Futuramaths!

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Futurama?

What is Futurama?

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"The Futurama was an exhibit/ride at the 1939-40 New York World's Fair held in the United States, designed by Norman Bel Geddes that tried to show the world 20 years into the future (1959-1960), including automated highways and vast suburbs. The exhibit was sponsored by General Motors."

-Wikipedia.

Futurama?

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"*the best TV program ever created. Evily canceled by fox.*"

-tranquil_demon, Urban Dictionary.

Why Futurama?

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Why Futurama?

So. . .what exactly makes Futurama so awesome?

- Witty dialogue?
- Well-woven plot?
- Scathingly satirical social commentary?
- Other stuff?

I'm a psychic.

Now, I know what you're *all* thinking:

I'm a psychic.

Now, I know what you're *all* thinking:

"But that's not why people watch TV. Clever things make people feel stupid, and unexpected things make them feel scared."

-Fry

Exploration

So to truly answer this question, it is imperative that we carefully analyse a RANDOMLY selected episode of Futurama. Here's one I prepared earlier. . .

The problem

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- How can we switch people back into their bodies?

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- Mind-switching machine.
- Any pair may only switch once due to immune response.
- How can we switch people back into their bodies?

To get us motivated on why we want to switch their bodies back, here's a quote:

"*Bodies are for hookers and fat people!*"

Bender.

Base case easy

Let's first try the easiest case: Amy and Bender have switched their bodies.

Base case easy

Let's first try the easiest case: Amy and Bender have switched their bodies.

It's possible for them to switch back with the help of 2 new people.

Also easy

Let's now try a slightly harder case: Amy and Bender switched, then Fry and Hermes switched.

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It's obviously possible with 4 new people.

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Let's now try a slightly harder case: Amy and Bender switched, then Fry and Hermes switched.

It's obviously possible with 4 new people.

It's still possible with just 2 new people.

By induction, it's all easy!

Is it always possible to switch back in general?

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Is it always possible to switch back in general?

Yes. In fact, I claim that:

Given that n people have their minds in the wrong body, it's always possible to switch everyone back if we have $2n-2$ new people.

By induction, it's all easy!

Is it always possible to switch back in general?

Yes. In fact, I claim that:

Given that n people have their minds in the wrong body, it's always possible to switch everyone back if we have $2n - 2$ new people.

Proof.

Let's say that person 1's mind is stuck in person 2. Then we can use 2 new people to switch the minds of these two bodies. Then we have at most $n - 1$ people with their minds in the wrong body. Repeat for everyone-else. $\square$

Now that we know that it's always possible, at most how many new people do we need?

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It turns out that we will only ever need 2 new people, and this is called the Futurama theorem.

To prove this theorem, let's figure out a way to phrase our problem mathematically.

Notation

Given that n people have used the mind-swapper, we label these bodies from 1 to n. Let (j, k) represent the swapping of the minds inhabiting body j and body k. We can thinking of (j, k) as a bijective function from $\{1, 2, \ldots, n\}$ to itself. For example:

$$(1, 5) \cdot 1 = 5 \text{ and } (1, 5) \cdot 5 = 1, \text{ whereas } (1, 5) \cdot 4 = 4.$$

We call (j, k) a 2-cycle.

Extending this notation, we can define (i, j, k) to represent a sequence of legitimate mind-swaps which resulted in person i's mind inhabiting person j's body, and person j's mind inhabiting person k's body and person k's mind inhabiting person i's mind. That is, the minds cycle through these bodies in the order presented in the brackets.

Likewise, we can define $(i_1, i_2, i_3, i_4, \ldots, i_m)$ to represent this cycling of m minds through m-bodies. We call this a m-cycle.

Notice that as these are functions, we may compose them. So, $(1,2)\cdot(1,4,5)$ should be understood as applying the function $(1,4,5)$ and then the function $(1,2)$. And so we have:

$$(1,2)\cdot(1,4,5)\cdot 5 = 2.$$

Notice that we resolve everything from the right first.

Having developed the necessary notation, let's prove the Futurama theorem.

"Voice on T.V.: *Is today's hectic lifestyle making you tense and impatient?*

Bender: *Shut up and get to the point!*"

T.V. vs. Bender

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T.V. vs. Bender

Let π be a sequence of mind-swaps that have resulted in the minds of k people inhabiting the wrong body.

Fact: it is possible to represent π as a sequence of cycles such that no pairs of cycles have any shared numbers.

e.g.: $(234) \cdot (345) = (45) \cdot (23)$.

Consider the case when π is made up of precisely one k-cycle. We may choose the labeling so that

$$\pi = (1, 2, 3, \dots, k).$$

Then consider the following sequence of mind-swaps (read from right to left):

$$\sigma = (x, y) \cdot (1, y) \cdot (2, x) \cdot (2, y) \cdot (3, x) \cdot \ldots \cdot (k-1, x) \cdot (k, x),$$

where x and y correspond to two new people. Observe that this is a legitimate sequence of mindswaps, as no swap between two bodies is repeated. Moreover, as every swap involves either x or y, none of these swaps were performed as a part of π.

Voila!

I claim that performing σ after π gets everybody back to their original bodies.

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"If we hit that bullseye, the rest of the dominos will fall like a house of cards. Checkmate."

Zapp Brannigan

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So, that was pretty straight-forward, right? But what if Professor Farnsworth made a machine such that:

- there are 3 seats and each the minds are switched as per a 3-cycle, and
- no collection of 3 people are allowed to use the machine together more than once.

Is it still possible to get everyone back to their original bodies? If so, at most how many new people are needed?

And another thought...

What about general m-seated mind-swappers?

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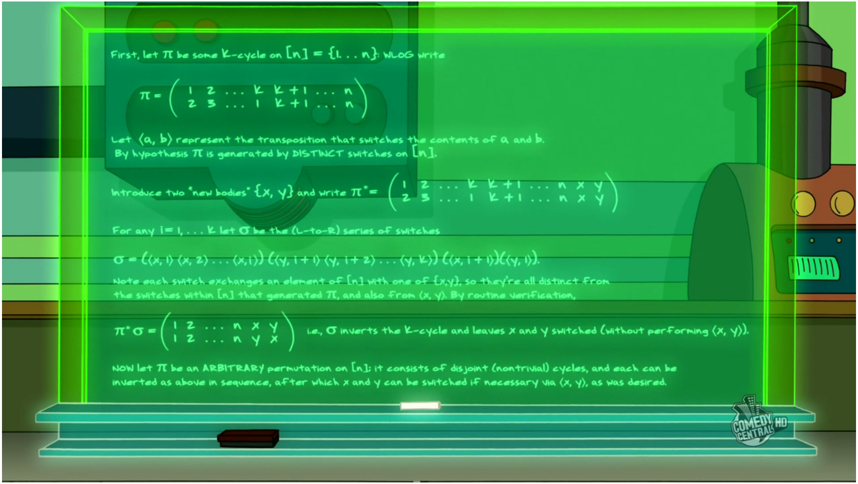
What about general m-seated mind-swappers?

It's a simple exercise to show that for a m-seated mind-swapper (where $m > 2$), at most $m - 2$ new people are needed. In particular, this means that in the 3-seater case, at most 1 new person is needed!

Back to the show!

The original proof by Ken Keeler.



"As the candy hearts poured into the fiery quasar, a wondrous thing happened, why not? They vaporised into a mystical love radiation that spread across the universe, destroying many, many planets - including two gangster planets and a cowboy world. But one planet was exactly the right distance to see the romantic rays, but not be destroyed by them - Earth. So all over the world, couples stood together in joy. And me, Zoidberg! And no one could've been happier, unless it would've also been Valentine's Day. What? It was? Hooray!"

Zoidberg

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In Russell Love Theatre.