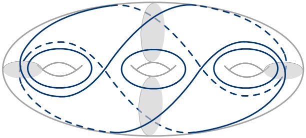


Fundamental shadow link complement

- Take truncated tetrahedra $\Delta_1, \ldots, \Delta_n$.



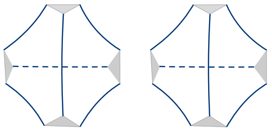
- Glue along triangles to get a handlebody.



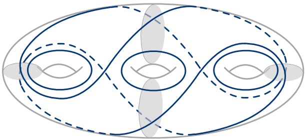
- DH = double of handlebody.

Fundamental shadow link complement

- Take truncated tetrahedra $\Delta_1, \ldots, \Delta_n$.



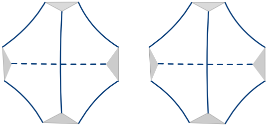
- Glue along triangles to get a handlebody.



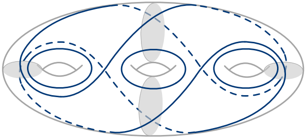
- DH = double of handlebody. L = union of edges.

Fundamental shadow link complement

- Take truncated tetrahedra $\Delta_1, \dots, \Delta_n$.



- Glue along triangles to get a handlebody.



- DH = double of handlebody. L = union of edges.
- $DH \setminus L$ = fundamental shadow link complement.

- Gram matrix function: For $\mathrm{u} = (u_1, \ldots, u_6) \in \mathbb{C}^6$,

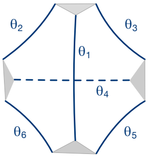
- Gram matrix function: For $\mathrm{u} = (u_1, \ldots, u_6) \in \mathbb{C}^6$,

$$\mathrm{G}(\mathrm{u}) = \begin{bmatrix} 1 & -\cosh u_1 & -\cosh u_2 & -\cosh u_6 \\ -\cosh u_1 & 1 & -\cosh u_3 & -\cosh u_5 \\ -\cosh u_2 & -\cosh u_3 & 1 & -\cosh u_4 \\ -\cosh u_6 & -\cosh u_5 & -\cosh u_4 & 1 \end{bmatrix}.$$

- Gram matrix function: For $\mathrm{u} = (u_1, \ldots, u_6) \in \mathbb{C}^6$,

$$\mathrm{G}(\mathrm{u}) = \begin{bmatrix} 1 & -\cosh u_1 & -\cosh u_2 & -\cosh u_6 \\ -\cosh u_1 & 1 & -\cosh u_3 & -\cosh u_5 \\ -\cosh u_2 & -\cosh u_3 & 1 & -\cosh u_4 \\ -\cosh u_6 & -\cosh u_5 & -\cosh u_4 & 1 \end{bmatrix}.$$

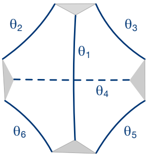
- If $\mathrm{u} =$ dihedral angles of a hyperbolic tetrahedron $\times i$,



- Gram matrix function: For $\mathrm{u} = (u_1, \ldots, u_6) \in \mathbb{C}^6$,

$$\mathrm{G}(\mathrm{u}) = \begin{bmatrix} 1 & -\cosh u_1 & -\cosh u_2 & -\cosh u_6 \\ -\cosh u_1 & 1 & -\cosh u_3 & -\cosh u_5 \\ -\cosh u_2 & -\cosh u_3 & 1 & -\cosh u_4 \\ -\cosh u_6 & -\cosh u_5 & -\cosh u_4 & 1 \end{bmatrix}.$$

- If $\mathrm{u} =$ dihedral angles of a hyperbolic tetrahedron $\times i$,

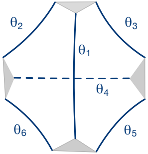


then $\mathrm{G}(\mathrm{u})$ coincides with Gram matrix of the tetrahedron.

- Gram matrix of tetrahedron:

$$G = \begin{bmatrix} 1 & -\cos\theta_1 & -\cos\theta_2 & -\cos\theta_6 \\ -\cos\theta_1 & 1 & -\cos\theta_3 & -\cos\theta_5 \\ -\cos\theta_2 & -\cos\theta_3 & 1 & -\cos\theta_4 \\ -\cos\theta_6 & -\cos\theta_5 & -\cos\theta_4 & 1 \end{bmatrix}.$$

- If $u =$ dihedral angles of a hyperbolic tetrahedron $\times i$,



then $G(u)$ coincides with Gram matrix of the tetrahedron.

- Theorem 3 (Wong-Y. '21) M obtained from a fundamental shadow link complement (DH, L) by a hyperbolic surgery.

- Theorem 3 (Wong-Y. '21) M obtained from a fundamental shadow link complement (DH, L) by a hyperbolic surgery. Then the Reidemeister torsion twisted by the adjoint action of the holonomy representation

- Theorem 3 (Wong-Y. '21) M obtained from a fundamental shadow link complement (DH, L) by a hyperbolic surgery. Then the Reidemeister torsion twisted by the adjoint action of the holonomy representation

$$\mathrm{Tor}(M) = \det\left(\frac{\partial \mu_i}{\partial u_j}\right) \cdot \prod_{i=1}^{|L|} \frac{1}{\sinh^2 \frac{\gamma_i}{2}} \cdot \prod_{k=1}^{n} \sqrt{\det \mathrm{G}\left(\frac{\mathrm{u}_k}{2}\right)},$$

- Theorem 3 (Wong-Y. '21) M obtained from a fundamental shadow link complement (DH, L) by a hyperbolic surgery. Then the Reidemeister torsion twisted by the adjoint action of the holonomy representation

$$\mathrm{Tor}(M) = \det\left(\frac{\partial\mu_i}{\partial u_j}\right) \cdot \prod_{i=1}^{|L|} \frac{1}{\sinh^2\frac{\gamma_i}{2}} \cdot \prod_{k=1}^{n} \sqrt{\det \mathrm{G}\left(\frac{\mathrm{u}_k}{2}\right)},$$

where u_i is the holonomy of the meridian of the i-th component

- Theorem 3 (Wong-Y. '21) M obtained from a fundamental shadow link complement (DH, L) by a hyperbolic surgery. Then the Reidemeister torsion twisted by the adjoint action of the holonomy representation

$$\mathrm{Tor}(M) = \det\left(\frac{\partial \mu_i}{\partial u_j}\right) \cdot \prod_{i=1}^{|L|} \frac{1}{\sinh^2 \frac{\gamma_i}{2}} \cdot \prod_{k=1}^{n} \sqrt{\det \mathrm{G}\left(\frac{\mathrm{u}_k}{2}\right)},$$

where u_i is the holonomy of the meridian of the i-th component and $\mathrm{u}_k = (u_{k_1}, \dots, u_{k_6})$ are the holonomy of the meridians intersecting the tetrahedron Δ_k;

- Theorem 3 (Wong-Y. '21) M obtained from a fundamental shadow link complement (DH, L) by a hyperbolic surgery. Then the Reidemeister torsion twisted by the adjoint action of the holonomy representation

$$\mathrm{Tor}(M) = \det\left(\frac{\partial \mu_i}{\partial u_j}\right) \cdot \prod_{i=1}^{|L|} \frac{1}{\sinh^2 \frac{\gamma_i}{2}} \cdot \prod_{k=1}^{n} \sqrt{\det \mathrm{G}\left(\frac{\mathrm{u}_k}{2}\right)},$$

where u_i is the holonomy of the meridian of the i-th component and $\mathrm{u}_k = (u_{k_1}, \ldots, u_{k_6})$ are the holonomy of the meridians intersecting the tetrahedron Δ_k; μ_i is the holonomy of the curve on the i-th component along which you do the surgery,

- Theorem 3 (Wong-Y. '21) M obtained from a fundamental shadow link complement (DH, L) by a hyperbolic surgery. Then the Reidemeister torsion twisted by the adjoint action of the holonomy representation

$$\mathrm{Tor}(M) = \det\left(\frac{\partial \mu_i}{\partial u_j}\right) \cdot \prod_{i=1}^{|L|} \frac{1}{\sinh^2 \frac{\gamma_i}{2}} \cdot \prod_{k=1}^{n} \sqrt{\det \mathrm{G}\left(\frac{\mathrm{u}_k}{2}\right)},$$

where u_i is the holonomy of the meridian of the i-th component and $\mathrm{u}_k = (u_{k_1}, \dots, u_{k_6})$ are the holonomy of the meridians intersecting the tetrahedron Δ_k; μ_i is the holonomy of the curve on the i-th component along which you do the surgery, γ_i is the holonomy of the core curve of the solid torus filled in,

- Theorem 3 (Wong-Y. '21) M obtained from a fundamental shadow link complement (DH, L) by a hyperbolic surgery. Then the Reidemeister torsion twisted by the adjoint action of the holonomy representation

$$\mathrm{Tor}(M) = \det\left(\frac{\partial \mu_i}{\partial u_j}\right) \cdot \prod_{i=1}^{|L|} \frac{1}{\sinh^2 \frac{\gamma_i}{2}} \cdot \prod_{k=1}^{n} \sqrt{\det \mathrm{G}\left(\frac{\mathrm{u}_k}{2}\right)},$$

where u_i is the holonomy of the meridian of the i-th component and $\mathrm{u}_k = (u_{k_1}, \dots, u_{k_6})$ are the holonomy of the meridians intersecting the tetrahedron Δ_k; μ_i is the holonomy of the curve on the i-th component along which you do the surgery, γ_i is the holonomy of the core curve of the solid torus filled in, and G is the Gram matrix function.

- Conjecture (Ohtsuki '16, Gang-Romo-Yamazaki '16)

$$\mathrm{RT}_r(M) = \frac{1}{\sqrt{\mathrm{Tor}(M)}} e^{\frac{r}{4\pi}(\mathrm{Vol}(M) + i\mathrm{CS}(M))} \Big(1 + O\big(\frac{1}{r}\big)\Big)$$

as $r \to \infty$, odd.

- Conjecture 2 (Wong-Y. '21)

$$\mathrm{RT}_r(M, L, \mathrm{n}^{(r)}) = \frac{e^{\frac{1}{2}\sum \mathrm{l}_k}}{\sqrt{\mathrm{Tor}(M_{L_\theta})}} e^{\frac{r}{4\pi}(\mathrm{Vol}(M_{L_\theta}) + i\mathrm{CS}(M_{L_\theta}))} \Big(1 + O\big(\frac{1}{r}\big)\Big)$$

as $r \to \infty$, odd.

- Theorem 3 (Wong-Y. '21) M obtained from a fundamental shadow link complement (DH, L) by a hyperbolic surgery. Then the Reidemeister torsion twisted by the adjoint action of the holonomy representation

$$\mathrm{Tor}(M) = \det\left(\frac{\partial \mu_i}{\partial u_j}\right) \cdot \prod_{i=1}^{|L|} \frac{1}{\sinh^2 \frac{\gamma_i}{2}} \cdot \prod_{k=1}^{n} \sqrt{\det \mathrm{G}\left(\frac{\mathrm{u}_k}{2}\right)},$$

where u_i is the holonomy of the meridian of the i-th component and $\mathrm{u}_k = (u_{k_1}, \dots, u_{k_6})$ are the holonomy of the meridians intersecting the tetrahedron Δ_k; μ_i is the holonomy of the curve on the i-th component along which you do the surgery, γ_i is the holonomy of the core curve of the solid torus filled in, and G is the Gram matrix function.

Thank You !