A cohomology theory for discrete G-set

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August 4, 2014

Abstract

We provide a cohomology theory for studying the (left-)action of a group G on a space M with coefficients in a commutative ring R.

1 A rough analogy

This cohomology theory is designed for discrete groups G acting on a discrete space M, although the theory does apply more generally (with somewhat strange results). We begin be interpreting a (discrete) G-set M as a *discrete smooth manifold* with local G-structure. We use the discrete differential geometry we develop in this analogy and de Rham cohomology to motivate a cohomology theory for M (as a G-set).

The action of a discrete group G on a discrete space M naturally renders each G-orbit $G \cdot m \subset M$ a homogeneous space. We first use this G-action to designate "paths" on M, and hence regard the orbit-decomposition of M to as a decomposition of M into "*path-connected components*".

Definition 1 (Paths)**.** We define paths on M as a sequence of points $\{m_i\}$ "joined" by elements of G:

$$\ldots m_{-1} \xrightarrow{g_{-1}} m_0 = g_{-1} \cdot m_{-1} \xrightarrow{g_0} m_1 = g_0 \cdot m_0 \xrightarrow{g_1} m_2 = g_1 \cdot m_1 \ldots$$

Recall that tangent vectors at some point $m \in M$ may be regarded as equivalence classes of paths going through m. Thus, the group G is one natural candidate for a discretised version of a tangent space at m. We opt instead to take its Abelianisation G_{Ab} as the tangent space at m, for the simple reason that we'd like to be able to commutatively add tangent vectors. We denote Abelianised group elements by $[g]$ and adopt additive summation $+$ for the binary operator on G_{Ab}. Thus: $[g] + [h] = [gh] = [hg]$. We also choose to keep the tangent bundle simple — it's the product space $M \times G_{Ab}$. Philosophically, this means that we're thinking of each G-orbit in M as being modelled on "*open sets*[1]" in G.

Smooth vector fields on open submanifolds of $\mathbb{R}^n$ are (globally) expressed in the form:

$$\xi = f_1 \tfrac{\partial}{\partial x_1} + f_2 \tfrac{\partial}{\partial x_2} + \ldots + f_k \tfrac{\partial}{\partial x_k},$$

where the f_i are smooth $\mathbb{R}$-functions. To emulate this expression, we first fix a (unital) commutative ring R and refer to it as the *coefficient ring*[2]. This allows us to smooth R-functions on M as follows:

[1] Possibly "open neighbourhoods of subvarieties".

[2] We often choose the coefficient ring to be a subgroup of G_{Ab}, just as $\mathbb{R}$ is a subgroup of $\mathbb{R}^n$.

Definition 2. A *differentiable* or a C^1 R-function on M is a function $f : M \to R$ such that

1. the expression $f([g] \cdot m)$ is well-defined. That is: for any $g_1, g_2 \in G$ and $m \in M$, the following expressions are equivalent:

$$f(g_1 g_2 \cdot m) = f(g_2 g_1 \cdot m);$$

2. both the left and right "limits" used to define the derivative at m agree:

$$f([g] \cdot m) - f(m) = f(m) - f([g^{-1}] \cdot m).$$

Given a differentiable A-function, we call the function $[g](f) : M \to R$ defined pointwise by:

$$[g](f)(m) := f([g] \cdot m) - f(m) = f(m) - f([g^{-1}] \cdot m)$$

the directional derivative of f along $[g]$. We say that a R-function f is C^{k+1} if all of its directional derivatives $[g](f)$ are C^k. We say that a R-function is *smooth* or C^∞ if f can be iteratively directionally differentiated with respect to any sequence of elements of G_{Ab}.

Note 1. It's easy to check that if G_{Ab} is cyclic, then C^1 functions are smooth.

We're now equipped to define smooth vector fields:

Definition 3. A *vector field* on M is an element of the tensor product:

$$R^M \bigotimes_{\mathbb{Z}} G_{Ab}" = " \left\{ \sum f_i \otimes [g_i] \mid f_i : M \to R, [g_i] \in G_{Ab} \right\}.$$

A vector field is called *smooth* if each of the coefficient R-functions f_i constituting a vector field is smooth.

The next step in defining a discrete analogue of de Rham cohomology is to have differential forms, and so we discretise the notion of a cotangent bundle. The cotangent space over a point $m \in M$ is the dual space

$$G^*_{Ab} := \mathrm{Hom}(G_{Ab}, R)$$

to the tangent space G_{Ab} at m. Accordingly, the *cotangent bundle* is the product space $M \times G^*_{Ab}$.

Definition 4. We define the space of $(0, n)$-tensor fields on the G-space M with R coordinates as the following tensor product:

$$R^M \bigotimes_{R} (G^*_{Ab})^{\otimes n}" = " \left\{ \sum f_i \otimes [g_{i1}]^* \otimes \ldots \otimes [g_{in}]^* \mid f_i : M \to R, [g_{ij}]^* \in G^*_{Ab} \right\}.$$

Just as with vector fields, a tensor-field is smooth if the coefficient functions f_i are smooth. We further say that a $(0, n)$-tensor field ϕ is a smooth n-*form* if it is *alternating*:

$$\phi(m, g_1, \ldots, g_n) = -\phi(m, g_1, \ldots, g_{i-1}, g_{i+1}, g_i, g_{i+2}, \ldots, g_n)$$

We denote the collection of smooth n-forms on a G-space M by $\Omega^n(G, M; R)$.

There are limitations to this theory of "discrete differential geometry". A key issue being that the product of two smooth functions isn't smooth. This makes it difficult for us to define Lie brackets, so we take the cop-out and set Lie brackets to zero. Thus, we define the differential $d : \Omega^n(G, M; A) \to \Omega^{n+1}(G, M; R)$

$$\begin{aligned}&(d\varphi)(m, g_1, \dots, g_{n+1})\\ &= \sum_{i=1}^{n} [(-1)^{i+1}\varphi(g_i \cdot m, g_1, \dots, \hat{g_i}, \dots, g_n) + (-1)^i \varphi(m, g_1, \dots, \hat{g_i}, \dots, g_n)]\end{aligned}$$

motivated by setting Lie brackets to zero in the usual exterior derivative. It's straightforward to check that $d \circ d = 0$. The collection of cocycles $Z^*(G, M; R)$ modulo the coboundaries $B^*(G, M; R)$ of this chain complex then define the cohomology groups

$$H^*(G, M; R) := Z^*(G, M; R)/B^*(G, M; R).$$

The fact that multiplying smooth functions does not produce smooth functions means that this cohomology is not naturally endowed with the structure of a graded ring. On the other hand, pullbacks work:

Lemma 1. *Given a pair of maps,*

$$\phi : G_1 \to G_2,\ F : X_1 \to X_2,$$

so that ϕ *is a homomorphism and* F *is equivariant with respect to the group action, i.e.:* $F(g \cdot x) = \phi(g) \cdot F(x)$*, then the chain pullback map*

$$F^* : Z^i(G_2, X_2; A) \to Z^i(G_1, X_1; A),$$

given by

$$(F^*\varphi)(x, g_1, \dots, g_i) = \varphi(F(x), \phi(g_1), \dots, \phi(g_i))$$

commutes with the differential map d*.*

Proof. This is actually a fairly straight-forward definition bash. I'll leave it as an exercise. □

In addition, this cohomology satisfies four of the Eilenberg-Steenrod axioms (Excision, Dimension, Additivity and Exactness), assuming that we interpret distinct G-orbits in M as different connected components of M and that when defining relative cohomologies, the relative subspace need to also be a G-space. We're not sure how the Homotopy axiom should be interpreted in this discrete setting. Note that given our setup, the Excision, Additivity and Exactness axioms are essentially equivalent. Perhaps we can define more generalised G-spaces in the future when trying to formulate a theory of discrete differential geometry.

Here're a few simple results:

let $G = n\mathbb{Z}$ acting on $M = \mathbb{Z}$ by addition, then

$$H^i(n\mathbb{Z}, \mathbb{Z}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^n & \text{if } i = 0, 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

In fact, it should be fairly easy to show that if G is finitely generated, then the number of free generators of G_{Ab} - using the characterisation of finitely generated Abelian modules, determines the top non-zero dimension of the cohomology. And the $\mathbb{Z}$-dimension of the 0-th cohomology specifies how many orbits there are in M.

Oh, and here are two more results that I computed:

let $G = m\mathbb{Z} \times n\mathbb{Z}$ acting on $X = \mathbb{Z}$ by addition, then

$$H^i(m\mathbb{Z} \times n\mathbb{Z}, \mathbb{Z}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{\gcd(m,n)} & \text{if } i = 0, 2 \\ \mathbb{Z}^{2\gcd(m,n)} & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Now let $G = m\mathbb{Z} \times n\mathbb{Z}$ acting on $X = \mathbb{Z} \times \mathbb{Z}$ by addition, then

$$H^i(m\mathbb{Z} \times n\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{mn} & \text{if } i = 0 \\ \mathbb{Z}^{3mn} & \text{if } i = 1 \\ \mathbb{Z}^{2mn} & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

So yea, here we have this very natural, simple and rudimentary version of cohomology that seems to give some info about group actions, well. . . simple at least in the case when G is simple enough. It'd be interesting to see what classical theorems in group theory we could prove with this? And it'd be interesting to maybe do some computations with p-adic group actions and maybe Lie groups and stuff? Although the coefficients will definitely need to change, because something like

$$H^i(\mathbb{R}, \mathbb{R}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

does suggest that one should try again with different coefficients, and I'm fairly sure that the universal coefficient theory doesn't work in this world. Especially because using $R = \mathbb{Z}$ kills the torsion in G_{Ab} when considering this cohomology, whereas using cyclic groups as coefficients can keep the torsion alive.

2 Computations with $R = \mathbb{Z}$

So, thanks to Dougal, I can now do $H^2(G, M)$ of $G = \mathbb{Z} = \langle t \rangle$, but I didn't really write it down, so I should probably rederive it and stuff and figure out how it works. So, basically, we want to compute what the first two group cohomologies for a $G = \mathbb{Z}$ action on some $\mathbb{Z}G$-module M looks like.

2.1 $G = \mathbb{Z}$

2.1.1 $H^0(G, M)$

A 0-cochain is a homomorphism $\varphi_0 : \mathbb{Z} \to M$, and this is completely determined by the image of the generator of $\mathbb{Z}$. Therefore, the cochains are isomorphic to M, and since every 0-cochain is a cocycle, we see that:

$$H^0(G, M) \cong M \text{ as } \mathbb{Z}G\text{-modules.}$$

2.1.2 $H^1(G, M)$

A 1-cochain is a homomorphism $\varphi_1 : \mathbb{Z}G \to M$. This is completely determined by the restriction of φ_1 to G, and we now consider this in the case when φ_1 is a cocycle. Note that this means that:

$$0 = d\varphi_1(t^i, t^j) = t^i \cdot \varphi_1(t^j) - \varphi_1(t^{i+j}) + \varphi_1(t^i)$$

for every combination of i and j. Setting $i = j = 0$, we see that:

$$1 \cdot \varphi_1(1) - \varphi_1(1) + \varphi_1(1) = \varphi_1(1) = 0$$

In addition, we get that:

$$\begin{aligned} t^i \cdot \varphi_1(t^j) - \varphi_1(t^{i+j}) &= t^i \cdot \varphi_1(t^k) - \varphi_1(t^{i+k}) \\ \varphi_1(t^{i+j}) - \varphi_1(t^{i+k}) &= t^i \cdot (\varphi_1(t^j) - \varphi_1(t^k)). \end{aligned}$$

This in turn means that, if $n \geqslant 0$:

$$\begin{aligned} \varphi_1(t^a) &= \sum_{n=1}^{a} (\varphi_1(t^n) - \varphi_1(t^{n-1})) \\ &= \sum_{n=1}^{a} t^{n-1}(\varphi_1(t) - \varphi_1(1)) \\ &= \sum_{n=1}^{a} t^{n-1}\varphi_1(t) = \frac{t^a - 1}{t - 1} \cdot \varphi_1(t) \end{aligned}$$

Similar calculations apply for $n < 0$, and we end up getting that:

$$\varphi_1(t^a) = \frac{t^a - 1}{t - 1} \cdot \varphi_1(t), \text{ for any } a \in \mathbb{Z}. \tag{5}$$

So yes, we see that the cocycles $Z^1(G, M)$ is isomorphic as a $\mathbb{Z}G$-module to M. As for the coboundaries, we know that

$$d\varphi_0(t) = t \cdot \varphi_0(1) - \varphi_0(1),$$

hence $B^1(G, M) \cong \operatorname{im}(t - 1)$, and we obtain that the first homology

$$H^1(G, M) \cong M/\operatorname{im}(t - 1).$$

2.1.3 $H^2(G, M)$

As before, a 2-cochain is a homomorphism $\varphi_2 : C_2 \to M$, where C_2 is the free $\mathbb{Z}\mathbb{G}$-module generated by $G \times G$, satisfying the condition that $d\varphi_2 = 0$. It is completely determined by how it evaluates on $G \times G$, and the cocycle condition tells us that for any i, j, k:

$$d\varphi_2(t^i, t^j, t^k) = t^i \cdot \varphi_2(t^j, t^k) - \varphi_2(t^{i+j}, t^k) + \varphi_2(t^i, t^{j+k}) - \varphi_2(t^i, t^j) = 0$$

Given an arbitrary 2-cocycle φ_2, we will construct a 1-cochain σ_1 such that $d\sigma_1 = \varphi_2$. Note that this means that any cocycle in $Z^2(G, M)$ is a coboundary, hence $Z^2(G, M) = B^2(G, M)$, and

$$H^2(G, M) = 0.$$

So, let's see, we know that an arbitrary 1-cocycle resembles:

$$\varphi_1(t^a) = \frac{t^a - 1}{t - 1} \cdot \varphi_1(t),$$

therefore, whatever we choose σ_1 to be, we may always subtract by an appropriate φ_1 so that $\sigma_1(t) = 0$. Now, let's assume that there exists such a σ_1 so that $d\sigma_1 = \varphi_2$, and we derive what properties are required of σ_1 for this to be true. In some sense, this is like saying that we have a differential equation between φ_2 and σ_1, and we want to find σ_1 given the initial condition that $\sigma(t) = 0$. So let's see, we know that:

$$d\sigma_1(t^i, t^j) = t^i \cdot \sigma_1(t^j) - \sigma_1(t^{i+j}) + \sigma_1(t^i)$$

which means that:

$$\sigma_1(1) = d\sigma_1(1, 1) = \varphi_2(1, 1).$$

Now, setting $j = 1$, we see that we require that:

$$\begin{aligned} \varphi_2(t^{i-1}, t) = d\sigma_1(t^{i-1}, t) &= t^{i-1} \cdot \sigma_1(t) - \sigma_1(t^i) + \sigma_1(t^{i-1}) \\ \Rightarrow \sigma_1(t^i) &= \sigma_1(t^{i-1}) - \varphi_2(t^{i-1}, t). \end{aligned}$$

therefore, testing out the first few values of i, we obtain the following specifications on σ_1:

$$\begin{aligned} \sigma_1(t) &= \sigma_1(1) - \varphi_2(1, t) = \varphi_2(1, 1) - \varphi_2(1, t) \\ \sigma_1(t^{-1}) &= \sigma_1(1) + \varphi_2(1, t) = \varphi_2(1, 1) + \varphi_2(1, t) \\ \sigma_1(t^2) &= \sigma_2(1) - \varphi_2(1, t) = \varphi_2(1, 1) - \varphi_2(1, t) - \varphi_2(t, t) \\ \sigma_1(t^{-2}) &= \varphi_2(1, 1) + \varphi_2(1, t) + \varphi_2(t, t) \\ \sigma_1(t^3) &= \varphi_2(1, 1) - \varphi_2(1, t) - \varphi_2(t, t) - \varphi_(t^2, t) \\ \sigma_1(t^{-3}) &= \varphi_2(1, 1) + \varphi_2(1, t) + \varphi_2(t, t) + \varphi_2(t^t, t) \\ \sigma_1(t^3) &= \varphi_2(1, 1) - \varphi_2(1, t) - \varphi_2(t, t) - \varphi_(t^2, t) - \varphi_2(t^3, t) \\ \sigma_1(t^{-3}) &= \varphi_2(1, 1) + \varphi_2(1, t) + \varphi_2(t, t) + \varphi_2(t^2, t) + \varphi_2(t^3, t). \end{aligned}$$

We might be worried at this point that $\sigma_1(t) \neq 0$ - which would certainly contradict the fact that we chose σ_1 so that $\sigma_1(t) = 0$. However, we know that:

$$\begin{aligned} 0 = d\varphi_2(1, 1, t^i) = \varphi_2(1, t^i) - \varphi_2(1, t^i) + \varphi_2(1, t^j) - \varphi_2(1, 1) \qquad (6) \\ \Rightarrow \varphi_2(1, 1) = \varphi(1, t^j). \qquad (7) \end{aligned}$$

Hence, this is consistent with our specifications on $\sigma_1(t)$. As for the rest of the proof, we use induction to show the following:

$$\sigma_1(t^a) = \varphi_2(1, 1) - \sum_{i=0}^{a-1} \varphi_2(t^i, t), \text{ if } a \geqslant 0$$

and

$$\sigma_1(t^a) = \varphi_2(1,1) + \sum_{i=0}^{|a-1|} \varphi_2(t^{-i}, t), \text{ if } a \leqslant 0.$$

Moreover, we know that that for any $a \in \mathbb{Z}$,

$$\sigma_1(t^a) = \sigma_1(t^{a-1}) - \varphi_2(t^{a-1}, t).$$

Thus, we see that if our claim is true at all, this spells out our unique normalized candidate for σ_1 such that $d\sigma_1 = \varphi_2$. We now show that this is indeed true - that is, we shall show that for any $(t^i, t^j) \in \mathbb{Z} \times \mathbb{Z}$, then $d\sigma_1(t^i, t^j) = \varphi_2(t^i, t^j)$. We do this by induction.

Proof. Our base case that $d\sigma_1(1,1) = \varphi_2(1,1)$ is true because we chose $\sigma_1(1) = \varphi_2(1,1)$. We first do induction over the left coordinate: assume that $d\sigma_1(t^k, 1) = \varphi_2(t^k, 1)$ for some $k \geqslant 0$. Then we know that:

$$\begin{aligned}
d\sigma_1(t^{k+1}, 1) &= t^{k+1} \cdot \sigma_1(1) - \sigma_(t^{k+1}) + \sigma(t^{k+1}) \\
&= t^{k+1} \cdot \sigma_1(1) = t \cdot (t^k \cdot \sigma_1(1) - \sigma_(t^k) + \sigma(t^k)) \\
&= t \cdot d\sigma_1(t^k, 1) = t \cdot \varphi_2(t^k, 1) \\
&= t \cdot \varphi_2(t^k, 1) - \varphi_2(t^{k+1}, 1) + \varphi_2(t, t^k) - \varphi_2(t, t^k) + \varphi_2(t^{k+1}, 1) \\
&= \varphi_2(t^{k+1}, 1),
\end{aligned}$$

as desired. The induction for $k < 0$ then proceeds as follows:

$$\begin{aligned}
d\sigma_1(t^{k-1}, 1) &= t^{k-1} \cdot \sigma_1(1) - \sigma_(t^{k-1}) + \sigma(t^{k-1}) \\
&= t^{k-1} \cdot \sigma_1(1) = t^{-1} \cdot (t^k \cdot \sigma_1(1) - \sigma_(t^k) + \sigma(t^k)) \\
&= t^{-1} \cdot d\sigma_1(t^k, 1) = t^{-1} \cdot \varphi_2(t^k, 1) \\
&= t^{-1} \cdot \varphi_2(t^k, 1) - \varphi_2(t^{k-1}, 1) + \varphi_2(t^{-1}, t^k) - \varphi_2(t^{-1}, t^k) + \varphi_2(t^{k-1}, 1) \\
&= \varphi_2(t^{k-1}, 1),
\end{aligned}$$

and our induction is complete, thereby establishing that $d\sigma_1(t^i, 1) = \varphi_2(t^i, 1)$ for any $i \in \mathbb{Z}$. Let us now assume that $d\sigma_1(i, k) = \varphi_2(i, k)$ for some $k \geqslant 0$, then:

$$\begin{aligned}
d\sigma_1(t^i, t^{k+1}) &= t^i \cdot \sigma_1(t^{k+1}) - \sigma_1(t^{i+k+1}) + \sigma_1(t^i) \\
&= t^i \cdot \sigma_1(t^k) - t^i \cdot \varphi_2(t^k, t) - \sigma_1(t^{i+k}) + \varphi_2(t^{i+k}, t) + \sigma(t^i) \\
&= d\sigma_1(t^i, t^k) + \varphi_2(t^{i+k}, t) - t^i \cdot \varphi_2(t^k, t) \\
&= \varphi_2(t^i, t^k) + \varphi_2(t^{i+k}, t) - t^i \cdot \varphi_2(t^k, t) \\
&= \varphi_2(t^i, t^{k+1}).
\end{aligned}$$

Now we assume that $d\sigma_1(i, k) = \varphi_2(i, k)$ holds for some $k \leqslant 0$, then:

$$\begin{aligned}
d\sigma_1(t^i, t^{k-1}) &= t^i \cdot \sigma_1(t^{k-1}) - \sigma_1(t^{i+k-1}) + \sigma_1(t^i) \\
&= t^i \cdot \sigma_1(t^k) + t^i \cdot \varphi_2(t^{k-1}, t) - \sigma_1(t^{i+k}) - \varphi_2(t^{i+k-1}, t) + \sigma(t^i) \\
&= d\sigma_1(t^i, t^k) - \varphi_2(t^{i+k-1}, t) + t^i \cdot \varphi_2(t^{k-1}, t) \\
&= \varphi_2(t^i, t^k) - \varphi_2(t^{i+k-1}, t) + t^i \cdot \varphi_2(t^{k-1}, t) \\
&= \varphi_2(t^i, t^{k-1}).
\end{aligned}$$

We have therefore shown that $\varphi_2 = d\sigma_1$, and hence

$$Z^2(G, M) = B^2(G, M) \Rightarrow H^2(G, M) = 0.$$

$\square$

So, that was helpful, right? Let us now compute the homologies of $G = \mathbb{Z} \times \mathbb{Z} = \langle s \rangle \oplus \langle t \rangle$ acting on some module M. This is, in some sense, what we care about, because I want to compute stuff to do with the representation variety associated with a torus - which has fundamental group $\pi_1 \cong \mathbb{Z} \times \mathbb{Z}$.

2.2 $G = \mathbb{Z} \times \mathbb{Z}$

2.2.1 $H^0(G, M)$

A 0-cochain is a homomorphism $\varphi_0 : \mathbb{Z} \times \mathbb{Z} \to M$, and this is completely determined by the image of the generators of each $\mathbb{Z}$. Therefore, the module of cochains is isomorphic to $M \times M$; since every 0-cochain is a cocycle, we see that:

$$H^0(G, M) \cong M \times M \text{ as } \mathbb{Z}G\text{-modules.}$$

2.2.2 $H^1(G, M)$

A 1-cochain is a homomorphism $\varphi_1 : \mathbb{Z}G \to M$. This is completely determined by the restriction of φ_1 to G, and we now consider this in the case when φ_1 is a cocycle. Note that this means that:

$$0 = d\varphi_1((s^i, t^j), (s^m, t^n)) = (s^i, t^j) \cdot \varphi_1(s^m, t^n) - \varphi_1(s^{i+m}, t^{j+n}) + \varphi_1(s^i, t^j)$$

for every combination of (i, j) and (m, n). Setting $(i, j) = (m, n) = (0, 0)$, we see that:

$$(1, 1) \cdot \varphi_1(1, 1) - \varphi_1(1, 1) + \varphi_1(1) = \varphi_1(1, 1) = 0$$

In addition, we get that:

$$\begin{aligned} &(s^i, t^j) \cdot \varphi_1(s^{m_1}, t^{n_1}) - \varphi_1(s^{i+m_1}, t^{j+n_1}) = (s^i, t^j) \cdot \varphi_1(s^{m_2}, t^{n_2}) - \varphi_1(s^{i+m_2}, t^{j+n_2}) \\ &\Rightarrow \varphi_1(s^{i+m_1}, t^{j+n_1}) - \varphi_1(s^{i+m_2}, t^{j+n_2}) = (s^i, t^j) \cdot (\varphi_1(s^{m_1}, t^{n_1}) - \varphi_1(s^{m_2}, t^{n_2})). \end{aligned}$$

This in turn means that, if $n \geqslant 0$:

$$\begin{aligned} \varphi_1(t^a) &= \sum_{n=1}^{a} (\varphi_1(t^n) - \varphi_1(t^{n-1})) \\ &= \sum_{n=1}^{a} t^{n-1}(\varphi_1(t) - \varphi_1(1)) \\ &= \sum_{n=1}^{a} t^{n-1}\varphi_1(t) = \frac{t^a - 1}{t - 1} \cdot \varphi_1(t) \end{aligned}$$

Similar calculations apply for $n < 0$, and we end up getting that:

$$\varphi_1(t^a) = \frac{t^a - 1}{t - 1} \cdot \varphi_1(t), \text{ for any } a \in \mathbb{Z}. \tag{8}$$

So yes, we see that the cocycles $Z^1(G, M)$ is isomorphic as a $\mathbb{Z}G$-module to M. As for the coboundaries, we know that

$$d\varphi_0(t) = t \cdot \varphi_0(1) - \varphi_0(1),$$

hence $B^1(G, M) \cong \mathrm{im}(t-1)$, and we obtain that the first homology

$$H^1(G, M) \cong M/\mathrm{im}(t-1).$$

2.2.3 $H^2(G, M)$

As before, a 2-cochain is a homomorphism $\varphi_2 : C_2 \to M$, where C_2 is the free $\mathbb{ZG}$-module generated by $G \times G$, satisfying the condition that $d\varphi_2 = 0$. It is completely determined by how it evaluates on $G \times G$, and the cocycle condition tells us that for any i, j, k:

$$d\varphi_2(t^i, t^j, t^k) = t^i \cdot \varphi_2(t^j, t^k) - \varphi_2(t^{i+j}, t^k) + \varphi_2(t^i, t^{j+k}) - \varphi_2(t^i, t^j) = 0$$

Given an arbitrary 2-cocycle φ_2, we will construct a 1-cochain σ_1 such that $d\sigma_1 = \varphi_2$. Note that this means that any cocycle in $Z^2(G, M)$ is a coboundary, hence $Z^2(G, M) = B^2(G, M)$, and

$$H^2(G, M) = 0.$$

So, let's see, we know that an arbitrary 1-cocycle resembles:

$$\varphi_1(t^a) = \frac{t^a - 1}{t - 1} \cdot \varphi_1(t),$$

therefore, whatever we choose σ_1 to be, we may always subtract by an appropriate φ_1 so that $\sigma_1(t) = 0$. Now, let's assume that there exists such a σ_1 so that $d\sigma_1 = \varphi_2$, and we derive what properties are required of σ_1 for this to be true. In some sense, this is like saying that we have a differential equation between φ_2 and σ_1, and we want to find σ_1 given the initial condition that $\sigma(t) = 0$. So let's see, we know that:

$$d\sigma_1(t^i, t^j) = t^i \cdot \sigma_1(t^j) - \sigma_1(t^{i+j}) + \sigma_1(t^i)$$

which means that:

$$\sigma_1(1) = d\sigma_1(1, 1) = \varphi_2(1, 1).$$

Now, setting $j = 1$, we see that we require that:

$$\begin{aligned} \varphi_2(t^{i-1}, t) = d\sigma_1(t^{i-1}, t) &= t^{i-1} \cdot \sigma_1(t) - \sigma_1(t^i) + \sigma_1(t^{i-1}) \\ &\Rightarrow \sigma_1(t^i) = \sigma_1(t^{i-1}) - \varphi_2(t^{i-1}, t). \end{aligned}$$

therefore, testing out the first few values of i, we obtain the following specifications on σ_1:

$$\begin{aligned}
\sigma_1(t) &= \sigma_1(1) - \varphi_2(1,t) = \varphi_2(1,1) - \varphi_2(1,t) \\
\sigma_1(t^{-1}) &= \sigma_1(1) + \varphi_2(1,t) = \varphi_2(1,1) + \varphi_2(1,t) \\
\sigma_1(t^2) &= \sigma_2(1) - \varphi_2(1,t) = \varphi_2(1,1) - \varphi_2(1,t) - \varphi_2(t,t) \\
\sigma_1(t^{-2}) &= \varphi_2(1,1) + \varphi_2(1,t) + \varphi_2(t,t) \\
\sigma_1(t^3) &= \varphi_2(1,1) - \varphi_2(1,t) - \varphi_2(t,t) - \varphi_(t^2,t) \\
\sigma_1(t^{-3}) &= \varphi_2(1,1) + \varphi_2(1,t) + \varphi_2(t,t) + \varphi_2(t^t,t) \\
\sigma_1(t^3) &= \varphi_2(1,1) - \varphi_2(1,t) - \varphi_2(t,t) - \varphi_(t^2,t) - \varphi_2(t^3,t) \\
\sigma_1(t^{-3}) &= \varphi_2(1,1) + \varphi_2(1,t) + \varphi_2(t,t) + \varphi_2(t^2,t) + \varphi_2(t^3,t).
\end{aligned}$$

We might be worried at this point that $\sigma_1(t) \neq 0$ - which would certainly contradict the fact that we chose σ_1 so that $\sigma_1(t) = 0$. However, we know that:

$$0 = d\varphi_2(1,1,t^i) = \varphi_2(1,t^i) - \varphi_2(1,t^i) + \varphi_2(1,t^j) - \varphi_2(1,1) \tag{9}$$
$$\Rightarrow \varphi_2(1,1) = \varphi(1,t^j). \tag{10}$$

Hence, this is consistent with our specifications on $\sigma_1(t)$. As for the rest of the proof, we use induction to show the following:

$$\sigma_1(t^a) = \varphi_2(1,1) - \sum_{i=0}^{a-1} \varphi_2(t^i,t), \text{ if } a \geqslant 0$$

and

$$\sigma_1(t^a) = \varphi_2(1,1) + \sum_{i=0}^{|a-1|} \varphi_2(t^{-i},t), \text{ if } a \leqslant 0.$$

Moreover, we know that that for any $a \in \mathbb{Z}$,

$$\sigma_1(t^a) = \sigma_1(t^{a-1}) - \varphi_2(t^{a-1},t).$$

Thus, we see that if our claim is true at all, this spells out our unique normalized candidate for σ_1 such that $d\sigma_1 = \varphi_2$. We now show that this is indeed true - that is, we shall show that for any $(t^i,t^j) \in \mathbb{Z} \times \mathbb{Z}$, then $d\sigma_1(t^i,t^j) = \varphi_2(t^i,t^j)$. We do this by induction.

Proof. Our base case that $d\sigma_1(1,1) = \varphi_2(1,1)$ is true because we chose $\sigma_1(1) = \varphi_2(1,1)$. We first do induction over the left coordinate: assume that $d\sigma_1(t^k,1) = \varphi_2(t^k,1)$ for some $k \geqslant 0$. Then we know that:

$$\begin{aligned}
d\sigma_1(t^{k+1},1) &= t^{k+1} \cdot \sigma_1(1) - \sigma_(t^{k+1}) + \sigma(t^{k+1}) \\
&= t^{k+1} \cdot \sigma_1(1) = t \cdot (t^k \cdot \sigma_1(1) - \sigma_(t^k) + \sigma(t^k)) \\
&= t \cdot d\sigma_1(t^k,1) = t \cdot \varphi_2(t^k,1) \\
&= t \cdot \varphi_2(t^k,1) - \varphi_2(t^{k+1},1) + \varphi_2(t,t^k) - \varphi_2(t,t^k) + \varphi_2(t^{k+1},1) \\
&= \varphi_2(t^{k+1},1),
\end{aligned}$$

as desired. The induction for $k < 0$ then proceeds as follows:

$$\begin{aligned}
d\sigma_1(t^{k-1}, 1) &= t^{k-1} \cdot \sigma_1(1) - \sigma_(t^{k-1}) + \sigma(t^{k-1}) \\
&= t^{k-1} \cdot \sigma_1(1) = t^{-1} \cdot (t^k \cdot \sigma_1(1) - \sigma_(t^k) + \sigma(t^k)) \\
&= t^{-1} \cdot d\sigma_1(t^k, 1) = t^{-1} \cdot \varphi_2(t^k, 1) \\
&= t^{-1} \cdot \varphi_2(t^k, 1) - \varphi_2(t^{k-1}, 1) + \varphi_2(t^{-1}, t^k) - \varphi_2(t^{-1}, t^k) + \varphi_2(t^{k-1}, 1) \\
&= \varphi_2(t^{k-1}, 1),
\end{aligned}$$

and our induction is complete, thereby establishing that $d\sigma_1(t^i, 1) = \varphi_2(t^i, 1)$ for any $i \in \mathbb{Z}$. Let us now assume that $d\sigma_1(i, k) = \varphi_2(i, k)$ for some $k \geqslant 0$, then:

$$\begin{aligned}
d\sigma_1(t^i, t^{k+1}) &= t^i \cdot \sigma_1(t^{k+1}) - \sigma_1(t^{i+k+1}) + \sigma_1(t^i) \\
&= t^i \cdot \sigma_1(t^k) - t^i \cdot \varphi_2(t^k, t) - \sigma_1(t^{i+k}) + \varphi_2(t^{i+k}, t) + \sigma(t^i) \\
&= d\sigma_1(t^i, t^k) + \varphi_2(t^{i+k}, t) - t^i \cdot \varphi_2(t^k, t) \\
&= \varphi_2(t^i, t^k) + \varphi_2(t^{i+k}, t) - t^i \cdot \varphi_2(t^k, t) \\
&= \varphi_2(t^i, t^{k+1}).
\end{aligned}$$

Now we assume that $d\sigma_1(i, k) = \varphi_2(i, k)$ holds for some $k \leqslant 0$, then:

$$\begin{aligned}
d\sigma_1(t^i, t^{k-1}) &= t^i \cdot \sigma_1(t^{k-1}) - \sigma_1(t^{i+k-1}) + \sigma_1(t^i) \\
&= t^i \cdot \sigma_1(t^k) + t^i \cdot \varphi_2(t^{k-1}, t) - \sigma_1(t^{i+k}) - \varphi_2(t^{i+k-1}, t) + \sigma(t^i) \\
&= d\sigma_1(t^i, t^k) - \varphi_2(t^{i+k-1}, t) + t^i \cdot \varphi_2(t^{k-1}, t) \\
&= \varphi_2(t^i, t^k) - \varphi_2(t^{i+k-1}, t) + t^i \cdot \varphi_2(t^{k-1}, t) \\
&= \varphi_2(t^i, t^{k-1}).
\end{aligned}$$

We have therefore shown that $\varphi_2 = d\sigma_1$, and hence

$$Z^2(G, M) = B^2(G, M) \Rightarrow H^2(G, M) = 0.$$

$\square$