ISTANBUL TECHNICAL UNIVERSITY

FACULTY OF SCIENCE&LETTERS

BACHELOR'S THESIS

First Order Scalar Wave Equations and Applications in Traffic Flow

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Semester	:	**2010-2011 Spring**
Department	:	**Mathematical Engineering**
Matriculation Number	:	**090060080**

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ACKNOWLEDGEMENTS

In the preparation process of this thesis; first, I want to greatly thank to my supervisor Prof. Dr. Mevlüt TEYMÜR for spending his valuable time and knowledge generously for narrowing, deciding and introducing the subject in order to make me ready for academic studies. Second, I also deeply thank to Research Assistant MSc. Ali DEMİRCİ for giving aid in some problems that I faced. Third, I want to thank to my mother and father for supporting me all in my life. Finally, I want to thank to my friends Yasin Serdar ÇALIK for helping me to sketch the figures that I used in this thesis and Yusuf Evrim KILIÇ for helping me literary.

Gökhan GÖKSU

June 2011

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ABSTRACT

In this thesis, the aim was to offer an introductive review of wave motion and its applications. In first section, I want to give compact further information about Partial Differential Equations (PDEs), the method of characteristics, Cauchy Problem and Wave Equations. In second section, the first order scalar wave equation, shock structure for first order scalar wave equation, weak solutions for shocks and shock fitting are tried to be analyzed. In the last section, introductive information about traffic flow is given, discussed and further readings about this topic are offered.

1. INTRODUCTION

Differential Equation is an equation which contains an unknown function of one or several variables and its derivatives with respect to its arguments. This subject is a fairly crucial and fundamental subject in applied mathematics and plays an important role of various disciplines such as engineering, physics, mechanics etc…

A differential equation demonstrates a relation between the unknown function and the rate of the function according to the change of its arguments.

Differential equations are classified into two categories: ordinary differential equations (ODEs), which are differential equations of an unknown function depending on a single independent variable and partial differential equations (PDEs), which are differential equations of an unknown function depending on two or more variables. If a differential equation contains its *n*th order derivative, it is called *n*th order differential equation.

Wave equations are PDEs, which model wave motions, for example; sound waves, light waves, water waves etc… Wave equations have widespread applications on fluid mechanics, acoustics, electromagnetic and so on… In section 2, I am going to try to present compact further information about the discontinuities, weak solutions and shocks for first order scalar wave equations.

Traffic flow is an essential subject in fluid mechanics. Haberman asserts that, traffic problems bother the scientists, before the invention of the automobile. He also mention that installing traffic lights or signs; determining the period of the traffic lights; building entrances, exits and overpasses; deciding the lane number of the highways are the main topics of the traffic or transportation theory. Here, the aim is to optimize the traffic flow in order to prevent congestion, accidents, air pollution or serious disasters [2]. This topic will be investigated in the third section.

In the forthcoming subsection I give brief information about structural classification of the first order PDEs, method of characteristics, Cauchy Problem and the wave equation.

1.1 Structural Classification of first order PDEs

First order PDEs are classified as follows: first order linear PDEs, first order semi-linear PDEs, first order quasi-linear PDEs and first order non-linear PDEs. In this section our unknown function is taken as $u(x, y)$, which is a function of x and y.

1.1.1 First Order Linear PDEs

A PDE is called a first order linear PDEs, if the defining function is a linear function of u, u_x and u_y

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y) \qquad (1.1)$$

where u_x and u_y represent partial derivatives of u with respect to x and y.

1.1.2 First Order Semi-Linear PDEs

A PDE is called a first order semi-linear PDE, if it has the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u) \tag{1.2}$$

where a and b are functions of x and y and c is a non-linear function of u.

1.1.3 First Order Quasi-Linear PDEs

A PDEs is called a first order quasi-linear PDE, if has the form

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \tag{1.3}$$

where a, b and c are functions of u, x and y.

1.1.4 First Order Non-Linear PDEs

A PDE is called a first order non-linear PDE, if the defining function is not a linear function of u_x, u_y and u;

$$F(x,t,u,u_x,u_y) = 0. \tag{1.4}$$

1.2 The Method of Characteristics

The method of characteristics is a solution technique for PDEs [3]. It is mainly about finding relation between the unknown function and the arguments by using characteristic lines.

The characteristic equations for a first order linear PDE (1.1) are

$$\begin{aligned} &\frac{dx}{dt} = a(x,y), \\ &\frac{dy}{dt} = b(x,y) \; and \\ &\frac{du}{dt} + c(x,y)u = d(x,y); \end{aligned} \tag{1.5}$$

the characteristic equations for a first order semi-linear PDE (1.2) are

$$\begin{aligned} &\frac{dx}{dt} = a(x,y), \\ &\frac{dy}{dt} = b(x,y) \; and \\ &\frac{du}{dt} = c(x,y,u); \end{aligned} \tag{1.6}$$

the characteristic equations for a first order semi-linear PDE (1.3) are

$$\frac{dx}{dt} = a(x, y, u),$$
$$\frac{dy}{dt} = b(x, y, u) \; and \tag{1.7}$$
$$\frac{du}{dt} = c(x, y, u)$$

where t defines the changes along a characteristic line. However, the characteristic equations for a first order non-linear PDE are defined a little bit differently. They are (see [1]);

$$\begin{aligned} \frac{dx}{dt} &= F_p, \\ \frac{dy}{dt} &= F_q, \\ \frac{du}{dt} &= pF_p + qF_q, \\ \frac{dp}{dt} &= -F_x - pF_u, \\ \frac{dq}{dt} &= \; -F_y - qF_u. \end{aligned} \tag{1.8}$$

1.3 Cauchy Problem

Let us define a curve $\Gamma \in \mathbb{R}^2$ by parametric equations;

$$x = x_0(s), y = y_0(s) \text{ where } \alpha \leq s \leq \beta \text{ and } \alpha, \beta \in \mathbb{R}. \tag{1.9}$$

Finding the solution (1.1) or (1.2) or (1.3) or (1.4) satisfying the condition

$$on \; \Gamma: u = u_0(s) \tag{1.10}$$

is called a Cauchy Problem. Sometimes $u = u_0(s)$ is called an initial condition.

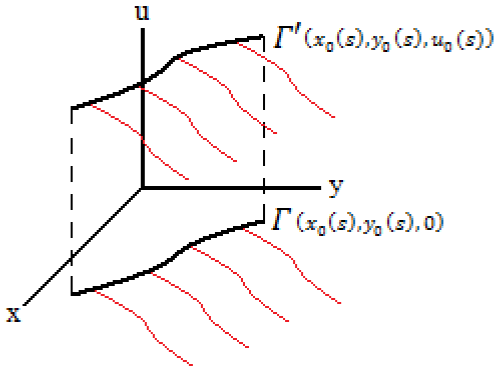

Figure 1.1 Cauchy Problem

Cauchy problem could also been defined on a curve in $\mathbb{R}^3$. Let us define a curve $\Gamma' \in \mathbb{R}^3$ and on this curve let $x = x_0(s)$, $y = y_0(s)$ and $u = u_0(s)$ where $\alpha \leq s \leq \beta$ and $\alpha, \beta \in \mathbb{R}$. A problem

$$\Gamma': \begin{cases} x = x_0(s) \\ y = y_0(s) \\ u = u_0(s) \end{cases} \tag{1.11}$$

denotes also a Cauchy Problem. Note that the projection of Γ' onto xy-plane is going to be Γ.

It is also need to be stressed that the unique solution exists only in the case

$$\frac{\partial(x,y)}{\partial(s,t)} \neq 0.$$

If

$$\frac{\partial(x,y)}{\partial(s,t)} = 0,$$

we have to control two things. There exists infinitely many solutions if and only if u_x and u_y are determined from

$$u_y(bx_s - ay_s) = cx_s - au_s,$$

$$u_x(ay_s - bx_s) = cy_s - bu_s,$$

where a, b and c are the components obtained from (1.2) or (1.3). Else there won't be a solution.

1.4 Wave equation

The wave equation is a PDE and plays a very fundamental role in applied mathematics. Its general linear form is

$$\frac{\partial^2 u}{\partial t^2} = {c_0}^2 \nabla^2 u \qquad or \qquad {c_0}^2 \nabla^2 u - \frac{\partial^2 u}{\partial t^2} = 0 \tag{1.12}$$

where $c_0 \in R$ and ∇^2 denotes the Laplacian operator i.e.

$$\nabla^2 = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

if $u = u(x_1, x_2, \ldots, x_n, t)$. Its general non-linear form is

$$\frac{\partial^2 u}{\partial t^2} = [c(u)]^2 \nabla^2 u \qquad or \qquad [c(u)]^2 \nabla^2 u - \frac{\partial^2 u}{\partial t^2} = 0 \tag{1.13}$$

where c is a continuous function of u. In one dimension, this equation would be

$$\frac{\partial^2 u}{\partial t^2} = [c(u)]^2 \frac{\partial^2 u}{\partial x^2} \qquad or \qquad [c(u)]^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0. \tag{1.14}$$

The one dimensional form of (1.12) is going to be

$$\frac{\partial^2 u}{\partial t^2} = {c_0}^2 \frac{\partial^2 u}{\partial x^2} \qquad or \qquad {c_0}^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0. \tag{1.15}$$

For (1.15), the components are $a = {c_0}^2$, $b = 0$ and $c = -1$. The discriminant is

$$b^2 - 4ac = 0 - 4\,({c_0}^2)(-1) = 4\,{c_0}^2 > 0$$

so the equation is hyperbolic. The characteristic equations for (1.15) will be

$$\frac{dt}{dx} = \pm \frac{1}{c_0}. \tag{1.16}$$

If these equations are integrated we obtain

$$\begin{aligned} c_0 dt = dx \Rightarrow\ c_0 t = x + constant => \ \Phi(x,t) = x - c_0 t\ \ and \\ c_0 dt = -dx \Rightarrow\ c_0 t = -x + constant => \ \Psi(x,t) = x + c_0 t \end{aligned} \tag{1.17}$$

where $\Phi(x,t)$ and $\Psi(x,t)$ are the characteristic lines of the equation (1.15). If we calculate the derivatives

$$u_x = u_\Phi \Phi_x + u_\Psi \Psi_x = u_\Phi + u_\Psi \Rightarrow u_{xx} = (u_\Phi)_x + (u_\Psi)_x = u_{\Phi\Phi}\Phi_x + u_{\Psi\Psi}\Psi_x = u_{\Phi\Phi} + u_{\Psi\Psi}$$

$$u_t = u_\Phi \Phi_t + u_\Psi \Psi_t = -c_0 u_\Phi + c_0 u_\Psi \Rightarrow u_{tt} = -c_0 (u_\Phi)_t + c_0 (u_\Psi)_t$$

$$\Rightarrow u_{tt} = {c_0}^2 u_{\Phi\Phi} - 2{c_0}^2 u_{\Psi\Phi} + {c_0}^2 u_{\Psi\Psi}.$$

If the derivatives are replaced with the new characteristic derivatives the canonical form of the wave equation

$$u_{\Phi\Psi} = 0 \tag{1.18}$$

is obtained. If the canonical form will be integrated twice with respect to Φ and Ψ

$$u(\Psi,\Phi) = u(x,t) = F(\Phi) + G(\Psi) = F(x - c_0 t) + G(x + c_0 t) \tag{1.19}$$

will be obtained and this is the general solution of the wave equation, where $F, G \in C^2$ and are arbitrary functions.

If the linear form (1.15) is factorized in one dimension, we obtain

$$\left(c_0 \frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)\left(c_0 \frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) u = 0. \tag{1.20}$$

If we hold on the one component of this factorization, say $c_o u_x + u_t = 0$, the general solution will be

$$u = F(x - c_0 t). \tag{1.21}$$

If we do the same factorization to the non-linear equation (1.14) we get

$$c(u)u_x + u_t = 0. \tag{1.22}$$

2. FIRST ORDER SCALAR WAVE EQUATIONS

Whitham states that, even though the classical problems are derived from (1.14), numerous wave motions been analyzed by modeling the simplest form (1.22) [1]. This equation (1.22) will be analyzed in this section. We will analyze the continuous solutions, where the discontinuities happen, is it possible to construct a solution at discontinuities, where and when the shock occur and can we fit a continuous solution for shocks and discontinuities.

From now on it is useful to indicate that $q(x,t)$ means the flow, $\rho(x,t)$ means the flow density, $u(x,t)$ means the velocity field or flow velocity and $c(\rho)$ means the propagation velocity, a function of ρ.

While analyzing the equation (1.22) it is a common use to replace u with ρ. Whitham tells that it is better to denote the unknown function as ρ, since the unknown function is the density of something in general [1]. So the equation (1.22) takes the form in linear case

$$c_0\rho_x + \rho_t = 0, \tag{2.1}$$

and in non-linear case

$$c(\rho)\rho_x + \rho_t = 0. \tag{2.2}$$

2.1 Continuous Solutions

Let us analyze the structure of the equation (2.2). It is already known that $\rho(x,t)$ is a function of two independent variables x and t. If we calculate the total derivative of ρ with respect to t we obtain

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{dx}{dt}\frac{\partial\rho}{\partial x}. \tag{2.3}$$

If we correlate (2.2) and (2.3), these results are seen:

$$\frac{d\rho}{dt} = 0 \; and \; \frac{dx}{dt} = c(\rho)\,. \tag{2.4}$$

This means that ρ is not dependant to x and the relation between x and t will be linear since c is a continuous function of ρ and ρ is not dependant to x along these lines.

Let us consider a line $\mathcal{C}$ with the initial value problem (IVP)

$$\rho(x,0) = f(x) \;\; where \; x\epsilon\mathbb{R}. \tag{2.5}$$

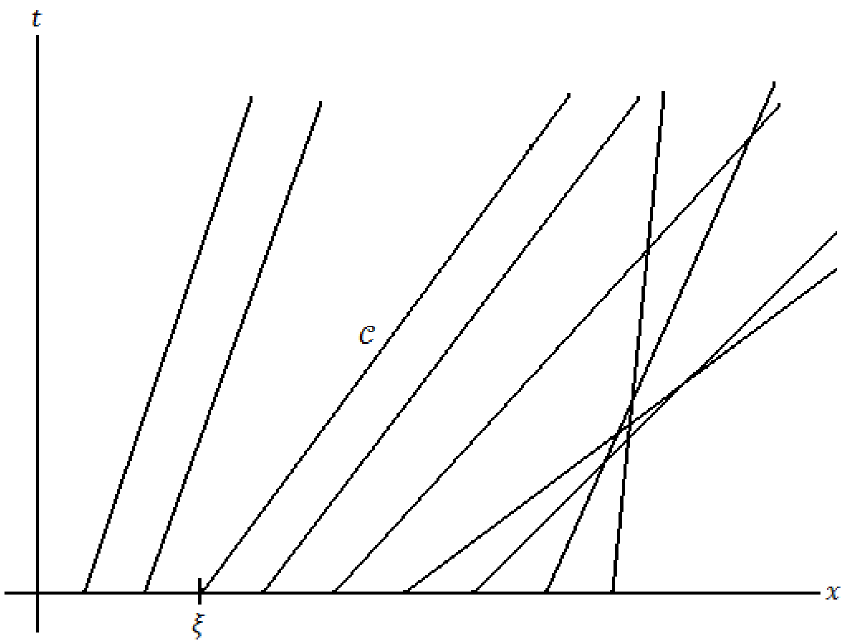

Figure 2.1 Characteristics for the general IVP (2.5)

Ponder the initial condition shown above in Figure 2.1. The selected line $\mathcal{C}$ intersects the x-axis at $x=\xi$ and that brings us to the solution $\rho=f(\xi)$ on the line $\mathcal{C}$. We already know from (2.4)

$$\frac{dx}{dt}=c(\rho)=c\big(f(\xi)\big)=c(constant)=constant\Rightarrow\ x=c\big(f(\xi)\big)t+constant$$

and if we apply the initial condition and if we denote $c=F(\xi)\overset{\text{def}}{=}c(f(\xi))$ we get the equation of the line $\mathcal{C}$

$$x=\xi+F(\xi)t. \tag{2.6}$$

If we do not take ξ as a constant, but as a function of x and t, we could find out whether this structure satisfies the equation (2.2) and the initial condition (2.5).

Let us start by taking the partial derivatives of ρ with respect to x and t:

$$\begin{gathered}\rho_t=f'(\xi)\xi_t\Rightarrow\xi_t=\frac{\rho_t}{f'(\xi)},\\ \rho_x=f'(\xi)\xi_x\Rightarrow\xi_x=\frac{\rho_x}{f'(\xi)}.\end{gathered} \tag{2.7}$$

If we take the partial derivatives of the equation (2.6) and substitute ξ_x and ξ_t from (2.7), the partial derivatives of ρ, ρ_x and ρ_t, can be supplied

$$\begin{gathered}(x)_t=\xi_t+F'(\xi)\xi_t t+F(\xi)\Rightarrow 0=F(\xi)+[1+F'(\xi)t]\xi_t\\ \underset{(2.7)}{\Longrightarrow}\rho_t=-\frac{F(\xi)f'(\xi)}{1+F'(\xi)t},\end{gathered} \tag{2.8a}$$

$$(x)_x = \xi_x + F'(\xi)\xi_x t \Rightarrow 1 = [1 + F'(\xi)t]\xi_x \underset{(2.7)}{\Longrightarrow} \rho_x = \frac{f'(\xi)}{1 + F'(\xi)t} \tag{2.8b}$$

and these results satisfies the equation (2.2) and the initial condition (2.5).

The condition which make ρ_x and ρ_t infinite in (2.8a) and (2.8b) is the breaking condition and that is

$$1 + F'(\xi_B)t_B = 0 \Rightarrow t_B = -\frac{1}{F'(\xi_B)}. \tag{2.9}$$

and the breaking picture can be seen in Figure 2.2.

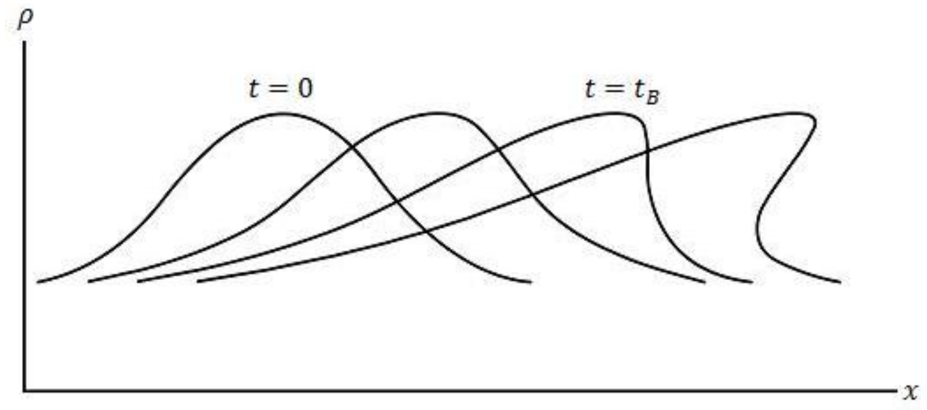

Figure 2.2 Breaking wave

Let us consider the initial functions, first of which is namely initial step function, for initial conditions

$$f(x) = \begin{cases} \rho_1, & x > 0 \\ \rho_2, & x < 0 \end{cases}, F(x) = \begin{cases} c_1 = c(\rho_1), & x > 0 \\ c_2 = c(\rho_2), & x < 0 \end{cases} \tag{2.9}$$

and let $c_2 > c_1$, that means the slope of the lines coming from the x-axis is greater than the slope of the lines coming from the t-axis. The characteristic diagram is going to be as in Figure 2.3.

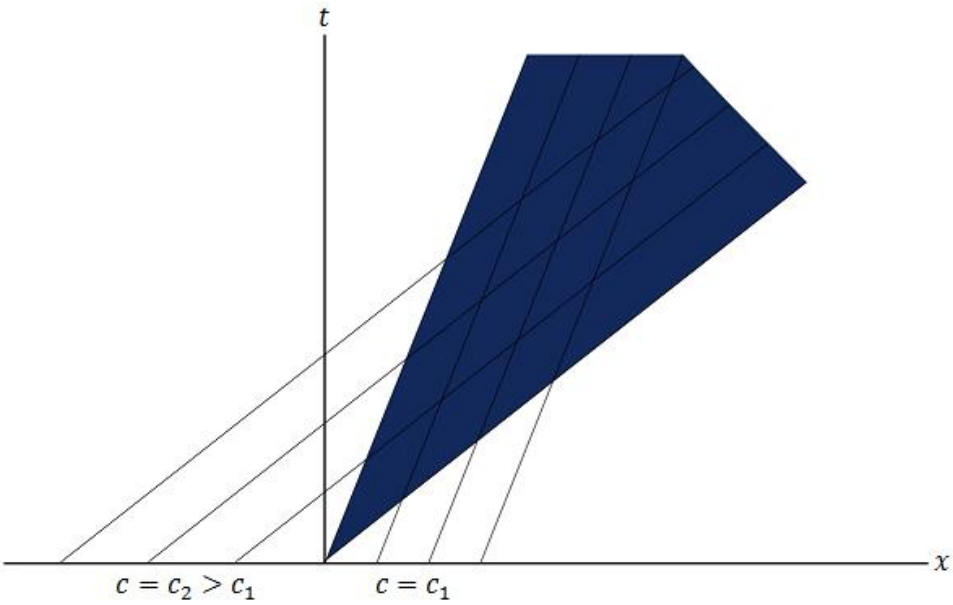

Figure 2.3

It is need to be underlined that the breaking occurs at the coloured area shown in Figure 2.3. So the wave profile can be seen in Figure 2.4.

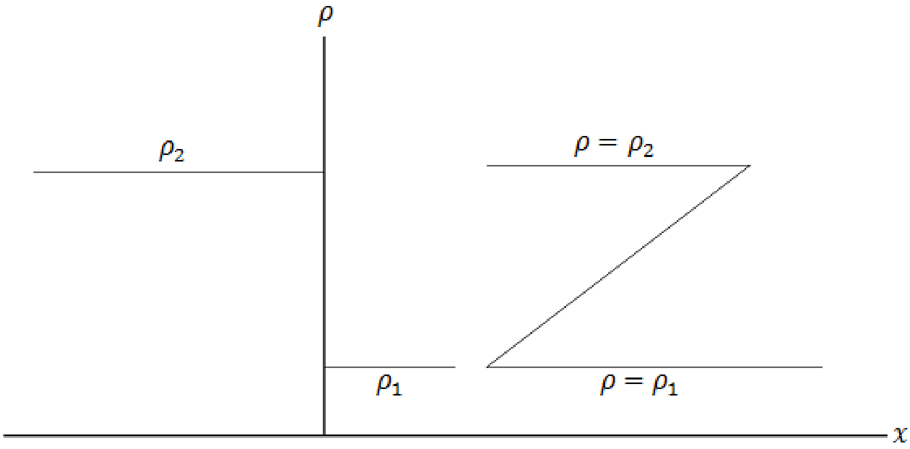

Figure 2.4

Let us examine the condition $\xi = 0$ where $c_2 < F < c_1$. This leads to a fan solution which can be seen in Figure 2.5. Taking $\xi = 0$ makes (2.6) as follows:

$$x = Ft \Rightarrow F = \frac{x}{t} \; where \; c_2 < F < c_1.$$

Since $c = F$ we have the solution namely centered expansion wave

$$c = \begin{cases} c_1, & \frac{x}{t} > c_1 \\ \frac{x}{t}, & c_2 < \frac{x}{t} < c_1, \\ c_2, & \frac{x}{t} < c_2 \end{cases} \tag{2.10}$$

and the characteristic diagram can be sketched and the slope fan can be seen in Figure 2.5.

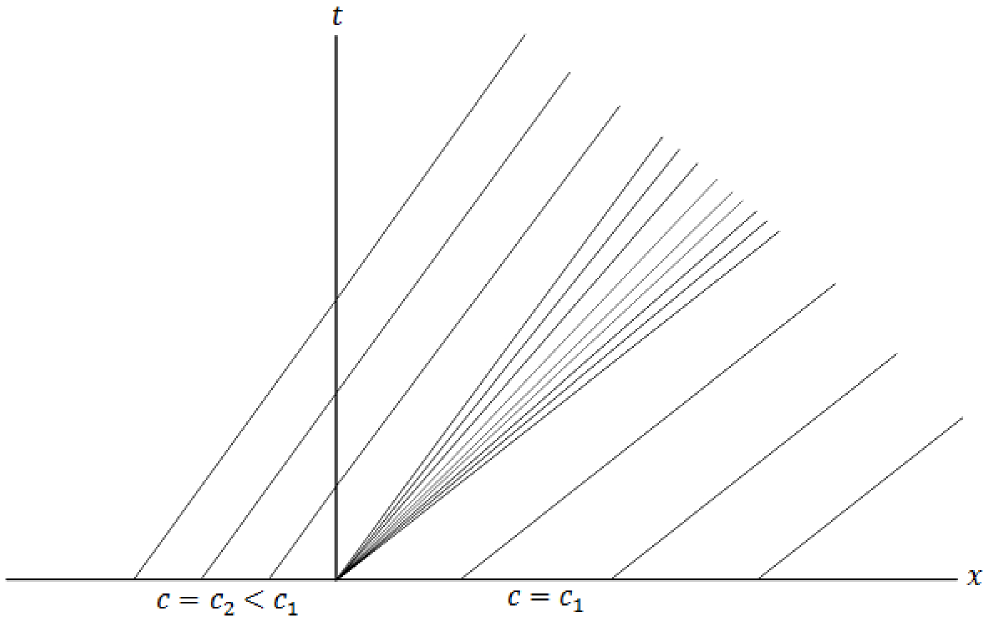

Figure 2.5

Since there are no overlaps there are no breakings as we can see in Figure 2.6.

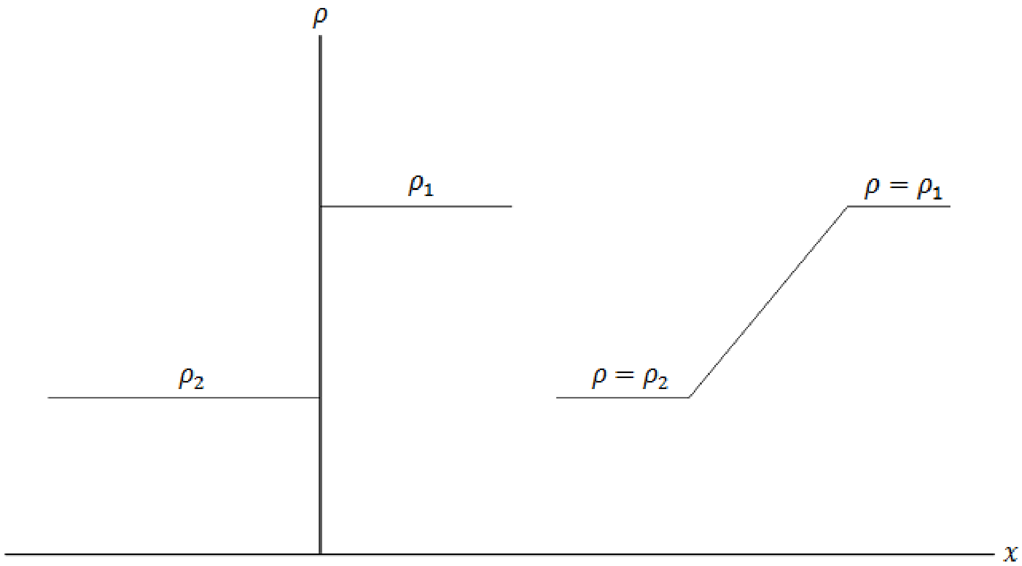

Figure 2.6

All in all, when breaking occurs, the equation (2.2) is inefficient, since there are multivalued solutions. In such cases, it is needed to seek a weak solution and that is investigated later.

2.2 Conservation Law for Traffic Flow

Consider a conservative highway with the traffic density $\rho(x,t)$, flow $q(x,t)$ and the velocity field (was propagation velocity previously) $v(x,t) \stackrel{\text{def}}{=} c(\rho(x,t))$ which are known functions of x and t. According to Haberman, traffic flow will be (see [2])

$$traffic\ flow = (traffic\ density)(velocity\ field).$$

That is

$$q(x,t) = \rho(x,t)v(x,t). \qquad (2.11)$$

We already know that from the equation (2.4)

$$\frac{dx}{dt} = v(x,t)$$

and with $x(0) = x_0$, this will show us the position of cars according to time. We have also found from (2.11) that

$$v(x,t) = \frac{q(x,t)}{p(x,t)}$$

and if we substitute and integrate the equation (2.4)

$$\frac{dx}{dt} = \frac{q(x,t)}{p(x,t)} \Rightarrow \int_{t_0}^{\tau} q(x,t)dt = \int_a^b \rho(x,t)dx$$

where t_0 denotes the initial time, τ denotes the future time and where a and b denote the initial and future positions of the car shown in Figure 2.7. Haberman asserts that, the left side of the equation gives the number of cars N on some interval [2]. So we obtain the equation

$$N = \int_a^b \rho(x,t)dx. \qquad (2.12)$$

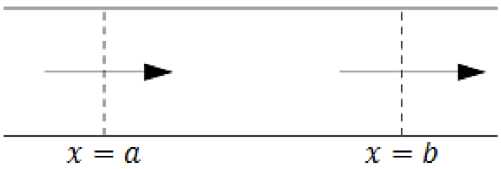

Figure 2.7

We have assumed that the highway is conservative that means any car neither enters nor leaves the highway. However, the rate of the number of passing cars per hour at $x = a$ and $x = b$ may not be the same. If there is any difference, that should be the flow difference between $x = a$ and $x = b$, i.e.

$$\frac{dN}{dt} = q(a,t) - q(b,t). \qquad (2.13)$$

If we substitute the N in equation (2.12) with the N in (2.13) and if we replace the full derivative by the partial derivative since as Haberman asserts $x = a$ and $x = b$ are considered as additional variables which is $\partial/\partial\, t$ will become d/dt when $x = a$ and $x = b$ are fixed [2]. Then we obtain

$$\frac{d}{dt}\int_a^b \rho(x,t)dx = q(a,t) - q(b,t) \ or$$
$$\frac{\partial}{\partial t}\int_a^b \rho(x,t)dx = q(a,t) - q(b,t). \tag{2.14}$$

At (2.14) taking $b = a + \Delta a$, dividing the both part by Δa and taking limit as $\Delta a \to 0$:

$$\lim_{\Delta a \to 0} \frac{\partial}{\partial t}\frac{1}{\Delta a}\int_a^{a+\Delta a} \rho(x,t)dx = \lim_{\Delta a \to 0} \frac{q(a,t) - q(a+\Delta a)}{\Delta a} = -\lim_{\Delta a \to 0} \frac{q(a+\Delta a,t) - q(a,t)}{\Delta a}$$
$$= -\frac{\partial q(a,t)}{\partial a}.$$

If we sketch the integral seen in Figure 2.8, this approximation will be yielded

$$\frac{1}{\Delta a}\int_a^{a+\Delta a} \rho(x,t)dx \approx \rho(a,t)$$

since the number of cars between a and $a + \Delta a$ can be approximated $\Delta a \rho(a,t)$. Finally, when we replace a with x, this equation is achieved:

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}q(x,t) = 0 \tag{2.15}$$

which is the well known conservation equation.

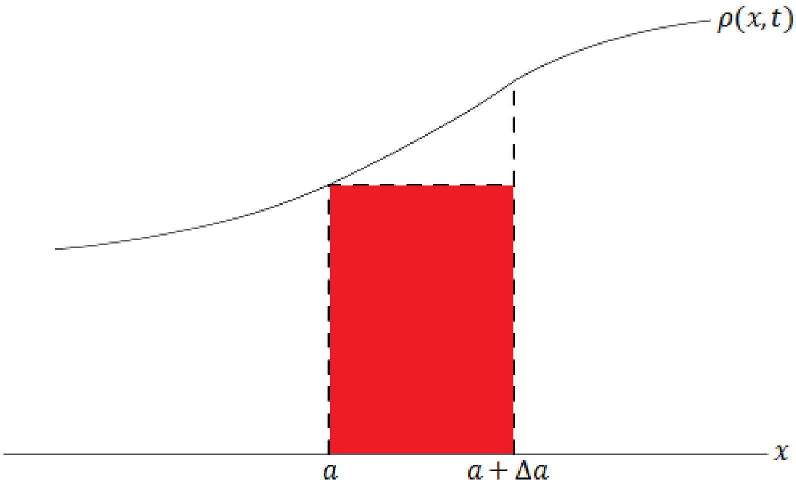

Figure 2.8

On the other hand, the relation between ρ and q could be functional. In such cases, for example $q = Q(\rho)$, if we use $Q(\rho)$ instead of q in the conservation equation (2.15) we get

$$\rho_t + Q'(\rho)\rho_x = 0 \tag{2.17}$$

and if this result is compared with (2.2), this result can be rewritten

$$\rho_t + c(\rho)\rho_x = 0 \;\; where \; c(\rho) = Q'(\rho) \tag{2.18}$$

2.3 Shock Waves

From now on we are going to consider the case that breaking occurs and also whether ρ and q are differentiable. Whitham: "Shock waves appears as discontinuities in ρ" [1]. That is why we are going to investigate the solutions on discontinuities. First of all we should know that, as Whitham affirms the conservation equation (2.14) is valid whether discontinuities exist[1].

Let us consider a jump discontinuity at $x = s(t)$ where $x_2 < s(t) < x_1$ which can be seen as in Figure 2.9.

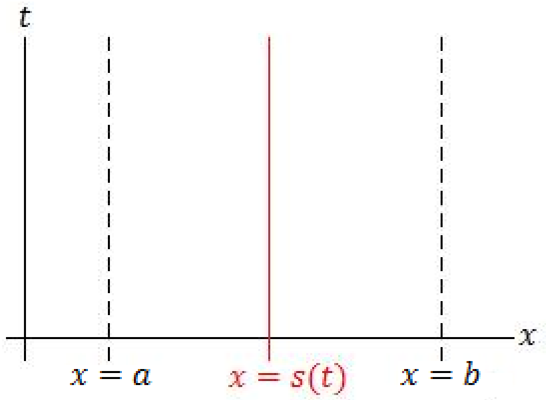

Figure 2.9 The line where jump discontinuity is seen.

Also assume that $\rho, q \in C^1$ in $x_2 \leq x < s(t)$ and $s(t) < x \leq x_1$. That allows us to write the equation

$$q(a,t) - q(b,t) \; = \frac{d}{dt}\int_a^{s(t)} \rho(x,t)dx + \frac{d}{dt}\int_{s(t)}^{b} \rho(x,t)dx$$

When we evaluate the total derivative (analog to (2.3)) of the two integrals of the right side of the equation and when we apply the Leibniz's formula, will find

$$\frac{d}{dt}\int_a^{s(t)} \rho(x,t)dx = \frac{\partial}{\partial t}\int_a^{s(t)} \rho(x,t)dx + \frac{dx}{dt}\left[\rho(s^-,t)\frac{ds}{dx} - 0\right],$$

$$\frac{d}{dt}\int_{s(t)}^{b} \rho(x,t)dx = \frac{\partial}{\partial t}\int_{s(t)}^{b} \rho(x,t)dx + \frac{dx}{dt}\left[0 - \rho(s^+,t)\frac{ds}{dx}\right].$$

where s^- are the values in front of the shock, s^+ are the values back of the shock.

Since

$$\frac{ds}{dx} = \frac{ds}{dt}\frac{dt}{dx} = \dot{s}\frac{dt}{dx}$$

total derivatives will be

$$\frac{d}{dt}\int_a^{s(t)}\rho(x,t)dx = \frac{\partial}{\partial t}\int_a^{s(t)}\rho(x,t)dx + \rho(s^-,t)\dot{s},$$

$$\frac{d}{dt}\int_{s(t)}^{b}\rho(x,t)dx = \frac{\partial}{\partial t}\int_{s(t)}^{b}\rho(x,t)dx - \rho(s^+,t)\dot{s}.$$

Here is $\dot{s}$ is not unfamiliar to us but the shock velocity since it is the difference of shock position with respect to time. Then the conservation equation has a new form:

$$q(a,t) - q(b,t) = \frac{\partial}{\partial t}\int_a^{s(t)}\rho(x,t)dx + \frac{\partial}{\partial t}\int_{s(t)}^{b}\rho(x,t)dx + \rho(s^-,t)\dot{s} - \rho(s^+,t)\dot{s}.$$

When we calculate the limit as $a \to s^-$ and $b \to s^+$ the integral incline to zero so

$$q(s^-,t) - q(s^+,t) = [\rho(s^-,t) - \rho(s^+,t)]\dot{s}.$$

Let us represent the flow and density behind the shock with subscript 1 and ahead the shock with subscript 2, the shock velocity with U. Therefore we have the form:

$$q_2 - q_1 = U(\rho_2 - \rho_1). \tag{2.18}$$

Since $\rho, q \in C^1$ in $x_2 \leq x < s(t)$ and $s(t) < x \leq x_1$, we can state that $q = Q(\rho)$, which makes $q_1 = Q(\rho_1)$ and $q_2 = Q(\rho_2)$ behind and ahead the shock. So we obtain

$$U = \frac{Q(\rho_2) - Q(\rho_1)}{\rho_2 - \rho_1}. \tag{2.19}$$

Consider the step situation in (2.9). The flow picture seems like in Figure 2.10.

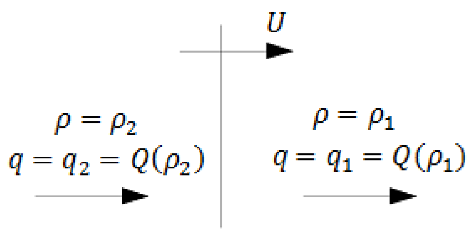

Figure 2.10

2.4 Shock Structure

Whitham says, the problem of finding the shock structure is to seek more precise explanation to the simple discontinuous solution shown in Figure 2.10 [1].We have already looked for the structure given in (2.18). There is also a new structure that we assume

$$q = Q(\rho) - \nu\rho_x \tag{2.20}$$

i.e. q is not only a function of ρ but also a function of density gradient ρ_x, where ν is a positive constant.

This new $-\nu\rho_x$ term tells us that, in traffic flow the drivers decrease their speed when traffic density increases, conversely it is also convincing. The assumption (2.20) makes (2.15)

$$\rho_t + c(\rho)\rho_x = \nu\rho_{xx} \ \ where\ c(\rho) = Q'(\rho). \tag{2.21}$$

Whitham asserts that the term $\nu\rho_{xx}$ presents diffusion similarly to the heat equation (see [1])

$$\rho_t = \nu\rho_{xx}. \tag{2.22}$$

For this equation, when we consider the initial step condition (2.9) the solution has already found not only by Whitham but also by Haberman (see [1] and [5])

$$\rho = \rho_2 + \frac{\rho_1 - \rho_2}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4\nu t}}} e^{-\xi^3} d\xi. \tag{2.23}$$

This result characterizes a transition step with slope $1/\sqrt{\nu t}$ coming from $\rho \to \rho_2$ as $x \to -\infty$, going to $\rho \to \rho_1$ as $x \to +\infty$.

Let us explore the shock structure by defining a new variable such as:

$$\rho = \rho(X), \qquad X = x - Ut.$$

When we write the differentials,

$$\frac{\partial X}{\partial x} = 1, \qquad \frac{\partial X}{\partial t} = -U, \qquad \rho_t = \rho_X X_t = -U\rho_X, \qquad \rho_x = \rho_X X_x = \rho_X,$$
$$\rho_{xx} = \rho_{XX} X_x = \rho_{XX},$$

and substitute into equation (2.21) we get

$$[c(\rho) - U]\rho_X = \nu\rho_{XX}. \tag{2.22}$$

Integrating (2.22) with respect to X

$$Q(\rho) - U\rho + c = \nu\rho_X \tag{2.23}$$

where c∈ ℝ. Integrating (2.23) again

$$Q(\rho) - U\rho + c = \nu\frac{d\rho}{dX} \Rightarrow \frac{1}{\nu}\int dX = \int \frac{d\rho}{Q(\rho) - U\rho + c}$$

an implicit relation between ρ and X could be obtained by calculating the integral, when Q is known, as

$$\frac{X}{\nu} = \int \frac{d\rho}{Q(\rho) - U\rho + c}. \tag{2.24}$$

But the general tendency of the solution is prescribed and sketched by Whitham as in Figure 2.11 [1]. Here $Q(\rho) - U\rho - c$ is negative and of course the slope ρ_X is negative between two steady states. This is possible when $c'(\rho) > 0$.

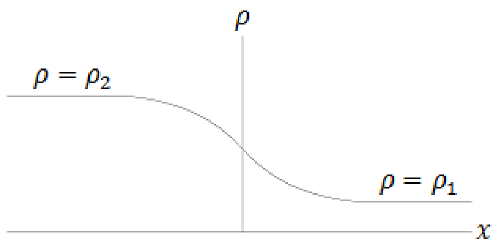

Figure 2.11

If $Q(\rho) - U\rho - c$ is positive, the slope ρ_X becomes positive between two steady states and that condition can be pictured in Figure 2.12. This is possible when $c'(\rho) < 0$.

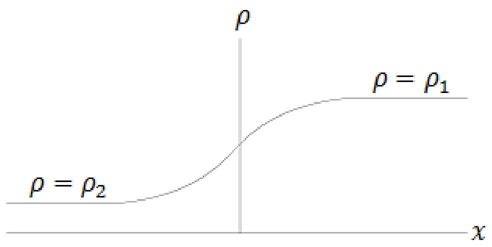

Figure 2.12

In both conditions, since $\rho_X \to 0$ as $X \to \pm\infty$, the right side tends to be zero as $X \to +\infty$(or as $\rho \to \rho_1$) and as $X \to -\infty$(or as $\rho \to \rho_2$) so

$$Q(\rho_1) - U\rho_1 + c = Q(\rho_2) - U\rho_2 + c = 0 \tag{2.25}$$

should be satisfied and U can be found as in (2.19).

It is important to stress that the transition tends to be flattened as ν decreases and limit to a step function while $\nu \to 0$. Conversely, when ν increases the transition profile tends to be splayed and flow picture will be in Figure 2.13.

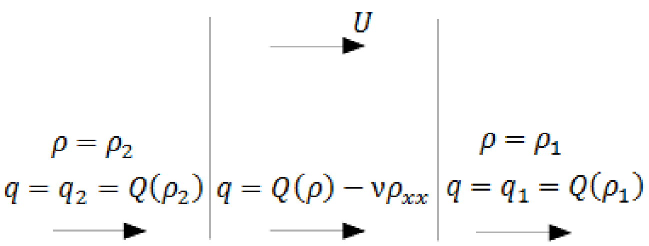

Figure 2.12

For given a quadratic $Q(\rho)$ such as $Q(\rho) = a_1\rho^2 + a_2\rho + a_3$. It is obvious to see

$$Q(\rho) - U\rho + c = a_1\rho^2 + (a_2 - U)\rho + (a_3 + c). \tag{2.26}$$

Since (2.25), means the expression $Q(\rho) - U\rho + c$ has roots ρ_1 and ρ_2 we can express $Q(\rho)$ as follows

$$Q(\rho) - U\rho + c = a_1(\rho - \rho_1)(\rho - \rho_2) = a_1\rho^2 - a_1(\rho_1 + \rho_2)\rho + a_1\rho_1\rho_2. \quad (2.27)$$

When (2.26) and (2.27) are compared

$$a_1 = a_1,$$

$$a_2 - U = -a_1(\rho_1 + \rho_2) \; and$$

$$a_3 + c = a_1\rho_1\rho_2$$

i.e.

$$U = a_2 + a_1(\rho_1 + \rho_2) \; and \; c = a_1\rho_1\rho_2 - a_3. \quad (2.28)$$

Finding a and b from

$$\frac{1}{(\rho - \rho_1)(\rho - \rho_2)} = \frac{a}{(\rho - \rho_1)} + \frac{b}{(\rho - \rho_2)} = \frac{(a + b)\rho - (ap_2 + b\rho_1)}{(\rho - \rho_1)(\rho - \rho_2)},$$

$$a = \frac{1}{(\rho_1 - \rho_2)} \; and \; b = \frac{-1}{(\rho_1 - \rho_2)}$$

(2.24) becomes

$$\frac{X}{\nu} = \int \frac{d\rho}{a_1(\rho - \rho_1)(\rho - \rho_2)} = \frac{1}{a_1}\frac{1}{(\rho_1 - \rho_2)}\left[\int \frac{d\rho}{(\rho - \rho_1)} - \int \frac{d\rho}{(\rho - \rho_2)}\right]$$

$$= \frac{1}{a_1}\frac{1}{(\rho_1 - \rho_2)}\ln\frac{(\rho - \rho_1)}{(\rho - \rho_2)}$$

and

$$X = \frac{\nu}{a_1(\rho_1 - \rho_2)}\ln\frac{(\rho - \rho_1)}{(\rho - \rho_2)}. \quad (2.28)$$

Whitham asserts that when the shock is strong, that is the transition happen rapid, the shock can be approximated by a discontinuity [1]. If the shock is weak, i.e. $(\rho_2 - \rho_1)/\rho_1 \to 0$, they should not be considered as a discontinuity. Whitham emphasizes that it is effective to make approximations for weak shocks [1]. For example, since $(\rho_2 - \rho_1)/\rho_1 \to 0$ the shock velocity U in (2.19) can be expanded into Taylor series. Before that he says that the shock velocity U tends to the propagation velocity $c(\rho) = Q'(\rho)$. The expansion will be as follows:

$$U = Q'(\rho_1) + \frac{1}{2}(\rho_2 - \rho_1)Q''(\rho_1) + O(\rho_2 - \rho_1)^2.$$

Also if the propagation velocity would be expanded:

$$c(\rho_2) = c(\rho_1) + (\rho_2 - \rho_1)Q''(\rho_1) + O(\rho_2 - \rho_1)^2.$$

If these two expansions are combined:

$$U = \frac{1}{2}(c_1 + c_2) + O(\rho_2 - \rho_1)^2 \tag{2.29}$$

where $c_1 = c_1(\rho_1)$ and $c_2 = c_2(\rho_2)$.

2.5 Weak Solutions

We already investigated the solution when discontinuities occur. In (2.18), we have written the shock velocity with respect to the flow quantities behind and ahead the shock. It is also need to be said that, this notation can also written as

$$-U[\rho] + [q] = 0. \tag{2.29}$$

This brackets shows us the jump in the quantity. When we replace the functional relation $q = Q(\rho)$ into these equations, we get for continuous parts

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q(\rho)}{\partial x} = 0, \tag{2.30}$$

for discontinuous parts

$$-U[\rho] + [Q(\rho)] = 0. \tag{2.31}$$

The expression (2.31) can be contemplated as a weak solution for (2.30).

If we multiply (2.30) by an arbitrary test function Φ, which is in $C^1(R)$, R denotes an arbitrary rectangle region in xt-plane and $\Phi = 0$ on the boundary of R and if we integrate with respect to x and t

$$\iint_R \left[\frac{\partial \rho}{\partial t} + \frac{\partial Q(\rho)}{\partial x}\right] \Phi dxdt = 0. \tag{2.32}$$

If ρ and $Q(\rho)$ are in $C^1(R)$, then we integrate this by parts. We get

$$-\iint_R [\rho \Phi_t + Q(\rho)\Phi_x]\, dxdt. \tag{2.33}$$

This result (2.33) makes the solution finding process easier since ρ and $Q(\rho)$ do not have to be differentiable. The weak solutions are $\rho(x,t)$ which satisfies (2.33) for all arbitrary continuous Φ test functions.

Knowing this let us expand this theory on a region R that consist a discontinuity illustrated in Figure 2.13.

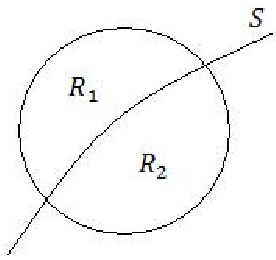

Figure 2.13

Here the solution $\rho(x,t)$ is continuously differentiable in R_1 and R_2 but not in S and there is a jump discontinuity across S. Let us write the expression (2.32) in such case

$$\iint_R \left[\frac{\partial\rho}{\partial t}+\frac{\partial Q(\rho)}{\partial x}\right]\Phi dxdt = \iint_{R_1} \left[\frac{\partial\rho}{\partial t}+\frac{\partial Q(\rho)}{\partial x}\right]\Phi dxdt + \iint_{R_2} \left[\frac{\partial\rho}{\partial t}+\frac{\partial Q(\rho)}{\partial x}\right]\Phi dxdt$$

$$+\iint_S [l[\rho]+m[Q(\rho)]]\Phi dxdt$$

where $\boldsymbol{v} = l\mathbf{i} + m\mathbf{j}$ denote the normal vectors to the line S. Since it is always perpendicular to S,

$$l[\rho]+m[Q(\rho)] = 0 \ on\ S.$$

This leads to a shock condition analog to (2.31) since $U = -l/m$. Also since $\rho(x,t)$ is continuously differentiable in R_1 and R_2, we know that (2.30) is valid for such case,

$$\iint_{R_1} \left[\frac{\partial\rho}{\partial t}+\frac{\partial Q(\rho)}{\partial x}\right]\Phi dxdt = \iint_{R_2} \left[\frac{\partial\rho}{\partial t}+\frac{\partial Q(\rho)}{\partial x}\right]\Phi dxdt = 0.$$

2.6 Shock Fitting for Quadratic $\boldsymbol{Q(\rho)}$

In this part, we want to seek a good fit for discontinuous shocks satisfying (2.19) into continuous solutions as

$$\rho = f(\xi),$$
$$x = \xi + F(\xi)t. \qquad (2.34a)$$

For this process we will consider $Q(\rho)$ is quadratic i.e. $Q(\rho) = a_1\rho^2 + a_2\rho + a_3$.

Before introducing the idea let us ponder a shock in a breaking wave in Figure 2.14. In this figure ξ_1 and ξ_2 are the values behind and ahead the shock.

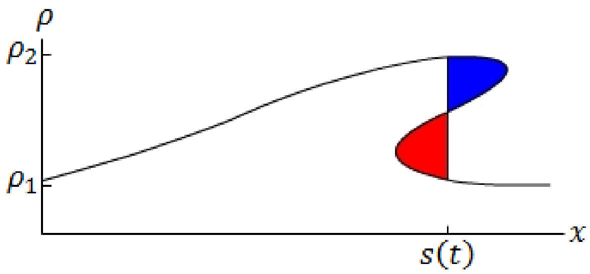

Figure 2.14

In shock fitting, the problem is determining the position of the line that splits the overhang. Whitham finds a good solution for this problem by separating the overhang with two lobes, which have equal areas [1]. The propagation velocity will be $c(\rho) = Q'(\rho) = 2a_1\rho + a_2$. If $\int \rho dx$ is conserved $2a_1 \int \rho dx + a_2 \int dx = \int 2a_1\rho + a_2 = \int cdx$ will also be conserved, because the relation between c and ρ is linear. So the general form of the ρx-curve and cx-curve will be the same as seen in Figure 2.15.

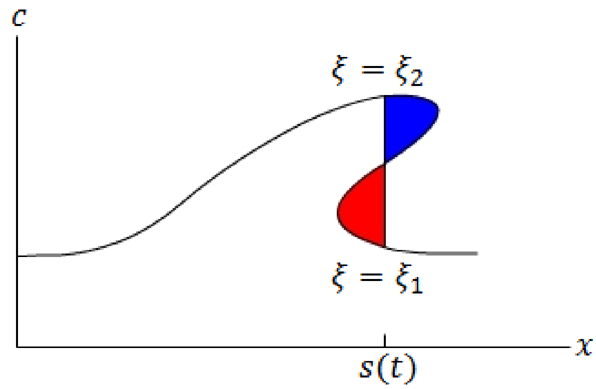

Figure 2.15

In order to simplify the situation, we will get help from continuous solution. When we adapt

$$c = F(\xi) = c(f(\xi)) \ and$$
$$x = \xi + F(\xi)t \tag{2.34b}$$

into shock which has a velocity (2.29), we get

$$U = \frac{1}{2}(c_1 + c_2) = \frac{1}{2}(F(\xi_1) + F(\xi_2)). \tag{2.35}$$

Whitham offers that c is also a solution for (see [1])

$$c_t + cc_x = 0\,. \tag{2.36}$$

Here taking $Q(c) = 1/2\, c^2$ also satisfies the conservation equation

$$c_t + (\frac{1}{2}c^2)_x = 0, \tag{2.37}$$

which is analog to (2.15) and whose weak solution is c. The same solution for shock velocity will be also obtained

$$U = \frac{\frac{1}{2}c_2^2 - \frac{1}{2}c_1^2}{(c_2 - c_1)} = \frac{1}{2}(c_1 + c_2). \tag{2.38}$$

In (2.34b), the formulation between x and ξ is given as $x = \xi + F(\xi)t \Rightarrow \xi = x - F(\xi)t$. So if we resketch the Figure 2.15 by dislocating x with $-F(\xi)t$ we infer the Figure 2.16. The vertical axis will be preserved, since $c = F(\xi)$.

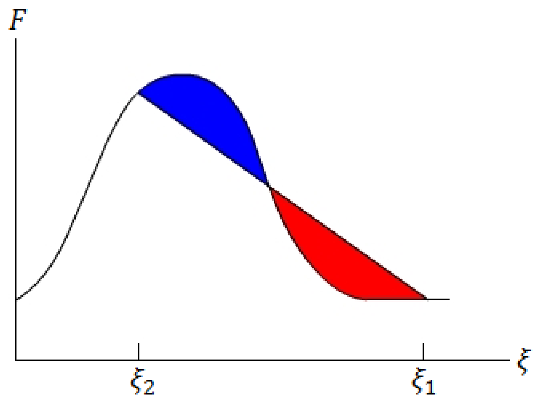

Figure 2.16

Subsequently, under the mapping the area is invariant, that is $A_1 = A_2$. So when we want to calculate the integral ξ_1 to ξ_2, it is helpful to see that the integral is equal to the area of the trapezoid in Figure 2.16 when we replace A_1 with A_2. That is the equal area property, which can analytically be expressed

$$\int_{\xi_2}^{\xi_1} F(\xi)d\xi = \frac{1}{2}[F(\xi_1) + F(\xi_2)](\xi_1 - \xi_2). \tag{2.39}$$

We also have

$$x = s(t) = \xi_1 + F(\xi_1)t \tag{2.40}$$

and

$$x = s(t) = \xi_2 + F(\xi_2)t \tag{2.41}$$

Now we have three functions $s(t)$, $\xi_1(t)$ and $\xi_2(t)$ with three equations.

3. APPLICATIONS IN TRAFFIC FLOW

As said before the main aim of the traffic engineer is to maximize the traffic flow by setting up traffic lights, stop signs, lane width, and maximum speed. Only the fundamental concepts will be discussed in this chapter.

3.1 Traffic Flow

In traffic flow, the velocity is defined as a function of the density, i.e. the ratio between the flow $q = Q(\rho)$ and the traffic density ρ:

$$V(\rho) = \frac{Q(\rho)}{\rho} \ \ or\ Q(\rho) = \rho V(\rho). \tag{3.1}$$

In traffic, we can infer some constraints. The density should have a maximum value since only a limited amount of cars can fill the road. This maximum flow is called the capacity of the road. At maximum value of density say $\rho = \rho_{max}$, cars are bumper to bumper like in Istanbul Traffic, the velocity will attain its minimum value which will be zero. This relation is shown in Figure 2.17.

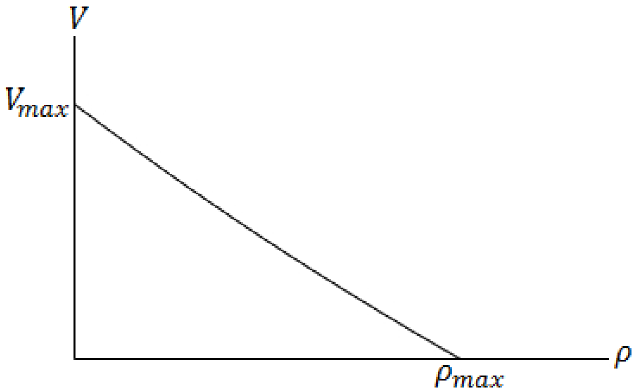

Figure 2.17

The flow may be zero in two cases. If there is no traffic say $\rho = 0$ or the traffic is not moving say $\rho = \rho_{max}$, the flow will be zero. For the other cases, $0 < \rho < \rho_{max}$, the flow must be positive. The relation between flow and density is illustrated in Figure 2.18.

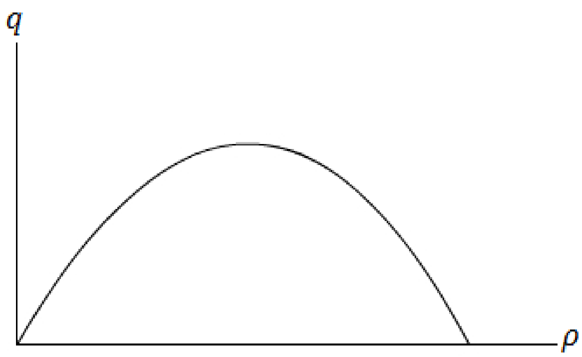

Figure 2.17

If we derive the flow $q = Q(\rho)$ with respect to ρ we get the propagation velocity as

$$c(\rho) = Q'(\rho) = V(\rho) + \rho V'(\rho). \tag{3.2}$$

Since $V'(\rho)$ is negative, can easily be seen in Figure 2.17, the propagation velocity will be less than the car velocity. That means drivers tend to decrease their velocity, when disturbances come ahead.

3.2 Discussions and Further Readings

As Haberman told the mankind has trouble with transportation problems. After the invention of the automobile new problems arise and transportation problems increase [2]. Where and how frequently the traffic engineers should set up the traffic lights, what should the period of the green-red traffic light be, what should the green and red light period rate be, how many lanes should be on the road? All these questions are considered for increasing the efficient traffic flow.

In order to increase the traffic efficiency, the cars should be forced to drive at a speed which makes the flow maximum which is considered in 3.1. This can be done in various ways. For example, red light stops the traffic in a while. After the light turns to green the cars flow more fluently. Bottlenecks can be given as another example. The density is very high before crossing the bridge and the density decreases after crossing the bridge, so the flow becomes more efficient.

Analysis of the shock structure and traffic light problems of traffic theory is discussed highly in [1] and [2].

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