Moduli Spaces
and
Weil-Petersson Volumes

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Abstract

Maryam Mirzakhani's work in moduli space theory has yielded much surprising and beautiful mathematics. In this Honours thesis, we introduce moduli space theory and present three major results of her work: the classification of geodesics, the Generalised McShane Identity, and the Mirzakhani Recursion formula, all contributing to the calculation of the Weil-Petersson volume of moduli spaces.

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Chapter 1

Background

Introduction

The concept of moduli spaces first arose as a space of parameters for a system. For example, a feather in the *real* world has parameters specifying its geographical position, its size, the direction in which it is pointed…etc. In this Honours thesis, we focus on moduli spaces of bordered Riemann surfaces: an object arising naturally from string theory.

The aim of this first chapter is to be a self-contained introduction to moduli space theory, with definitions and results selected to allow us to understand the latter exposition of Mirzakhani's work [8] on calculating Weil-Petersson volumes of moduli spaces.

1.1 Differentiable Manifolds

We begin by providing a basic introduction to differential topology and geometry. The notions of smoothness are defined, followed by analyticity. We then define metrics and sectional curvature: two concepts central to defining Riemannian and hyperbolic manifolds. For the sake of brevity, there is little to no discussion of the motivating ideas behind the following definitions.

1.1.1 Smoothness and Analyticity

Definition 1.1.1. Given a function $f : U \to \mathbb{R}^k$ mapping from an open subset of $\mathbb{R}^n$, we call f **smooth** if its partial derivatives of all orders exist and are continuous. Equivalently, we say that f is differentiable of class C^∞.

Definition 1.1.2. Consider a topological space M, such that M is Hausdorff and has a countable basis. If M comes equipped with a collection of open sets and functions, denoted by $\{(U_\alpha, \alpha)\}_{\alpha\in\mathcal{A}}$, where

- $\alpha : U_\alpha \subset M \to \mathbb{R}^n$, are homeomorphisms between U_α and $\alpha(U_\alpha) \subset \mathbb{R}^n$, and
- for any two pairs (U_{α_1}, α_1) and (U_{α_2}, α_2), the **transition function**

  $$\alpha_2 \circ \alpha_1^{-1} : \alpha_1(U_{\alpha_1} \cap U_{\alpha_2}) \subset \mathbb{R}^n \to \alpha_2(U_{\alpha_1} \cap U_{\alpha_2}) \subset \mathbb{R}^n \text{ is a smooth map,}$$

- $$\bigcup_{\alpha\in\mathcal{A}} U_\alpha = M,$$

then we call M a **smooth n-manifold**. In addition, $\alpha : U_\alpha \to \mathbb{R}^n$ is called a **coordinate chart**, and $\{(U_\alpha, \alpha)\}_{\alpha \in \mathcal{A}}$ is called a **smooth atlas**, we say that it defines a smoothness structure on M.

Definition 1.1.3. Given a smooth m-manifold M and n-manifold N, with respective atlases $\mathcal{A}_M$ and $\mathcal{A}_N$; a function $f : M \to N$ is **smooth at** $x \in M$ if: for all coordinate charts $(U_\mu, \mu) \in \mathcal{A}_M$ and $(U_\nu, \nu) \in \mathcal{A}_N$ such that $x \in U_\mu$ and $f(x) \in U_\nu$, the map

$$\nu \circ f \circ \mu^{-1} : \mu(f^{-1}(U_\nu) \cap U_\mu) \subset \mathbb{R}^m \to \mathbb{R}^n$$

has continuous derivatives of all orders at $\mu(x)$.

If f is smooth at all points in M, then we say that f is a **smooth function**.

Note 1.1.1. In the above definitions, the notation f^{-1} is taken to mean the preimage of f, whereas μ^{-1} and α_1^{-1} are the inverse homeomorphisms of μ and α_1.

Definition 1.1.4. A smooth function $f : M \to N$ is called a **diffeomorphism** if there exists a smooth function $g : N \to M$ inverse to f. That is: $g \circ f$ is the identity function on M and $f \circ g$ is the identity function on N. If such a function exists, we say that M and N are **diffeomorphic**.

We will be studying moduli spaces of Riemann surfaces, which, instead of locally resembling $\mathbb{R}^2$, look like $\mathbb{C}$ in a small enough patch.

Definition 1.1.5. Given an open subset U of $\mathbb{C}$, a function $w : U \to \mathbb{C}$ is called **analytic** or **holomorphic**[1] if at each point $u \in U$, the limit

$$\lim_{z \to u} \frac{w(z) - w(u)}{z - u} \text{ exists.}$$

Note 1.1.2. The division in this limit is the natural division inherited from $\mathbb{C}$ as a field. It is easy to see that being holomorphic will guarantee that a function is differentiable, when considered as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. It is also a standard fact from complex analysis that any holomorphic function will be smooth when considered as an $\mathbb{R}^2$ function.

Definition 1.1.6. Two open subsets U and V of $\mathbb{C}$ are **biholomorphic** if there exists a bijective function $w : U \to V$ such that both w and its inverse w^{-1} are holomorphic functions.

Definition 1.1.7. A **Riemann surface** R is a connected Hausdorff space with a system of coordinate neighbourhoods $\{(U_i, z_i)\}_{i \in I}$, satisifying:

- every U_i is an open subset of R, and $R = \cup_{i \in I} U_i$,
- every z_i is a homeomorphism of U_i onto an open subset $D_i \subset \mathbb{C}$,
- if $U_i \cap U_j \neq \emptyset$, then the transition mapping

$$z_{ji} = z_j \circ z_i^{-1} : z_i(U_i \cap U_j) \to z_j(U_i \cap U_j)$$

is a biholomorphic mapping. Such a collection of coordinate neighbourhoods defines a **complex structure** on R.

[1] We ignore the fact that analyticity and holomorphism are two different concepts that happen to coincide.

Definition 1.1.8. A Riemann surface homeomorphic to a sphere with g handles is called a **closed Riemann surface of genus** g.

Note 1.1.3. It is known that every (orientable) closed surface without boundary has finite genus and can be classified, up to homeomorphism, by its genus. As a consequence, every bordered compact Riemann surface can be classified by its genus and the number of boundary components it has.

Lemma 1.1.1. *An orientable real 2-manifold can be given a complex structure and hence made into a Riemann surface.*

Note 1.1.4. For our purposes, we will **only deal with oriented manifolds**. This fact will be assumed for the remainder of this thesis, and unless stated otherwise, **all functions will be assumed to be orientation preserving**. Having said this, we provide no rigorous definitions for orientability or orientedness, and will be content with the intuitive idea that if we start from a point and travel over a path in our surface, returning to the starting point, then we have not 'flipped' over. In other words: there are no embedded Möbius strips on our surface.

1.1.2 Differential Geometry and Riemannian Manifolds

In moduli spaces of Riemann surfaces, we understand two Riemann surfaces to represent the same point by seeing if there exists an angle-preserving map between them. The concept of an 'angle' is formalised via the identity:

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = ||\boldsymbol{u}|| \cdot ||\boldsymbol{v}|| \cdot \cos(\theta),$$

where $\langle \ , \ \rangle$ is an inner product. We introduce angles and lengths to manifolds by placing inner products on the tangent bundle of the manifold.

Definition 1.1.9. Given $p \in M$, where M is a smooth m-manifold, let $\mathrm{C}^\infty(\{p\})$ denote all functions $f : M \to \mathbb{R}$ such that f is smooth at x. Then a $\mathbb{R}$-linear map $X : \mathrm{C}^\infty(\{p\}) \to \mathbb{R}$ is called a **derivation** if for all $f, g \in \mathrm{C}^\infty(\{p\})$,

$$X(fg) = X(f) \cdot g(p) + f(p) \cdot X(g).$$

Note 1.1.5. A derivation at a point p is one abstract formulation of a tangent vector in a manifold's tangent space at the point p. More specifically, each derivation corresponds to the tangent vector (at p) of a (locally) smooth path running through p: given a tangent vector at a point, we can take a directional derivative (derivation) along this smooth path. Conversely, given a directional derivative, we can set up a system of differential equations, to which we know there is a solution in a small neighbourhood around p, thereby obtaining a locally smooth path.

Definition 1.1.10. Given a smooth m-manifold M, let $\{(U_\alpha, \alpha)\}_{\alpha \in \mathcal{A}}$ denote the smooth atlas given the smoothness structure on M. The **tangent space** T_pM is the collection of all derivations at p. Observe that T_pM is a real vector space ismorphic to $\mathbb{R}^m$.

Definition 1.1.11. The **tangent bundle** of M, as specified above is the set

$$TM = \coprod_{p \in M} T_pM,$$

where x specifies the tangent space y lies on, and $\boldsymbol{v}$ is a derivation in T_xM. Each point y in TM may be written as $(x, \boldsymbol{v})$, and define the map:

$$\pi : TM \to M, \text{ given by } \pi(x, \boldsymbol{v}) = x,$$

Note that T_xM is spanned by derivations ∂_i, $1 \le i \le m$ corresponding $\frac{\partial}{\partial e_i}$ in $T_{\alpha(x)}(\mathbb{R}^m)$, where e_i denotes the i-th coordinate in $\mathbb{R}^m$. Specifically, $\boldsymbol{v} = \sum v_i \partial_i$ is defined by:

$$\boldsymbol{v}(f)|_x = \sum_{i=1}^{m} v_i \frac{\partial (f \circ \alpha^{-1})}{\partial e_i}|_{\alpha(x)}.$$

We topologise the tangent bundle as follows: equip TM with the smoothness structure $\{(\tilde{U}_{\tilde{\alpha}}, \tilde{\alpha})\}_{\alpha \in \mathcal{A}}$, where $\tilde{U}_{\tilde{\alpha}}$ is defined as $\pi^{-1}(U_\alpha)$, and $\tilde{\alpha} : \pi^{-1}(U_\alpha) \to \mathbb{R}^{2m}$ is given by:

$$\tilde{\alpha}(x, \boldsymbol{v} = \sum_{i=1}^{m} v_i \partial_i) = (\alpha(x), v_1, \ldots, v_m).$$

Defined in this way, TM is itself a smooth manifold.

Definition 1.1.12. Any smooth map $f : M \to N$ induces a smooth map $df : TM \to TN$ given by:

$$df(x, \boldsymbol{v}) = (f(x), \boldsymbol{u}), \text{ such that } \boldsymbol{u}(g) = \boldsymbol{v}(g \circ f).$$

Moreover, if f is a diffeomorphism then so too is df.

Definition 1.1.13. A differentiable n-manifold M, where each tangent space T_xM is equipped with an inner-product g_x, such that the g_x vary smoothly with respects to x, is called a **Riemannian n-manifold**. We denote it by (M, g), where $g|_{T_xM} = g_x$.

Definition 1.1.14. Given a smooth m-manifold M, if each tangent space T_pM is equipped with an inner-product g_p, such that the function

$$g : TM \to \mathbb{R}, \; g|_{T_pM} = g_p,$$

varies smoothly with respect to p, then we call M a **Riemannian m-manifold** equipped with the metric g. We denote this symbolically by (M, g), and g is referred to as the **Riemannian metric** on M.

Note 1.1.6. The **metric** used here differs from a distance function as referred to when dealing with **metric spaces**. Where there is ambiguity, we shall refer to these collection inner products as a **metric tensor**. Metric tensors induce a natural distance function, and hence any Riemannian manifold may be turned into a metric space.

Definition 1.1.15. Given two manifolds M and N, where N is a Riemann surface equipped with a metric h. A diffeomorphism $f : M \to N$ induces a **pullback metric** $f^*(h)$ given by:

$$\langle (df)^{-1}\boldsymbol{u}, (df)^{-1}\boldsymbol{v} \rangle_{f^*(h)} = \langle \boldsymbol{u}, \boldsymbol{v} \rangle_h.$$

Definition 1.1.16. Given two Riemannian manifolds (M, g) and (N, h), a diffeomorphism $f : M \to N$ is called (an)

- **isometric (isometry)**, if the pullback metric $f^*(h) \equiv g$, and

- **conformal (conformal equivalence)**, if the pullback metric $f^*(h) \equiv ug$, where $u : TM \to \mathbb{R}^+$ is constant on each tangent space in TM. Conformal maps are sometimes called angle-preserving maps.

Note 1.1.7. In with complex analysis, a function between two domains in $\mathbb{C}$ is biholomorphic if and only if it gives a conformal equivalence. Therefore, as Riemannian surfaces can be made into Riemann surfaces via their metric, we will often confuse the notion of biholomorphism and conformal equivalence between these surfaces.

Definition 1.1.17. A **geodesic** is a curve whose tangent vectors are parallel transported along it. Or, expressed in terms of affine connections:

$$\nabla_{\frac{d\gamma}{dt}} \frac{d\gamma}{dt} = 0.$$

A geodesic is called **simple** if it does not self-intersect, and a **closed geodesic** can be thought of as an map of S^1 which still satisfies the parallel transport condition.

Note 1.1.8. We will talk about geodesics exclusively in the context of hyperbolic manifolds, in which case the Affine connection given in the definition will always be assumed to be the Levi-Civita connection. In addition, we will often refer to the image of a geodesic as the geodesic itself.

1.1.3 Canonical Representation of Riemann Surfaces

We later represent conformal equivalence classes of Riemann surfaces with hyperbolic surfaces and now state some results which make this possible. For proofs of these results, consider [4].

Theorem 1.1.2 (Uniformization theorem)**.** *Every simply connected Riemann surface without boundary is biholomorphic to one of the three Riemann surfaces $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{C}$, $\mathbb{H}$.*

In particular, the universal covering space of any Riemann surface without boundary is biholomorphic to one of the above. Hence, we have the following corollary:

Corollary 1.1.3. *For every Riemann surface R without boundary, there exists a universal covering surface that is biholomorphic to one of the following: $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{C}$, $\mathbb{H}$. Further, the universal covering group, denoted by Γ, acts freely, proper discontinuously, and holomorphically. Thus, Γ is isomorphic to the fundamental group $\pi_1(R)$.*

Theorem 1.1.4. *Given a Riemann surface R which is not biholomorphic to $\hat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{C}-\{0\}$ or a torus. Then R may be equipped with a complete hyperbolic structure, with the resulting hyperbolic surface labelled as R^h. Moreover, any two Riemann surfaces R_1 and R_2 are biholomorphic if and only if R_1^h is isometric to R_2^h.*

1.2 Hyperbolic Surfaces

The following section will require the notion of sectional curvature, a definition may be found in the appendix (definition A.1.4).

Definition 1.2.1. A Riemannian m-manifold with constant sectional curvature -1 is called a **hyperbolic** manifold. In our work, we will deal primarily with **bordered hyperbolic surfaces**, which are hyperbolic surfaces with locally length-minimising geodesic boundaries.

Note 1.2.1. By the Uniformization theorem, every Riemann surface is in the same conformal class as a Hyperbolic surface. The Gauss-Bonnet formula A.1.1 then tells us that all Riemann surfaces with positive constant curvature are conformally equivalent to the 2-spheres, and the torus is the only closed Riemann surface with constant zero curvature.

It is known that the universal cover of a Hyperbolic surface is isometric to the quotient of a simply connected region in $\mathbb{H}$. We will use this fact for a proof of the Generalised McShane Identity in section 2.2. For now, let us take a look at three models of the hyperbolic plane.

1.2.1 Hyperbolic Plane

Poincaré Disk Model

Consider the unit disk in the complex plane:

$$B = \{z \in \mathbb{C} : |z| \leq 1\},$$

equipped with the metric:

$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}.$$

Note 1.2.2. By a **model of the hyperbolic plane**, we mean a set (usually in $\mathbb{C}$) equipped with a hyperbolic structure so that it is isometric to $\mathbb{H}$. For our chosen sample of models, the underlying set usually comes equipped with a natural metric (e.g.: Euclidean for B) which has a nice relationship to its hyperbolic metric (e.g.: conformal equivalence).

Theorem 1.2.1 (Schwarz-Pick lemma)**.** *Let* $\mathrm{Isom}^+(\mathbb{H})$ *denote the group of all orientation-preserving self-diffeomorphisms of* $\mathbb{H}$ *which are isometric with respect to the hyperbolic metric. Moreover, let* $\mathrm{M\ddot{o}b}(B)$ *denote the group of all linear fractional transformations which map* B *to itself. Then:*

$$\mathrm{Isom}^+(\mathbb{H}) \textit{ can be identified with } \mathrm{M\ddot{o}b}(B).$$

Further, $\mathrm{Isom}^+(\mathbb{H}) = \mathrm{M\ddot{o}b}(B)$ *is the group of all orientation preserving self-diffeomorphisms of* $\mathbb{H}$ *which preserve hyperbolic distance and angle.*

Lemma 1.2.2. *The element-wise identity map on* B, id_B, *gives a conformal map between* (B, e) *and* (B, ds^2) *where* e *is the Euclidean metric and* ds^2 *the hyperbolic metric stated above.*

Proof. Consider $\boldsymbol{u} = (u_1, u_2), \boldsymbol{v} = (v_1, v_2) \in T_pB$, and let θ_e and θ_{ds^2} respectively denote the angles corresponding to the Euclidean and hyperbolic metrics. Then:

$$\begin{aligned}\cos(\theta_{ds^2}) &= \frac{\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{ds^2}}{\|\boldsymbol{u}\|_{ds^2}\cdot\|\boldsymbol{v}\|_{ds^2}} = \frac{4(u_1v_1+u_2v_2)}{(1-|p|^2)^2} \times \frac{(1-|p|^2)^2}{4\sqrt{(u_1^2+u_2^2)(v_1^2+v_2^2)}} \\ &= \frac{u_1v_1+u_2v_2}{\sqrt{(u_1^2+u_2^2)(v_1^2+v_2^2)}} = \frac{\langle \mathrm{id}_B(\boldsymbol{u}), \mathrm{id}_B(\boldsymbol{v})\rangle_e}{\|\mathrm{id}_B(\boldsymbol{u})\|_e\cdot\|\mathrm{id}_B(\boldsymbol{v})\|_e} = \cos(\theta_e)\end{aligned}$$

□

Note 1.2.3. We put $\cos(\theta_{ds^2})$ and $\cos(\theta_e)$ in, to demonstrate that id_B is angle-preserving.

We now take a look at the Poincaré half-plane model, which we'll denote by $(\mathbb{C}^+, dh^2)$. This is the model we will use when presenting figures of universal covers of hyperbolic manifolds.

Poincaré Half-plane Model

The Poincaré half-plane model, may be defined as:

$$\mathbb{C}^+ = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\},$$

equipped with the following metric:

$$dh^2 = \frac{|dz|^2}{\mathrm{Im}(z)^2}.$$

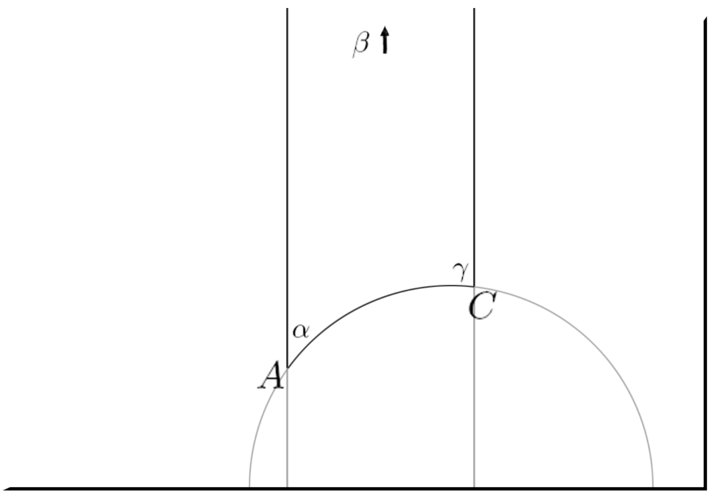
Note 1.2.4. It is a standard result that in the half-plane model, geodesics are precisely all the vertical paths and all semi-circles centered along the real axis.

On both the Euclidean and the hyperbolic plane, we define a geodesic triangle to be any region bounded by three geodesics. In $\mathbb{R}^2$, the area of a geodesic triangle may be aribitrarily large. Whereas in the hyperbolic plane, their area is bounded above by π. We will later see that owing to this fact, hyperbolic pants have area 2π (lemma 1.3.8). In addition, we later prove that all bordered hyperbolic surfaces with g genera and n boundaries can be decomposed into $3g - 3 + n$ pairs of pants, therefore, any two bordered hyperbolic surfaces of the same signature must have the same area.

Lemma 1.2.3. *The area of a geodesic triangle $\triangle \subset \mathbb{H}$ is equal to $\pi - (\alpha + \beta + \gamma)$ where $\alpha, \beta,$ and γ respectively denote the interior angles of $\triangle$.*

Proof. In the case that $\beta = 0$, that is, if one of the vertices of our triangle lies 'at' ∞, then:

$$\mathrm{Area}(\triangle) = \int\int_{\triangle} \frac{dx\ dy}{y^2}.$$



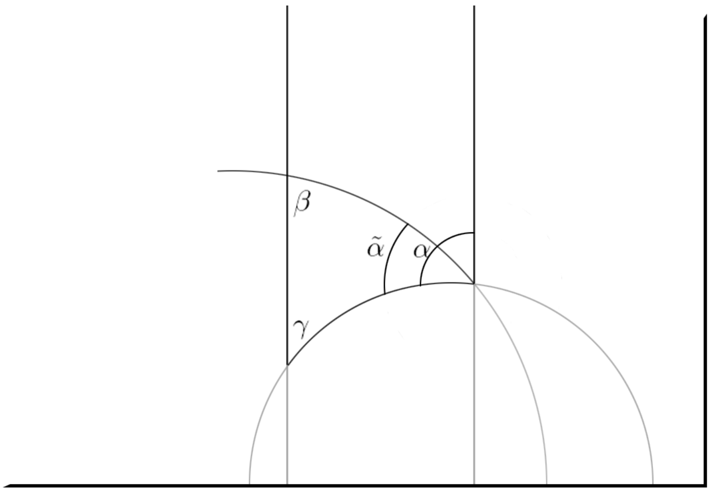
If we use Stokes' theorem, we have:

$$\int\int_{\triangle} \frac{dx\ dy}{y^2} = \int_{\partial\triangle} \frac{dx}{y} = \int_{\boldsymbol{AC}} \frac{dx}{y}$$

with the last equality given by the fact that the two 'verticle' sides make no contribution when integrating with respect to x. Then, parametrising the path $\boldsymbol{AC}$ by the semicircle as its geodesic boundary, we have:

$$\int_{\boldsymbol{AC}} \frac{dx}{y} = -\int_{\pi-\alpha}^{\gamma} d\theta = \pi - \alpha - \gamma.$$

Now consider a geodesic triangle $\triangle$ with angles $\tilde{\alpha}, \beta$ and γ.



We see that the area of $\triangle$ is the difference of the areas of the two triangles with a vertex at $\{\infty\}$. Therefore:

$$\text{Area}(\triangle) = [\pi - \alpha - \gamma] - [\pi - (\pi - \beta) - (\alpha - \tilde{\alpha})] = \pi - (\tilde{\alpha} + \beta + \gamma),$$

as desired. Finally, any triangle in $\mathbb{H}$ is either of the above form(s), or can be cut along a vertical line into two triangles of this form. By doing so and adding up the areas of the two subtriangles, we once again obtain the desired formula. $\square$

Using the fact that B may be biholomorphically mapped to $\mathbb{C}^+$ by the Möbius transformation:

$$f : B \to \mathbb{C}^+,\ f(z) = \frac{z-i}{z+i},$$

we may re-express the Schwarz-Pick theorem for the half-plane model as:

Theorem 1.2.4 (Schwarz-Pick Theorem)**.**

$$\text{Isom}^+(\mathbb{H}) = \text{Möb}(\mathbb{C}^+) = \left\{ \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{R},\ ad - bc = 1 \right\}$$

which we can identify with:

$$\text{PSL}_2(\mathbb{R}) = \text{SL}_2(\mathbb{R})/\{\pm I\},\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Hyperboloid Model

Define the bilinear form:

$$h(\boldsymbol{x}, \boldsymbol{y}) = x_1y_1 + x_2y_2 - x_3y_3,\ \boldsymbol{x} = (x_1, x_2, x_3), \boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}^3.$$

Then, consider the set

$$H = \{\boldsymbol{x} \in \mathbb{R}^3 : h(\boldsymbol{x}, \boldsymbol{x}) = -1, x_3 > 0\}.$$

It is known that H, equpped with the metric

$$h^1 = dx_1^2 + dx_2^2 - dx_3^2$$

induced by h, is a model of the hyperbolic plane, $\mathbb{H}$. observe that, as a set in $\mathbb{R}^3$, H is a hyperboloid, hence the name.

Theorem 1.2.5. *Consider the linear transformations $R_\theta, T_\delta \in GL(3, \mathbb{R})$, given by:*

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; T_\delta = \begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix}.$$

Let Ω denote the group in $GL(3, \mathbb{R})$ generated by these two elements, then Ω is the group of orientation preserving isometries of (H, h^1). That is:

$$\Omega = \mathrm{Isom}^+(H, h^1).$$

Note 1.2.5. Observe that transformations of the form R_θ act on H by rotating about the axis $\{(0,0,z) : z \in \mathbb{R}\}$ by θ, and T_δ is a translation along H, parallel to the plane $\{(x,y,0) : x,y \in \mathbb{R}\}$ by a hyperbolic distance of δ. This in turn means that any point $p \in (H, h^1)$ can be written as $R_{\theta_p} T_{\delta_p}(p_0)$, where $p_0 = (0,0,1)$. Thus, we may place a coordinate system (θ, δ) on H, where $\theta \in [0, 2\pi)$ or, similarly, $\mathbb{S}^1$ and $\delta \in \mathbb{R}_0^+$. Given such a coordinate system, the corresponding metric h^1 takes the form $ds^2 = d\delta^2 + \sinh^2\delta d\theta^2$, which is the polar-coordinates form of the Poincaré disk model.

We will soon see how the hyperboloid model may be used to derive hyperbolic trigonometric identities for geodesic polygons.

1.3 Hyperbolic Pants

A whole section is dedicated to the study of Hyperbolic pants because they are the building blocks of all compact hyperbolic surfaces with geodesic borders, and will be used to define Fenchel-Nielsen coordinates on Teichmüller spaces.

1.3.1 Hyperbolic Trigonometry

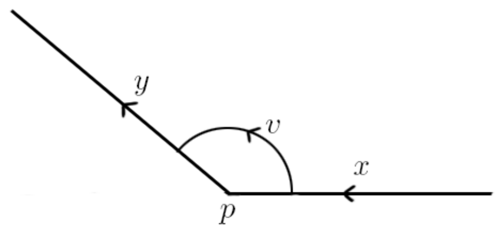
We begin by outlining a technique used in the hyperboloid model to obtain trigonometric identities. We then demonstrate this technique, and obtain several trigonometric identities which will be used to show the uniqueness of hyperbolic pants with fixed border lengths, and also for establishing the explicit formula for $\mathcal{D}$ and $\mathcal{R}$ found in section 2.2.

Definition 1.3.1. A **hyperbolic polygon** is a piecewise geodesic **oriented** closed curve in $\mathbb{H}$, possibly with self-intersections. We call the geodesic arcs of a hyperbolic polygon its **sides**.

Definition 1.3.2. Let (x, y) be an ordered pair of adjoining sides of a hyperbolic polygon P, so that going along x to y is along the orientation of P. In addition, let p be the unique common vertex of x and y, and let $R_v \in \Omega$ denote the rotation which fixes p and rotates y until it overlaps with x. We call v the **angle of rotation** corresponding to p. Moreover, we say that v is the **subsequent angle** of x, and that y is the **subsequent side** of v.

Definition 1.3.3. Given $(x, y) \in AS(P)$, where $AS(P)$ is the set of ordered pairs consisting of one angle and one side, not necessarily in that order. We say that (x, y) is of **angle type** if:

1. y is the subsequent angle of side x, or,
2. y is the subsequent side of angle x.



Definition 1.3.4. Given an ordered pair (x, y) of angle type

1. then, define $N_x \in \Omega$ to be:

$$N_x = T_x = \begin{pmatrix} \cosh x & 0 & \sinh x \\ 0 & 1 & 0 \\ \sinh x & 0 & \cosh x \end{pmatrix}.$$

2. then, define $N_x \in \Omega$ to be:

$$N_x = R_{x+\pi} = \begin{pmatrix} -\cos x & +\sin x & 0 \\ -\sin x & -\cos x & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Definition 1.3.5. An **angle type polygon** is a hyperbolic n-polygon P equipped with a set $X_P = \{s_1, \dots, s_{2n}\}$, comprised of angles and sides, such that

$$(s_1, s_2), (s_2, s_3), \dots, (s_{2n-1}, s_{2n}), (s_{2n}, s_1) \in AS(P)$$

are all of angle type.

Note 1.3.1. All this is really saying is that our X_P is ordered in such a way as to alternatingly describe the length of sides and sizes of adjacent angles of P.

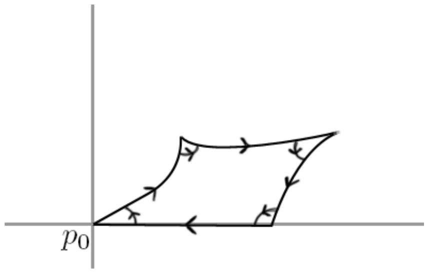
Definition 1.3.6. Let μ denote the geodesic in (H, h^1) given by $\{T_\delta(p_o) \in H : \delta \in \mathbb{R}\}$. Then, we say that (x, y) is in **standard position** if it is an ordered pair of angle type

1. and side x is lying on μ, with the vertex not corresponding to angle y lying on $p_0 = (0, 0, 1)$.
2. and the point corresponding to angle x is lying on p_0, with the side that x is subsequent to lying on μ.

Theorem 1.3.1. *For any angle type polygon P equipped with sides and angles $X_P = \{s_1, \dots, s_n\}$,*

$$N_{s_{2n}} N_{s_{2n-1}} \dots N_{s_2} N_{s_1} = id_\Omega.$$

Proof. Consider the image of P under some isometry $S_{s_1} \in \Omega$ so that (s_1, s_2) is mapped to standard position.



We then apply N_{s_1} to $S_{s_1} \cdot P$, and observe that the pair (s_2, s_3) is now is standard position. Iterating this process, we have:

$$\begin{aligned} N_{s_{2n}} N_{s_{2n-1}} \dots N_{s_2} N_{s_1} S_{s_1} \cdot P &= S_{s_1} \cdot P \\ \Rightarrow S_{s_1}^{-1} N_{s_{2n}} N_{s_{2n-1}} \dots N_{s_2} N_{s_1} S_{s_1} \cdot P &= id_\Omega \cdot P \end{aligned}$$

The only orientation preserving isometry in $\mathbb{H}$ to preserve more than 3 points is the identity isomorphism, therefore

$$\begin{aligned} S_{s_1}^{-1} N_{s_{2n}} N_{s_{2n-1}} \dots N_{s_2} N_{s_1} S_{s_1} &= id_\Omega \\ \Rightarrow N_{s_{2n}} N_{s_{2n-1}} \dots N_{s_2} N_{s_1} &= id_\Omega . \end{aligned}$$

□

We will now demonstrate how theorem 1.3.1 can be used to derive hyperbolic trigonometric identities.

Example 1.3.1. *Consider a angle type quadrilateral P with oriented sides a, b, c, d, and corresponding subsequent angles $\alpha, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$. That is, $X_P = \{a, \alpha, b, \frac{\pi}{2}, c, \frac{\pi}{2}, d, \frac{\pi}{2}\}$. By the above theorem, we know that:*

$$N_{\frac{\pi}{2}} N_b N_\alpha N_a = N_c^{-1} N_{\frac{\pi}{2}}^{-1} N_d^{-1} N_{\frac{\pi}{2}}^{-1}.$$

We proceed to explicitly calculate the left and right hand sides of this equation:

$$\begin{aligned} \text{LHS} &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cosh b & 0 & \sinh b \\ 0 & 1 & 0 \\ \sinh b & 0 & \cosh b \end{pmatrix} \cdot \begin{pmatrix} -\cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & -\cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\cdot \begin{pmatrix} \cosh a & 0 & \sinh a \\ 0 & 1 & 0 \\ \sinh a & 0 & \cosh a \end{pmatrix} = \begin{pmatrix} -\cosh a \sin\alpha & -\cos\alpha & \sinh a \sin\alpha \\ \dots & -\sin\alpha \cosh b & \dots \\ \dots & \sin\alpha \sinh b & \dots \end{pmatrix}; \\ \text{RHS} &= \begin{pmatrix} \cosh c & 0 & sinhc \\ 0 & 1 & 0 \\ -\sinh c & 0 & \cosh c \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cosh d & 0 & -\sinh d \\ 0 & 1 & 0 \\ -\sinh d & 0 & \cosh d \end{pmatrix} \\ &\cdot \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\cosh c & -\sinh c \sinh d & -\sinh c \cosh d \\ 0 & -\cosh d & -\sinh d \\ \sinh c & \cosh c \sinh d & \cosh c \cosh d \end{pmatrix}. \end{aligned}$$

By equating matrix coordinates, we obtain several trigonometric identities specific to P, in particular,

Lemma 1.3.2. *Given P as specified above,*

$$\cos\alpha = \sinh c \sinh d.$$

Note 1.3.2. We will use this formula later as part of the proof for the Generalised McShane Identity, and will consider the $\alpha = 0$ case in particular.

Example 1.3.2. *Consider now a right-angled hyperbolic hexagon P with sides a, b, c, d, e and$f$, listed with respect to orientation. Then, P, equipped with*

$$X_P = \{a, \frac{\pi}{2}, b, \frac{\pi}{2}, c, \frac{\pi}{2}, d, \frac{\pi}{2}, e, \frac{\pi}{2}, f, \frac{\pi}{2}\}$$

is a angle type polygon. Then, by theorem 1.3.1,

$$N_{\frac{\pi}{2}} N_c N_{\frac{\pi}{2}} N_b N_{\frac{\pi}{2}} N_a = N_d^{-1} N_{\frac{\pi}{2}}^{-1} N_e^{-1} N_{\frac{\pi}{2}} N_f^{-1} N_{\frac{\pi}{2}}.$$

After explicitly calculating both sides in term of matrices and equating coefficients, we obtain:

$$\cosh e = \sinh a \cosh b \sinh c - \cosh a \cosh c,$$

and

$$\frac{\sinh a}{\sinh d} = \frac{\sinh c}{\sinh f} = \frac{\sinh e}{\sinh b}.$$

From these two relations, it is easy to see that given the lengths of any three sides of a right-angled hyperbolic hexagon. We can determine what all the other sides are. That is:

Lemma 1.3.3. *The lengths of any three sides in a right-angled hyperbolic hexagon uniquely specifies the hexagon up to isometry.*

1.3.2 Hyperbolic Pants

Definition 1.3.7. We say that a polygon in $\mathbb{H}$ is **geodesic** if it can be written as the intersection of a finite number of regions bounded by a geodesic.

Note 1.3.3. It is important, for the purpose of this thesis, to draw the distinction between hyperbolic polygons and geodesic polygons. In any case, we will not encounter any hyperbolic polygons from this point onwards.

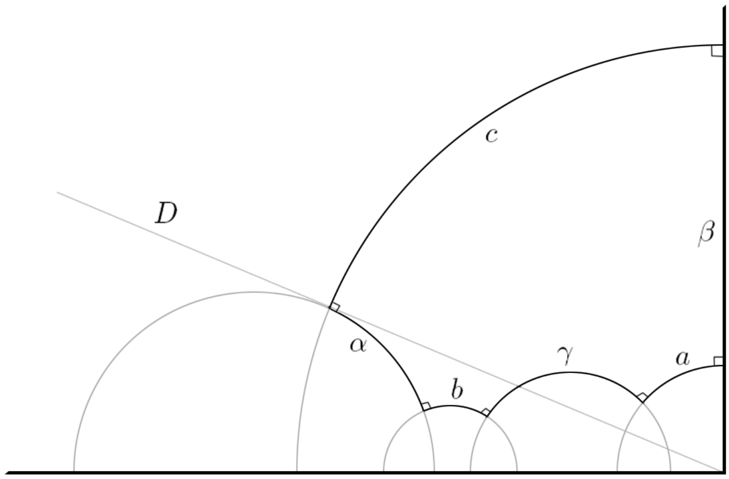
Definition 1.3.8. A **hyperbolic pair of pants**, or simply a **pair of pants**, is a compact Riemannian surface of signature $(0, 3)$, equipped with the hyperbolic metric, so that each of its three boundary components is a simple closed (distance minimising) geodesics.

Theorem 1.3.4. *For any triple of positive real numbers (ℓ_1, ℓ_2, ℓ_3), there exists a unique pair of pants (up to isometry) with boundary geodesics $\gamma_1, \gamma_2, \gamma_3$ of lengths $\ell(\gamma_i) = \ell_i,\ i = 1, 2, 3$.*

We state the above theorem to motivate the lemmata that we are about to prove.

Lemma 1.3.5. *For any positive triplet (a, b, c), there exists a unique right-angled geodesic hexagon in the hyperbolic plane with three alternating sides of length (a, b, c).*

Proof. We first prove existence by construction.



Let β, a and γ be the lengths of the correspondingly marked geodesics in the diagram. We fix a, but allow β and γ to vary for now. Then let the line D denote all points in $\mathbb{H}$ so that the hyperbolic distance from β to D is always equal to c. Observe that D is indeed a Euclidean straight line in the half-plane model, because the Möbius transformation $z \mapsto \lambda z,\ \lambda \in \mathbb{R}_+$ is an isometry by the Schwarz-Pick lemma. We then consider a geodesic α tangent to D moved along D until the distance between α and γ is precisely b. By continuity, this is always possible.

On the other hand, the uniqueness of such a hexagon has been proven in lemma 1.3.3. □

Lemma 1.3.6. *For any simple curve γ in a bordered hyperbolic surface X, there exists a shortest geodesic in the homotopy class of this curve. Moreover, if either end-points of γ lies on a boundary component, then the shortest geodesic representative of γ must meet this boundary component perpendicularly.*

Proof. [1] provides two different strategies for proving this lemma. One method is to build an equicontinuous family of geodesics of decreasing length and then use the Arzelà Ascoli theorem to show that a limiting curve of minimal length exists. A second method is to draw geodesics intersecting γ, and then to replace segments of γ with segments of the geodesics in a controlled manner, and thereby providing a limiting construction of the shortest geodesic. In the boundary meeting case, the angle of intersection between this geodesic and the boundaries must be $\frac{\pi}{2}$. Otherwise, we could construct a piece-wise geodesic path (and hence a geodesic) with shorter distance. □

Lemma 1.3.7. *Given a pair of pants, for any unordered pair of boundaries, there exists a unique simple complete geodesic path perpendicular to both of them. We'll call the unique geodesic going from α to β the* α, β**-link**. Moreover, we'll call the endpoint of an α, β-link that lies on α, the α**-link point** and similarly, the β**-link point** is the endpoint lying on β.

Proof. The existence of a shortest geodesic between any two boundary components is guaranteed by lemma 1.3.6. As for uniqueness, we will prove this via the Gauss-Bonnet formula (A.1.1). Observe that on a pair of pants, any two simple paths running from one boundary component to another are homotopic. Hence, if there were two geodesics running from one boundary to the other, then we can find some simple region bounded by geodesics. If the two geodesics do not meet, this region will be a right-angled hyperbolic rectangle (denoted by R). And if they do meet: a hyperbolic triangle (denoted by $\triangle$) with (at least) two right-angles. In the former case, all four interior angles are right angles, and by the Gauss-Bonnet formula, this would force the area of the region to be

$$\text{Area}(R) = \int_R dA = -\int_R K\ dA = -[2\pi\chi(R) - \int_{\partial R} \kappa_g] = -(2\pi - 2\pi - 0) = 0,$$

where χ is the Euler characteristic, $K = -1$ is the sectional curvature of R and κ_g is the geodesic curvature of the boundary. Which, in this case, is equal to the sum of all the angles at the corners of ∂R by which the geodesics sides turn. The fact that the area is 0 gives us a contradiction in this first case. In the second case, there are two right angles and by Gauss-Bonnet, the area of the region is

$$\text{Area}(\triangle) = -[2\pi - (\frac{\pi}{2} + \frac{\pi}{2} + \alpha_3) - 0] = -(\pi - \alpha_3) \leq 0,$$

hence another contradiction. □

Note 1.3.4. Since there are three unordered pairs of boundaries, there exist three such canonical links on any pair of pants. Now, if we cut along these three links, then we obtain two right-angled hexagons. As these two hexagons share three boundaries (and hence boundary lengths), by lemma 1.3.3, they must be isometric.

We now prove theorem 1.3.4.

Proof. By lemma 1.3.5, we know that there exists a hyperbolic hexagon of side-lengths $\frac{\ell_1}{2}, \frac{\ell_3}{2}, \frac{\ell_3}{2}$. If we glue any such two hexagons together, making sure that hyperbolicity is retained along the glued paths, then we obtain a hyperbolic pair of pants with desired boundary length components.

Now for uniqueness.

Consider two hyperbolic pairs of pants P and $\tilde{P}$ with boundary components $\gamma_i, \tilde{\gamma}_i,\ i = 1, 2, 3$, such that the lengths of the corresponding boundary components are equal. Thus, when we respectively divide our pair of pants P and $\tilde{P}$ into two hexagons along their three boundary-perpendicular geodesics, we will have four isometric hexagons. Now, using the gluing lemma we can turn a pair of these isometries into an isometry between our pairs of pants. □

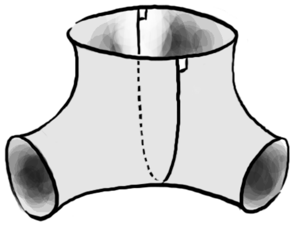
Lemma 1.3.8. *The area of any hyperbolic pair of pants is* 2π.

Proof. From lemma 1.2.3, we know that the area of each constuent right-angled hexagon is $4 \times \pi -$ interior angles $= 4\pi - 6 \times \frac{\pi}{2} = \pi$. Then, the total area of the pair of pants follows. Alternatively, we could prove this result via the Gauss-Bonnet formula. □

Note 1.3.5. Since the surface area is fixed, intuitively, we then expect that hyperbolic pants with larger boundary lengths must be *thinner*. Although this may seem like a fairly arbitrary comment to make at this stage, it does have implications when we study embedded pairs of pants in bordered hyperbolic surfaces: the thinner a pair of pants is, the more times it is able to wrap around a surface.

We now state and prove a few other facts about hyperbolic pants which will be used for the characterisation of geodesics needed to prove the Generalised McShane Identity.

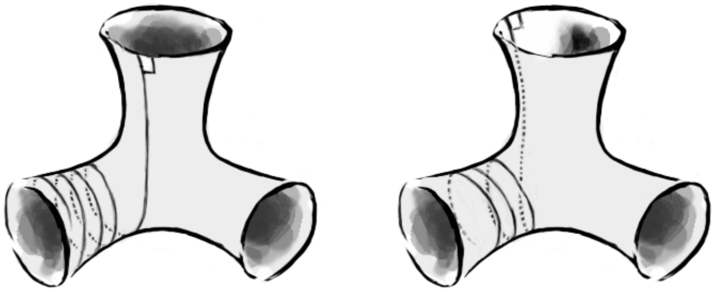
Lemma 1.3.9. *Given a hyperbolic pair of pants P, with geodesic boundary components $\{\alpha_i\}_{1\leq i\leq 3}$, for each α_i, there exists a unique simple complete geodesic γ_i with both end-points in α_i, such that γ_i is perpendicular to α_i at both ends. We call these geodesics* **self-links**, and the two end-points of a self-link are called **self-link points**.



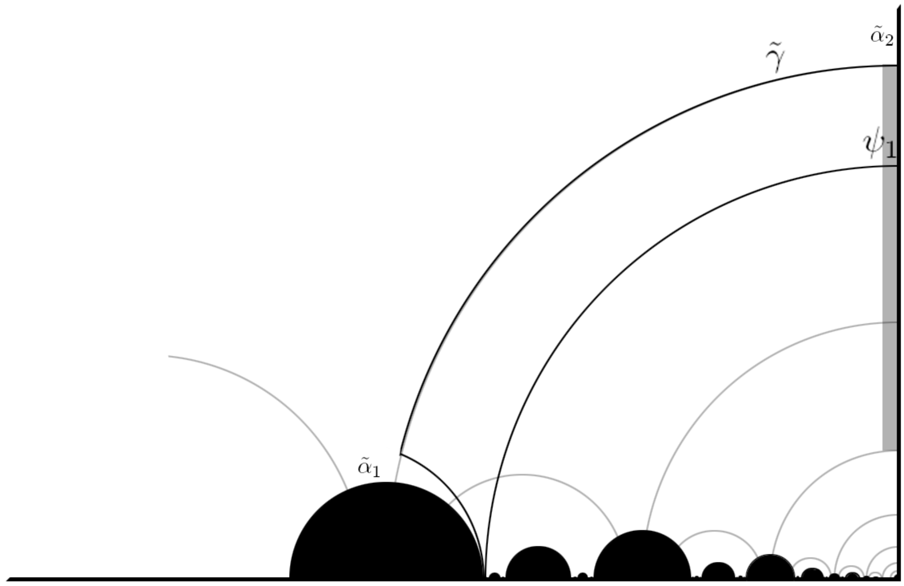
Proof. The argument is almost identical to that of lemma 1.3.7. □

Lemma 1.3.10. *There exists only two simple infinite geodesics $\gamma : [0,\infty) \to P$ such that:*

- $\gamma(0) \in \alpha_1$, *and* γ *intersects* α_1 *perpendicularly,*
- $\gamma(t)$ *spirals to* α_2.



Proof. Consider the following universal cover of a pair of hyperbolic pants P, where $\tilde{\alpha}_1$ is a lift of α_1 and $\tilde{\alpha}_2$ is a lift of α_2 closest to $\tilde{\alpha}_1$. There are countably infinite number of these, one for each segment on $\tilde{\alpha}_1$ covering α_1 exactly once. We'll restrict ourselves to the shaded region along $\tilde{\alpha}_1$ corresponding to a single lift of α_1.



We begin by showing the existence of a simple geodesic ray emanating perpendicularly from α_1 and spiraling around α_2. It is known that there is a small collar neighbourhood around α_2 diffeomorphic to $\mathbb{S}^1 \times [0,1]$, where the diffeomorphism h carries α_2 to $\mathbb{S}^1 \times \{1\}$. Now, consider the following path $\sigma : [0,\infty) \to \mathbb{S}^1 \times [0,1]$ given by:

$$\sigma(t) = (\exp(it), \frac{t}{1+t}).$$

Then, $h \circ \sigma$ is a simple and infinitely long path in P, and we can choose its starting point to lie along the unique α_1, α_2-link in P. Then, consider the geodesic γ which goes along the α_1, α_2-link in P from α_1 until it reaches $h \circ \sigma(0)$, and then follows $h \circ \sigma$. Then, γ is a simple and infinitely long path in P. From the same *cutting* and shortening argument as used for the proof of lemma 1.3.6, we can construct a simple geodesic homotopic to γ and going going from α_1 to α_2. Moreover, we know from the length minimising process that this geodesic meets α_1 perpendicularly. Hence, we know that there exists a simple geodesic ray from α_1 so that it spirals around α_2. Now, we know that the lift of this geodesic ray must be homotopic to $\tilde{\gamma}$ - the lift of γ. But there's only one possible candidate for such a lift: ψ_1. Notice that if we caused σ to spiral the other way, that is, $\sigma(-t)$, then we would have created a simple path homotopic to ψ_2 (not drawn, it's the geodesic tangent to the other side of $\tilde{\alpha}_2$). We now try to prove that there are no other such geodesic rays.

Consider a simple infinite geodesic ray ψ_3 meeting γ_1 perpendicularly, and spiraling to γ_2. Then, let β denote the segment of ψ_3 before it permanently stays in the small collar neighbourhood around α_2 mentioned previously. We extend β to a simple path going from α_1 to α_2, and know that the shortest geodesic representative of the homology class of β is also simple, and must be the α_1, α_2-link. Thus, β is homotopic to the path give by travelling down the α_1, α_2-link until reaching the small collar neighbourhood around α_2, and then going slightly around until reaching the endpoint of β. This in turn means that ψ_3 is homotopic to a infinite simple curve η given by travelling down the α_1, α_2-link, across to the end point of β, and then going along the rest of ψ_3 and thereby staying in the small collar neighbourhood around α_2. However, then η must lift to something that is asymptotic to $\tilde{\alpha}_2$. Hence, the geodesic representative of its homotopic class (which we know is ψ_3) must be one of the two previously stated geodesics.

□

Lemma 1.3.11. *Given any self-link γ on a bordered hyperbolic surface X going from β_1 to β_1, there exist a unique embedded pair of pants $P \subset X$, so that γ lies in P.*

Proof. Let x and y denote the self-link points of γ self-linking α. Then, α can be divided into two disjoint segments by removing x and y. Take the union of one of these segments and γ and we have a closed loop η on X. Then there exists a unique shortest geodesic representative $\tilde{\eta}$, of the homotopy class of η. Similarly, taking the other disjoint segment union γ, denoted by δ, will give us a shortest geodesic representative $\tilde{\delta}$, of the homotopy class of δ. Then, α, $\tilde{\delta}$ and $\tilde{\eta}$ bound a pair of pants.

γ lies in P, because it's the shortest geodesic representative of the unique self-link contained in P. □

Note 1.3.6. Much of the above proof did not necessitate that γ be a self-link, merely that it is a simple path joining α to itself. Thus, any such path corresponds to an embedded pair of pants. However, in the case that γ is not a self-link, the induced pair of pants, in general, will not contain parts of γ.

Lemma 1.3.12. *For any two boundary components α_1 and α_2 joined by a simple geodesic γ, there exists a unique embedded pair of pants containing γ.*

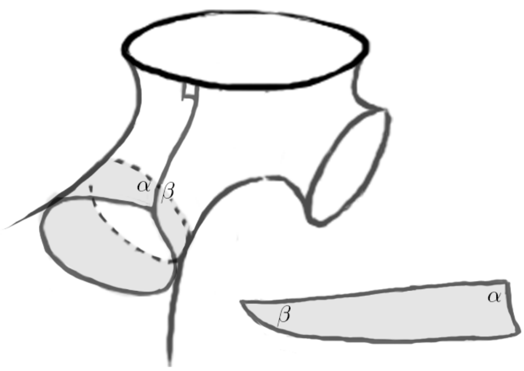
Proof. The argument is essentially the same as for the previous lemma, we only *sketch* the proof.

□

Note 1.3.7. In the above proof, we have assumed, without loss of generality, that γ is the α_1, α_2-link.

Lemma 1.3.13. *Consider a bordered hyperbolic surface X, and β a geodesic boundary component of X. Let $x \in \beta$ be a point on the boundary so that the geodesic perpendicularly emanating from x, denoted by γ_x, intersects itself in X. Furthermore, denote the segment of γ_x up to the first time it intersects itself by $\tilde{\gamma}_x$. Then, there is a unique embedded pair of pants in X containing $\tilde{\gamma}_x$.*

Proof. Notice that $\tilde{\gamma}_x$ is essentially a loop on the end of a leash. If we travel down this leash, go around the loop once but stopping just before getting back to the start of the loop, and the back up along the leash, remaining close but not actually touching the leash. Then we have simple curve going from β to β, and by the proof of lemma 1.3.11, this curve induces an embedded pair of pants P, with boundaries denoted by β, α_1 and α_2. We know without loss of generality that the loop at the end of the leash is homotopic to α_1. It also cannot intersect α_1, otherwise, we would have a hyperbolic geodesic 2-gon forming, which is easily contradicted by the Gauss-Bonnet formula. Thus, either the loop is outside P or inside. If outside, then the following figure shows that we obtain a geodesic triangle with two internal angles w and v adding up to π, which cannot be possible. On the other hand, if the loop is inside of P, then so too must the leash, otherwise, it would intersect with one of the boundary geodesics twice, thereby giving us a hyperbolic 2-gon.



Note 1.3.8. Special note to Craig, the symbols α and β in the diagram refer to angles, and are not to be confused with the sides of the pair of pants.

□

1.4 Teichmüller Spaces

There exist various approaches to defining Teichmüller spaces of a riemann surfaces S. For example: markings can be defined in terms of the generators of the fundamental group of S (Fricke Space), or we might assign to each marking homeomorphism a function measuring at each point how much the image of a infinitesimal circle is skewed into an infinitesimal ellipse (Beltrami coefficients).

The two formulations that we have selected each serves a different purpose. The first is chosen because it is conceptually similar to the definition of a moduli space as conformal classes of Riemann surfaces. The second is chosen because we will construct Fenchel-Nielsen coordinates via hyperbolic pants decompositions. Readers wanting a more complete picture are might wish to consult [4] or [3].

1.4.1 Riemann Surface Definition

Theorem 1.4.1. *Any two homeomorphic Riemann surfaces are diffeomorphic.*

Definition 1.4.1. For each pair of non-negative integers (g, n), where $2g + n \geq 3$, there exist complete oriented Riemann surfaces of with g genera and n marked points. Fix one such $S_{g,n}$ for each (g, n). A **marked surface** is defined to be a pair: (S, φ), such that S is a Riemann surface with g genera and n removed points, and

$$\varphi : S_{g,n} \to S$$

is a homeomorphism taking each marked point to the corresponding marked point. We say that φ gives a **marking homeomorphism** on S by $S_{g,n}$.

One way of thinking about (S, φ) as a marking is to think about by pulling back the atlas on S to $S_{g,n}$, thereby giving the topological space $S_{g,n}$ a Riemann surface structure. In this sense, the homeomorphism φ *marks* $S_{g,n}$.

Definition 1.4.2. We say that two marked Riemann surfaces (S_1, φ_1) and (S_2, φ_2) are **Teichmüller equivalent** if there exists a biholomorphism (conformal map) $c : S_1 \to S_2$ such that:

- φ_1 and $c \circ \varphi_2 : S_{g,n} \to S_1$ are isotopic,
- let $\varphi^t : [0, 1] \times S_{g,n} \to S_1$ denote the isotopy referred to above, then for each $t \in [0, 1]$, $\varphi^t : S_{g,n} \to S_1$ leaves the marked points fixed.

We denote the Teichmüller equivalence class of any marked surface (S, φ) by $(S, \varphi)_{\mathcal{T}}$.

Note 1.4.1. Given any two isotopic maps $\varphi_1, \varphi_2 : S_{g,n} \to S$, then $(S, \varphi_1)_{\mathcal{T}} = (S, \varphi_2)_{\mathcal{T}}$. However, the converse is not true: for example, take a non-trivial isometry $f : S \to S$, then $(S, \varphi)_{\mathcal{T}} = (S, f \circ \varphi)_{\mathcal{T}}$ but φ is not isotopic to $f \circ \varphi$.

Here is one way to think about the above definition: take an atlas $\{(U_1, \alpha_1)\}$ on S_1. The isotopy condition tells us that on S_2, there needs to be some atlas $\{U_2, \alpha_2)\}$, so that when $\{(U_1, \alpha_1)\}$ is dragged about by the isotopy and pulled back to S_1 by a map which preserves angles, it isometrically 'agrees' with $\{(U_2, \alpha_2)\}$. The isotopy ensures that we haven't taken a collection of coordinate neighbourhoods and wrapped them around the surface multiple times, before regluing them to eachother along the patches that they were originally glued along. Permitting such behaviour would in fact result in the Moduli space.

Definition 1.4.3 (Teichmüller 1)**.** Let $\mathcal{T}(S_{g,n})$ denote the set of all Teichmüller equivalent marked surfaces, then we call $\mathcal{T}(S_{g,n})$, also denoted by $\mathcal{T}_{g,n}$, the **Teichmüller space of** $S_{g,n}$.

1.4.2 Hyperbolic Surface Definition

From theorem 1.4.1, in each Teichmüller class of marked surfaces, there exist marked surfaces where the marking homeomorphism is a diffeomorphism. Therefore, it is equivalent to restrict ourselves to only look at marked surfaces equipped with a diffeomorphism. Moreover, thanks to 1.1.4, in each Teichmüller equivalence class, we can always find a hyperbolic marked surface. These observations motivate the following equivalent formulation for Teichmüller spaces.

Definition 1.4.4 (Teichmüller 2)**.** Let $\mathcal{H}$ be the set of all smooth hyperbolic structures on $S_{g,n}$, $2g + n \geq 3$, and let Diff^0 denote the group of marked points fixing, self-diffeomorphisms of $S_{g,n}$ which are isotopic to an isometry of $S_{g,n}$. Then the Teichmüller space $\mathcal{T}(S_{g,n})$ can be defined as:

$$\mathcal{H}/\mathrm{Diff}^0.$$

Looking at hyperbolic structures on $S_{g,n}$ is analogous to the idea of pulling back, by the marking homeomorphism, Riemann surface structures to $S_{g,n}$, as in our first definition of Teichmüller spaces. Let us now try to understand how quotienting by Diff^0 is analogous to Teichmüller equivalence.

Observe that two hyperbolic surfaces (S_1, φ_1) and (S_1, φ_2) are Teichmüller equivalent if and only if $\varphi_1 \circ \varphi_2^{-1}$ is isotopic to a conformal map c. If we pull back (S_1, φ_1) and (S_2, φ_2) onto $S_{g,n}$, we would say that the induced hyperbolic structures are Teichmüller equivalent if $\varphi_1 \circ \varphi_2^{-1}$ is isotopic to a conformal map. Observe further that conformal maps between compact hyperbolic surfaces are isometries (a corollary of theorem 1.1.4), hence the given definition of Diff^0.

1.4.3 Teichmüller Spaces with Lengths

We will now deal with Teichmüller and moduli spaces of bordered hyperbolic surfaces with specified lengths. Instead of looking at a hyperbolic surface $S_{g,n}$ with g genera and n marked points, Mirzakhani generalises the definition to hyperbolic surfaces homeomorphic to Riemann surfaces with g genera and n discs removed, and with fixed boundary lengths.

Note 1.4.2. Unless stated otherwise, we will henceforth use the notation $S_{g,n}$ to mean a surface with g genera and n boundaries, as opposed to marked points.

Definition 1.4.5 (Teichmüller with Lengths)**.** Given a bordered Riemann surface S, consider the collection of marked hyperbolic surfaces (X, f) that have geodesic boundary components of equal and fixed length. Specifically, let $\partial S = \bigcup_{i=1}^{n} \alpha_i$, where each α_i is a boundary components of S, then we fix $\boldsymbol{L} \in \mathbb{R}_+^n$ such that, for every marked surface (X, f) in our collection, $\ell_{f(\alpha_i)}(X) = L_i$.

We then say that two marked surfaces (X, f) and (Y, g) are **Teichmüller equivalent** if $f \circ g^{-1} : Y \to X$ is isotopic to a conformal map (and hence an isometry). The set of all marked hyperbolic surfaces under Teichmüller equivalence is denoted by $\mathcal{T}(S, \boldsymbol{L})$. We will sometimes denote $\mathcal{T}(S_{g,n}\boldsymbol{L})$ by $\mathcal{T}_{g,n}(\boldsymbol{L})$.

Note 1.4.3. If $\mathcal{T}(S, \boldsymbol{L})$ is to be a generalisation of $\mathcal{T}_{g,n}$, it is important that we introduce a fixed length to the boundaries. If no boundary lengths were fixed, then the Teichmüller space for a hyperbolic pair of pants would be $\mathbb{R}_+^3$. Where as Consider $\mathcal{T}_{0,3}$ is a single point.

1.5 Fenchel-Nielsen Coordinates

In their 1948 paper, Fenchel and Nielsen constructed a coordinate system for Teichmüller spaces of closed hyperbolic surfaces. By introducing a fixed length to the boundary components of a bordered hyperbolic surface, Mirzakhani has generalised their definition to include $\mathcal{T}(S, \boldsymbol{L})$.

We adopt some unconventional definitions for this section, namely: pants covers and τ-twists, so as to avoid providing preparatory work needed for other methods of rigorously describing the twisting parameter in Fenchel-Nielsen coordinates. Readers may wish to consult pages 143-144 of [1].

1.5.1 Pants Covers and Decompositions

The following definitions are to help formalise the intuitive idea of cutting bordered surfaces into pants.

Definition 1.5.1. Given a (possibly bordered) Riemann surface S , let $\{(P_i, \iota_i)\}$ be a collection of pairs of pants P_i and holomorphisms $\iota_i : P_i \to S$, such that:

1. the closed sets $\{\iota_i(P_i)\}$ cover S;
2. $\iota_i|_{\mathrm{int}(P_i)} : \mathrm{int}(P_i) \to \iota_i(\mathrm{int}(P_i)) \subset S$ is a biholomorphism;
3. for each boundary component $\partial_j P_i$ of P_j $(j = 1, 2, 3)$, $\iota_i|_{\partial_j P_i} : \partial_j P_i \to S$ is injective.
4. any connected component of the intersection of any two distinct $\{\iota_i(P_i)\}$ is the image of one of the boundary components of both pairs of pants.

We say that $\{(P_i, \iota_i)\}$ is a **pants cover** of S.

Definition 1.5.2. A **pants decomposition** of a Riemann surface S is a set of disjoint simple closed curves $\{\gamma_i\}$, such that: if we denote each connected component of $S - \bigcup \gamma_i$ by P_j°, and $\hat{\iota}_j$ its natural embedding, then the there exists a pants cover $\{(P_j, \iota_j)\}$ for S, where

$$\mathrm{int}(P_j) = P_j^\circ, \text{ and } \iota_j|_{\mathrm{int}(P_j)} = \hat{\iota}_j.$$

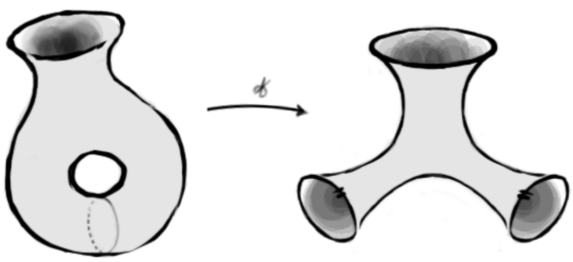
Although we have defined pants covers and decompositions for Riemann surfaces, in practice, we seldom deal with anything other than hyperbolic surfaces (thanks to our chosen definition of Teichüller space). In which case, we make the following special definitions:

Definition 1.5.3. Given a hyperbolic surface S, a **hyperbolic pants cover** is a pants collection $\{(P_i, \iota_i)\}$, where each P_i is a hyperbolic pair of pants, each ι_i is an isometric immersion, and each $\iota_i|_{\mathrm{int}(P_i)}$ and $\iota_i|_{\partial_j P_i}$ is an isometry. Similarly, any pants decomposition with a corresponding pants cover that is hyperbolic is called a **hyperbolic pants decomposition**.

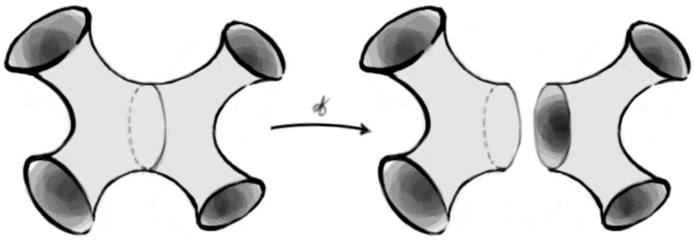
Note 1.5.1. Given a hyperbolic pants cover of a hyperbolic surface S, every boundary component of a hyperbolic pair of pants P_i isometrically maps to a geodesic in S, thus, we may determine its length. Then, since hyperbolic pants are uniquely (up to isometry) specified by their boundary lengths, the corresponding pants cover for any hyperbolic pants decomposition must be unique up to isometry.

We now present two examples of hyperbolic pants decompositions, each showcases a different type of decomposition that can occur.

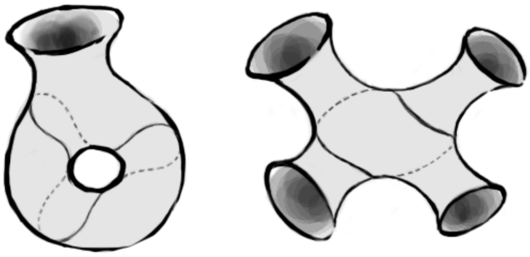
Example 1.5.1. *Consider the hyperbolic surface $S_{1,1}$, and the depicted pants decomposition. We see that it is possible for a pair of pants to join itself. In this particular case, the resulting pair of pants (from the 'cut') must be rotationally and reflectively symmetrical.*



Example 1.5.2. *This second example shows a different type of pants decomposition. Unlike the previous example in which the cut results in the removal of a genus, in this pants decomposition, we sever the surface into two disjoint pieces.*



Example 1.5.3. *In general, pants decompositions are not unique. Consider the following simple geodesics on $S_{1,1}$ and $S_{0,4}$.*



If we decompose the surface along this curve for $S_{1,1}$, then we obtain a surface of genus 0 with 3 geodesic boundary components: a hyperbolic pair of pants. On the other hand, since a sphere and has betti number 0 (that is, the total number of cuts that can be made without separating the surface into disconnected pieces), cutting along the simple geodesic shown for $S_{0,4}$ must result in two disconnected surfaces. Each of these two disjoint pieces must each contain two of the original boundary components, and one new geodesic boundary resulting from the cut.

Lemma 1.5.1. *Let $\{(P_i, \iota_i)_{1\leq i\leq n})\}$ be a hyperbolic pants decomposition for the bordered hyperbolic surface X. Then, X is isometric to*

$$\mathcal{S}(\{(P_i, \iota_i)\}) := \coprod P_i/[x] \sim [y] \text{ iff. } \iota_a(x) = \iota_b(y),$$

where a, b are any two natural numbers so that $x \in P_a$ and $y \in P_b$, and $[x], [y]$ are their quotient space representatives.

Proof. We will sketch a proof by constructing a function from $\mathcal{S}(\{(P_i, \iota_i)\})$ to X. Consider g given by:

$$g([x]) = \iota_i(x), \text{ if } x \in P_i.$$

Observe that the equivalence relation on $\mathcal{S}(\{(P_i, \iota_i)\})$ ensures that g is well-defined. In addition, g is isometric by the isometric property of the ι_i [2]. Then consider the function $h : X \to \mathcal{S}(\{(P_i, \iota_i)\})$ given by:

$$h(x) = [y], \text{ where } y \text{ is any element in the preimage of } x \in P_i.$$

h too is well-defined and isometric for similar reasons. Then, $g \circ h = \mathrm{id}_X$ and $h \circ g = \mathrm{id}_{\mathcal{S}(\{(P_i,\iota_i)\})}$, and we're done. □

Lemma 1.5.2. *For any compact Riemann surface $S_{g,n}$ with g genera and n boundary components, there exists a pants decomposition. Moreover, any pants decomposition will be comprised of $3g-3+n$ closed loops.*

Proof. We begin by sketching an algorithm for obtaining a pants decomposition.

For each genus in $S_{g,n}$, there is a homotopy class $[\alpha_i]$, $1 \le i \le g$ that corresponds to 'going around' the genus. In each homotopy class, there is a unique geodesic representative γ_i, and if we cut along these g number of 'γ_i's, then we obtain a connected surface with no genus and $2g+n$ geodesic boundary components. We know from lemma 1.3.12 that for any pair of these two boundary components, we may join them up by a geodesic, and this geodesic will help give us a hyperbolic pair of pants containing our chosen two boundary components as well as this geodesic. The third boundary of this hyperbolic pair of pants must be a geodesic embedded in our S_{2g+n}. Thus, if we choose it as a γ_{g+1}, and cut along it, we end up with 1 hyperbolic pair of pants and a connected surface with $2g+n-2+1 = 2g+n-1$ geodesic boundary components. Iterating this process for S_{2g+n-1}, we obtain k hyperbolic pairs of pants and S_{2g+n-k}. This process continues until $2g+n-k=3$, that is, we're left with a hyperbolic pair of pants, and a hyperbolic pants decomposition $\{\gamma_i\}$, $1 \le i \le g+k = 3g-3+n$.

Using the Euler Characteristic, we'll prove that any pants decomposition must contain $3g-3+n$ simple closed geodesics.

Given a pants decomposition $\{\gamma_i\} 1 \le i \le m$, we would like to show that $m = 3g-3+n$. Observe that we can recover the original surface $S_{g,n}$ via connect sums of a corresponding hyperbolic pants cover to our pants decomposition, where each connect sum is along one of the γ_i. Let p denote the number of hyperbolic pairs of pants in an arbitrary corresponding pants cover. Each pair of pants has 3 boundaries, and when we cut along any γ_i, we obtain 2 boundaries which belong on a pair of pants. Therefore,

$$3p = 2m + n.$$

Since the Euler characteristic of each γ_i is 0, then using the following result regarding Euler characteristics:

$$\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N),$$

we see that

$$2 - 2g - n = \chi(S_{g,n}) = \sum_{k=1}^{p} \chi(P_k) = -p,$$

where each P_k is a hyperbolic pair of pants. Therefore, $6 - 6g - 3 = -(2m+n) \Rightarrow m = 3g-3+n$, as desired. □

1.5.2 Twisting

We now define τ-twists, which are actions taking one hyperbolic surface to another, by cutting it along a specified geodesic loop, twisting it by $\frac{\tau}{2\pi}$ times the length of the loop, and then regluing.

[2]Strictly speaking, it is also because of the constant sectional curvature of hyperbolic manifolds, and arguably, the transitivity of $\mathrm{Isom}^+(\mathbb{H})$.

Given a hyperbolic pants decomposition $\{\gamma_j\}$ with corresponding hyperbolic pants cover $\{(P_i, \iota_i)\}$ for the bordered hyperbolic surface $X = \mathcal{S}(\{(P_i, \iota_i)\})$ (up to isometry), consider $\gamma \in \{\gamma_j\}$, with total length ℓ. We know that $\tilde{\gamma}_k$ lifts to two boundaries of (one or two) pairs of pants in $\{(P_i, \iota_i)\}$. Denote these boundaries by α_1 and α_2, with the pants leg surrounding α_1 embedding to the left, with respect to the orientation, of γ_k, and pants leg surrounding α_2 to the right.

We then parametrise $\alpha_n,\ n = 1, 2$ in terms of length in the positive orientation so that

$$\alpha_1(0) = \alpha_2(0),$$

and extend $\alpha_2 : [0, \ell) \to X$ to all of $\mathbb{R}$ by defining

$$\tilde{\alpha}_n(t) = \alpha\left(\frac{\ell t}{2\pi} \mod \ell\right).$$

Note 1.5.2. $\tilde{alpha_2}$ is no longer parametrised in terms of length, but rotation angle.

Definition 1.5.4. Finally, for $\tau \in [0, 2\pi)$, we define the **τ-twist** around γ, denoted by tw_γ^τ to be a diffeomorphism taking the surface $X = \mathcal{S}(\{(P_i, \iota_i)\})$ to the following hyperbolic surface:

$$\mathrm{tw}_\gamma^\tau(X) = \coprod P_i / \sim,$$

where the equivalence relation $\sim$ is given by:

$$\begin{aligned}[x] \sim [y] \text{ iff. } &\iota_a(x) = \iota_b(y) \text{ for } x, y \notin \alpha_1 \cup \alpha_2, \text{ and} \\ &\iota_a(x = \alpha_1(t)) = \iota_b(y = \alpha_2(t + \tau)) \text{ otherwise.}\end{aligned}$$

Where a, b are any two natural numbers so that $x \in P_a$ and $y \in P_b$.

Moreover, since τ-twists on disjoint simple closed geodesics are commutative, given a collection $\Gamma = \{\gamma_m\}_{1 \le m \le \mu}$ of disjoint closed geodesics, and a vector $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_\mu)$, we define the $\boldsymbol{\tau}$-twist around Γ to be:

$$\mathrm{tw}_\Gamma^{\boldsymbol{\tau}} = \mathrm{tw}_{\gamma_\mu}^{\tau_\mu} \circ \ldots \circ \mathrm{tw}_{\gamma_1}^{\tau_1}.$$

Note 1.5.3. For $\tau \neq 0$, the hyperbolic structure given by X is not Teichmüller equivalent to that of $\mathrm{tw}_\gamma^\tau(X)$. We will prove this when constructing Fenchel-Nielsen coordinates by considering what happens to the X and $\mathrm{tw}_\gamma^\tau(X)$-images of self-link points on γ_1 and γ_2.

Definition 1.5.5. Given a simple closed geodesic γ in a bordered hyperbolic surface X, let A denote a collar neighbourhood of γ. Then, A is an annulus and there is some smooth embedding:

$$h : \mathbb{S}^1 \times [0, 1] \to X,$$

where $\mathbb{S}^1 = \{e^{i\theta} : \theta \in [0, 2\pi)\}$, and h is a diffeomorphism between $\mathbb{S}^1 \times [0, 1]$ and A. We define a function $\phi_\gamma(X) : X \to X$ given by:

$$\phi_\gamma(X) = \begin{cases} h \circ f \circ h^{-1}(x), & x \in A, \\ x, & \text{otherwise,} \end{cases}$$

where $f : \mathbb{S}^1 \times [0, 1] \to \mathbb{S}^1 \times [0, 1]$ is given by:

$$f(s, t) = (se^{i2\pi t}, t).$$

We call ϕ_γ a **Dehn twist** around γ.

Moreover, since a series of Dehn twists around Γ (as per the previous definition) is commutative, for $\boldsymbol{n} = (n_1, \ldots, n_\mu) \in \mathbb{Z}^\mu$, we define the $\boldsymbol{n}$-Dehn twist around Γ to be:

$$\phi_\Gamma^{\boldsymbol{n}} = \phi_{\gamma_\mu}^{n_\mu} \circ \ldots \circ \phi_{\gamma_1}^{n_1},$$

where ϕ_γ^n is applying the Dehn twist n times.

Note 1.5.4. The conventional definition of Dehn twists is not restricted to bordered hyperbolic surfaces and geodesics. We make that choice so as to simplify our language for the upcoming proof.

As defined here, a Dehn twist is in general a self-homeomorphism, but not necessarily a self-diffeomorphism. For the proof below, we specify the collar neighbourhood of γ to be

$$C(\gamma) = \{p \in X | \mathrm{dist(p, \gamma) \leq w}\},$$

where w is chosen to be small enough to be disjoint to any other closed geodesics in a hyperbolic pants decomposition containing γ. For example:

$$w = \operatorname{arcsinh}(1/\sinh(\frac{1}{2}\ell(\gamma))).$$

In addition, if we alter $f(s,t)$ slightly (for example, reparametrise t to be a bump function from 0 to 1, for $t = 0$ to $t = \frac{1}{2}$, and t=1 for the rest), we end up with a Dehn twist which is a self-diffeomorphism fixing everything to the right of γ. When discussing Dehn twists in the context of Fenchel-Nielsen coordinates we will exclusively mean such a self-diffeomorphism.

1.5.3 Fenchel-Nielsen Coordinates

Theorem 1.5.3 (Fenchel-Nielsen Coordinates). *Given a bordered Riemann surface $S_{g,n}$, with g genera, and n boundary components $\{\alpha_i\}_{1\leq i\leq n}$ of length corresponding to the entries in $\boldsymbol{L} = (L_1, L_2, \ldots, L_n)$. Fix a pants decomposition*

$$\Gamma = \{\gamma_j\},\ 1 \leq j \leq k = 3g - 3 + n.$$

For any marked hyperbolic surface $(X, f) \in \mathcal{T}(S_{g,n})$, consider $[f(\gamma_j)]$, the homotopy class of $f(\gamma_j)$ in X. Let γ_j^X denote the unique geodesic representative of $[f(\gamma_j)]$, and let

$$\Gamma^X = \{\gamma_j^X\},\ 1 \leq j \leq k = 3g - 3 + n$$

denote the hyperbolic geodesic pants decomposition in X induced by Γ. Furthermore, respectively call $\ell_j(X) \in \mathbb{R}_+$ and $\theta_j(X) \in \mathbb{R}$ the **length and twist parameters** of γ_j^X. Then, for each Teichmüller equivalence class $(X, f)_{\mathcal{T}}$, there exists a unique vector

$$(\ell_1(X), \ell_2(X), \ldots, \ell_k(X), \theta_1(X), \theta_2(X), \ldots, \theta_k(X)) \in \mathbb{R}_+^k \times \mathbb{R}^k$$

corresponding to Γ^X. Conversely, for every vector $\boldsymbol{v} \in \mathbb{R}_+^k \times \mathbb{R}^k$, there exists a unique $(X, f)_{\mathcal{T}} \in \mathcal{T}_{g,n}(\boldsymbol{L})$ with Γ^X inducing $\boldsymbol{v}$. That is:

there is a bijection between $\mathcal{T}_{g,n}(\boldsymbol{L})$ and $\mathbb{R}_+^k \times \mathbb{R}^k$.

This congruence gives us an atlas, on $\mathcal{T}_{g,n}(\boldsymbol{L})$, which we call the **Fenchel-Nielsen coordinates** of $\mathcal{T}_{g,n}(\boldsymbol{L})$.

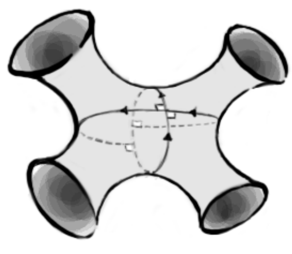
Note 1.5.5. The coordinates described here differs from what Mirzakhani describes in [8]. We parametrise the θ_i in terms of the angle of revolution, whereas she parametrises in terms of the lengths of the twisted closed geodesic. We make this choice so as to set up Fenchel-Nielsen coordinates in accordance with the original Fenchel-Nielsen paper. A consequence of which, is that we will need to reparametrise when stating Wolpert's theorem in 1.7.2.

Prior to proving this result, we first define what twist parameters are.

Definition 1.5.6. We say that a marked hyperbolic surface (X, f) is in **standard position with respect to** Γ if, for each γ_j^X,

- at least one pair of self-link points coming from the two pairs of pants γ_j^X lies line up, and
- the (potentially closed) geodesic built from the self-links running perpendicularly through γ_j^X is consistently oriented, and its orientation takes it from the right side of γ_j^X to the left.

A point $(X, f)_{\mathcal{T}}$ in the Teichmüller space is said to be in **standard position** if all its marked surface representatives are in standard position.



Lemma 1.5.4. *For any Teichmüller class $(X, f)_{\mathcal{T}}$, and some fixed $\boldsymbol{v} \in \mathbb{R}_+^k \times \mathbb{R}^k$, there exists a unique $\boldsymbol{\tau}$-twist ($\boldsymbol{\tau} \in [0, 2\pi)^k$), such that*

$$(\mathrm{tw}_\Gamma^{\boldsymbol{\tau}}(X), \mathrm{tw}_\Gamma^{\boldsymbol{\tau}} \circ f)_{\mathcal{T}}$$

is in standard position.

Proof. We will first show existence by construction. Consider (X, f) representing $(X, f)_{\mathcal{T}} \in \mathcal{T}_{g,n}(\boldsymbol{L})$ and let $\{P_i, \iota_i\}$ be the pants cover corresponding to Γ^X. Because twists on any $\gamma_j^X \in \Gamma^X$ are transitive, we can always twist *the* two self-link points, so that when the two points join up, they will give a oriented geodesic running from the right of γ_j^X to the left. Denote this twist by $\mathrm{tw}_{\gamma_j}^{\tau_j}$. Doing this for all geodesics in Γ^X (there are only finitely many), the $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_k)$-twist will bring (X, f) into standard position. Now, for any (Y, g) representing the same Teichmüller class, $g \circ f^{-1}$ will be isotopic to a isometry, and therefore will preserve the joined geodesic.

□

Performing a Dehn-twist around any geodesic in Γ^X gives a non-Teichüller equivalent marked surface in standard position, therefore, $\boldsymbol{\tau}$-twists alone cannot generate all non-Teichmüller equivalent marked surfaces. However, we have the following result:

Theorem 1.5.5. *$\boldsymbol{\tau}$-twists and $\boldsymbol{n}$-Dehn-twists generate all Teichmüller classes of marked hyperbolic surfaces $(X, f)_{\mathcal{T}}$ where the lengths of Γ^X are fixed. Moreover, each pair of $\boldsymbol{\tau}$-twists and $\boldsymbol{n}$-Dehn-twists generates a different Teichmüller class of marked hyperbolic surfaces.*

Note 1.5.6. The above statement assumes that the length of Γ^X is always well-defined in a Teichmüller class. We will prove this fact shortly.

Definition 1.5.7. The **total twisting** of (X_1, f_1) with respect to (X_2, f_2) is the vector $\boldsymbol{\theta}$ given by:

$$\boldsymbol{\theta} = \boldsymbol{\tau} + 2\pi\boldsymbol{n},$$

where $\mathrm{tw}_{\Gamma^{X_1}}^{\boldsymbol{\tau}}$ and $\phi_{\Gamma^{X_2}}^{\boldsymbol{n}}$ is the unique pair of twists so that

$$(\mathrm{tw}_{\Gamma_{X_1}}^{\boldsymbol{\tau}} \circ \phi_{\Gamma_{X_1}}^{\boldsymbol{n}}(X_1),\ \mathrm{tw}_{\Gamma_{X_1}}^{\boldsymbol{\tau}} \circ \phi_{\Gamma_{X_1}}^{\boldsymbol{n}} \circ f_1)_{\mathcal{T}} = (X_2, f_2)_{\mathcal{T}}.$$

We turn now, to the proof of theorem 1.5.3 (Fenchel-Nielsen Coordinates).

Proof. Consider a marked hyperbolic surface (X, f), we will first prove that for $(X, f)_{\mathcal{T}}$, the length of each geodesic in Γ^X is well-defined.

Consider $(X, f)_{\mathcal{T}} = (Y, g)_{\mathcal{T}}$, then $f \circ g^{-1} : Y \to X$ is isotopic to an isometry. Therefore, the lengths of each geodesic γ_j^Y in Γ^Y is the same as the geodesic representative of

$$[f \circ g^{-1}(\gamma_j^Y)] = [f \circ g^{-1}(g(\gamma_j))] = [f(\gamma_j)],$$

which is, by definition, γ_j^X. Since we chose (X, f) arbitrarily, this tells us that we are able to extract a well-defined length vector for each point in the Teichmüller space.

Then, we wish to associate twisting parameters to each point. Although this may be unorthodox, we fix for each length vector in $\boldsymbol{a} \in \mathbb{R}_+^k$, a point $(X_{\boldsymbol{a}}, f_{\boldsymbol{a}})_{\mathcal{T}}$ [3]. Then, for any point $(Z, h)_{\mathcal{T}}$, let its twist parameter be the unique total twisting vector of $(Z, h)_{\mathcal{T}}$ with respect to $(X_{\boldsymbol{a}}, f_{\boldsymbol{a}})_{\mathcal{T}}$, where $\boldsymbol{a}$ is length vector for $(Z, h)_{\mathcal{T}}$. Thus, by fixing this collection of sets, we are able to give each point in the Teichmüller space a distinct coordinate of the form

$$\boldsymbol{\ell} \times \boldsymbol{\theta} \in \mathbb{R}_+^k \times \mathbb{R}^k.$$

On the other hand, we have covered all possible combinations in $\mathbb{R}_+^k \times \mathbb{R}^k$, because by varying pants covers we are able to obtain arbitrary length choices (hence fixed representatives for every length vector exists), and by performing inverse τ-twists and Dehn twists, we are able to obtain all total twistings.

Thus, we have a 1 to 1 correspondence between points

$$\boldsymbol{\ell} \times \boldsymbol{\theta} \in \mathbb{R}_+^k \times \mathbb{R}^k$$

and points in the Teichmüller space.

□

1.6 Moduli Spaces of Riemann Surfaces

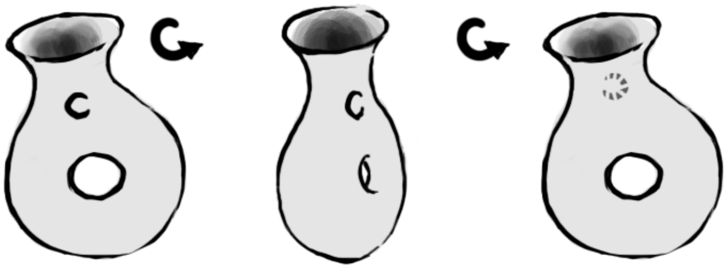
We now define the moduli space of Riemann surfaces.

Definition 1.6.1. Given a surface S, the **mapping class group** of S is the group comprised of isotopy classes of orientation preserving self-homeomorphisms of S, leaving each boundary component set-wise fixed. We denote the mapping class group by $\mathrm{Mod}(S)$.

We now introduce a few examples of orientation preserving self-homeomorphism, in the hopes of building some intuition.

Example 1.6.1. *Dehn twists, as defined in the previous section are representatives of isotopy classes in the mapping class group.*

Example 1.6.2. *Consider a torus with one disc removed. The following figure shows an orientation preserving rotation that corresponds to a non-trivial mapping class in* $\mathcal{M}_{1,1}(L)$.



Notice that if we perform the rotation twice, then we obtain the identity map, and hence the induced mapping class of this homeomorphism is of order 2.

Finally, we define moduli spaces.

Definition 1.6.2. The quotient space

$$\mathcal{M}(S) = \mathcal{T}(S)/\mathrm{Mod}(S)$$

is called the **moduli space** of S. Similarly, we define the length-specified moduli space of $S_{g,n}(\boldsymbol{L})$ to be:

$$\mathcal{M}(S_{g,n}, \boldsymbol{L}) = \mathcal{T}(S_{g,n}, \boldsymbol{L})/\mathrm{Mod}(S_{g,n}).$$

We will often denote $\mathcal{M}(S_{g,n}, \boldsymbol{L})$ and $\mathrm{Mod}(S_{g,n})$ respectively by $\mathcal{M}_{g,n}(\boldsymbol{L})$ and $\mathrm{Mod}_{g,n}$.

[3] It is assumed that we choose the $(X_{\boldsymbol{a}}, f_{\boldsymbol{a}})_{\mathcal{T}}$, so that, by smoothly varying $\boldsymbol{a}$, we obtain hyperbolic surfaces that *deform* smoothly.

Note 1.6.1. Dehn twists are representatives of elements in the mapping class group, and it can be shown that all characteristic zero elements of the Mapping class group are generated by Dehn-twists.

We will now consider a few examples of moduli spaces.

Example 1.6.3. *Consider a hyperbolic pair of pants $S_{0,3}(\boldsymbol{L})$ with boundaries β_i, $i = 1, 2, 3$. If the lengths of the β_i are fixed by some vector $\boldsymbol{L}$, then by theorem 1.3.4, this pair of pants must be unique. Therefore, there is only point in $\mathcal{T}(S_{0,3}, \boldsymbol{L})$, and by extension, only one point in the moduli space $\mathcal{M}(S_{0,3}, \boldsymbol{L})$.*

Example 1.6.4. *Consider $\mathcal{M}(S_{0,4})$, that is: the moduli space of a sphere with 4 marked points. Then for any two such surfaces, we can use using Möbius transformations an the uniformization theorem, to send each of the marked points 1,2,3 to their corresponding marked points. However, the image of the 4-th point is uniquely specified by the Möbius transformation, and depending on the initial surface, the 4-th point can go anywhere except to the corresponding marked points for marked points 1,2 and 3. Therefore, the Moduli space of $S_{0,4}$ is the 2-sphere minus 3 points.*

1.7 Weil-Petersson Metric

We present everything in this section without proof, a rigorous account of most of these results may be found in [4][4].

1.7.1 Symplectic Structure

Readers unfamiliar with differential forms are suggested to first consult the appendices B.1.

Definition 1.7.1. A **symplectic form** on a real vector space V is a bilinear form $\omega : V \times V \to \mathbb{R}$ such that ω is

- **skew-symmetric**: $\forall\ \boldsymbol{u}, \boldsymbol{v} \in V,\ \omega(\boldsymbol{u}, \boldsymbol{v}) = -\omega(\boldsymbol{v}, \boldsymbol{u})$,
- **non-degenerate**: if $\omega(\boldsymbol{u}, \boldsymbol{v}) = 0$ for all $\boldsymbol{v} \in V$, then $\boldsymbol{u} = 0$.

We call a vector space equipped with a symplectic form, a **symplectic vector space**.

Note 1.7.1. The skew-symmetry property of a symplectic form means that if V is odd dimensional, then $\omega \equiv 0$. Which, contradicts the non-degeneracy condition. Therefore, symplectic forms exist only on even dimensional vector spaces.

Definition 1.7.2. A differential form $\omega : TM \to \mathbb{R}$ on a manifold M is called **closed** if its exterior derivative (or differential) $d\omega \equiv 0$.

Definition 1.7.3. A **symplectic manifold** is a smooth manifold M equipped with a smooth closed 2-form $\omega : TM \to \mathbb{R}$, such that for every $p \in M$, $\omega|_{T_pM}$ is a simplectic form on T_pM.

Lemma 1.7.1. *Any symplectic form ω induces a non-degenerate volume form:*

$$\omega^n = \frac{1}{n!} \underbrace{\omega \wedge \ldots \wedge \omega}_{n \text{ times}}.$$

[4]Strictly speaking, the results proven in [4] are for $\mathcal{T}(S_{g,n})$ and $\mathcal{M}(S_{g,n})$ for surfaces with g genera and n marked points, as opposed to $\mathcal{T}_{g,n}(\boldsymbol{L})$ and $\mathcal{M}_{g,n}(\boldsymbol{L})$.

Note 1.7.2. Notations vary with regard to volume forms depending on the definition of the wedge product used: factorials are sometimes omitted, and/or powers of -1 introduced. We adopt the notation consistent with [8].

1.7.2 The Weil-Petersson Metric

We will not define the Weil-Petersson metric, as the definitions the author has seen need extra background such as Beltrami forms, or their dual - holomorphic quadratic differentials. In any case, the explicit details are not central to our purpose of understanding Mirzakhani's volume calculations. All we need to know are the following facts:

- There exists a metric g_{WP} on $\mathcal{T}(S_{g,n}, \boldsymbol{L})$ called the **Weil-Petersson metric**, such that g_{WP} is invariant under the action of $S_{g,n}$. Thus, induces a metric on $\mathcal{M}(S_{g,n}, \boldsymbol{L}) = \mathcal{T}(S_{g,n}, \boldsymbol{L})/\text{Mod}_{g,n}$.
- Ahlfors showed that the Weil-Petersson metric is Kählerian[5]. One of the consequences of this being: there exists a symplectic form ω_{WP} such that:

  $$\omega_{WP}(\boldsymbol{u}, \boldsymbol{v}) = \langle J\boldsymbol{u}, \boldsymbol{v}\rangle_{g_{WP}}.$$

  Where J is the tangent bundle isomorphism inducing the almost complex structure on $\mathcal{T}(S_{g,n}, \boldsymbol{L})$. We call O_{WP} the **Weil-Petersson symplectic form**. Note that O_{WP} is also $\text{Mod}_{g,n}$ invariant.
- The Weil-Petersson symplectic form in turn induces a non-degenerate volume form ω_{WP}, which we call the **Weil-Petersson volume form**. It turns out that the volume of $\mathcal{T}(S_{g,n}, \boldsymbol{L})$ with respect to ω_{WP} is infinite, but the volume of $\mathcal{M}(S_{g,n}, \boldsymbol{L})$ is finite, we compute it with the assistance of the following theorem.

Theorem 1.7.2 (Wolpert). *Given a pants decomposition $\mathcal{P} = \{\gamma_i\}_{i=1}^{3g-3+n}$, of the Riemann surface R, and let the Fenchel-Nielsen coordinates associated with $\mathcal{P}$ be denoted by $(\ell_1, \ldots \ell_{3g-3+n}, \theta_1, \ldots, \theta_{3g-3+n})$; set $\tau_j = \frac{\ell_j}{2\pi}\theta_j$. Then,*

$$\omega_{WP} = \sum_{j=1}^{3g-3+n} d\ell_j \wedge d\tau_j.$$

Definition 1.7.4. Since the twist parameters in the Fenchel Nielsen coordinates need to be length-adjusted for this presentation of the Weil-Petersson symplect form, we sometimes refer to $(\ell_1, \ldots \ell_{3g-3+n}, \tau_1, \ldots, \tau_{3g-3+n})$ as the **denormalised** or **augmented Fenchel Nielsen coordinates**.

[5] This fact is stated purely for completeness, in practice, we shall only need the consequence that there exists such a symplectic form.

Chapter 2

Mirzakhani's Volume Calculations

The second half of this document will be dedicated to introducing Mirzakhani's work in moduli space theory. In particular, we will demonstrate her technique of integrating Weil-Petersson volumes of moduli spaces, while providing exposition of the theory accompanying the calculations.

2.1 $\mathrm{Vol}(\mathcal{M}_{1,1}(\boldsymbol{L}))$ and $\mathrm{Vol}(\mathcal{M}_{0,4}(\boldsymbol{L}))$

Starting off, we compute the Weil-Petersson volumes of $\mathcal{M}_{1,1}(\boldsymbol{L})$ and $\mathcal{M}_{0,4}(\boldsymbol{L})$. There are two main reasons for doing this:

- we will identify a possible strategy to handle the computation of Weil-Petersson volumes of more generalised moduli spaces of bordered Riemann surfaces.
- the actual results of these computations will serve as initial conditions for Mirzakhani's recursion, as well as being the 'first case' for an induction proof later on.

2.1.1 The volume of $\mathcal{M}_{1,1}$

One of the key results in [8] is the Generalized McShane Identity. As we will soon see, it is central to our strategy for calculating volumes. Here is the original form of the theorem, as proven by McShane in [7].

Theorem 2.1.1 (McShane Identity)**.** *Let X be a one-punctured hyperbolic torus, and let $\mathcal{F}$ denote the set of all simple closed geodesics γ on X, then:*

$$\sum_{\gamma \in \mathcal{F}} (1 + e^{\ell_\gamma(X)})^{-1} = \frac{1}{2}.$$

This formula is arguably *the* key ingredient for the calculation of $\mathrm{Vol}(\mathcal{M}_{1,1})$, the steps of which we will now demonstrate. The essential features of this calculation and the computation of $\mathrm{Vol}(\mathcal{M}_{1,1}(\boldsymbol{L}))$ are very similar, and this demonstration will pave the way for the more general calculation.

Theorem 2.1.2. *The volume of $\mathcal{M}_{1,1}$ with respect to the Weil-Petersson metric is:*

$$\mathrm{Vol}(\mathcal{M}_{1,1}) = \frac{\pi^2}{6}.$$

Proof. Observe that on $S_{1,1}$, the hyperbolic 1-torus with one marked point, any simple closed geodesic γ is a hyperbolic pants decomposition. Fix such a decomposition γ on $S_{1,1}$ and consider the space:

$$\mathcal{M}^*_{1,1} = \{([X], \gamma^{[X]}) |\ [X] \in \mathcal{M}(S_{1,1})\} / \sim_{\mathcal{M}^*}, \text{ where}$$

- we think of X as the set $S_{1,1}$ equipped with a hyperbolic structure, and $[X]$ as the isometry class of X;
- we think of γ as a simple closed loop on X (not necessarily a geodesic), γ^X as the shortest geodesic representative of the homotopy class of γ, and $\gamma^{[X]}$ as the class of geodesics on $[X]$ corresponding to γ^X;
- the equivalence relation $\sim_{\mathcal{M}^*}$ is given by: $([X], \gamma^{[X]}) \sim_{\mathcal{M}^*} ([Y], \gamma^{[Y]})$ if and only if there exists an isometry $f : X \to Y$ taking the homotopy class of γ^X to the homotopy class of γ^Y.

$\mathcal{M}^*_{1,1}$ is a natural covering space for $\mathcal{M}_{1,1}$, with the projection map $\pi_1 : \mathcal{M}^*_{1,1} \to \mathcal{M}_{1,1}$ given by $\pi_1([X], \gamma^{[X]}) = [X]$. In particular, this means that the Weil-Petersson induced volume form on $\mathcal{M}_{1,1}$, which we shall denote by $d[X]$, pulls backs to a non-degenerate volume form $\pi_1^* d[X]$ on $\mathcal{M}^*_{1,1}$. Take special note that $\pi_1^* d[X]$ is mapping class group invariant.

Note 2.1.1. $\mathcal{M}^*_{1,1}$ is larger space than $\mathcal{M}_{1,1}$, and it keeps track of extra information in the following way: in general, hyperbolic pants decompositions are not unique. Consider the same surfaces X equipped with two pants decompositions of different lengths[1]. In $\mathcal{M}_{1,1}$, X is the same as itself, so $[X] = [X]$. However, by keeping track of $\gamma_1^{[X]}$ and $\gamma_2^{[X]}$, we are able to tell these apart in $\mathcal{M}^*_{1,1}$.

Furthermore, $\mathcal{M}^*_{1,1}$ is symplectomorphic to $\mathcal{T}_{1,1}/\mathrm{Stab}(\gamma)$, therefore, we have a quotient map

$$\pi_2 : \mathcal{T}_{1,1} \to \mathcal{M}^*_{1,1}$$

given by $\pi_2(\ell, \tau) \sim_{\mathcal{M}^*_{1,1}} \pi_2(\ell, \tau + \ell)$, since adding ℓ to the twisting parameter is equivalent to performing a right Dehn twist, which happens to be the generator for the stabiliser group of α. Since $\mathrm{Stab}(\gamma)$ is in the mapping class group, the Weil-Petersson volume form ω_{WP} induces a mapping class group invariant volume form $d([X], \gamma^{[X]})$ on $\mathcal{M}^*_{1,1}$. In addition,

$$d([X], \gamma^{[X]}) = \pi_1^*(d[X]),$$

because $\pi_1 \circ \pi_2 : \mathcal{T}_{1,1} \to \mathcal{M}_{1,1}$ is precisely the quotient map corresponding to quotienting by the mapping class group: $\mathrm{Mod}(S_{1,1})$.

Armed with this information, we can now happily integrate with respect to the volume form $d[X]$, induced by the Weil-Petersson form. Observe:

$$\begin{aligned}\int_{\mathcal{M}_{1,1}} d[X] &= 2 \int_{\mathcal{M}_{1,1}} \sum_{\gamma^{[X]} \subset [X]} (1 + e^{\ell_\gamma([X])})^{-1} d[X] \\ &= 2 \int_{\mathcal{M}_{1,1}} \sum_{\gamma^{[X]} \subset \pi_1([X], \gamma^{[X]})} (1 + e^{\ell_\gamma(\pi_1([X], \gamma^{[X]}))})^{-1} d[X] \\ &= 2 \int_{\mathcal{M}^*_{1,1}} (1 + e^{\ell([X], \gamma^{[X]})})^{-1} d([X], \gamma^{[X]}),\end{aligned}$$

[1] Just take them from different homology classes.

where the sum is over all isometry class of geodesics in $[X]$, and $\ell([X],\gamma^{[X]})$ is well defined, because the length of γ^X is the same for each X. Now, if we pull-back the integration form to the Weil-Petersson volume form, we have:

$$\begin{aligned}\int_{\mathcal{M}_{1,1}} d[X] &= 2\int_{\mathcal{T}_{1,1}} (1+e^{\ell([X],\gamma^{[X]})})^{-1}\omega_{WP} \\ &= 2\int_{\mathcal{T}_{1,1}} (1+e^{\ell})^{-1}d(\ell,\tau), \text{ since } \ell = \ell([X],\gamma^{[X]}) \\ &= 2\int_0^\infty \int_0^\ell (1+e^\ell)^{-1} d\tau\, d\ell \\ &= 2\int_0^\infty \ell(1+e^\ell)^{-1} = 2\int_0^\infty \frac{\ell}{1+e^\ell}d\ell = \frac{\pi^2}{6}.\end{aligned}$$

□

2.1.2 The volume of $\mathcal{M}_{1,1}(L)$

There are three key steps in the above calculation. We now adjust each of these steps to compute the Weil-Petersson volume of $\mathcal{M}_{1,1}(\boldsymbol{L}=(L))$.

1. Some sort of McShane Identity is needed, such as:

   Corollary 2.1.3 (Corollary to the Generalised McShane Identity)**.** *Let X be a fixed hyperbolic torus with geodesic boundary component of length L, and let $\mathcal{F}$ be the collection of all simple closed geodesics on X, then:*

   $$\sum_{\gamma\in\mathcal{F}} \mathcal{D}(L,\ell_\gamma(X),\ell_\gamma(X)) = L, \text{ where } \mathcal{D}(x,y,z) = 2\log\left(\frac{e^{\frac{x}{2}}+e^{\frac{y+z}{2}}}{e^{\frac{-x}{2}}+e^{\frac{y+z}{2}}}\right).$$

   Although this new formula might not resemble the McShane identity, by differentiating both sides with respect to L, we have:

   $$\frac{\partial}{\partial L}\mathcal{D}(L,x,x) = \frac{1}{1+e^{x-\frac{L}{2}}} + \frac{1}{1+e^{x+\frac{L}{2}}} = 1 = \frac{\partial L}{\partial L}.$$

   Notice that if we take $L\to 0$, which intuitively corresponds to making the boundary as small as a hole (a marked point), we obtain the McShane identity.

2. The second key step is to set up the series of pullbacks of $d[X]$ to ω_{WP}. In this case, the setup is essentially the same as for $\mathcal{M}_{1,1}$, so we leave out the details, only altering the notation slightly.

$$
\begin{aligned}
L \cdot \mathrm{Vol}_{1,1}(\boldsymbol{L} = L) &= \int_{\mathcal{M}_{1,1}(L)} L \; d[X] \\
&= \int_{\mathcal{M}_{1,1}(L)} \sum_{\gamma^{[X]} \in \mathcal{F}} \mathcal{D}(L, \ell_{\gamma^{[X]}}([X]), \ell_{\gamma^{[X]}}([X])) \; d[X] \\
&= \int_{\mathcal{M}^{*}_{1,1}(L)} \mathcal{D}(L, \ell_{\gamma^{[X]}}([X]), \ell_{\gamma^{[X]}}([X])) \; d([X], \gamma^{[X]}) \\
&= \int_{\mathcal{T}_{1,1}(L)} \mathcal{D}(L, \ell, \ell) \; \omega_{WP} \\
&= \int_0^\infty \int_0^\ell 2 \log \left(\frac{e^{\frac{L}{2}} + e^{\frac{\ell+\ell}{2}}}{e^{\frac{-L}{2}} + e^{\frac{\ell+\ell}{2}}} \right) d\tau \; d\ell \\
&= \int_0^\infty 2\ell \log \left(\frac{e^{\ell} + e^{\frac{L}{2}}}{e^{\ell} + e^{\frac{-L}{2}}} \right) \; d\ell
\end{aligned}
$$

3. The last step is to differentiate both sides by L, and thereby giving us integrals which may be calculated in terms of some even integer values of the Riemann-zeta function.

$$
\begin{aligned}
\frac{\partial}{\partial L} L \cdot \mathrm{Vol}_{1,1}(L) &= \int_0^\infty \ell \left(\frac{1}{1 + e^{\ell - \frac{L}{2}}} + \frac{1}{1 + e^{\ell + \frac{L}{2}}} \right) d\ell \\
&= \int_{\frac{L}{2}}^\infty \frac{x - \frac{L}{2}}{1 + e^x} dx + \int_{\frac{-L}{2}}^\infty \frac{y + \frac{L}{2}}{1 + e^y} dy, \\
&\qquad \text{with } x = \ell + \frac{L}{2}, \text{ and } y = \ell - \frac{L}{2}, \\
&= 2 \int_0^\infty \frac{x}{1 + e^x} dx - \int_0^{\frac{L}{2}} \frac{x - \frac{L}{2}}{1 + e^x} dx - \int_0^{\frac{-L}{2}} \frac{y + \frac{L}{2}}{1 + e^y} dy \\
&= \frac{\pi^2}{6} - \int_0^{\frac{L}{2}} (x - \frac{L}{2}) \left(\frac{1}{1 + e^x} + \frac{1}{1 + e^{-x}} \right) dx \\
&= \frac{\pi^2}{6} - \int_0^{\frac{L}{2}} x - \frac{L}{2} \; dx = \frac{\pi^2}{6} + \frac{L^2}{8}. \\
\Rightarrow \; L \cdot \mathrm{Vol}_{1,1}(L) &= \int \frac{\pi^2}{6} + \frac{L^2}{8} \; dL = \frac{\pi^2 L}{6} + \frac{L^3}{24} \\
\mathrm{Vol}_{1,1}(L) &= \frac{\pi^2}{6} + \frac{L^2}{24}
\end{aligned}
$$

2.1.3 The volume of $\mathcal{M}_{0,4}(\boldsymbol{L})$

As a last example, we compute $\mathrm{Vol}_{0,4}(\boldsymbol{L})$ via the above three step procedure. We make special mention of two differences between this calculation and the above which must be taken into account in the generalised case.

- The $S_{0,4}$ McShane identity is expressed in a function $\mathcal{R}$, which does not immediately resemble $\mathcal{D}$. The difference in the functions used has to do with the fact that any hyperbolic pants decomposition for $S_{1,1}$ cuts through a genus, and hence the pair of pants with β_1 as a boundary also has both other boundaries coming from non-peripheral geodesics. Whereas for $S_{0,4}$, any hyperbolic pants decomposition severs the surface into two pairs of pants, and the pair which contains β_1 also contains one of the other boundary geodesics. To see this relationship made explicit, please turn to section 2.2.

- We need to perform the pullback of $d[X]$ to ω_{WP} three times. Each of these three times corresponds to one of the three partitions of four marked boundaries $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ into two sets of two marked boundaries. For example: $\{\beta_1, \beta_2\} \cup \{\beta_3, \beta_4\}$.

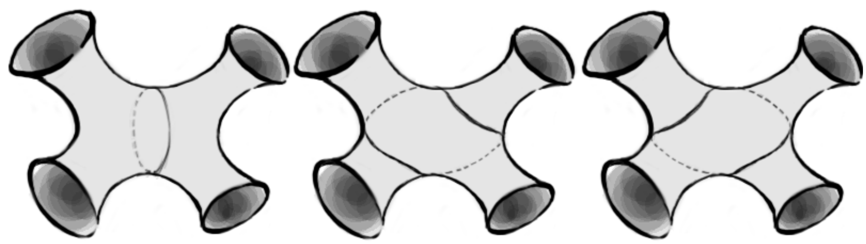
1. For $\mathcal{M}_{0,4}(\boldsymbol{L} = (L_1, L_2, L_3, L_4))$, the generalised McShane identity tells us:

Corollary 2.1.4. *Consider a hyperbolic sphere X with geodesic boundary components β_1 , β_2, β_3 and β_4 of length L_1, L_2, , L_3 and L_4. Let $\mathcal{F}_{1,i}$ denote the set of all geodesics which, in conjunction with β_1 and β_i, form the boundaries of a pair of pants in a hyperbolic pants cover of X. Then,*

$$\sum_{\gamma \in \mathcal{F}_{1,2}} \mathcal{R}(L_1, L_2, \ell_\gamma(X)) + \sum_{\gamma \in \mathcal{F}_{1,3}} \mathcal{R}(L_1, L_3, \ell_\gamma(X)) + \sum_{\gamma \in \mathcal{F}_{1,4}} \mathcal{R}(L_1, L_4, \ell_\gamma(X)) = L_1,$$

where $\mathcal{R}(x, y, z) = x - \log\left(\frac{\cosh(\frac{x}{2})+\cosh(\frac{y+z}{2})}{\cosh(\frac{x}{2})+\cosh(\frac{y-z}{2})}\right)$.

2. Just as with the $\mathcal{T}_{1,1}(L)$ case, a hyperbolic pants decomposition of $\mathcal{T}_{0,4}(\boldsymbol{L})$ is comprised of a single simple closed geodesic, and hence the Fenchel-Nielsen coordinate for $\mathcal{T}_{0,4}(\boldsymbol{L})$ is also $\mathbb{R}_+ \times \mathbb{R}$. However, there are three types of pants decompositions, each partitioning the boundaries of $S_{0,4}$ differently.



Let us ignore this for now, and try to set up our integrals as before, and see what problems, if any, that we run into.

$$\begin{aligned}
L_1 \cdot \mathrm{Vol}_{0,4}(\boldsymbol{L}) &= \int_{\mathcal{M}_{0,4}(\boldsymbol{L})} L_1 \; d[X] \\
&= \int_{\mathcal{M}_{0,4}(\boldsymbol{L})} \sum_{i=2}^{4} \sum_{\gamma^{[X]} \in \mathcal{F}_{1,i}} \mathcal{R}(L_1, L_i, \ell_{\gamma^{[X]}}([X])) \; d[X] \\
&= \sum_{i=2}^{4} \int_{\mathcal{M}_{0,4}(\boldsymbol{L})} \sum_{\gamma^{[X]} \in \mathcal{F}_{1,i}} \mathcal{R}(L_1, L_i, \ell_{\gamma^{[X]}}([X])) \; d[X]
\end{aligned}$$

In order to compute each of the three integrals in the same way as the previous examples, we pull $d[X]$ back to $\mathcal{M}_{0,4}(\boldsymbol{L})^{i*}$, which we set up as a covering space with one copy of $\mathcal{M}_{0,4}(\boldsymbol{L})$ for each geodesic isometry class in $\mathcal{F}_{1,i}$. In so doing, we are able to break up the integral of

$$\sum_{\gamma^{[X]}\in\mathcal{F}_{1,i}} \mathcal{R}(L_1, L_i, \ell_{\gamma^{[X]}}([X]))$$

over one fundamental domain (or one copy of $\mathcal{M}_{0,4}(\boldsymbol{L})$), to the integral of

$$\mathcal{R}(L_1, L_i, \ell_{\gamma^{[X]}}([X]))$$

over all of $\mathcal{M}_{0,4}(\boldsymbol{L})^{i*}$. It turns out that each of these intermediate spaces happens to agree with a quotient of $\mathcal{T}_{0,4}(\boldsymbol{L})$ with augmented Fenchel-Nielsen coordinates, with the quotient being with respect to a subgroup of the mapping class group of $S_{0,4}$. Each of our three integrals then pulls back to integrals of smooth functions on $\mathcal{T}_{0,4}(\boldsymbol{L})$, which we are able to compute explicitly.

Each element of $\mathcal{F}_{1,i}$ is an isometry class of geodesics, each of which is a pants decomposition so that there exists a pair of pants in the corresponding pants cover bounded by β_1 and β_i and our pants decomposition. Therefore, we define the intermediate spaces as follows, allowing us to redistribute each term in our sum over $\mathcal{F}_{1,i}$ to a different fundamental domain.

$$\mathcal{M}_{0,4}(\boldsymbol{L})^{i*} = \{([X], \gamma^{[X]}) | X \in \mathcal{M}_{0,4}(\boldsymbol{L}), \gamma^{[X]} \in \mathcal{F}_{1,i}\}.$$

As before, $\mathcal{M}_{0,4}(\boldsymbol{L})^{i*} \cong \mathcal{T}_{0,4}(\boldsymbol{L})/\mathrm{Stab}(\alpha_i)$, since $\mathrm{Stab}(\alpha_i) \subset \mathrm{Mod}_{0,4}$, $d([X], \gamma^{[X]})$ pulls back to ω_{WP}. Let us continue with the calculations:

$$\begin{aligned}
L_1 \cdot \mathrm{Vol}_{0,4}(\boldsymbol{L}) &= \sum_{i=2}^{4} \int_{\mathcal{M}_{0,4}(\boldsymbol{L})} \sum_{\gamma^{[X]}\in\mathcal{F}_{1,i}} \mathcal{R}(L_1, L_i, \ell_{\gamma^{[X]}}([X]))\ d[X] \\
&= \sum_{i=2}^{4} \int_{\mathcal{M}_{0,4}(\boldsymbol{L})^{i*}} \mathcal{R}(L_1, L_i, \ell_{\gamma^{[X]}}([X]))\ d([X], \gamma^{[X]}) \\
&= \sum_{i=2}^{4} \int_{\mathcal{T}_{0,4}(\boldsymbol{L})} \mathcal{R}(L_1, L_i, \ell)\ \omega_{WP} \\
&= \sum_{i=2}^{4} \int_0^\infty \int_0^\ell L_1 - \log\left(\frac{\cosh(\frac{L_1}{2}) + \cosh(\frac{L_i+\ell}{2})}{\cosh(\frac{L_1}{2}) + \cosh(\frac{L_i-\ell}{2})}\right)\ d\tau\ d\ell \\
&= \sum_{i=2}^{4} \int_0^\infty \ell\left[L_1 - \log\left(\frac{\cosh(\frac{L_1}{2}) + \cosh(\frac{L_i+\ell}{2})}{\cosh(\frac{L_1}{2}) + \cosh(\frac{L_i-\ell}{2})}\right)\right]\ d\ell.
\end{aligned}$$

3. As with previous cases, we take a partial derivative with respect L_1 to obtain:

$$\frac{\partial}{\partial L_1} L_1 \mathrm{Vol}_{0,4}(\boldsymbol{L}) =$$
$$\sum_{i=2}^{4} \ell\left[\frac{1}{1+e^{\frac{\ell+L_1+L_2}{2}}} + \frac{1}{1+e^{\frac{\ell-L_1+L_2}{2}}} + \frac{1}{1+e^{\frac{\ell+L_1-L_2}{2}}} + \frac{1}{1+e^{\frac{\ell-L_1-L_2}{2}}}\right],$$

and with similar substitutions and cancellations, we obtain:

$$\begin{aligned}\frac{\partial}{\partial L_1} L_1 \mathrm{Vol}_{0,1}(\boldsymbol{L}) &= 2\pi^2 + \frac{1}{2}(3L_1^2 + L_2^2 + L_3^2 + L_4^2)\\ \Rightarrow \; L_1 \cdot \mathrm{Vol}_{0,4} &= 2\pi^2 L_1 + \frac{1}{2}(L_1^3 + L_1L_2^2 + L_1L_3^2 + L_1L_4^2)\\ \mathrm{Vol}_{0,4}(\boldsymbol{L}) &= 2\pi^2 + \frac{1}{2}(L_1^2 + L_2^2 + L_3^2 + L_4^2).\end{aligned}$$

2.2 Generalised McShane Identity

As we have seen from the previous section, the McShane identity gives a partition of 1 into a countably infinite sum of smooth functions that we pullback and redistribute when calculating the volume of $\mathcal{M}_{1,1}$. Similarly, the Generalised McShane Identity partitions a boundary length into an infinite sum. We derive the identity for a few basic cases, without fulling justifying all of our arguments, then go on to prove the Generalised McShane Identity by characterising all possible types of behaviours of geodesics which emanate perpendicularly from X.

2.2.1 General Setup

Definition 2.2.1. Given a hyperbolic surface X with geodesic boundary β_i, let $E(X) \subset X$ be the union of all simple complete geodesics that either:

- meet one of the β_i perpendicularly, and do not meet any other boundary components, or;
- meet two boundary components perpendicularly.

Then, we define $E_i(X)$ to be $E(X) \cap \beta_i$. We will often refer to $E_i(X)$ simply as E_i.

Definition 2.2.2. Given $x \in \beta$, where β is a boundary, we define γ_x to be the unique complete (possibly self-intersecting) geodesic perpendicular to β so that $x \in \gamma_x$.

Definition 2.2.3. Consider a metric space (X, d), where d is the distance function on M. For $Y \subset X$ and $\delta \in [0, \infty)$, define the δ-dimensional **Hausdorff content** of Y to be:

$$C_H^\delta(Y) = \inf\left\{\sum_i r_i^\delta : \text{ there exists a cover of } Y \text{ with balls of radii } r_i > 0\right\}.$$

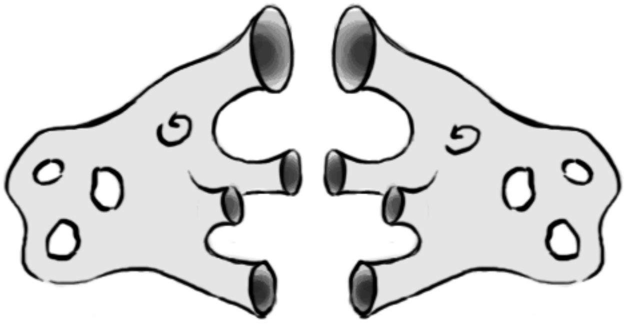
Then the **Hausdorff dimension** of X is:

$$\dim_H(X) = \inf\{\delta \geq 0 : C_H^\delta(X) = 0\}.$$

Theorem 2.2.1 (Birman and Series)**.** *Let G be the set of all simple complete geodesics on a closed hyperbolic surface, then the union of the elements of G forms a set of points with Hausdorff dimension 1.*

Corollary 2.2.2. *$E_i \subset \beta_i$ has Lebesgue measure 0.*

Proof. (of the corollary) Given such a hyperbolic surface X, let X' denote the mirror image of X. Glue X and X' along their corresponding boundaries to from a closed hyperbolic surface $\tilde{X}$, making sure that the orientations matched up.



Let D_i be the set of all the points on geodesics eminating from E_i. Since everything in D_i meets the boundary perpendicularly, the geodesics in D_i and its mirror twin D_i' in X' combine to give a collection $\tilde{D}_i$ of simple complete geodesics in $\tilde{X}$. Since $\tilde{D}_i$ is a subset of the set of all simple complete geodesics of Z, which has Hausdorff dimension 1, we can cover $\tilde{D}_i$ with a countable set of balls whose radii $\{r_i\}$ add up to an arbitrarily small number ϵ. Thus, for $r_i < 1$, the Lebesgue measure of

$$m(D_i) = \frac{1}{2}m(\tilde{D}_i) \leq \sum_{\mathbb{N}} \pi r_i^2 \leq \sum_{\mathbb{N}} \pi r_i = \pi\epsilon.$$

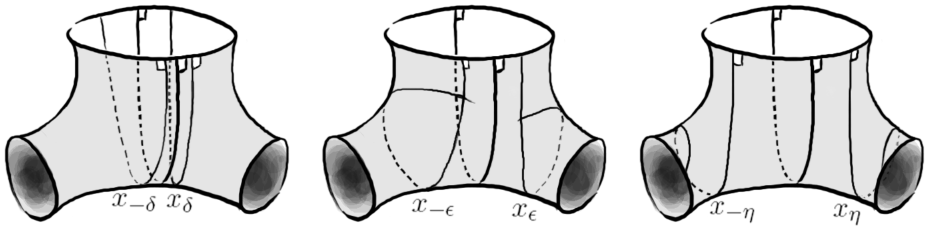
Since ϵ can be made to be as small as we want, $m(D_i) = 0$. As for the Lebesgue measure of E_i treated as a subset of the boundary: we can see that a collar neighbourhood around E_i has measure $\sinh(r) \times m(E_i)$, where r is the width of the collar neighbourhood taken. Since this is a subset of D_i, it has measure 0. Therefore, $m(E_i) = 0$. □

2.2.2 $S_{0,3}$, $S_{1,1}$ and $S_{0,4}$

We begin by examining different types of behaviours exhibited by geodesics on a given pair of hyperbolic pants $S_{0,3}$, followed by $S_{1,1}$ and $_{0,4}$.

$S_{0,3}$ Case

Given a hyperbolic pair of pants Σ, with boundaries A, B and C respectively of lengths a, b and c, we know that on Σ, there exists a unique simple closed geodesic γ from $x \in A$ to $y \in A$ called a self-link. Since the self-link meets A perpendicular at both points, $\gamma = \gamma_x = \gamma_y$. Now, consider moving slightly away from x to $x_{-\delta}$ and x_δ. The perpendicular geodesics emanating from these points will also meet A, but not perpendicularly because we've proven that the self-link is unique. If we move further out to $x_{-\epsilon}$ and x_ϵ, $\gamma_{x_{-\epsilon}}$ and γ_{x_ϵ} will intersect itself. And if we keep moving our 'x's outward, they continue to self-intersect until $x_{(1)}$ and $x_{(2)}$: the two points whose corresponding geodesics spiral to A and B respectively. Moving on past $x_{(1)}$ and $x_{(2)}$ to $x_{-\eta}$ and x_η, we find that our geodesics each spiral around a pants leg before intersecting B and C respectively. However, they do not intersect perpendicularly, as that is only true of the unique A, B-link x_B and the unique A, C-link x_C.



The exact same dynamics occur (modulo reflection) if we start from y, going outwards, and we similarly obtain the special points $y_{(1)}$ and $y_{(2)}$ corresponding to geodesics spiraling to A and B respectively, and eventually, $y_B = x_B$ and $y_C = x_C$.

We now define two functions that we have encountered previously:

$$\mathcal{D}, \mathcal{R} : \mathbb{R}^3_+ \to \mathbb{R}.$$

Definition 2.2.4. For $(a, b, c) \in \mathbb{R}^3_+$ corresponding to the pair of pants described above. $\mathcal{D}$ is defined to be:

$$\mathcal{D}(a, b, c) = d(x_{(1)}, x_{(2)}) + d(y_{(1)}, y_{(2)}),$$

where d is the geodesic distance from $x_{(1)}$ to $x_{(2)}$ containing x, and similarly for the 'y's. Conversely, we define $\mathcal{R}$ to be:

$$\mathcal{R}(a, b, c) = \bar{d}(x_{(1)}, y_{(1)}), \text{ and } \mathcal{R}(a, c, b) = \bar{d}(x_{(2)}, y_{(2)}),$$

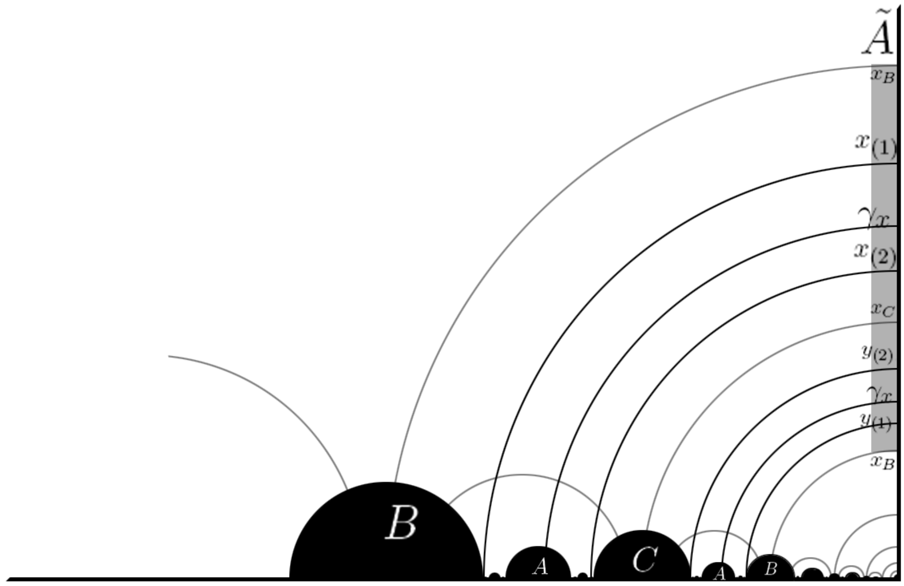
where $\bar{d}$ denotes taking the longer geodesic distance along A. That is: $\bar{d}(x_{(1)}, y_{(1)})$ is the length of the geodesic segment along A containing $x, x(2), y$ and $y_{(2)}$. Observe that we have the following identity:

$$a + \mathcal{D}(a, b, c) = \mathcal{R}(a, b, c) + \mathcal{R}(a, c, b),$$

by considering how the '$\mathcal{D}$'s and '$\mathcal{R}$'s cover A.

Note 2.2.1. Since hyperbolic pants have reflection symmetry, $\mathcal{D}(a, b, c) = \mathcal{D}(a, c, b)$, although this is in general not true of $\mathcal{R}$.

We will now derive explicit expressions for $\mathcal{D}$ and $\mathcal{R}$ in terms of elementary functions. Consider the figure below: a picture of the universal cover for a hyperbolic pair of pants viewed in the Poincarè half-plane model. The vertical axis corresponds to a lift of A, which we shall denote by $\tilde{A}$. The shaded segment of $\tilde{A}$ projects to a path going around A precisely once. In addition, $A, B, C, \gamma_x, \gamma_y, x_B, x_C, x_{(1)}, x_{(2)}, y_{(1)}$ and $y_{(2)}$, as shown, represent their lifts.



Note 2.2.2. We have assumed without loss of generality that a lift of A can always be placed along the vertical axis: if not, send $\tilde{A}$ to the imaginary axis via an orientation-preserving Mo̤bius transformation, obtain the result and then back via the inverse transformation.

First observe that that the the short interval (which we'll call I) from $x_{(2)}$ to $y_{(2)}$ projects to precisely the region on A completement to what's measured by by $\mathcal{R}(a, c, b)$. That is:

$$a = \ell(I) + \mathcal{R}(a, c, b). \tag{2.1}$$

Now, imagine for a second that Σ was embedded in some hyperbolic surface X, so that A and B were boundaries, but C was some non-peripheral geodesic in X. Then, outside of I, any perpendicularly emanating geodesic, apart from $\gamma_x, \gamma_{x_{(1)}}, \gamma_{y_{(1)}}, \gamma_{x_{(2)}}, \gamma_{y_{(2)}}$ and γ_{x_B} will either hit boundaries A or B non-perpendicularly, or self-intersect. Therefore, nothing outside of I, apart from a few points, lies in E_A. This means that, modulo these few 'strange' points, we can partition $A - E_A$ into I and a region of length $\mathcal{R}(a, c, b)$. This is the basic idea for deriving the Generalised McShane identity: we first partition A into I and this other set with length expressed in either $\mathcal{D}$ or $\mathcal{R}$. We then go on and analyse what's happening inside of I, in the hopes of getting out more $\mathcal{D}$ and $\mathcal{R}$ terms. It turns out that this is always possible, and one way to do it is by considering different pairs of embedded pants in X containing the boundary A. We have seen that $\mathcal{R}$ is useful for measuring a partitioned set in the case that two of the boundaries of Σ is peripheral, however, should A be the only boundary component of Σ embedded in X, then both the interval I and another one corresponding to geodesics going through B would need further analysis, and the partitioned region in this case has length $\mathcal{D}(a, b, c)$.

Lemma 2.2.3. *The function $\mathcal{R} : \mathbb{R}^3_+ \to \mathbb{R}_+$ is explicitly given by:*

$$\mathcal{R}(a, b, c) = a - \log\left(\frac{\cosh(\frac{a}{2}) + \cosh(\frac{b+c}{2})}{\cosh(\frac{a}{2}) + \cosh(\frac{b-c}{2})}\right).$$

Proof. We prove this by considering two hyperbolic quadrilaterals with two right-angles. The first has and boundaries $\tilde{A}, \gamma_{x_{(2)}}, C$ and γ_{x_C}, the second has sides $\tilde{A}, \gamma_{y_{(2)}}, C$ and γ_{x_C}.

Let the respective segments on $\tilde{A}$ bounding these two quadrilaterals be denoted by a_1 and a_2. Then, we know that $\ell(I) = a_1 + a_2$ and by lemma 1.3.2:

$$\cos(0) = 1 = \sinh(a_1)\sinh(\ell(\gamma_{x_C})) = \sinh(a_2)\sinh(\ell(\gamma_{x_C})).$$

Therefore, by equation (2.1), we know that:

$$\mathcal{R}(a, c, b) = a - 2\text{arcsinh}\left(\frac{1}{\ell(\gamma_{x_c})}\right).$$

Now, by decomposing Σ into two right-angled hexagons with side lengths $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}, \ell(\gamma_{x_A}), \ell(\gamma_{x_B})$ and $\ell(\gamma_{x_C})$ (not in that order), we can use the hyperbolic cosine formula derived in example 1.3.2 to express $\ell(\gamma_{x_C})$ in terms of a, b and c. Substituting these values in to equation (2.1) and performing some algebraic simplification gives us the desired identity. □

Corollary 2.2.4. *The function $\mathcal{D} : \mathbb{R}^3_+ \to \mathbb{R}_+$ is explicitly given by:*

$$\mathcal{D}(a, b, c) = 2\log\left(\frac{e^{\frac{a}{2}} + e^{\frac{b+c}{2}}}{e^{\frac{-a}{2}} + e^{\frac{b+c}{2}}}\right).$$

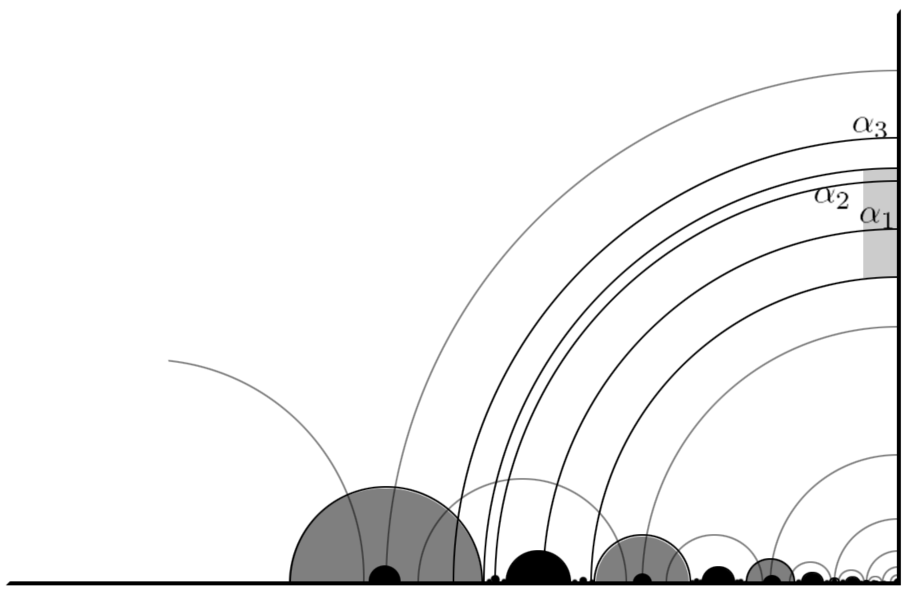
Proof. We know that

$$a + \mathcal{D}(a, b, c) = \mathcal{R}(a, b, c) + \mathcal{R}(a, c, b),$$

substituting in the derived expressions for $\mathcal{R}(a, b, c)$ and $\mathcal{R}(a, c, b)$ gives the desired result (modulo some tedious simplifications). □

$S_{1,1}$ Case

Let us turn to the universal cover of a hyperbolic surface $S_{1,1}$, where the pants decomposition has the shortest length, and $S_{1,1}$ is in standard position. This is a minor detail that will make the universal cover look slightly more *symmetrical*. The observations and arguments that we present however, still apply for any arbitrary $S_{1,1}$.



Note 2.2.3. The darker grey section is essentially where the author has decided not to include any detail.

We know the following:

- there are countably infinite simple closed geodesics on $S_{1,1}$: one for each (positive) homotopy class of its fundamental group generated by a single element.
- any simple closed geodesic in $S_{1,1}$ is a hyperbolic pants decomposition, and on its corresponding pants cover, we have a unique self-link for the boundary A.

Therefore, we have countably infinitely many self-links of A on $S_{1,1}$.

Notice that self-links are simple geodesics in $S_{1,1}$, because they're simple in the pants cover and do not meet any boundaries other than A, so will remain simple in the quotient. In addition, self-links meet A perpendicularly, therefore self-link points lie in $E_A(S_{1,1})$. Now, as seen with $S_{0,3}$, for each of these countably many self-link points x, there's an open neighbourhood surrounding it, where any geodesics perpendicularly emanating from it will either meet A again (not perpendicularly), or self-intersect. Therefore, these points cannot be in E_A. Moreover, the boundary points of this open neighbourhood correspond to infinite simple geodesic rays which shoot from A perpendicularly and end up spiraling to some simple closed geodesic embedded in $S_{1,1}$. The lighter shaded segment on the above figure corresponds to one such neighbourhood. We will later prove that the points in E_A minus all these self-link points actually form a set homeomorphic to the Cantor set. But for now, we consider lifts of these self-link points, the neighbourhoods surrounding them, and attempt to see why E_A should be homeomorphic to the Cantor set plus these self-link points.

Consider the lifts of all the self links of A, where their corresponding self-link points are lifted to the lighter region (as opposed to the lighter shaded region) on $\tilde{A}$ which projects to a path going around A precisely once. We can see that there are many geodesics perpendicular to $\tilde{A}$ which project to geodesics meeting $\tilde{A}$ perpendicularly at both ends, such as $\alpha_1, \alpha_2, \alpha_3$. However, not all of these project to self-links; in particular, α_2 lies in the lighter shaded region corresponding to geodesics which either meet A non-perpendicularly or self-intersect - α_2 must belong to the latter type. To ascertain if α_3 corresponds to a self-link, we make the following observations:

- let the lighter region on $\tilde{A}$ be denote by $\tilde{A}_0$, and for any $x \in \tilde{A}_0$, let γ_x denote the geodesic emanating from x that's perpendicular to $\tilde{A}_0$. Then, for the shortest self-link of A in $S_{1,1}$, the two endpoints of this self-link lift to two points $x, y \in \tilde{A}_0$.
- Since the self-link which lifts to γ_x and γ_y is the shortest in $S_{1,1}$, γ_x and γ_y must be the shortest geodesics in the universal cover joining $\tilde{A}_0$ to any other lift of A.
- We then remove the two open neighbourhoods around x and y corresponding to paths in A which either meet A again (non-perpendicularly) or self-intersect. Label this left over set by $\tilde{A}_1$, notice that $\tilde{A}_1$ is closed. Consider now the second shortest self-link in $S_{1,1}$; it also lifts to x_1 and y_1 in $\tilde{A}_1$, and by similar arguments as before, γ_{x_1} and γ_{x_2} must be the shortest geodesics in the universal cover joining $\tilde{A}_1$ to any other lift of A.

This argument can be iterated indefinitely and in this way, we can determine if γ_{α_3} projects to a self-link. Notice that, in going from $\tilde{A}_0$ to $\tilde{A}_1$, we remove two open regions, and we're left with a few closed intervals. However, if we keep taking the shortest remain geodesic and removing a neighbourhood of it, it's not hard to see that eventually, any closed interval will have some open region within it removed (in fact, the open region removed will be the neighbourhood of the point in this interval that has emanating geodesic with least distance). This process of removing sets is similar to more generalised constructions of Cantor-like sets, such as the Smith-Volterra-Cantor set, thus, it is certainly plausible that the left over points in $\tilde{A}_0$ will be homeomorphic to the Cantor set. We have shown in corollary 2.2.2 that E_A has measure zero in A, and hence the length of A is the sum of all these open neighbourhoods around self-links. For each self-link, this length is

$$\mathcal{D}(\ell(A), \ell_\gamma(S_{1,1}), \ell_\gamma(S_{1,1})),$$

where γ is the pants decomposition corresponding to this self-link. Thus, we have the $S_{1,1}$ Generalised McShane Identity:

$$\sum_\gamma \mathcal{D}(\ell(A), \ell_\gamma(S_{1,1}), \ell_\gamma(S_{1,1})) = \ell(A),$$

where $\ell_\gamma(S_{1,1})$ denotes the length of γ in $S_{1,1}$, and the sum is over all geodesics (and hence pants decompositions) in $S_{1,1}$.

Note 2.2.4. Although we have explored the above 'derivation' in some technicality via the universal cover of $S_{1,1}$, one useful point to note is this: if we shoot off a path γ_x from the boundary A, for x not in E_A, it either self-intersects or hits A non-perpendicularly. However, $\mathcal{R}$ is related to partitioned regions which not only have these type of points, but also have points going from one boundary to another. Therefore, in the generalised McShane identity for this case, we expect that all terms in the sum must be $\mathcal{D}$ terms.

$S_{0,4}$ Case

Let $S_{0,4}$ be a bordered hyperbolic surface with boundaries A, B, C and D - respectively of lengths a, b, c and d. We begin by looking at the universal cover of $S_{0,4}$. First observe that just as with $S_{1,1}$, any simple closed geodesic on $S_{0,4}$ is a hyperbolic pants decomposition. Therefore, to each geodesic we associate a unique self-link of A. Just as before, we can remove open neighbourhoods around the two self-link points of length $\mathcal{D}(\ell(A), \ell(B), \ell_\gamma(S_{0,4}))$ (for example) and take the sum of all of these. However, in this case, there will still be closed intervals remaining, because the neighbourhoods around self-links only contain points which self-intersect or come back and hit A. However, in $S_{0,4}$, we have addition neighbourhoods of points which correspond to geodesics perpendicularly shooting from A and hitting B, C or D. In fact, any pair of pants in a decomposition of $S_{0,4}$ will contain two boundary geodesics, and one non-peripheral geodesic. We have seen previously that this will partition A into two sets, one of which has length expressed as the function $\mathcal{R}$. Specifically, for every self-link, there are three corresponding regions in A where only 3 points lie in E_A. The sum of the length of these three areas is precisely

$$\mathcal{R}(a, x, \ell_\gamma(S_{0,4})),$$

where $x = b, c$ or d depending on the pants cover, and γ is the hyperbolic pants decomposition in question. Now, for each hyperbolic pants decomposition, we obtain a unique pair of embedded pants containing the boundary A, and for each such pair, there is an unique self-link on A surrounded by a region of length $\mathcal{R}$(something) not lying in E_A. It turns out that these regions partition everything in A save some points in E_A, and so we have:

$$\sum_{\gamma} \mathcal{R}(a, x, \ell_\gamma(S(0,4))) = a,$$

where the sum is over all pants decompositions in $S_{0,4}$ and x is chosen appropriately. If we then group the collection of all pants decompositions in $S_{0,4}$ into the sets $\mathcal{F}_{A,B}$, $\mathcal{F}_{A,C}$ and $\mathcal{F}_{A,D}$, depending on which boundary geodesic is on the same pair of pants in the corresponding pants cover, then we can break up the sum into three 'smaller' sums, and explicitly say what x is in each resulting partition, giving:

$$\sum_{\gamma \in \mathcal{F}_{A,B}} \mathcal{R}(a, b, \ell_\gamma(S_{0,4})) + \sum_{\gamma \in \mathcal{F}_{A,C}} \mathcal{R}(a, c, \ell_\gamma(S_{0,4})) + \sum_{\gamma \in \mathcal{F}_{A,D}} \mathcal{R}(a, d, \ell_\gamma(S_{0,4})) = a.$$

Which is precisely the Generalised McShane Identity for the $S_{0,4}$ case.

Note 2.2.5. $\mathcal{R}$ is not symmetric in its last two coordinates, thus, it is important that we take $\mathcal{R}(a, x, \ell_\gamma(S_{0,4}))$. $\ell\gamma(S_{0,4})$ is in the last coordinate, because the interval on A not covered by $\mathcal{R}$ is always closer to γ.

2.2.3 Geodesic Characterisation

Definition 2.2.5. A **geodesic lamination** Ω is a collection of disjoint complete simple geodesics, such that Ω is closed. Further, a **leaf** is any complete geodesic contained in the lamination. A lamination is called **minimal** if no proper subset is a lamination. Lastly, a **non-trivial lamination** is a geodesic lamination that is minimal, compact, and has more than one leaf.

Definition 2.2.6. Given a path γ, we say that it **spirals to a lamination** $\Omega(\gamma)$ if and only if $\Omega(\gamma)$ is in the closure of γ.

Note 2.2.6. Observe that for any $x \in E_i$, the induced simple complete geodesic γ_x either:

- meets a boundary component β_j (i not necessarily distinct from j) perpendicularly, or
- spirals to a compact minimal lamination $\Omega(\gamma_x)$.

We will later show that E_i is the union of a Cantor-like set with a countable set of isolated points. To do so, we partition points in E_i in terms of the behaviours of geodesics perpendicularly emanating from them. Although the statement of following the theorem assumes that we know E_i to be such a union, rest assured that the proof is not circular. We will prove that points are:

- **isolated points** by showing that there is an open neighbourhood in the complement which surrounds it,
- **end points** by showing that there is an open neighbourhood in the complement on one side of the point, but not on the other side (mirroring the end-points at each stage of the Cantor set construction), and
- **inside points** by showing that any arbitraily small neighbourhood around it is full of points in E_i (in fact, full of points in E_i minus all isolated points).

Theorem 2.2.5 (Characterisation of Geodesics)**.** *Given any $x \in E_i$, one of the following occurs:*

1. *γ_x either meets a boundary component perpendicularly, or spirals to one if and only if x is an* **isolated point** in E_i;

2. $\Omega(\gamma_x)$ is a simple closed non-boundary geodesic if and only if x is an **end point** of E_i;

3. $\Omega(\gamma_x)$ is not a simple closed curve if and only if, it is an **inside point** of E_i.

Note 2.2.7. By end point, and inside point, it is meant that these points correspond respectively to the boundary points are each stage of the Cantor set construction and to the resulting points that lie in the final intersection in the Cantor set construction but were not on the boundary at any stage.

We will now provide a proof of this result, one case at a time.

Case1

We begin by showing that any pair of self-link points x_1 and x_2 are isolated. In particular, we show that the maximum connected neighbourhood of x_1 in β_1 disjoint to $E_i - \{x_1\}$ is bounded by two points y_1 and y_2, such that γ_{y_1} and γ_{y_2} are infinite geodesic rays. This then tells us that the length of this maximal open interval around x_1 is equal to $\frac{1}{2}\mathcal{D}(\ell_{\beta_1}, \ell_{\alpha_1}, \ell_{\alpha_2})$, where the geodesics α_1, α_2 and β_i bound the unique embedded pair of pants containing this self-link.

Proof. Let P denote the unique embedded hyperbolic pair of pants containing the self-link γ_{x_1}, and let α_1 and α_2 denote the other two non-peripheral[2] boundary components of P. We have shown previously that there are exactly four simple complete infinite geodesics rays in P meeting β_i perpendicularly (two for each α_j). Let y_1, y_2 and z_1, z_2 respectively refer to these four corresponding meeting points on E_i, such that:

- γ_{y_i} and γ_{z_i} spiral around α_i, and
- if we denote the shorter geodesic path from y_1 to y_2 on β_i by (y_1, y_2), then $x_1 \in E_1 \cap (y_1, y_2)$. Similarly, $x_2 \in E_2 \cap (z_1, z_2)$.

We want to show that $x_1 = E_i \cap (y_1, y_2)$, and $x_2 = E_j \cap (z_1, z_2)$.

We know that $\{\gamma_{x_k}, \gamma_{y_k}, \gamma_{z_k}\}_{k=1,2}$ are the only geodesics (in P) emanating from β_i perpendicularly without meeting the other two boundary components. Therefore, given $z \in E_i - \{x_1, x_2, y_1, y_2, z_1, z_2\}$, γ_z cannot be fully contained in P, and must meet α_1, without loss of generality. Now, if restrict ourselves to P, and consider the segment $\tilde{\gamma}_z$ of γ_z, prior to it first leaving P, we can see that since $\tilde{\gamma}_z$ is simple, it must lie in the same homotopy class as the β_i, α_1-link. This in turn means that every lift of $\tilde{\gamma}_z$ in the universal cover of P is homotopic to some lift of the β_i, α_1-link. But then, we can do the same construct as that in the proof of the existence of infinite simple rays to to turn $\tilde{\gamma}_z$ into an infinite simple curve that's homotopic to either a lift of γ_{y_1} or γ_{z_1}. Therefore, the lift of z must lie in between the lift of y_1 and the lift of the β_i, α_1-link point, or between the lift of z_1 and the lift of the link point. In either case, $z \in (y_1, z_1)$, and not in (y_1, y_2) or (z_1, z_2).

Hence, we have: $x_1 = E_i \cap (y_1, y_2)$ and $x_2 = E_j \cap (z_1, z_2)$, as desired.

[2]The case when only one of these two are non-peripheral is argued identically.

As for the scenario where we're given a simple complete geodesic γ_{x_1} meeting two distinct boundaries perpendicularly, or an infinitely ray spiraling to a boundary component of X, let P denote the unique hyperbolic pair of pants containing γ_{x_1}, and let α denote the non-peripheral geodesic bounding $P \subset X$. As before, there are four simple complete geodesic rays in P meeting β_i perpendicularly. In particular, two of them, γ_{y_1} and γ_{z_1} emanating from y_1 and z_1 respectively, spiral to α. Then consider $z \in E_i - \{\text{special points}\}$, γ_z must meet α and by the same arguments as above, we can prove that $z \in (y_1, z_1)$. Thus, $x_1 = E_i \cap (y_1, y_2)$, and x_1 is isolated. □

It should be noted that case 1 is sufficient for the derivation of the Generalised McShane Identity[3]. Thus, we take a minor detour to prove the main result of this section.

Theorem 2.2.6 (Generalised McShane Identity for Bordered Surfaces)**.** *For any $X \in \mathcal{T}_{g,n}(L_1, \ldots, L_n)$ with $3g - 3 + n > 0$,*

$$\sum_{\{\gamma_1,\gamma_2\}\in\mathcal{F}_i} \mathcal{D}(L_i, \ell_{\gamma_1}(X), \ell_{\gamma_2}(X)) + \sum_{j\neq i}\sum_{\gamma\in\mathcal{F}_{i,j}} \mathcal{R}(L_i, L_j, \ell_\gamma(X)),$$

where $\mathcal{F}_i$ denotes all pairs of closed non-peripheral geodesics which, along with β_i, bound an embedded pair of pants, and $\mathcal{F}_{i,j}$, $i \neq j$ is the collection of all non-peripheral geodesics which bound an embedded pair of pants along with β_i and β_j.

Proof. Let I denote the set of isolated points in E_i. Then, we know the there are three types of points in I: self-links (denoted by I_s), β_i, β_j-links (denoted by I_l), and infinite geodesic rays. For each point $x \in I_l$, there is a unique pair of pants containing the link γ_x. In addition, there is a uniquely specified connected neighbourhood in $\beta_i - (E_i - I)$ containing x of length $\mathcal{R}(L_i, L_j, \ell_\gamma(X))$, denote this neighbourhood by U_x. Similarly, for each point $y \in I_s$, there exists a uniquely specified neighbourhood $V_y \subset \beta_i - (E_i - I)$ of length $\mathcal{D}(L_i, \ell_{\gamma_1}(X), \ell_{\gamma_2}(X))$. We know that these neighbourhoods are mutually disjoint, because by lemmata 1.3.11, 1.3.12 and 1.3.13, the maximal connected non-intersecting part of any self-intersecting geodesic, or any geodesic going from one peripheral boundary to another, lies in a unique pair of pants. Therefore, the sum of all these open regions surrounding points in I_s and I_l is:

$$\sum_{\{\gamma_1,\gamma_2\}\in\mathcal{F}_1} \mathcal{D}(L_1, \ell_{\gamma_1}(X), \ell_{\gamma_2}(X)) + \sum_{i=2}^{n}\sum_{\gamma\in\mathcal{F}_{1,i}} \mathcal{R}(L_1, L_i, \ell_\gamma(X)).$$

Note 2.2.8. $\bigcup_{x\in I_l} U_x$ already contains all the infinite geodesic rays.

If we can show that

$$K = \beta_i - \left(\bigcup_{x\in I_l} U_x \;\cup\; \bigcup_{y\in I_s} V_y\right)$$

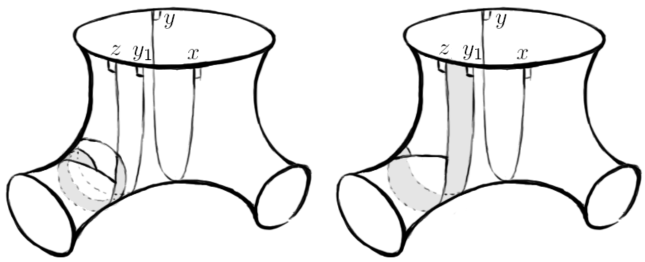
lies in $E_i - I$, then we have:

$$\begin{aligned} L_i = m(\beta_i) &= m(K) + m(\beta_i - K) \leq m(E_i - I) + m(\beta_1 - K) = m(\beta_1 - K) \\ \Rightarrow L_i = m(\beta_i) &= m(\beta_i - K) \\ &= \sum_{\{\gamma_1,\gamma_2\}\in\mathcal{F}_i} \mathcal{D}(L_i, \ell_{\gamma_1}(X), \ell_{\gamma_2}(X)) + \sum_{j\neq i}\sum_{\gamma\in\mathcal{F}_{i,j}} \mathcal{R}(L_i, L_j, \ell_\gamma(X)), \end{aligned}$$

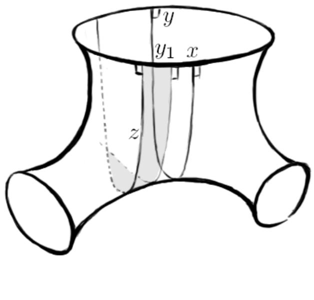
[3] It is also sufficient to prove that E_i is the union of isolated points with a Cantor-like set.

since $0 = m(E_i) \geq m(E_i - I) \geq 0$.

Therefore, let us consider $z \in K$, if $z \notin E_i$, then it either self-intersects or meets a boundary non-perpendicularly. In the self-intersecting case, we know from lemma 1.3.13 that everything up to the first point that γ_z hits itself lies in a pair of pants, which we'll denote by P. Let x and y denote the self-link points of P, and y_1, z_1, y_2 and z_2 be points on β_i such that the geodesics perpendicularly shooting from them are the four canonical simple geodesic rays in P. By assumption, z does not lie in (y_1, y_2) or (z_1, z_2), therefore, it must, without loss of generality, lie in (y_1, z_1). Now, since γ_z self-intersects, it loops around the pants leg ending in α_1. Thus, γ_{y_1}, the geodesic ray lying on the same side of the β_i, α_1-link point as z must intersect the loop at the end of the initial non-self-intersecting segment of γ_z. Then, trace out a geodesic polygon in the following way: we go along γ_z until we first meet γ_{y_1}, then, follow γ_{y_1} back up to β_1. If along the way, we meet γ_z again, then follow γ_z back down to obtain a geodesic 2-gon, which is impossible. Otherwise, follow γ_{y_1} all the way back up and sidle across to z along β_i to obtain a geodesic triangle with two right angles, which we know is impossible, because the area of a hyperbolic triangle is π minus the sum of its angles.



Having obtained a contradiction for the self-intersecting case, we now consider the scenario when γ_z is a geodesic going from β_i to a boundary. If γ_z goes from β_i to a different boundary β_j, then by the same arguments as used for the proof of case 1, we can show that z lies in (y_1, z_1) (without loss of generality) of a uniquely associated pair of pants. But then, this interval is accounted for by the $\mathcal{R}$ term associated with the unique β_i, β_j-link lying in the same homotopy class as γ_z (that is, on this pair of pants). On the other hand, if γ_z goes from β_i to β_i, then by lemma 1.3.11, there exists a unique embedded pair of pants containing γ_z. Moreover, consider the piece-wise geodesic simple closed loop we get from travelling along γ_z and then along β_i until we get back to z. This loop must intersect the simple geodesic ray γ_{y_1}, and by the same argument as before, this would produce a geodesic triangle with two right angles: a contradiction. Thus, $K - E_i$ is empty, and the Generalised McShane Identity follows.



□

Case 2

We now continue with the remaining two cases of the classification of geodesics theorem. We will only outline the key ideas behind her proof.

Proof. We begin by showing that for $y \in E_i$, if $\Omega(\gamma_y)$ is a simple closed geodesic, then y is either an end point or an isolated point in E_i.

Observe that there exists a hyperbolic pair of pants P that includes γ_y, and has boundaries β_i, $\Omega(\gamma_y)$ and some other geodesic that we'll denote by α. Further, denote the two self-link points of β_i by $x_1, x_2 \in E_i$. Lastly, denote the two points which spiral to $\Omega(\gamma_y)$ by y and z, and the two points which spiral to α by y' and z'. Then, by case 1, we know that $E_i \cap (y, y') = x_1$. Therefore, y is not a both-sided limit point of E_i, thus, in the language of the Cantor set union isolated points structure of E_i, y must be either an end point or an isolated point.

To show that y is in fact the limit of a sequence of points in E_i, we do that following: consider an embedded geodesic Ω in X disjoint from β_i, $\Omega(\gamma_y)$ and α. Then, we find a simple path going from β_i, through $\Omega(\gamma_y)$ and then to Ω. We know from lemma 1.3.12 that this path induces a unique embedded pair of pants, denoted by P_0, in X with boundaries β_i, Ω and something-else. Then, we perform a Dehn-twist about $\Omega(\gamma_y)$, thus giving us a simple path in a different homotopy class, goin from β_i to Ω going through $\Omega(\gamma_y)$. This new path induces a new path of pants, denoted by P_1. By iterating this process, we build up a sequence of embedded hyperbolic pairs of pants $\{P_i\}_{j\in\mathbb{N}}$. It can be shown that the β_i, Ω-link point for this sequence of pants approaches y, but since Ω is non-peripheral, the β_i, Ω-link point is not an element of E_i and we're not done yet. Fortunately, either one of the geodesic rays γ_{v_j} going perpendicularly from β_i to spiral along Ω, does have starting point in E_i. Then, by making the observation that as the un-named boundary of P_i approaches infinity (due to the Dehn-twists), the distance between v_j and the β_i, Ω-link shrinks to zero (to prove, consider the corresponding $\mathcal{R}$-term approaching L_i), therefore, $\{v_j\}$ is a sequence in E_i approaching y. □

Note 2.2.9. We do not simply construct a sequence of self-link points going from β_i to β_j which approach y because, although these points do lie in E_i, they're isolated points and will be removed when proving that $E_i - \{\text{isolated points}\}$ is homeomorphic to the Cantor set.

Case 3

Once again, we will only sketch the proof for this case. Moreover, we adjust the original proof slightly, but still rely on the basic idea of finding a sequence of approximating quasi-geodesics, and then from this sequence of quasi-geodesics, find a sequence of approximating geodesics in E_i.

Definition 2.2.7. Let $d(x, y)$ denote the hyperbolic distance between $x, y \in \mathbb{H}$. Given a path $\alpha(t)$ parametrised by path-length. Then, we call α a **quasi-geodesic** if there exists $k > 0$ such that:

$$d(\alpha(s), \alpha(t)) > k|s - t|, \text{ for all } s \text{ and } t.$$

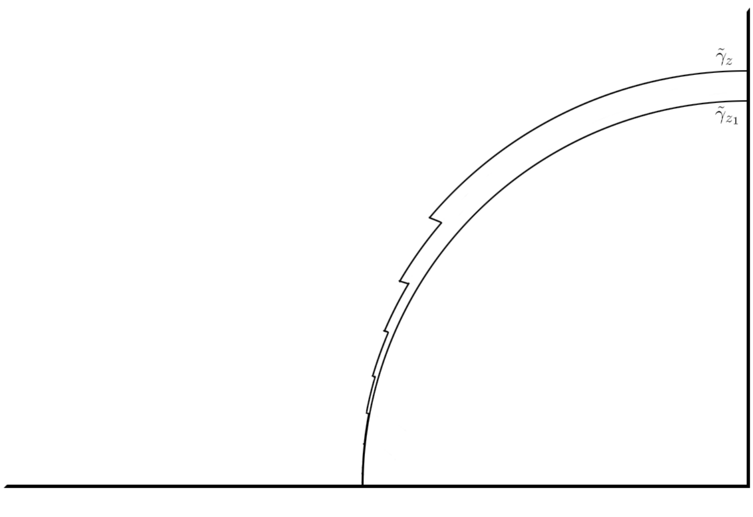
Proof. Firstly, it can be shown that for a geodesic ray γ_z, if $\Omega(\gamma_z)$ is not a closed geodesic loop, then it must be a (minimal) non-trivial lamination. Choose $y \in \Omega(\gamma_z)$ so that γ_z approximates the leaf of $\Omega(\gamma_z)$ containing y from both sides; this is always possible due to the uncountability of non-trivial laminations of geodesic rays. Then, we parametrise γ_z in terms of length, and find two points, $x_0 = \gamma_z(t_0)$ and $x_1 = \gamma_z(t_1)$, $t_0 < t_1$, such that:

- both points are less than $\frac{\epsilon}{2}$ away from y, and
- the tangent vectors at x_0 and x_1 are almost parallel, and both facing (roughly) the same way, and
- x_0 and x_1 can be joined by a small geodesic segment c, of length less than ϵ, which does not intersect with $\{\gamma_z(t) : t_o < t < t_1\}$.

Not only is this always possible, but it's actually always possible to find such points so that:

- the length of the segment $\{\gamma_z(t) : t_o < t < t_1\}$, which is $t_1 - t_0$, can be made to be arbitrarily long,
- the length of the segment c bridging x_0 and x_1 can be made arbitrarily short, or in other words, ϵ can be made to be arbitrarily small, and
- the tangent vectors at x_0 and x_1 can be made to be arbitrarily parallel.

Now, consider the path given by travelling along γ_z, all the way to x_1, then along c to x_0, then continuously looping along γ_z to c to $\gamma_z \ldots$etc., ad nauseum. The lift of this path, denoted by $\tilde{\gamma}_z$ in $\mathbb{H}$ is a quasi-geodesic, and is a bounded distance from a unique geodesic $\tilde{\gamma}_{z_1}$, which projects to a simple geodesic ray γ_{z_1} in X. We repeat the process and find two new points with larger $t_1 - t_0$, smaller ϵ and with more parallel tangent vectors, thereby eventually giving us a simple geodesic ray γ_{z_2}. The sequence $\{z_j\} \subset E_i$ has z as a limit point because they get closer and closer to the leaf in $\Omega(\gamma_z)$ containing y, therefore, z is not an isolated point. Moreover, by selecting which direction the tangent vectors of x_0 and x_1 face, we can control on which side of z the sequence $\{z_j\}$ lies. Therefore, we can approximate z from both sides, and z must be an 'inside point'.



$\square$

2.2.4 Corollaries

We will now prove that E_i is the union of a Cantor-like set and some isolated points.

Theorem 2.2.7 (Characterisation of Cantor-like Sets)**.** *Any perfect*[4]*, totally disconnected*[5]*, compact metric space is homeomorphic to the Cantor set.*

Corollary 2.2.8. *The set $E_i - I$, where I is the set of isolated points in E_i, is homeomorphic to the Cantor set.*

Proof. We will first argue that $E_i - I$ is closed. By the characterisation of geodesics, we know that any point not in $E_i - I \subset \beta_1$ must either self-intersect, meet a boundary (possibly perpendicularly), or spiral to a boundary component. For each such point, we can use the same arguments as used in the proof of theorem 2.2.6 to show that there is an open neighbourhood surrounding it not in $E_i - I$, therefore, the complement of $E_i - I$ is open in β_i. Since $E_I - I$ is closed and bounded in a metric space, it must be a compact metric subspace. In addition, we know from cases 2 and 3 that all points in $E_i - I$ are limit points in $E_i - I$, therefore, $E_i - I$ is a perfect set. Lastly, $E_i - I$ is measure 0 and must be totally disconnected. Therefore, by the characterisation of Cantor-like sets, $E_i - I$ is homeomorphic to the Cantor set. □

Although we have seen previously that the Generalised McShane Identity need not be proved via all three cases for the characterisation of geodesics, theorem 2.2.5 is an interesting result by itself, and lends a nice language for the proof of the Generalised McShane Identity. Which we state once more for closure.

Theorem 2.2.9 (Generalised McShane Identity for Bordered Surfaces)**.** *For any $X \in \mathcal{T}_{g,n}(L_1, \ldots, L_n)$ with $3g - 3 + n > 0$,*

$$\sum_{\{\gamma_1,\gamma_2\}\in\mathcal{F}_1} \mathcal{D}(L_1, \ell_{\gamma_1}(X), \ell_{\gamma_2}(X)) + \sum_{i=2}^{n} \sum_{\gamma\in\mathcal{F}_{1,i}} \mathcal{R}(L_1, L_i, \ell_\gamma(X)) = L_1.$$

Proof. The proof is a simplified version of the proof for theorem 2.2.5, the main idea is to rely on lemmata such as 1.3.9, 1.3.7, 1.3.12 and 1.3.11 to show that there is a 1 to 1 correspondence between self-links and embedded hyperbolic pairs of pants with two non-peripheral bounding geodesics and also that there is a 1 to 1 correspondence between -links and embedded hyperbolic pairs of pants with one non-peripheral bounding geodesics; and then relating each pair of pants with either a $\mathcal{D}$ or an $\mathcal{R}$ term. □

2.3 Mirzakhani's Recursion

We now present the main result of [8].

2.3.1 Mirzakhani Recursion Formula

The formulas presented here are essentially taken symbol for symbol from Mirzakhani's definitions. First observe that $\mathrm{Vol}(\mathcal{M}_{g,n}(\boldsymbol{L})) = V_{g,n}(L_1, \ldots, L_n)$ is symmetric with respect to the L_i. Thus, we will often write $V_{g,n}(\boldsymbol{L})$ to mean any $V_{g,n}(L_{\sigma(1)}, \ldots, L_{\sigma(n)})$, where $\sigma \in S_n$ is a permutation. In addition, we define the function

$$m(g, n) = \delta(g - 1) \times \delta(n - 1),$$

[4]All points are limit points.

[5]no non-trivial connected subsets

where the δ-function evaluates to 1 at 0 and 0 everywhere-else. We'll be using m to keep track of special cases in our recursion where we're dealing with $\mathcal{M}_{1,1}(\boldsymbol{L})$.

The Mirzakhani formula for $V_{g,n}(\boldsymbol{L})$ is computed recursively, and is defined as follows: let $\boldsymbol{L} = (L_1, \ldots, L_n)$, where L_k is the length of the k-th boundary β_k; and let $\widehat{L_i} = (L_1, \ldots, L_{i-1}, L_{i+1}, \ldots, L_n)$, that is: the vector $\boldsymbol{L}$ without the i-th coordinate. Then, the recursion relation is given by:

$$\frac{\partial}{\partial L_i}(L_i\ V_{g,n}(\boldsymbol{L})) = \mathcal{A}^{con}_{g,n}(L_i, \widehat{L_i}) + \mathcal{A}^{dcon}_{g,n}(L_i, \widehat{L_i}) + \mathcal{B}_{g,n}(L_i, \widehat{L_i}),$$

where each of the three terms are:

$$\mathcal{A}^{con}_{g,n}(L_i, \widehat{L_i}) = \frac{1}{2}\int_o^\infty \int_0^\infty x\ y\ \widehat{\mathcal{A}}^{con}_{g,n}(x, y, L_i, \widehat{L_i})\ dx\ dy,$$
$$\mathcal{A}^{dcon}_{g,n}(L_i, \widehat{L_i}) = \frac{1}{2}\int_o^\infty \int_0^\infty x\ y\ \widehat{\mathcal{A}}^{dcon}_{g,n}(x, y, L_i, \widehat{L_i})\ dx\ dy,$$
$$\mathcal{B}_{g,n}(L_i, \widehat{L_i}) = \int_0^\infty x\ \widehat{\mathcal{B}}_{g,n}(x, L_i, \widehat{L_i})\ dx.$$

The initial conditions are given by:

$$V_{0,3}(L_1, L_2, L_3) = 1,\ V_{1,1}(L_1) = \frac{L_1^2}{24} + \frac{\pi^2}{6}.$$

The definitions for each of the three functions $\widehat{\mathcal{A}}^{con}_{g,n}(x, y, L_i, \widehat{L_i})$, $\widehat{\mathcal{A}}^{dcon}_{g,n}(x, y, L_i, \widehat{L_i})$ and $\widehat{\mathcal{B}}_{g,n}(x, L_i, \widehat{L_i})$ is quite involved, and we will now define them one at a time.

The definition of $\widehat{\mathcal{A}}^{con}_{g,n}$

The function $\widehat{\mathcal{A}}^{con}_{g,n} : \mathbb{R}^{n+2}_+ \to \mathbb{R}_+$ is given by:

$$\widehat{\mathcal{A}}^{con}_{g,n}(x, y, L_i, \widehat{L_i}) = 2^{-m(g-1,n+1)} V_{g-1,n+1}(x, y, \widehat{L_i}) \cdot H(x+y, L_i),$$

where $H(a,b) = \frac{1}{1+e^{\frac{a+b}{2}}} + \frac{1}{1+e^{\frac{a-b}{2}}}$.

The definition of $\widehat{\mathcal{A}}^{dcon}_{g,n}$

Let $\mathcal{I}_{g,n}$ denote the set of ordered pairs

$$a = ((g_1, I_1), (g_2, I_2)),$$

where $I_1, I_2 \subset \{1, \ldots, i-1, i+1, \ldots, n\}$ and $0 \le g_1, g_2 \le g$ such that I_1 and I_2 partition $\{1, \ldots, i-1, i+1, \ldots, n\}$, and $g_1, g_2 \ge 0$ satisfy:

$$g_1 + g_2 = g,\ \text{and } 2 \le 2g_j + |I_j|,\ j = 1, 2,$$

where $|I_j|$ is the cardinality of I_j. Furthermore, for $I \subset \{1, \ldots, n\}$ with $|I| = k$, define L_I to be $(L_{j_1}, \ldots, L_{j_k})_{j_h \in I}$, and for each $a = ((g_1, I_1), (g_2, I_2)) \in \mathcal{I}_{g,n}$, let:

$$V(a, x, y, \widehat{L_i}) = 2^{-(m(g_1,n_1+1)+m(g_2,n_2+1))} \cdot V_{g_1,n_1+1}(x, L_{I_1}) \cdot V_{g_2,n_2+1}(y, L_{I_2}).$$

Then, the function $\widehat{\mathcal{A}}^{dcon}_{g,n} : \mathbb{R}^{n+2}_+ \to \mathbb{R}_+$ is given by:

$$\widehat{\mathcal{A}}^{dcon}_{g,n}(x, y, L_i, \widehat{L_i}) = \sum_{a \in \mathcal{I}_{g,n}} V(a, x, y, \widehat{L_i}) \cdot H(x+y, L_i).$$

The definition of $\widehat{\mathcal{B}}_{g,n}$

Lastly, $\widehat{\mathcal{B}}_{g,n} : \mathbb{R}_+^{n+1} \to \mathbb{R}_+$ is given by:

$$\widehat{\mathcal{B}}_{g,n}(x, L_i, \widehat{L_i}) = 2^{-m(g,n-1)} \sum_{j \neq i} \frac{1}{2}(H(x, L_i + L_j) + H(x, L_1 - L_j)) \cdot V_{g,n-1}(x, \widehat{L_{i,j}}),$$

where $\widehat{L_{i,j}}$ is just $\boldsymbol{L}$ without the i-th and j-th components.

2.4 Recursion Formula Proof

Earlier computations of $\mathrm{Vol}_{1,1}(L)$ and $\mathrm{Vol}_{0,4}(\boldsymbol{L})$ carried out via a three-step process showed that one way to compute the Weil-Petersson volume is to:

1. Generalise the McShane identity to the general case, which has been covered.
2. Figure out how, in the pullback process, each term in the infinite sum given by the Generalised McShane Identity is distributed to a distinct fundamental domain, first to the intermediate covering space $\mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma$, and then to the Teichmüller space.
3. Explicitly evaluate the resulting integral over the Teichmúller space. We'll do this via the Mirzakhani recursion formula.

2.4.1 Integration Over The Moduli Space

To work out how the Generalised McShane Identity function pulls back, Mirzakhani solves the related question of how we might pull back and distribute any moduli space space functions written in the form given by equation (2.2).

Definition 2.4.1. A **multicurve** $\gamma = \sum_{j=1}^{k} c_j \gamma_j,\ c_j \geq 0$ is a formal finite sum of disjoint, non-peripheral, simple closed curves on $S_{g,n}$.

Definition 2.4.2. Let $\Gamma = \{\gamma_i\}_{1 \leq i \leq k}$ be a pants decomposition for $S_{g,n}$. Given a multicurve $\gamma = \sum_{j=1}^{k} c_j \gamma_j$, we call Γ an **associated pants decomposition** of γ.

Note 2.4.1. In general, associated pants decompositions are not unique. For example: $\Gamma = \{\eta_1, \eta_2\}$ and $\tilde{\Gamma} = \{\tilde{\eta}_1, \tilde{\eta}_2\}$, where $\tilde{\eta}_1 = \eta_2$ and $\tilde{\eta}_2 = \eta_1$, are both hyperbolic pants decompositions for the multicurve $\gamma = 3\eta_1 + 3\eta_2 = 3\tilde{\eta}_2 + 3\tilde{\eta}_1$.

Observe that any isotopy class of self-homeomorphisms $h \in \mathrm{Mod}(S_{g,n})$ has a natural action on a multicurve γ given by:

$$h \cdot \gamma = \sum_{j=1}^{k} c_j \tilde{\gamma}_j,$$

where $\tilde{\gamma}_j$ is the shortest geodesic representative of the isotopy class $h(\gamma_j)$. Similarly, we define the action of h on an associated pants decomposition Γ to be:

$$h \cdot \Gamma = \{\eta_1, \ldots, \eta_k\},$$

where each η_j is the shortest geodesic representative of the isotopy class $h(\gamma_j)$.

Definition 2.4.3. Let $\mathcal{O}_\Gamma$ denote the set of homotopy classes of elements of $\mathrm{Mod}\cdot\Gamma$, then we define the following space:

$$\mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma = \{(X,\eta) : X \in \mathcal{M}_{g,n}(\boldsymbol{L}),\ [\eta] \in \mathcal{O}_\Gamma\},$$

where $\eta = (\eta_1, \ldots, \eta_k) = h \cdot \Gamma$ is the unique hyperbolic spants cover representing $[\eta] \in \mathcal{O}_\Gamma$. We call $\mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma$ the **Γ-pullback moduli space**.

Now consider a multicurve $\alpha = \sum_{i=1}^k c_i\gamma_i$ on $S_{g,n}$, with associated pants decomposition Γ as in the definition 2.4.2. For a marked hyperbolic surface surface (X, f) with $f : S_{g,n} \to X$, let

$$\ell_\alpha = \sum_{i=1}^k c_i \ell_{\gamma_i}(X),$$

where $\ell_{\gamma_i}(X)$ is the minimum length of any geodesic representative of the homology class of $f(\gamma_i)$ on X. Then, as long as the sum converges, any continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ induces a well-defined continuous function $f_\alpha : \mathcal{M}_{g,n}(\boldsymbol{L}) \to \mathbb{R}_+$ given by:

$$f_\alpha(X) = \sum_{[\tilde{\alpha}] \in \mathrm{Mod}\cdot[\alpha]} f(\ell_{\tilde{\alpha}}(X)). \tag{2.2}$$

Note 2.4.2. Although summands in the Generalised McShane Identity are seldom of this form, we will see in the next subsection that they can always be partitioned into sums which do take this form. Therefore, if we know how to integrate (2.2), we'll be able to pull back the integral of the Generalised McShane Identity to the Teichmüller space.

We now attempt to integrate a function f_γ of the specified form. The basic strategy we adopt is similar to that demonstrated for calculating the volumes of $\mathcal{M}_{1,1}(L)$ and $\mathcal{M}_{0.4}(\boldsymbol{L})$. We first pull back the volume form from $\mathcal{M}_{g,n}(\boldsymbol{L})$ to a intermediate space with symplectic structure, and pull back once more to $\mathcal{T}_{g,n}(\boldsymbol{L})$ equipped with Fenchel-Nielsen coordinates. We choose the Γ-pullback moduli space to be our desired intermediate space. But first, let us confirm that the symplectic form can be pulled back repeatedly via $\mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma$.

We know that:

- $\mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma$ is a covering space for $\mathcal{M}_{g,n}(\boldsymbol{L})$, with the covering map given by:

$$\pi^\Gamma : \mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma \to \mathcal{M}_{g,n}(\boldsymbol{L}),\ \pi^\Gamma(X,\eta) = X, \text{ and}$$

- $\mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma$ is also the quotient space of $\mathcal{T}_{g,n}(\boldsymbol{L})$ with respect to the group $G^\Gamma \subset \mathrm{Mod}(S_{g,n})$, of all mapping class group elements which fix each γ_i, and hence each η_i. That is, given:

$$G_\Gamma = \bigcap_{i=1}^k \mathrm{Stab}(\gamma_i),\ \text{then } \mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma \cong \mathcal{T}_{g,n}(\boldsymbol{L})/G_\Gamma.$$

Since the symplectic structure on the Teichmüller is invariant under the action of the mapping class group (and $G_\Gamma \subset \mathrm{Mod}_{g,n}$), this quotient induces the desired symplectic structure on $\mathcal{M}_{g,n}(\boldsymbol{L})$.

We commence with the integration:

$$\begin{aligned}\int_{\mathcal{M}_{g,n}(\boldsymbol{L})} f_\alpha(X)dX &= \int_{\mathcal{M}_{g,n}(\boldsymbol{L})} \sum_{[\tilde{\alpha}]\in \mathrm{Mod}\cdot[\alpha]} f(\ell_{\tilde{\alpha}}(X))dX \\ &= \int_{\mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma} \frac{1}{|\mathrm{Sym}(\tilde{\alpha})|} f(\ell_{\tilde{\alpha}}(X))d(X,\eta) \\ &= \frac{1}{|\mathrm{Sym}(\alpha)|} \int_{\mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma} f(\ell_{\tilde{\alpha}}(X))d(X,\eta).\end{aligned}$$

We divide by $|\mathrm{Sym}(\tilde{\alpha})|$ because the multicurve α often corresponds to more than one hyperbolic pants decomposition. Pulling out $\frac{1}{|\mathrm{Sym}(\alpha)|}$ is possible, since $|\mathrm{Sym}(\tilde{\alpha})|$ only depends on the coefficients - $\{c_j\}$.

In order to pullback further to the Teichmüller space, we would like to associate to each point $(X,\eta) \in \mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma$ a point in $\mathcal{T}_{g,n}(\boldsymbol{L})$ with symplectic Fenchel-Nielsen coordinates. That is: we would like to associate with (X,η) length parameters and twisting parameters. But recall that $f_\alpha(X)$ purely dependent on the length parameters, thus we consider re-expressing the above integral as:

$$\begin{aligned}\frac{1}{|\mathrm{Sym}(\alpha)|}\int_{\mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma} f(\ell_{\tilde{\alpha}}(X))d(X,\eta) &= \frac{1}{|\mathrm{Sym}(\alpha)|}\int_{\boldsymbol{a}\in\mathbb{R}^k_+}\int_{\mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma[\boldsymbol{a}]} f(|\boldsymbol{a}|)\ \nu[\boldsymbol{a}]\ d\boldsymbol{a}, \\ &= \frac{1}{|\mathrm{Sym}(\alpha)|}\int_{\boldsymbol{a}\in\mathbb{R}^k_+} f(|\boldsymbol{a}|)\int_{\mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma[\boldsymbol{a}]} \nu[\boldsymbol{a}]\ d\boldsymbol{a}\end{aligned}$$

where the inner integral is over $\mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma[\boldsymbol{a}]$: set of all points $(X,\eta) \in \mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma$ so that:

$$(\ell_{\eta_1}(X),\dots,\ell_{\eta_k}(X)) = \boldsymbol{a} \in \mathbb{R}^k_+,$$

and $\nu[\boldsymbol{a}]$ is the induced volume form on $\mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma$.

It is unclear to the author how one might rigorously demonstrate the existence of $\nu[\boldsymbol{a}]$, nonetheless, we now consider the integral

$$\int_{\mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma[\boldsymbol{a}]} \nu[\boldsymbol{a}].$$

Intuitively, we might expect this to evaluate to:

$$\int_0^{a_1}\dots\int_0^{a_k} d\boldsymbol{\tau} = a_1\dots a_k,$$

because we might expect that there is a 1 to 1 correspondence btween $\mathcal{M}_{g,n}(\boldsymbol{L})^\Gamma[\boldsymbol{a}]$ and the set

$$\{(a_1,\dots,a_k,\tau_1,\dots,\tau_k) \in \mathbb{R}^k_+ \times \mathbb{R}^k : 0 \le \tau_j \le a_j\}.$$

The rationale being that:

1. for each (X,η), we can extract its τ-twist component[6]. The Dehn twist component is ignored as Dehn twists are representative of elements in the mapping class group;
2. for each vector $\tau = (\tau_1,\dots,\tau_k)$, we can produce a distinct point from (X,η) by performing a sequence of $\frac{\tau_1}{2\pi}a_i$-twists on η_i.

However, this 1 to 1 correspondence is definitely not true in general. For example, in $\mathrm{Mod}_{1,1}(L)$, there is an order 2 rotation along the boundary as described in example 1.6.2. In this case, we have a 1 to 2 correspondence, because any τ-twist, where $\tau > \pi$ will give us a pre-existing point in the moduli space. In fact, for each genus, this correspondence doubles. Hence, we need to introduce an extra factor of 2^{-g}.

It may appear as though we've taken care of everything but often, certain points in the moduli space have extra (rotational) symmetry which are not shared by other points: take for example, a 2-torus with 1 disk removed whose pants decomposition geodesics are all of the same length. Then, when integrating, this 2-torus will be worth 'less' than all other points. Fortunately, the collection of such points (X, η) in $\mathcal{M}_{g,n}(\boldsymbol{L})^{\Gamma}[\boldsymbol{a}]$ is of measure zero.

Finally, by putting everything together, we obtain the following result:

Lemma 2.4.1. *Given a multicurve $\alpha = \sum_{i=1}^{k} c_i\gamma_i$ on $S_{g,n}$ with associated pants cover $\Gamma = (\gamma_1, \ldots, \gamma_k)$, and $|\boldsymbol{x}| = \sum_{i=1}^{k} c_i x_i$. Then, the integral of f_α over $\mathcal{M}_{g,n}(\boldsymbol{L})$, with respect to the Weil-Petersson volume form, is:*

$$\int_{\mathcal{M}_{g,n}(\boldsymbol{L})} f_\alpha(X)dX = \frac{2^{-M(\Gamma)}}{|\mathrm{Sym}(\alpha)|}\int_{\boldsymbol{x}\in\mathbb{R}_+^k} f(|\boldsymbol{x}|)\boldsymbol{x}\cdot d\boldsymbol{x},$$

where

$$M(\Gamma) = |\{i : \gamma_i \text{ separates off a one-handle from } S_{g,n}\}|,$$

and $\boldsymbol{x}\cdot d\boldsymbol{x} = x_1 \ldots x_k \cdot dx_1 \wedge \ldots \wedge dx_k$.

Note 2.4.3. We choose to use the notation $2^{-M(\Gamma)}$ instead of 2^{-g} to emphasise the relationship between this lemma and the following theorem.

In sketching the proof of the above lemma, we associated a canonical pants decomposition Γ to α. However, all arguments extend to the case, where, given a multicurve $\alpha = \sum_{i=1}^{k} c_i\gamma_i$, we associate to it $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$, and Γ isn't a pants decomposition. Doing so yields the following variant of the above lemma:

Theorem 2.4.2. *Given a multicurve $\alpha = \sum_{i=1}^{k} c_i\gamma_i$ on $S_{g,n}$, let: $\Gamma = (\gamma_1, \ldots, \gamma_k)$, $|\boldsymbol{x}| = |(x_1, \ldots, x_k)| = \sum_{i=1}^{k} c_i x_i$ and $\boldsymbol{x}\cdot d\boldsymbol{x} = x_1 \ldots x_k \cdot dx_1 \wedge \ldots \wedge dx_k$. Then, the integral of f_α over $\mathcal{M}_{g,n}(\boldsymbol{L})$, with respect to the Weil-Petersson volume form, is:*

$$\int_{\mathcal{M}_{g,n}(\boldsymbol{L})} f_\alpha(X)dX = \frac{2^{-M(\Gamma)}}{|\mathrm{Sym}(\alpha)|}\int_{\boldsymbol{x}\in\mathbb{R}_+^k} f(|\boldsymbol{x}|)V_{g,n}(\Gamma, \boldsymbol{x}, \beta, \boldsymbol{L})\ \boldsymbol{x}\cdot d\boldsymbol{x},$$

where $|\mathrm{Sym}(\alpha)|$ is the cardinality of the group of symmetries of α, and

$$M(\Gamma) = |\{i : \gamma_i \text{ separates off a one-handle from } S_{g,n}\}|,$$

and $V_{g,n}(\Gamma, \boldsymbol{x}, \beta, \boldsymbol{L})$ is defined as: $\mathrm{Vol}(\mathcal{M}(S_{g,n}(\Gamma), \ell_\Gamma = \boldsymbol{x}, \ell_\beta = \boldsymbol{L}))$, *where $S_{g,n}(\Gamma)$ is the surface resulting from cutting $S_{g,n}$ along Γ.*

Note 2.4.4. Notice that if Γ is a hyperbolic pants cover, then $V_{g,n}(\Gamma, \boldsymbol{x}, \beta, \boldsymbol{L}) = 1$, and we get back lemma 2.4.1.

---

[6] The *tau*-twist components must first be denormalised by dividing by 2π and multiplying by $\ell(\eta_i)$ so that they correspond to the *symplectic* Fenchel-Nielsen coordinates.

2.4.2 Proof of the Recursion Formula

We will now prove the Mirzakhani recursion formula:

Theorem 2.4.3. *For $(g, n) \neq (0, 3), (1, 1)$,*

$$\frac{\partial}{\partial L_i}(L_i \cdot V_{g,n}(\boldsymbol{L})) = \mathcal{A}^{con}_{g,n}(L_i, \widehat{L}_i) + \mathcal{A}^{dcon}_{g,n}(L_i, \widehat{L}_i) + \mathcal{B}_{g,n}(L_i, \widehat{L}_i).$$

Proof.

$$\text{Let } \tilde{\mathcal{R}}_j(X) = \sum_{\gamma \in \mathcal{F}_{1,j}} \mathcal{R}(L_1, L_j, \ell_\gamma(X)), \text{ and } \tilde{\mathcal{D}}(X) = \sum_{\{\gamma_1,\gamma_2\} \in \mathcal{F}_1} \mathcal{D}(L_1, \ell_{\gamma_1}(X), \ell_{\gamma_2}(X)),$$

where $\mathcal{F}_{1,j}$ is the collection of isotopy classes of simple closed geodesics on $[X] \in \mathcal{M}_{g,n}(\boldsymbol{L})$ which, together with the boundaries β_1 and β_j, bounds a hyperbolic pair of pants on (some) X. Similarly, $\mathcal{F}_1$ is the set of **unordered** pairs of isotopy classes of simple closed non-boundary geodesics on $[X]$ which, together with β_1 bound a hyperbolic pair of pants embedded in (some possibly other) X. Then the Generalised McShane identity can be expressed as:

$$L_1 = \tilde{\mathcal{D}}(X) + \sum_{j=2}^{n} \tilde{\mathcal{R}}_j(X).$$

Integrating both sides over $\mathcal{M}_{g,n}(\boldsymbol{L})$, we obtain:

$$L_1 \cdot \mathrm{Vol}(\mathcal{M}_{g,n}(\boldsymbol{L})) = \int_{\mathcal{M}_{g,n}(\boldsymbol{L})} \tilde{\mathcal{D}}(X)\ dX + \sum_{j=2}^{n} \int_{\mathcal{M}_{g,n}(\boldsymbol{L})} \tilde{\mathcal{R}}_j(X)\ dX.$$

We now evaluate these 'two' integrals, relating each to the $\mathcal{A}^{con}_{g,n}(.)$, $\mathcal{A}^{dcon}_{g,n}(.)$ and $\mathcal{B}_{g,n}(.)$ terms in the recursion.

$\sum_{j=2}^{n} \tilde{\mathcal{R}}_j(X)$ Case

First observe that for any geodesic representative γ_a of $[\gamma_a] \in \mathcal{F}_{1,j}$, by 'cutting along' γ_a in X and then regluing copies of γ_a to the two resulting boundaries (much like cutting a surface along pants decompositions), we obtain two disjoint pieces of X: one a hyperbolic pair of pants with boundaries β_1, β_j and γ_a.

Similarly, for any other γ_b of **the same length** as γ_a, representing any other $[\gamma_b] \in \mathcal{F}_{1,j}$, if we decompose the hyperbolic surface Y is on, then we obtain two pieces of Y: one a hyperbolic pair of pants isometric to the one formed by cutting along X. It can be shown that the two left-over regions, are surfaces with the same genus and number of boundary components. Therefore, they are homeomorphic, and hence diffeomorphic. Then, by aligning hyperbolic pants decompositions of these two remaining surfaces using this diffeomorphism, and then readjusting Y by changing the lengths of these pants decompositions (and twisting too). We obtain a new surface $\tilde{Y}$, so that Y is isometric to X, but with γ_a sent to a geodesic with isotopy class $[\gamma_b]$. Therefore, $\mathrm{Mod}_{g,n}$ acts transitively on $\mathcal{F}_{1,j}$.

Observe then, that:

$$\tilde{\mathcal{R}}_j(X) = \sum_{[\gamma]\in \mathrm{Mod}_{g,n}[\gamma_a]} \mathcal{R}(L_1, L_j, \ell_\gamma(X))$$

satisfies the conditions for theorem 2.4.2, and so we have:

$$\begin{aligned}&\int_{\mathcal{M}_{g,n}(\boldsymbol{L})} \tilde{\mathcal{R}}_j(X)\ d[X]\\ &= 2^{-m(g,n-1)} \int_0^\infty x \cdot \mathcal{R}(L_1, L_j, x) \cdot \mathrm{Vol}(\mathcal{M}(S_{g,n}(\gamma_j), \ell_{\gamma_j} = x, \boldsymbol{L}))\ dx\\ &= 2^{-m(g,n-1)} \int_0^\infty x \cdot \mathcal{R}(L_1, L_j, x) \cdot V_{g,n-1}(x, \widehat{L_{1,j}})\ dx,\end{aligned}$$

because $|\mathrm{Sym}(\gamma_j)| = 1$, and $S_{g,n}(\gamma_j)$ (that is: $S_{g,n}$ cut along γ_j) is symplectomorphic to $S_{g,n-1} \times S_{0,3}$, but the Weil-Petersson volume of $S_{0,3}$ is just 1. Then, taking a partial derivative with respect to L_1, we have:

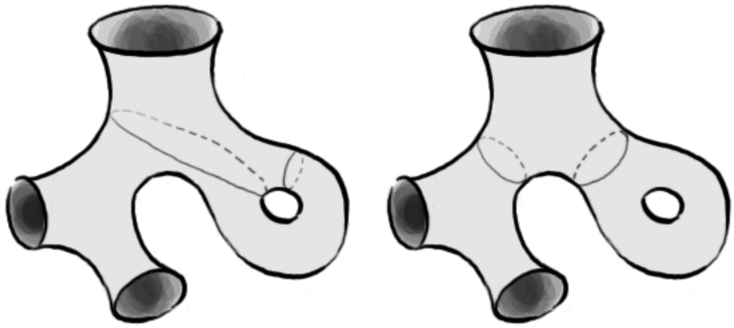
$$\begin{aligned}&\frac{\partial}{\partial L_1} \int_{\mathcal{M}_{g,n}(\boldsymbol{L})} \tilde{\mathcal{R}}_j(X)\ d[X]\\ &= \frac{1}{2 \cdot 2^{m(g,n-1)}} \int_0^\infty x(H(x, L_1 - L_j) + H(x, L_1 + L_j)) \cdot V_{g,n}(x, \widehat{L_{1,j}})\ dx.\end{aligned}$$

Then, we take the sum of each integral over j, to obtain:

$$\sum_{j=2}^{n} \frac{\partial}{\partial L_1} \int_{\mathcal{M}_{g,n}(\boldsymbol{L})} \tilde{\mathcal{R}}_j(X)\ d[X] = \mathcal{B}_{g,n}(\boldsymbol{L}).$$

2.4.3 $\tilde{\mathcal{D}}(X)$ Case

Ideally, we would like to be able to do the same trick as before: use the transitivity of $\mathrm{Mod}_{g,n}$ to express $\tilde{\mathcal{D}}$ as a sum over $\mathrm{Mod}_{g,n}[\gamma]$, where $\gamma = \gamma_1 + \gamma_2$ is a multicurve cobordering a pair of pants with β_1. However, $\mathrm{Mod}_{g,n}$ is, in general, not transitive over $\mathcal{F}_1$. Consider the following figure depicting two copies of $S_{1,3}$ which will have non-homeomorphic remnants once pairs of pants respectively bounded by $\{\beta_1, \alpha_1, \alpha_2\}$ and $\{\beta_1, \gamma_1, \gamma_2\}$ are removed. Mod(S) cannot act transitively over a collection of geodesics which don't give homeomorphic decompositions. Therefore, we first try to partition $\mathcal{F}_1$ according to the homeomorphism classes of the remnants.



Connected Remnant Case: if the removal of the pair of pants bounded by $\{\beta_1, \alpha_1, \alpha_2\}$ is still connected, then, we can can show, via similar arguments as for the transitivity of $\mathrm{Mod}_{g,n}$ on $\mathcal{F}_{1,j}$, that $\mathrm{Mod}_{g,n}$ acts transitively on all unordered pairs $\{\alpha_1, \alpha_2\}$ whose bounded pants, when removed, will leave a connected remnant.

Therefore, let A^{con} denote the set of isotopy classes of multicurves $\alpha = \alpha_1 + \alpha_2$ satisfying this connected remnant condition. Then, since $\mathcal{D}(L_1, \ell_{\alpha_1}(X), \ell_{\alpha_2}(X)) = \mathcal{D}(L_1, \ell_{\alpha_1}(X) + \ell_{\alpha_2}(X), 0)$, the function

$$\mathcal{D}^{con}([X]) = \sum_{[\alpha]\in A^{con}} \mathcal{D}(L_1, \ell_\alpha(X) = \ell_{\alpha_1}(X) + \ell_{\alpha_2}(X), 0)$$

is a sum which pulls back 'nicely', as shown by theorem 2.4.2, and tells us that:

$$\int_{\mathcal{M}_{g,n}(\boldsymbol{L})} \mathcal{D}^{con}([X])\ d[X]$$
$$= \frac{2^{-m(g-1,n+1)}}{|\mathrm{Sym}(\alpha)|} \int_0^\infty \int_0^\infty \mathcal{D}(L_1, x+y, 0)\mathrm{Vol}_{g-1,n+1}(\mathcal{M}(S_{g,n}(\alpha)), \ell_\alpha = (x,y), \boldsymbol{L})\ dx\ dy.$$

Then, taking a partial derivative with respect to L_1, we obtain:

$$\frac{\partial}{\partial L_1} \int_{\mathcal{M}_{g,n}(\boldsymbol{L})} \mathcal{D}^{con}([X])\ d[X]$$
$$= \frac{1}{2} \int_0^\infty \int_0^\infty 2^{-m(g-1,n+1)} H(x+y, L_1) \cdot V_{g-1,n+1}(x, y, \widehat{L_1})\ dx\ dy$$
$$= \mathcal{A}^{con}_{g,n}(\boldsymbol{L} = (L_1, \widehat{L_1})).$$

All that remains now, is the disconnected case.

Disconnected Remnant Case: Now considered unordered pairs $\{\gamma_1, \gamma_2\}$, $\{\delta_1, \delta_2\}$ bounding hyperbolic pants (along with β_1), so that when the pants are excised, the remnant is disconnected. Observe that if the remnants aren't homeomorphic, then the isotopy classes of the multicurves $\gamma_1 + \gamma_2$ and $\delta_1 + \delta_2$ are not related by any element of the mapping class group. On the other hand, we can use similar arguments as before to show that $\mathrm{Mod}_{g,n}$ acts transitively on all unordered pairs of geodesics yielding homeomorphic remnants. Of course, we can just classify the remnants by the number of genera and boundaries that they have. Therefore, we can partition $A^{dcon} = \mathcal{F}_1 - A^{con}$ into:

$$\bigcup_{a\in\tilde{\mathcal{I}}_{g,n}} A_a,$$

where, each element $a = \{(g_1, I), (g_2, J)\} \in \tilde{\mathfrak{I}}_{g,n}$ is the collection of all **unordered** pairs, where $\{(g_1, I), (g_2, J)\}$ tells us that one of the two connected components of the remnant has g_1 genera and contains the boundaries $\{\beta_i\}_{i \in I}$, and the other contains g_2 genera and the boundaries $\{\beta_i\}_{i \in J}$.

Immediately observe that we need the following:

- Since the pair of pants removed separated the surface, it could not have been a genus reducing pants excision. Therefore:

$$g_1 + g_2 = g, \ g_1, g_2 \geq 0$$

- connect component is a bordered hyperbolic surface and, by the Gauss Bonnet formula, must have negative Euler characteristic. Equivalently:

$$2g_1 + |I| + 1, \ 2g_2 + |J| + 1 \geq 3,$$

  where $|, |$ denotes cardinality. This extra $+1$ comes from the extra boundary given to each piece with the pants excision.

- $$I \cap J = \emptyset, \ I \cup J = \{2, \dots, n\},$$

  because the two sets partition the collection of boundaries.

In addition, given an unordered pair $\{(g_1, I), (g_2, J)\}$ satisfying the above conditions, there always exists a pants excision leaving two connected components with genera and boundary components corresponding to the pair. Now, it can be see that for any γ_a representing $[\gamma_a] \in A_a$, $|\mathrm{Sym}(\gamma_a)| = 1$, in general, because the two disjoint remnants aren't homeomorphic. However, for the case when $(g_1, I) = (g_2, J)$, we have $|\mathrm{Sym}(\gamma)| = 2$. Bearing this in mind, we can use theorem 2.4.2 to obtain something similar to the desired result. The difference lies in the fact that have taken unordered pairs $\{(g_1, I), (g_2, J)\}$, whereas Mirzakhani's recursions are for ordered pairs. Therefore, by breaking up terms where $(g_1, I) \neq (g_2, J)$, into two halves, we can pull out the necessary half out the front, as well as as sum over ordered pairs, thereby attaining the $\mathcal{A}^{dcon}_{g,n}(\boldsymbol{L})$ term desired.

$\square$

2.5 The Weil-Petersson Volume

As the last part of this thesis, we will show that the Weil-Petersson volume of the moduli space of any bordered Riemann surface is a always a polynomial in the squares of the lengths of the boundary components. We already know this to be true in the $\mathcal{M}_{1,1}(L)$ and $\mathcal{M}_{0,4}(\boldsymbol{L})$. Moreover, the Weil-Petersson volumes in appendix C.1 are also polynomials of squared lengths. In fact, we have the following result:

Theorem 2.5.1. *$V_{g,n}(\boldsymbol{L}) = \mathrm{Vol}(\mathcal{M}_{g,n}(\boldsymbol{L}))$ is a polynomial in $L_1^2, \dots, L_n^2$, such that*

$$V_{g,n}(\boldsymbol{L}) = \sum_{\alpha = (\alpha_1, \dots, \alpha_n)} C_\alpha \cdot \prod_{i=1}^{n} L_i^{2\alpha_i},$$

where $\alpha_i \in \mathbb{N}$, $|\alpha| = \Sigma \alpha_i \leq 3g + n - 3$, and $C_\alpha \in \pi^{6g+2n-6-2|\alpha|}\mathbb{Q}^+$.

We aim to prove this result by induction: we will take the Mirzakhani recursion formula, and show that each of the three terms $\mathcal{A}^{con}_{g,n}(\boldsymbol{L})$, $\mathcal{A}^{dcon}_{g,n}(\boldsymbol{L})$ and $\mathcal{B}_{g,n}(\boldsymbol{L})$ reduces to integrations over other polynomials which end up giving us a polynomial in the desired form; this observation implies the desired result. This theorem and argument will be restated later with necessary clarifications and qualifications, however, first, we need a few results in calculus and algebra.

Definition 2.5.1. For $i \in \mathbb{N}$, define $F_{2i+1} : \mathbb{R}_+ \to \mathbb{R}_+$by:

$$F_{2i+1}(t) = \int_0^\infty x^{2i+1} \cdot H(x,t)\ dx, \text{ where } H(x,t) = \frac{1}{1+e^{x+t}} + \frac{1}{1+e^{x-t}}.$$

Lemma 2.5.2. *For $m, n \in \mathbb{N}_0$,*

$$\int_0^T y^m (T-y)^n dy = \frac{m!n!}{(m+n+1)!} T^{m+n+1}.$$

Proof. Observe that for $n = 0$, this is true for all $m \in \mathbb{N}_0$. Therefore, we will prove this lemma by induction on n. Assume that for n the above lemma is true for all $m \in \mathbb{N}_0$, consider $n+1$:

$$\begin{aligned}\int_0^T y^m (T-Y)^{n+1} dy &= T\int_0^T y^m (T-Y)^n dy - \int_0^T y^{m+1}(T-Y)^n dy \\ &= \frac{m!n!}{(m+n+1)!} T^{m+n+1} - \frac{(m+1)!n!}{(m+n+2)!} T^{m+n+2} \\ &= \frac{m!(n+1)!}{(m+n+2)!} T^{m+n+2},\end{aligned}$$

as desired. □

Corollary 2.5.3. *For any $i, j \in \mathbb{N}_0$,*

$$\int_0^\infty \int_0^\infty x^{2i+1} y^{2j+1} H(x+y,t) dx\ dy = \frac{(2i+1)!(2j+1)!}{(2i+2j+3)!} F_{2i+2j+3}(t)$$

Proof. Let $Z = x + y$, then

$$\begin{aligned}\int_0^\infty \int_0^\infty x^{2i+1} y^{2j+1} H(x+y,t) dx\ dy &= \int_0^\infty \int_0^Z (Z-y)^{2i+1} y^{2j+1} H(Z,t) dy\ dZ \\ &= \frac{(2i+1)!(2j+1)!}{(2i+2j+3)!} \int_0^\infty Z^{2i+2j+3} H(Z,t)\ dZ, \text{ as desired.}\end{aligned}$$

□

Definition 2.5.2. The **Bernoulli numbers**, denote by B_n, are rational numbers given by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^\infty \mathrm{B}_n \frac{x^n}{n!}.$$

Definition 2.5.3. Given a complex number $z \in \mathbb{C}$ such that the real part of z is strictly greater than 1. Then the sum

$$\tilde{\zeta}(z) = \sum_{n=1}^\infty \frac{1}{n^s}$$

converges is the function is analytic where defined. It is possible to analytically continue $\tilde{\zeta}$ to $\mathbb{C} - \{1\}$ and the resulting function: $\zeta : \mathbb{C} - \{1\} \to \mathbb{C}$, is called the **Riemann zeta-function**, or simply as the **zeta-function**. In particular, $\zeta(0) = -\frac{1}{2}$.

Lemma 2.5.4. *Let $\zeta(z)$ denote the Riemann zeta-function, then for any positive integer $i \in \mathbb{N}$,*

$$\zeta(2i) = \frac{1}{(2i-1)!(1-2^{-2i+1})}\int_0^\infty \frac{x^{2i-1}}{1+e^x}dx = \frac{2^{2i-1}|\mathrm{B}_{2i}|\pi^{2i}}{(2i)!}.$$

Lemma 2.5.5. *Given $k \geq 0$, then*

$$\frac{F_{2k+1}(t)}{(2k+1)!} = \sum_{i=0}^{k+1}\zeta(2i)(2^{2i+1}-4)\frac{t^{2k+2-2i}}{(2k+2-2i)!}.$$

Proof.

$$\begin{aligned}
&\frac{F_{2k+1}(t)}{(2k+1)!}\\
&= \frac{1}{(2k+1)!}\int_0^\infty x^{2k+1}\left(\frac{1}{1+e^{\frac{x+t}{2}}} + \frac{1}{1+e^{\frac{x-t}{2}}}\right)dx\\
&= \frac{1}{(2k+1)!}\left[\int_{t/2}^\infty \frac{(2y-t)^{2k+1}}{1+e^y}\cdot 2dy + \int_{-t/2}^\infty \frac{(2y+t)^{2k+1}}{1+e^y}\cdot 2dy\right]\\
&= \frac{2}{(2k+1)!}\left[\int_0^\infty \frac{(2y-t)^{2k+1}+(2y+t)^{2k+1}}{1+e^y}dy\right.\\
&\qquad\qquad\left. - \int_0^{t/2}\frac{(2y-t)^{2k+1}}{1+e^y}dy + \int_{-t/2}^0 \frac{(2y+t)^{2k+1}}{1+e^y}dy\right].
\end{aligned}$$

Then, by binomial expanding $(2y \pm t)^{2k+1}$ and cancelling the odd terms, we have:

$$\begin{aligned}
&= \frac{4}{(2k+1)!}\sum_{j=1}^{k+1} t^{2(k-j+1)}\cdot\binom{2k+1}{2j-1}\cdot\int_0^\infty \frac{(2y)^{2j-1}}{1+e^y}dy - \frac{2}{(2k+1)!}\int_0^{t/2}(2y-t)^{2k+1}dy\\
&= \frac{4}{(2k+1)!}\sum_{j=1}^{k+1} 2^{2j-1}t^{2(k-j+1)}\cdot\binom{2k+1}{2j-1}\cdot\int_0^\infty \frac{y^{2j-1}}{1+e^y}dy + \frac{t^{2k+2}}{(2k+2)!}\\
&= \frac{4}{(2k+1)!}\sum_{j=1}^{k+1} 2^{2j-1}t^{2(k-j+1)}\cdot\binom{2k+1}{2j-1}\cdot(2j-1)!(1-2^{-(2j-1)})\zeta(2j) + \frac{t^{2k+2}}{(2k+2)!}\\
&= \sum_{j=1}^{k+1}\frac{t^{2k+2-2j}}{(2k+1-(2j-1))!}\zeta(2j)(2^{2j+1}-4) + \frac{t^{2k+2-2\times 0}}{(2k+2-2\times 0)!}\zeta(0)(2^{2\times 0+1}-4)\\
&= \sum_{i=0}^{k+1}\zeta(2i)(2^{2i+1}-4)\frac{t^{2k+2-2i}}{(2k+2-2i)!}.
\end{aligned}$$

□

This is the key lemma we needed to prove theorem 2.5.1, which we will state again for reference:

Theorem 2.5.1: $V_{g,n}(\boldsymbol{L}) = \mathrm{Vol}(\mathcal{M}_{g,n}(\boldsymbol{L}))$ *is a polynomial in* $L_1^2, \ldots, L_n^2$, *such that*

$$V_{g,n}(\boldsymbol{L}) = \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)} C_\alpha \cdot \prod_{i=1}^n L_i^{2\alpha_i},$$

where $\alpha_i \in \mathbb{N}$, $|\alpha| = \Sigma\alpha_i \leq 3g+n-3$, *and* $C_\alpha \in \pi^{6g+2n-6-2|\alpha|}\mathbb{Q}^+$.

Proof. Observe that if $\mathcal{A}^{con}_{g,n}(\boldsymbol{L})$, $\mathcal{A}^{dcon}_{g,n}(\boldsymbol{L})$ and $\mathcal{B}_{g,n}(\boldsymbol{L})$ are all polynomials in $\{L_i^2\}$, then by the mirzakhani recursion formula,

$$L_1 V_{g,n}(\boldsymbol{L}) = \int \mathcal{A}^{con}_{g,n}(\boldsymbol{L}) + \mathcal{A}^{dcon}_{g,n}(\boldsymbol{L}) + \mathcal{B}_{g,n}(\boldsymbol{L})\ dL_1$$

is a polynomial in $\{L_i^2\}_{i\neq 1}$ and positive odd powers of L_1. When divided through by L_1, we conclude that $V_{g,n}$ must be a polynomial in $\{L_i^2\}$. We would like, then, to show that $\mathcal{A}^{con}_{g,n}(\boldsymbol{L})$, $\mathcal{A}^{dcon}_{g,n}(\boldsymbol{L})$ and $\mathcal{B}_{g,n}(\boldsymbol{L})$ are all polynomials in $\{L_i^2\}$. We will prove this by induction on $3g+n$, simultaneously demonstrating that the coefficients C_α lie in $\pi^{6g+2n-6-2|\alpha|}\cdot\mathbb{Q}_+$.

For $3g+n=4$, I.E.: $(g,n) = (1,1),(0,4)$, we know the result to be true. Thus, assume that up to some k, the theorem is true for all (g',n') such that $3g'+n' \leq k$. Then, for (g,n) such that $3g+n = k+1$, we know from definition that:

$$\begin{aligned}
\mathcal{A}^{con}_{g,n}(\boldsymbol{L} = L_1,\widehat{L_1}) &= \frac{1}{2}\int_o^\infty \int_0^\infty x\ y\ \widehat{\mathcal{A}}^{con}_{g,n}(x,y,L_1,\widehat{L_1})\ dx\ dy,\\
\mathcal{A}^{dcon}_{g,n}(\boldsymbol{L}) &= \frac{1}{2}\int_o^\infty \int_0^\infty x\ y\ \widehat{\mathcal{A}}^{dcon}_{g,n}(x,y,L_1,\widehat{L_1})\ dx\ dy,\\
\mathcal{B}_{g,n}(\boldsymbol{L}) &= \int_0^\infty x\ \widehat{\mathcal{B}}_{g,n}(x,L_1,\widehat{L_1})\ dx,
\end{aligned}$$

where,

$$\begin{aligned}
\widehat{\mathcal{A}}^{con}_{g,n}(x,y,L_1,\widehat{L_1}) &= 2^{-m(g-1,n+1)}V_{g-1,n+1}(x,y,\widehat{L_1})\cdot H(x+y,L_1)\\
\widehat{\mathcal{A}}^{dcon}_{g,n}(x,y,L_1,\widehat{L_1}) &= \sum_{a\in\mathcal{I}_{g,n}} V(a,x,y,\widehat{L_1})\cdot H(x+y,L_1)\\
\widehat{\mathcal{B}}_{g,n}(x,L_1,\widehat{L_1}) &= 2^{-m(g,n-1)}\sum_{j\neq 1}\frac{1}{2}(H(x,L_1+L_j)+H(x,L_1-L_j))\cdot V_{g,n-1}(x,\widehat{L_{1,j}})
\end{aligned}$$

each only has terms which are products of monomials in $\{L_i^2\}$ and H terms in x, y and L_1. In addition, the coefficients of each such term lies in $\pi^{6g+2n-6-2m}\mathbb{Q}_+$ for m specified as per theorem 2.5.1. To simplify notation, for the remainder of this proof, we shall ignore all positive rational coefficients, as $\mathbb{Q}_+$ is closed under addition and multiplication.

Let us first consider a pair of terms in $\widehat{\mathcal{B}}_{g,n}(x,L_1,\widehat{L_1})$

$$\pi^{6g+2(n-1)-6-2m}x^{2h}\mu(L_1^2,\ldots,L_n^2)(H(x,L_1+L_j)+H(L_1-L_j),$$

where μ is a monomial, such that $2h+2\deg(\mu) = 2m$, where $\deg(\mu)$ considers μ as a polynomial with variables $L_1,\ldots,L_n$. Then,

$$\begin{aligned}
&\int_0^\infty x\cdot\pi^{6g+2(n-1)-6-2m}x^{2h}\mu(L_1^2,\ldots,L_n^2)(H(x,L_1+L_j)+H(L_1-L_j))\\
&\qquad = \pi^{6g+2n-8-2m}\mu(L_1^2,\ldots,L_n^2)(F_{2h+1}(L_1+L_j)+F_{2h+1}(L_1-L_j)),
\end{aligned}$$

which, by lemma 2.5.5, is a polynomial in $\{L_i^2\}$. Furthermore, each pair of terms:

$$\pi^{6g+2n-8-2m}\zeta(2i)\mu(L_1^2,\ldots,L_n^2)[(L_1+L_j)^{2h+2-2i}+(L_1-L_j)^{2h+2-2i}]$$

when expanded, becomes a polynomial in $\{L_i^2\}$. The coefficients lie in

$$\pi^{6g+2n-8-2m}\zeta(2i)\mathbb{Q}_+ = \pi^{6g+2n-6-2(m-i+1)}\mathbb{Q}_+$$

and since $\deg(\mu\cdot(L_1\pm L_j)^{2h+2-2i}) = 2m-2i+2$, it satisfies the coefficient condition for this theorem.

$\square$

Appendix A

A.1 Sectional Curvature

Definition A.1.1. Given a smooth manifold M, and $\Gamma(M, TM)$, the space of smooth vector fields on M; an **affine connection** is a bilinear function $\nabla : \Gamma(M, TM) \times \Gamma(M, TM) \to \Gamma(M, TM)$

$$(X, Y) \mapsto \nabla_X Y,$$

such that for any $f \in C^\infty(M, \mathbb{R})$, that is, the collection of smooth real functions on M:

1. $\nabla_{fX} Y = f \nabla_X Y$;
2. $\nabla_X (fY) = \mathrm{d}f(X)Y + f\nabla_X Y$.

Definition A.1.2. Given a Riemannian manifold (M, g), there exists a unique affine connection ∇ such that:

1. for any vector fields $X, Y, Z \in C^\infty(M, TM)$, $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$. I.E.: it preserves the metric,
2. $\nabla_X Y - \nabla_Y X = [X, Y]$. I.E.: the connection is torsion free.

We call such a connection the **Levi-Civita connection**.

Definition A.1.3. Given a Riemannian manifold (M, g), let ∇ denote its Levi-Civita connection. The **Riemann curvature tensor**, denoted by $R(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{w} : T_xM \times T_xM \times T_xM \to T_xM$, $x \in M$, is given by the following formula:

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w.$$

Definition A.1.4. Given a Riemannian manifold (M, g), and $\boldsymbol{u}, \boldsymbol{v} \in T_xM$, we define the **sectional curvature** of (M, g) at x to be:

$$K(\boldsymbol{u}, \boldsymbol{v}) = \frac{g(R(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{v}, \boldsymbol{u})}{g(\boldsymbol{u}, \boldsymbol{u})g(\boldsymbol{v}, \boldsymbol{v}) - g(\boldsymbol{u}, \boldsymbol{v})^2.}$$

It can be shown that the sectional curvature only depends on the 2-plane spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$.

Theorem A.1.1 (Gauss-Bonnet Formula)**.** *Given a Riemannian surface (M, h) with boundary ∂M (potentially empty), let K denote the sectional curvature of M, and κ_g the geodesic curvature of ∂M. Then,*

$$\int_M K \; dA + \int_{\partial M} \kappa_g \; ds = 2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic of M, dA is the area form over M and ds is the length form over ∂M.

Appendix B

B.1 Differential Forms and Exterior Algebra

Readers wanting particular information about differential forms or symplectic topology may wish to consult either [2] or [6].

B.1.1 Exterior Algebra

Definition B.1.1. Given a real vector space V, let $T : V^k \to \mathbb{R}$ be $\mathbb{R}$-linear with respect to each of the k copies of V, then we call T a k**-tensor**. Since finite sums and scalar multiples of k-tensors are also k-tensors, the collection of all k-tensors, denoted by $\mathfrak{T}^k(V^*)$ is a real vector space.

Definition B.1.2. Given an p-tensor T and a q-tensor S, we can construct a $p+q$-tensor $T \otimes S$ given by:

$$(T \otimes S)(\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}, \boldsymbol{v_{p+1}}, \ldots, \boldsymbol{v_{p+q}}) = T(\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}) \times S(\boldsymbol{v_{p+1}}, \ldots, \boldsymbol{v_{p+q}}).$$

We call $T \otimes S \in \mathfrak{T}^{p+q}(V^*)$ the **tensor product of** T **with** S, as opposed to the tensor product of S with T, which is, in general, different.

Note B.1.1. The notation for $\mathfrak{T}^k(V^*)$, where V^* is used instead of V probably highlights the strong connection between $\mathfrak{T}^k(V^*)$ and V^*, where V^* denotes the dual-space of V. Take for example, the fact that $\mathfrak{T}^1(V^*) = V^*$, which is a corollary of the following theorem.

Theorem B.1.1. *Let* $\{\phi_1, \ldots, \phi_k\}$ *be a basis for* V^**, then the* p*-tensors*

$$\{\phi_{i_1} \otimes \ldots \otimes \phi_{i_p} : 1 \leq i_1, \ldots, i_p \leq k\}$$

form a basis for $\mathfrak{T}^p(V^*)$*. In addition,* $\dim \mathfrak{T}^p(V^*) = k^p$*.*

Definition B.1.3. We call a tensor T **alternating** if, whenever two input vectors a switched, the sign of T reverses, that is:

$$T(\boldsymbol{v_1}, \ldots, \boldsymbol{v_i}, \ldots, \boldsymbol{v_j}, \ldots, \boldsymbol{v_p}) = -T(\boldsymbol{v_1}, \ldots, \boldsymbol{v_j}, \ldots, \boldsymbol{v_i}, \ldots, \boldsymbol{v_p}.)$$

Note B.1.2. Consider the action of the symmetric group S_k on V^k given by swapping coordinates, then for any $T \in \mathfrak{T}^k(V^*)$ and $\pi \in S_k$, we can define a p-tensor T^π given by:

$$T^\pi(\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}) = T(\boldsymbol{v_{\pi(1)}}, \ldots, \boldsymbol{v_{\pi(p)}}).$$

Given such a set up, the alternating p-tensors are precisely those satisfying:

$$T^\pi = (-1)^{|\pi|}T, \; \forall\, \pi \in \; S_p,$$

where $|\pi|$ denotes the parity of π as a permutation.

Lemma B.1.2. *Given any $T \in \mathfrak{T}^p(V^*)$, we can construct an alternating tensor, denoted* $\mathrm{Alt}(T)$*, given by:*

$$\mathrm{Alt}(T) = \frac{1}{p!} \sum_{\pi \in S_p} (-1)^{\pi} T^{\pi}.$$

Notice that if T is already an alternating function, the $\mathrm{Alt}(T) = T$*. Moreover, observe that the collection of alternating tensors forms a vector subspace of $\mathfrak{T}^p(V^*)$, which we'll denote by $\Lambda^p(V^*)$; $Alt : \mathfrak{T}^p(V^*) \to \Lambda^p(V^*)$ is a surjective homomorphism.*

Definition B.1.4. The **wedge product** of $T \in \Lambda^m(V^*)$ with $S \in \Lambda^n(V^*)$ is defined as:

$$T \wedge S = \mathrm{Alt}(T \otimes S) \in \Lambda^{m+n}(V^*).$$

The wedge product then makes the direction sum:

$$\Lambda(V^*) := \Lambda^0(V^*) \oplus \Lambda^1(V^*) \oplus \ldots \oplus \Lambda^k(V^*)$$

a non-commutative unital algebra. We call it the **exterior algebra** of V^*.

B.1.2 Differential Form

Differential forms are abstract constructions defined to give explicit meaning to symbolic quantities such as dx or dy used in integration.

Definition B.1.5. Given a smooth manifold X, a p-form on X is a function ω that assigns to each point $x \in X$ an alternating tensor $\omega(x) : T_x(X) \to \mathbb{R}$, that is: $\omega(x) \in \Lambda^p(T_x(X)^*)$.

Definition B.1.6. Given a smooth map $f : X \to Y$, and $\omega : T(Y) \to \mathbb{R}$, a p-form on Y, there exists an induced derivative map $df_x : T_x(X) \to T_{f(x)}(Y)$. Then, we can pullback $\omega(f(x))$ to $T_x(X)$ using the transpose $(df)^*$. Define:

$$f^*\omega(x) := (df_x)^*\omega[f(x)].$$

Definition B.1.7. Given a smooth k-form ω on X, a k-manifold. The closure of the set of all points where $\omega(x) \neq 0$ is called the **support** of ω. If this closure is compact, then we say that ω has **compact support**.

Note B.1.3. Since a k-form is a structured function $\omega : TX \to \mathbb{R}$, it makes sense to talk about it being smooth.

Theorem B.1.3. *Let $f : Y \to X$ be an orientation-preserving diffeomorphism, then for every compactly supported, smooth k-forms on X,*

$$\int_x \omega = \int_Y f^*\omega.$$

Appendix C

C.1 Table of Volumes

The following is a small table of some Weil-Petersson volumes taken from an upcoming Ph.D. thesis by Norman Do. However, the normalisation of the volume form used for the volumes presented here are in accordance to Mirzakhani's work, and occasionally differs from other sources by a power of 2. Moreover, sums are assumed to be taken over all unordered collection of indices.

(g,n)	$\mathrm{Vol}(\mathcal{M}_{g,n}(\boldsymbol{L}))$
$(0,3)$	1
$(0,4)$	$\frac{1}{2}\sum L_i^2 + 2\pi^2$
$(0,5)$	$\frac{1}{8}\sum L_i^4 + \frac{1}{2}\sum L_i^2 L_j^2 + 3\pi^2 \sum L_i^2 + 10\pi^4$
$(1,1)$	$\frac{1}{24}L^2 + \frac{\pi^2}{6}$
$(1,2)$	$\frac{1}{192}\sum L_i^4 + \frac{1}{96}L_1^2 L_2^2 + \frac{\pi^2}{12}\sum L_i^2 + \frac{\pi^4}{4}$
$(1,3)$	$\frac{1}{1152}\sum L_i^6 + \frac{1}{192}\sum L_i^4 L_j^2 + \frac{1}{96}\sum L_1^2 L_2^2 L_3^2 + \frac{\pi^2}{24}\sum L_i^4 + \frac{\pi^2}{8}\sum L_i^2 L_j^2 + \frac{13\pi^4}{24}\sum L_i^2 + \frac{14\pi^6}{9}$
$(2,1)$	$\frac{1}{442368}L^8 + + \frac{29\pi^2}{138240}L^6 + \frac{139\pi^4}{23040}L^4 + \frac{169\pi^6}{2880}L^2 + \frac{29\pi^8}{192}$

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