Conservation Law Models for Traffic Flow

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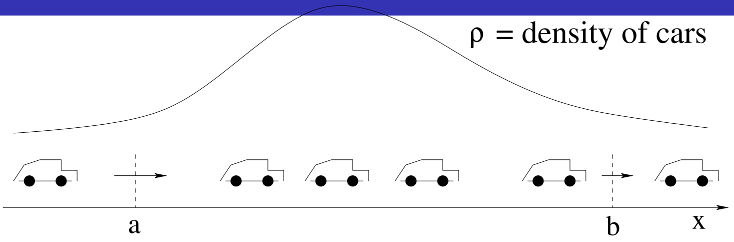
- Review of hyperbolic conservation laws
- Models of traffic flow, on a single road and on a network of roads
- Optimization problems
- Nash equilibria

A PDE model for traffic flow



$t=$ time, $\quad x=$ space variable along road, $\quad \rho=\rho(t,x)=$ density of cars

Total number of cars is conserved:

$$\frac{d}{dt}\int_a^b \rho(t,x)\,dx \;=\; [\text{flux of cars entering at } a] - [\text{flux of cars exiting at } b]$$

flux: $\quad = \;$ [number of cars crossing the point x per unit time]

$\quad = \;$ [density] $\times$ [velocity]

The Lighthill-Witham conservation law describing traffic flow

Assume: velocity of cars depends only on their density: $v = v(\rho)$

$$\frac{d}{dt}\int_a^b \rho(t,x)\,dx \;=\; [\text{flux of cars entering at } a] - [\text{flux of cars exiting at } b]$$

$$= \rho(t,a)\,v(\rho(t,a)) - \rho(t,b)\,v(\rho(t,b))$$

$$\int_a^b \frac{\partial}{\partial t}\rho\,dx \;=\; -\int_a^b \frac{\partial}{\partial x}[\rho\,v(\rho)]\,dx$$

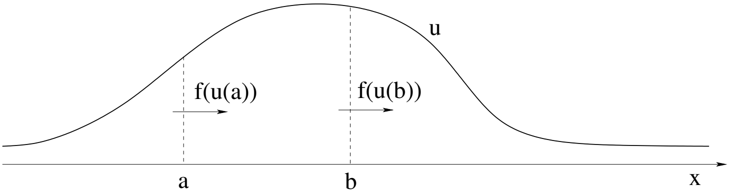
$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\big[\rho\,v(\rho)\big] \;=\; 0$$

The Scalar Conservation Law

$$\mathbf{u_t + f(u)_x = 0} \qquad \mathbf{u} = \text{conserved quantity}, \qquad \mathbf{f(u)} = \text{flux}$$

$$\frac{d}{dt}\int_a^b u(t,x)\,dx = \int_a^b u_t(t,x)\,dx = -\int_a^b f\big(u(t,x)\big)_x\,dx$$

$$= f\big(u(t,a)\big) - f\big(u(t,b)\big) = [\text{inflow at } a] - [\text{outflow at } b]$$



Weak solutions

$$\textbf{conservation equation:} \qquad \mathbf{u_t + f(u)_x \ = \ 0}$$

$$\textbf{quasilinear form:} \qquad \mathbf{u_t + f'(u) u_x \ = \ 0}$$

Conservation equation remains meaningful for $u = u(t, x)$ discontinuous, in distributional sense:

$$\iint \left\{ u\phi_t + f(u)\phi_x \right\} dxdt \ = \ 0 \qquad \text{for all} \quad \phi \in \mathcal{C}_c^1$$

Need only : $u, f(u)$ locally integrable

Convergence of weak solutions

$$\mathbf{u_t + f(u)_x \ = \ 0}$$

Assume: u_n is a solution, for every $n \geq 1$,

$$u_n \to u, \qquad f(u_n) \to f(u) \qquad \text{in} \quad \mathbf{L}^1_{loc}$$

then

$$\iint \left\{ u\phi_t + f(u)\phi_x \right\} dxdt \ = \ \lim_{n\to\infty} \iint \left\{ u_n\phi_t + f(u_n)\phi_x \right\} dxdt \ = \ 0$$

for all $\quad \phi \in \mathcal{C}^1_c$

(no need to check convergence of derivatives)

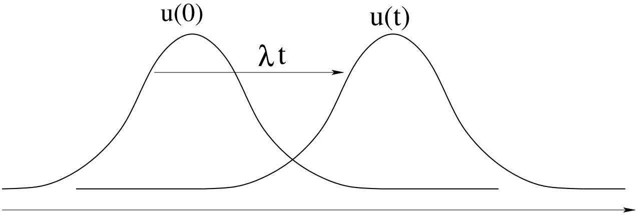
Scalar Equation with Linear Flux

$$u_t + f(u)_x \;=\; 0 \qquad\qquad f(u) = \lambda u$$

$$u_t + \lambda u_x \;=\; 0 \qquad\qquad u(0,x) = \phi(x)$$

Explicit solution: $\quad u(t,x) = \phi(x - \lambda t)$

traveling wave with speed $f'(u) \;=\; \lambda$



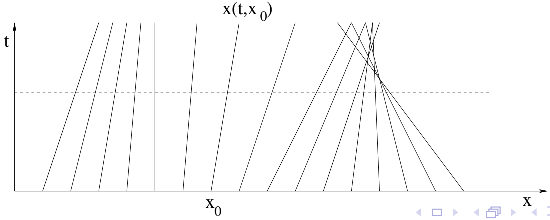
$$u_t + f'(u)u_x = 0 \qquad\qquad u(0,x) = \phi(x)$$

For each x_0, consider the straight line

$$t \mapsto x(t,x_0) = x_0 + tf'(\phi(x_0))$$

Set $u = \phi(x_0)$ along this line, so that $\dot{x}(t) = f'(u(t,x(t)))$. As long as characteristics do not cross, this yields a solution:

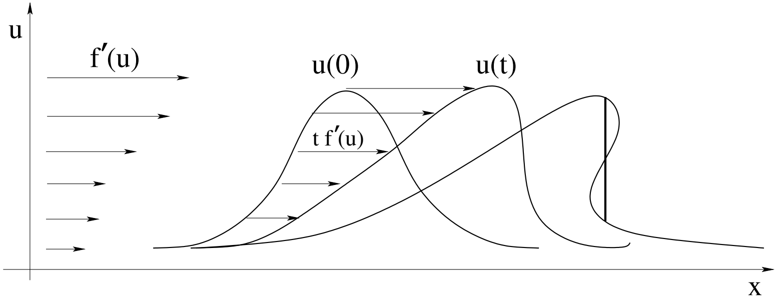
$$0 = \frac{d}{dt}u(t,x(t)) = u_t + \dot{x}u_x = u_t + f'(u)u_x$$



Loss of Regularity

$$u_t + f'(u)u_x \;=\; 0$$

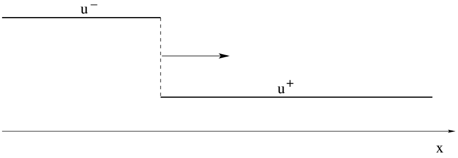
Assume: characteristic speed $f'(u)$ is not constant



Global solutions only in a space of discontinuous functions

$$u(t,\cdot) \in BV$$

Shocks



$$\mathbf{u_t + f(u)_x = 0}$$

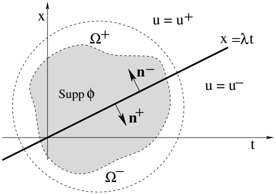
$$u(t,x) = \begin{cases} u^- & \text{if} \quad x < \lambda t \\ u^+ & \text{if} \quad x > \lambda t \end{cases}$$

is a weak solution if and only if

$$\lambda \cdot [u^+ - u^-] = f(u^+) - f(u^-)$$ **Rankine - Hugoniot equations**

[speed of the shock] $\times$ [jump in the state] $=$ [jump in the flux]

$$\iint \left\{ u\phi_t + f(u)\phi_x \right\} dxdt = 0 \qquad \text{for all} \quad \phi \in \mathcal{C}^1_c$$



$$\mathbf{v} \doteq \Big(u\phi\,,\ f(u)\phi \Big)$$

$$0 = \iint_{\Omega^+ \cup \Omega^-} \mathrm{div}\ \mathbf{v}\, dxdt = \int_{\partial\Omega^+} \mathbf{n}^+ \cdot \mathbf{v}\, ds + \int_{\partial\Omega^-} \mathbf{n}^- \cdot \mathbf{v}\, ds$$

$$= \int \big[\lambda u^+ - f(u^+)\big]\, \phi(t, \lambda t)\, dt + \int \big[- \lambda u^- + f(u^-)\big]\, \phi(t, \lambda t)\, dt$$

$$= \int \Big[\lambda(u^+ - u^-) - (f(u^+) - f(u^-))\Big]\, \phi(t, \lambda t)\, dt$$

Geometric interpretation

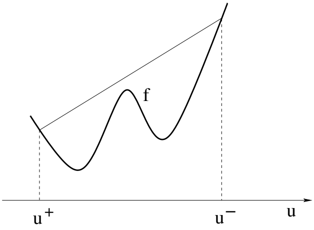
$$\lambda(u^+ - u^-) = f(u^+) - f(u^-) = \int_0^1 f'\big(\theta u^+ + (1-\theta)u^-\big) \cdot (u^+ - u^-)\, d\theta$$

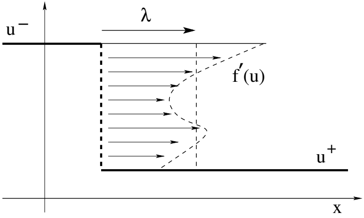
The Rankine-Hugoniot conditions hold if and only if the speed of the shock is

$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \int_0^1 f'\big(\theta u^+ + (1-\theta)u^-\big)\, d\theta$$

$$= \text{[average characteristic speed]}$$

scalar conservation law: $\quad u_t + f(u)_x = 0$





$$\lambda \;=\; \frac{f(u^+) - f(u^-)}{u^+ - u^-} \;=\; \frac{1}{u^+ - u^-}\int_{u^-}^{u^+} f'(s)\,ds$$

[speed of the shock] $=$ [slope of secant line through u^-, u^+ on the graph of f]

$=$ [average of the characteristic speeds between u^- and u^+]

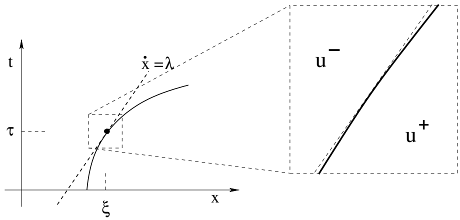
Points of approximate jump

The function $u = u(t,x)$ has an **approximate jump** at a point (τ,ξ) if there exists states $u^- \neq u^+$ and a speed λ such that, calling

$$U(t,x) \doteq \begin{cases} u^- & \text{if} \quad x < \lambda t, \\ u^+ & \text{if} \quad x > \lambda t, \end{cases}$$

there holds

$$\lim_{\rho\to 0+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi-\rho}^{\xi+\rho} \Big| u(t,x) - U(t-\tau,\ x-\xi) \Big| \, dxdt = 0$$



Theorem. *If u is a weak solution to a conservation law then the Rankine-Hugoniot equations hold at each point of approximate jump.*

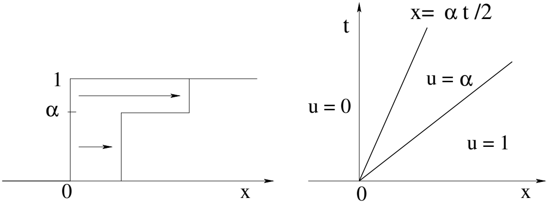
Weak solutions can be non-unique

Example: a Cauchy problem for Burgers' equation

$$u_t + (u^2/2)_x = 0 \qquad u(0,x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

Each $\alpha \in [0,1]$ yields a weak solution

$$u_\alpha(t,x) = \begin{cases} 0 & \text{if} \quad x < \alpha t/2 \\ \alpha & \text{if} \quad \alpha t/2 \leq x < (1+\alpha)t/2 \\ 1 & \text{if} \quad x \geq (1+\alpha)t/2 \end{cases}$$



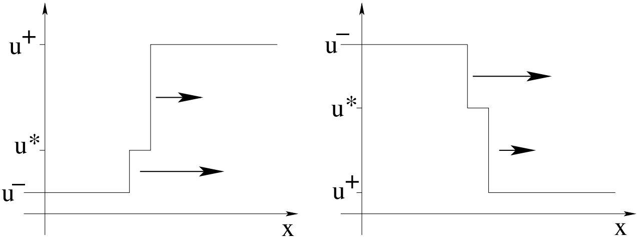
Stability conditions for shocks

Perturb the shock with left and right states u^-, u^+ by inserting an intermediate state $u^* \in [u^-, u^+]$

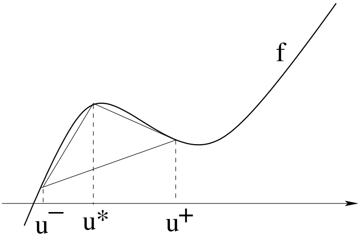
Initial shock is stable $\Longleftrightarrow$

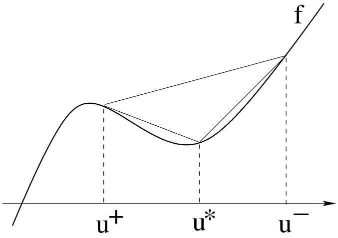
[speed of jump behind] $\geq$ [speed of jump ahead]

$$\frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^*)}{u^+ - u^*}$$



speed of a shock = slope of a secant line to the graph of f

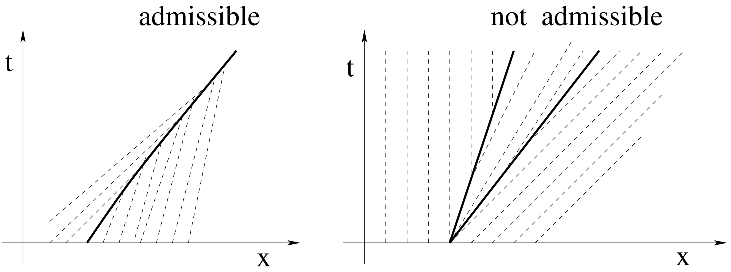




Stability conditions:

- when $u^- < u^+$ the graph of f should remain above the secant line
- when $u^- > u^+$, the graph of f should remain below the secant line

The Lax admissibility condition



A shock connecting the states u^-, u^+, travelling with speed $\lambda = \frac{f(u^+)-f(u^-)}{u^+-u^-}$ is *admissible* if

$$f'(u^-) \ \geq \ \lambda \ \geq \ f'(u^+)$$

i.e. characteristics do not move out from the shock from either side

Existence of solutions

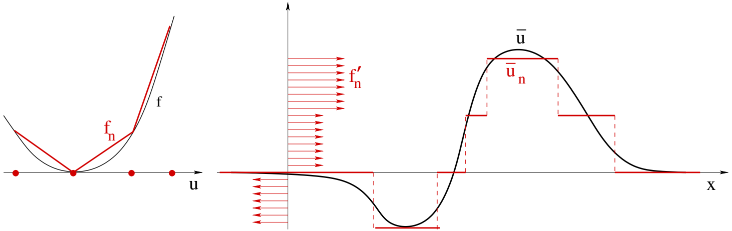
Cauchy problem: $\quad u_t + f(u)_x \;=\; 0\,, \qquad u(0,x) \;=\; \bar u(x)$

Polygonal approximations of the flux function (Dafermos, 1972)

Choose a piecewise affine function f_n such that

$$f_n(u) = f(u) \qquad u = j\cdot 2^{-n}\,, \qquad j\ \in \mathbb{Z}$$

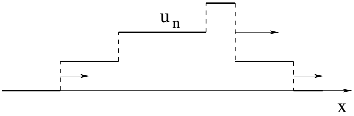
Approximate the initial data with a function $\bar u_n : \mathbb{R} \mapsto 2^{-n}\cdot\mathbb{Z}$

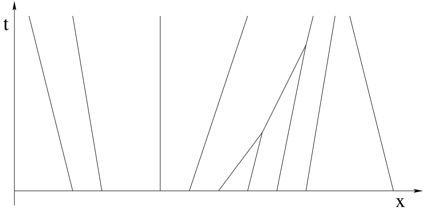


Front tracking approximations

piecewise constant approximate solutions: $u_n(t,x)$

$$(u_n)_t + f_n(u_n)_x \;=\; 0 \qquad\qquad u_n(0,x) = \bar u_n(x)$$





$$Tot.Var.(u_n(t,\cdot)) \;\le\; Tot.Var.(\bar u_n) \;\le\; Tot.Var.(\bar u)$$

$\Longrightarrow$ as $n\to\infty$, a subsequence converges in $\mathbf{L}^1_{loc}([0,\,T]\times\mathbb{R})$
to a weak solution $u = u(t,x)$

A contractive semigroup of entropy weak solutions

$$u_t + f(u)_x \;=\; 0$$

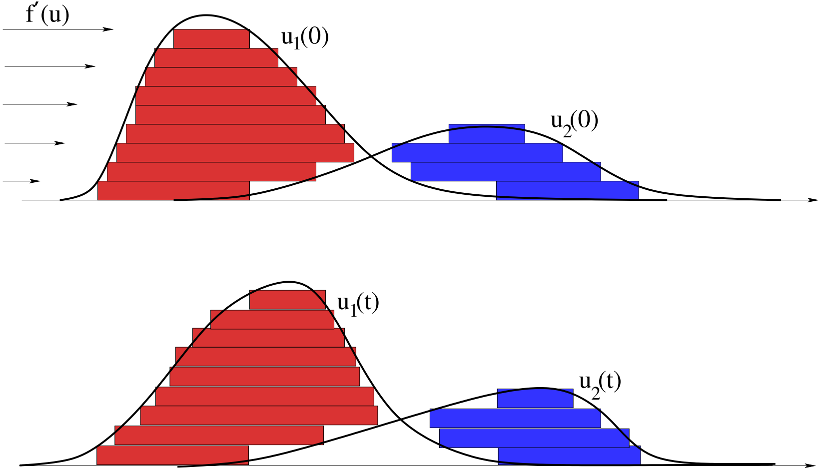
Two initial data in $\mathbf{L}^1(\mathbb{R})$: $\qquad u_1(0,x) = \bar u_1(x), \quad u_2(0,x) = \bar u_2(x)$

$\mathbf{L}^1$ - distance between solutions does not increase in time:

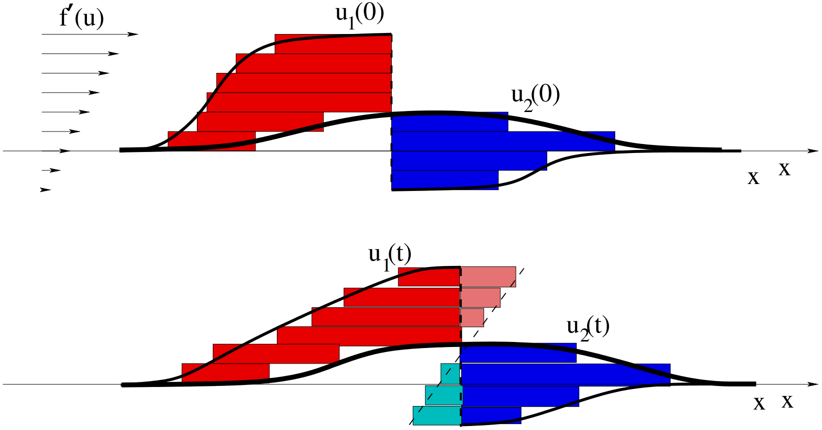
$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \;\leq\; \|\bar u_1 - \bar u_2\|_{\mathbf{L}^1(\mathbb{R})}$$

(not true for the $\mathbf{L}^p$ distance, $\; p > 1$)

The $\mathbf{L}^1$ distance between continuous solutions remains constant



The $\mathbf{L}^1$ distance decreases when a shock in one solution crosses the graph of the other solution



A related Hamilton-Jacobi equation

$$u_t + f(u)_x \;=\; 0 \qquad\qquad u(0,x) \;=\; \bar u(x)$$

$$U(t,x) \;=\; \int_{-\infty}^{x} u(t,y)\,dy$$

$$U_t + f(U_x) \;=\; 0 \qquad\qquad U(0,x) \;=\; \overline{U}(x) \;=\; \int_{-\infty}^{x} \bar u(y)\,dy$$

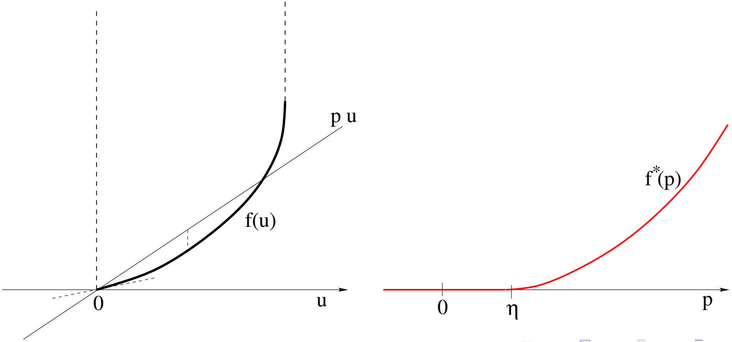
f convex $\qquad \Longrightarrow$

$U = U(t,x)$ is the value function for an optimization problem

Legendre transform

$$u \mapsto f(u) \in \mathbb{R} \cup \{+\infty\} \qquad \text{convex}$$

$$f^*(p) \doteq \max_u \{pu - f(u)\}$$

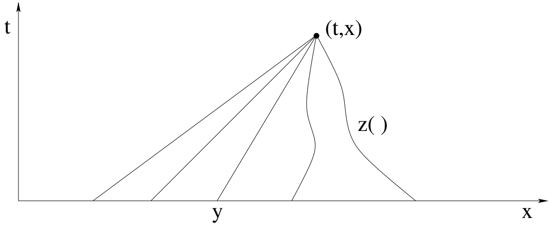


A representation formula

$$U_t + f(U_x) \;=\; 0 \qquad\qquad U(0,x) \;=\; \overline{U}(x)$$

$$U(t,x) \;=\; \inf_{z(\cdot)} \left\{ \int_0^t f^*(\dot z(s))\, ds + \overline{U}(z(0))\,; \qquad z(t) = x \right\}$$

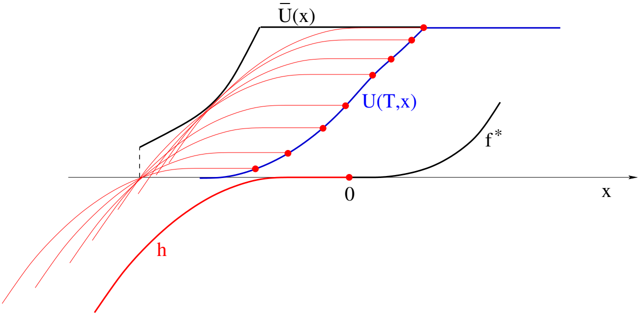
$$=\; \min_{y\in\mathbb{R}} \left\{ t\, f^*\left(\frac{x-y}{t}\right) + \overline{U}(y) \right\}$$



A geometric construction

$$U_t + f(U_x) \;=\; 0 \qquad\qquad U(0,x) \;=\; \overline{U}(x)$$

define $\quad h(s) \doteq -T\, f^*\left(\dfrac{-s}{T}\right)$



$$U(T,x) \;=\; \inf_y \left\{ \overline{U}(y) - h(y-x) \right\}$$

The Lax formula

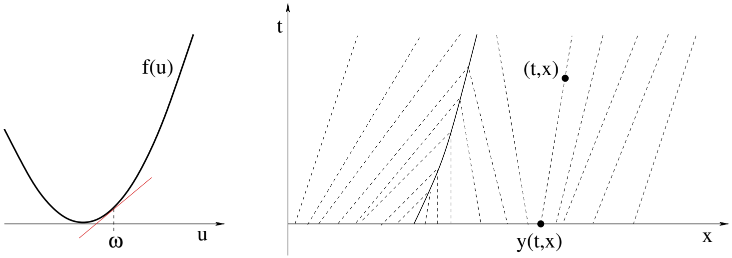
$$\text{Cauchy problem:} \qquad \begin{cases} u_t + f(u)_x & = & 0\,, \\ u(0,x) & = & \bar{u}(x) \end{cases}$$

For each $t > 0$, and all but at most countably many values of $x \in \mathbb{R}$, there exists a unique $y(t,\, x)$ s.t.

$$y(t,\, x) \;=\; \arg\min_{y\in\mathbb{R}} \left\{ t\, f^*\left(\frac{x-y}{t}\right) + \int_{-\infty}^{y} \bar{u}(s)\, ds \right\}$$

the solution to the Cauchy problem is

$$u(t,\, x) \;=\; (f')^{-1}\left(\frac{x - y(t,\, x)}{t}\right) \tag{1}$$



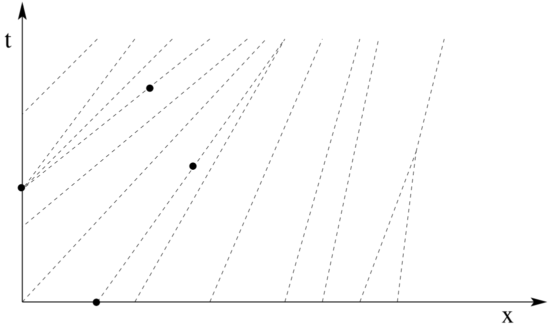
$$y(t,\, x) \;=\; \arg\min_{y\in\mathbb{R}} \left\{ t\, f^*\left(\frac{x-y}{t}\right) + \int_{-\infty}^{y} \bar{u}(s)\, ds \right\}$$

define the characteristic speed $\quad \xi \doteq \dfrac{x - y(t,x)}{t}$

if $\;f'(\omega) = \xi \qquad$ then $\qquad u(t,x) = \omega$

Initial-Boundary value problem

$$u_t + f(u)_x = 0 \qquad \begin{cases} u(0,x) = \bar u(x) & x > 0 \\ u(t,0) = b(t) & t > 0 \end{cases}$$



P. Le Floch, Explicit formula for scalar non-linear conservation laws with boundary condition, *Math. Models Appl. Sci.* (1988)

Systems of Conservation Laws

$$\begin{cases} \dfrac{\partial}{\partial t} u_1 + \dfrac{\partial}{\partial x} f_1(u_1, \ldots, u_n) & = & 0, \\ \qquad \cdot \quad \cdot \quad \cdot & & \\ \dfrac{\partial}{\partial t} u_n + \dfrac{\partial}{\partial x} f_n(u_1, \ldots, u_n) & = & 0 \end{cases}$$

$$u_t + f(u)_x = 0$$

$u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ conserved quantities

$f = (f_1, \ldots, f_n) : \mathbb{R}^n \mapsto \mathbb{R}^n$ fluxes

Hyperbolic Systems

$$u_t + f(u)_x = 0 \qquad\qquad u = u(t,x) \in \mathbb{R}^n$$

$$u_t + A(u)u_x = 0 \qquad\qquad A(u) = Df(u)$$

The system is **strictly hyperbolic** if each $n \times n$ matrix $A(u)$ has real distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u)$$

right eigenvectors	$r_1(u), \ldots, r_n(u)$	(column vectors)
left eigenvectors	$l_1(u), \ldots, l_n(u)$	(row vectors)

$$Ar_i = \lambda_i r_i \qquad\qquad l_i A = \lambda_i l_i$$

Choose bases so that $\quad l_i \cdot r_j = \begin{cases} 1 & \textit{if} \quad i = j \\ 0 & \textit{if} \quad i \neq j \end{cases}$

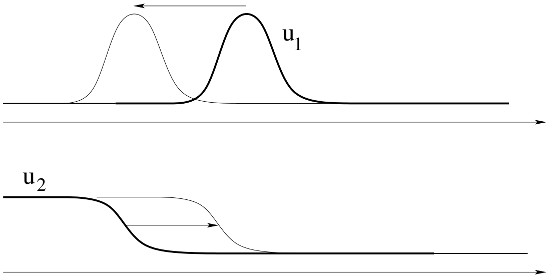
A linear hyperbolic system

$$u_t + Au_x = 0 \qquad\qquad u(0,x) = \phi(x)$$

$$\lambda_1 < \cdots < \lambda_n \quad \text{eigenvalues} \qquad\qquad r_1, \ldots, r_n \quad \text{eigenvectors}$$

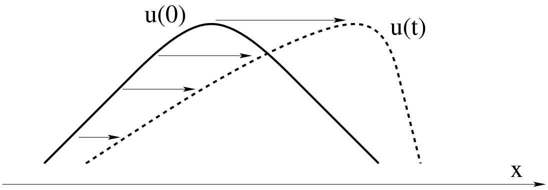
Explicit solution: **linear superposition of travelling waves**

$$u(t,x) = \sum_i \phi_i(x - \lambda_i t) r_i \qquad\qquad \phi_i(s) = l_i \cdot \phi(s)$$

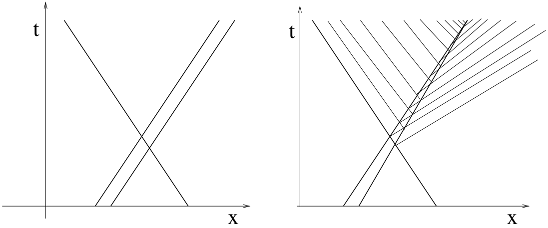


$$u_t + A(u)u_x = 0$$

eigenvalues depend on u $\Longrightarrow$ waves change shape



eigenvectors depend on u $\Longrightarrow$ nontrivial wave interactions



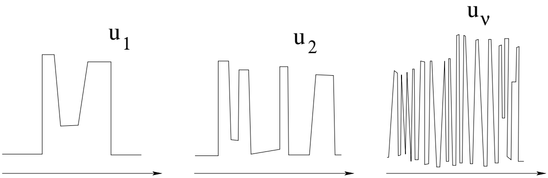
linear nonlinear

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar{u}(x)$$

- Construct a sequence of approximate solutions u_m

- Show that (a subsequence) converges: $u_m \to u$ in $\mathbf{L}^1_{loc}$

$$\Longrightarrow \quad u \text{ is a weak solution}$$



Need: a-priori bound on the total variation (J. Glimm, 1965)

Building block: the Riemann Problem

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$

B. Riemann 1860: 2×2 system of isentropic gas dynamics

P. Lax 1957: $n \times n$ systems (+ special assumptions)

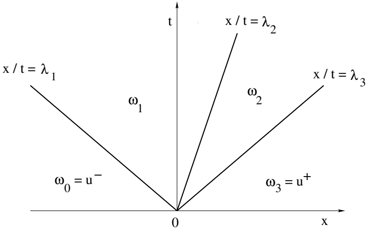
T. P. Liu 1975 $n \times n$ systems (generic case)

S. Bianchini 2003 (vanishing viscosity limit for general hyperbolic systems, possibly non-conservative)

invariant w.r.t. symmetry: $\quad u^\theta(t,x) \doteq u(\theta t, \theta x) \qquad \theta > 0$

Riemann Problem for Linear Systems

$$u_t + Au_x = 0 \qquad\qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$

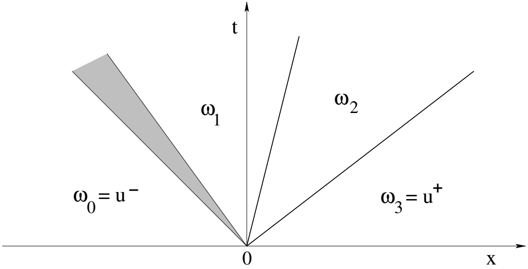


$$u^+ - u^- = \sum_{j=1}^{n} c_j r_j \qquad\qquad \text{(sum of eigenvectors of } A\text{)}$$

intermediate states : $\quad \omega_i \doteq u^- + \sum_{j \le i} c_j r_j$

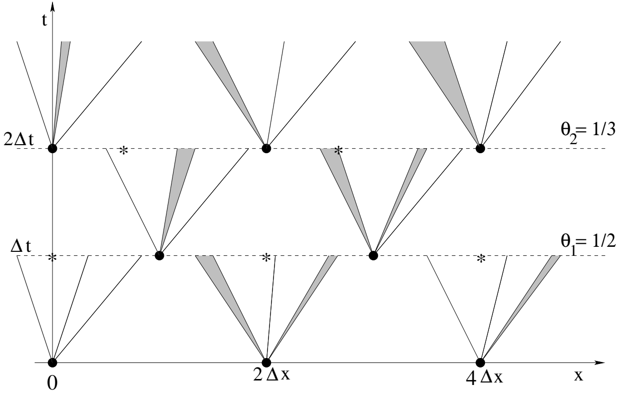
i-th jump: $\omega_i - \omega_{i-1} = c_i r_i$ travels with speed λ_i

General solution of the Riemann problem: concatenation of elementary waves

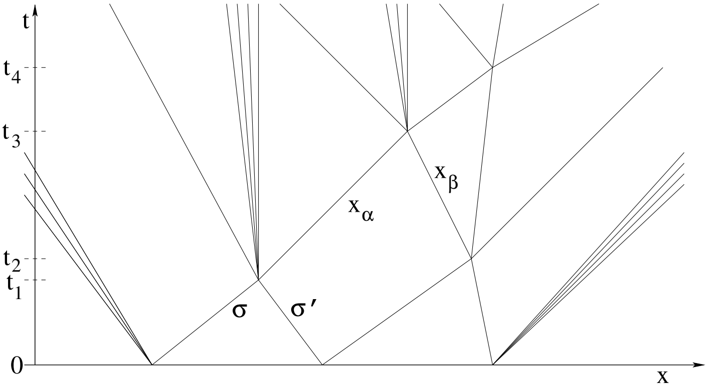


Construction of a sequence of approximate solutions

Glimm scheme: piecing together solutions of Riemann problems on a fixed grid in the *t*-*x* plane



Front tracking scheme: piecing together piecewise constant solutions of Riemann problems at points where fronts interact



Existence of solutions

$$u_t + f(u)_x = 0, \qquad\qquad u(0,x) = \bar{u}(x)$$

Theorem (Glimm 1965).

Assume:

• *system is strictly hyperbolic (+ some technical assumptions)*

Then there exists $\delta > 0$ *such that, for every initial condition* $\bar{u} \in \mathbf{L}^1(\mathbb{R};\ \mathbb{R}^n)$ *with*

$$Tot.Var.(\bar{u}) \leq \delta,$$

the Cauchy problem has an entropy admissible weak solution $u = u(t,x)$ *defined for all* $t \geq 0$.

Uniqueness and continuous dependence on the initial data

$$u_t + f(u)_x \;=\; 0 \qquad\qquad u(0,x) = \bar u(x)$$

Theorem (A.B.- R.Colombo, B.Piccoli, T.P.Liu, T.Yang, 1994-1998).

For every initial data $\bar u$ with small total variation, the front tracking approximations converge to a unique limit solution $u : [0,\infty[\mapsto \mathbf{L}^1(\mathbb{R})$.

The flow map $(\bar u, t) \mapsto u(t,\cdot) \doteq S_t \bar u$ is a uniformly Lipschitz semigroup:

$$S_0 \bar u = \bar u, \qquad S_s(S_t \bar u) = S_{s+t} \bar u$$

$$\big\| S_t \bar u - S_s \bar v \big\|_{\mathbf{L}^1} \;\leq\; L \cdot \big(\|\bar u - \bar v\|_{\mathbf{L}^1} + |t-s| \big) \qquad \text{for all} \quad \bar u, \bar v, \quad s,t \geq 0$$

Theorem (A.B.- P. LeFloch, M.Lewicka, P.Goatin, 1996-1998).

Any entropy weak solution to the Cauchy problem coincides with the limit of front tracking approximations, hence it is unique

Vanishing viscosity approximations

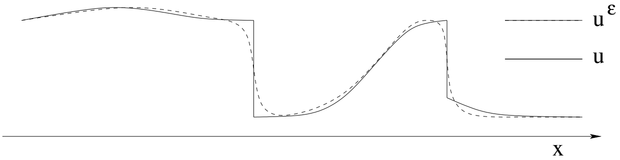
Claim: weak solutions of the hyperbolic system

$$u_t + f(u)_x \;=\; 0$$

can be obtained as limits of solutions to the parabolic system

$$u^\varepsilon_t + f(u^\varepsilon)_x \;=\; \varepsilon\, u^\varepsilon_{xx}$$

letting the viscosity $\varepsilon \to 0+$



Theorem (S. Bianchini, A. Bressan, *Annals of Math.* 2005)

Consider a strictly hyperbolic system with viscosity

$$u_t + A(u)u_x = \varepsilon\, u_{xx} \qquad\qquad u(0,x) = \bar u(x)\,. \tag{CP}$$

If Tot.Var.$\{\bar u\}$ is sufficiently small, then (CP) admits a unique solution $u^\varepsilon(t,\cdot) = S_t^\varepsilon \bar u$, defined for all $t \geq 0$. Moreover

$$\text{Tot.Var.}\big\{S_t^\varepsilon \bar u\big\} \;\leq\; C\,\text{Tot.Var.}\{\bar u\}\,, \tag{\textbf{BV bounds}}$$

$$\Big\|S_t^\varepsilon \bar u - S_t^\varepsilon \bar v\Big\|_{\mathbf{L}^1} \;\leq\; L\,\|\bar u - \bar v\|_{\mathbf{L}^1} \tag{$\mathbf{L^1}$ \textbf{stability}}$$

(Convergence) If $A(u) = Df(u)$, then as $\varepsilon \to 0$, the viscous solutions u^ε converge to the unique entropy weak solution of the system of conservation laws

$$u_t + f(u)_x \;=\; 0$$

Main open problems

- Global existence of solutions to hyperbolic systems for initial data $\bar{u}$ with **large total variation**

- Existence of entropy weak solutions for systems in **several space dimensions**

Part 2 - Modeling traffic flow

- engineering models
- microscopic models
- kinetic models
- macroscopic models

D. Helbing, A. Hennecke, and V. Shvetsov, Micro- and macro-simulation of freeway traffic. *Math. Computer Modelling* **35** (2002).

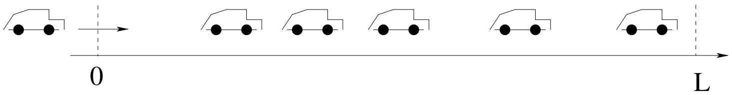
N. Bellomo, M. Delitala, V. Coscia, On the mathematical theory of vehicular traffic flow I. Fluid dynamic and kinetic modelling. *Math. Models Appl. Sci.* **12** (2002).

M. Garavello and B. Piccoli, *Traffic Flow on Networks. Conservation Laws Models.* AIMS Series on Applied Mathematics, Springfield, Mo., 2006.

A delay model (T. Friesz et al., 1993)

$X(t) \;=\;$ number of cars on a road at time t

If a new car enters at time t, it will exit at time $t + D(X(t))$



$D(X) =$ delay $=$ total time needed to travel along the road

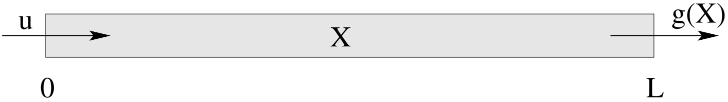
depends only on the total number of cars at the time of entrance

An ODE model (D. Merchant and G. Nemhauser, 1978)

$X(t)$ = total number of cars on a road at time t

$u(t)$ = incoming flux $\qquad$ g(X(t)) = outgoing flux

$$\dot{X}(t) = u(t) - g(X(t)) \qquad \text{conservation equation}$$



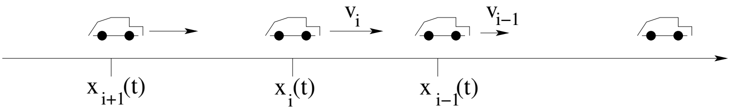
$$L = \text{ length of road,} \qquad \rho \approx \frac{X}{L} = \text{ density of cars}$$

$$g(X) = \rho\, v(\rho) = \frac{X}{L} \cdot v\left(\frac{X}{L}\right)$$

Models favored by engineers:

- simple to use, do not require knowledge of PDEs (or even ODEs)
- easy to compute, also on a large network of roads
- become accurate when the road is partitioned into short subintervals

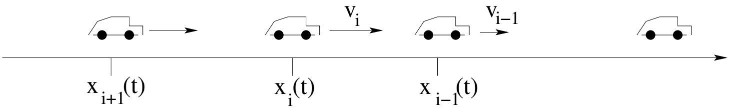
Microscopic models



$$\begin{aligned} x_i(t) &= \text{position of the } i\text{-th car} \\ v_i(t) &= \text{velocity of the } i\text{-th car} \end{aligned} \qquad i = 1, \ldots, N$$

Goal: describe the position and velocity of each car, writing a large system of ODEs

Car following models



Acceleration of i-th car depends on:

- its speed: v_i
- speed of car in front: v_{i-1}
- distance from car in front: $x_{i-1} - x_i$

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = a(v_i,\ v_{i-1},\ x_{i-1} - x_i) \end{cases} \qquad i = 1, \ldots, N$$

Microscopic intelligent driver model (Helbing & al., 2002)

$$i\text{-th driver} \quad \begin{cases} \text{accelerates, up to the maximum speed } \bar{v} \\ \text{decelerates, to keep a safe distance from the car in front} \end{cases}$$

$\bar{v}$ = maximum speed allowed on the road $\qquad v_i \in [0, \bar{v}]$

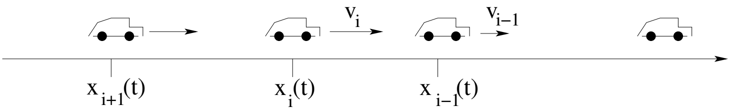
a = maximum acceleration

$$\dot{v}_i \;=\; a \cdot \left[1 - \left(\frac{v_i}{\bar{v}}\right)^{\delta}\right] - a \cdot \left(\frac{s^*(v_i,\ \Delta v_i)}{s_i}\right)^2$$

$$s_i \;=\; x_{i-1} - x_i \;=\; \text{actual gap from vehicle in front}$$

$$s_i^* \;=\; \text{desired gap}$$

Desired gap from the vehicle in front



$$s_i^* \;=\; \sigma_0 + \sigma_1 \sqrt{\frac{v_i}{\bar v}} + T v_i + \frac{v_i\,\Delta v_i}{2\sqrt{a\,b}} \;=\; \text{desired gap}$$

$\Delta v_i \;=\; v_i - v_{i-1} \;=\;$ speed difference with car in front

$\sigma_0 =$ jam distance (bumper to bumper)

$\sigma_1 =$ velocity adjustment of jam distance

$T =$ safe time headway

$b =$ comfortable deceleration

Equilibrium traffic

Assume: all cars have the same speed, constant in time.
Choose $\sigma_0 = \sigma_1 = 0$, $\delta = 1$

$$\dot{v}_i \;=\; a \cdot \left[1 - \frac{v_i}{\bar{v}}\right] - a \cdot \left(\frac{s^*(v_i,\ \Delta v_i)}{s_i}\right)^2 \;=\; 0$$

Equilibrium gap from vehicle in front

$$s_e(v) \;=\; s^*(v, 0) \cdot \left[1 - \frac{v_i}{\bar{v}}\right]^{-1/2}$$

Equilibrium velocity: $\qquad v_e(s) \;=\; \dfrac{s^2}{2\bar{v}T^2}\left(-1 + \sqrt{\dfrac{4T^2\bar{v}^2}{s^2}}\right)$

$$\Longrightarrow \qquad v_e \;=\; V_e(\rho) \qquad\qquad \rho \approx s^{-1} \;=\; \text{macroscopic density}$$

Statistical (kinetic) description

$f \;=\; f(t,x,V)$ statistical distribution of position and velocity of vehicles

$$f(t,x,V)\,dxdV \;=\; \begin{array}{l}\text{number of vehicles which at time } t\\ \text{are in the phase domain } [x,\ x+dx]\times[V,\ V+dV]\end{array}$$

local density: $$\rho(t,x) \;=\; \int_0^\infty f(t,x,V)\,dV$$

average velocity: $$v(t,x) \;=\; \frac{1}{\rho(t,x)}\int_0^\infty V\cdot f(t,x,V)\,dV$$

Evolution of the distribution function

$$\frac{\partial f}{\partial t} + V\,\frac{\partial f}{\partial x} + a(t,x)\frac{\partial f}{\partial V} ~=~ Q[f,\rho]$$

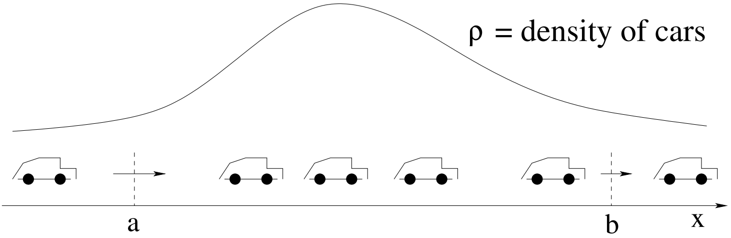
$a(t,x) =$ acceleration (may depend on the entire distribution f)

$Q(f,\rho)$ models a trend to equilibrium (as for BGK model in kinetic theory)

$$Q ~=~ c_r(\rho)\cdot\Big(f_e(V,\rho) - f(t,x,V)\Big)$$

$c_r =$ relaxation rate

A conservation law model (M. Lighthill and G. Witham, 1955)



$t =$ time, $x =$ space variable along road, $\rho = \rho(t,x) =$ density of cars

flux: $=$ [number of cars crossing the point x per unit time]

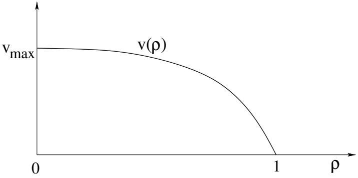
$$= \text{[density]} \times \text{[velocity]} = \rho \cdot v \qquad \color{red}{v = V(\rho)}$$

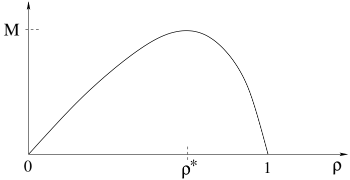
$$\rho_t + \big[\rho\, V(\rho)\big]_x = 0$$

Flux function

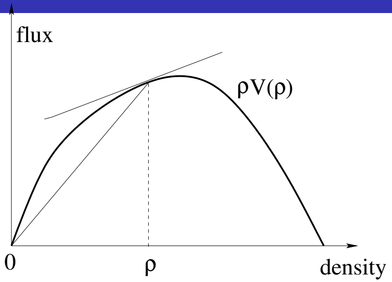
Assume: $\rho \mapsto \rho V(\rho)$ is concave

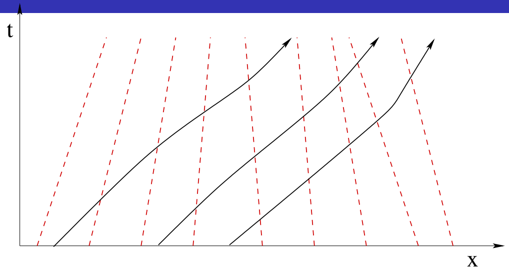
$$V'(\rho) \; < \; 0\,, \qquad\qquad 2V'(\rho) + \rho V''(\rho) \; < \; 0$$





Characteristics vs. car trajectories

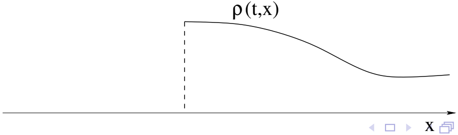




$$[\rho V(\rho)]' \quad = \quad V(\rho) + \rho V'(\rho) \quad < \quad V(\rho)$$

characteristic speed < speed of cars

Weak solutions can have upward shocks



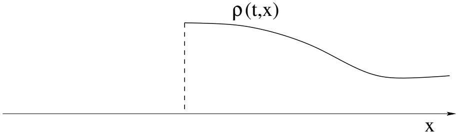
Adding a viscosity ?

$$\rho_t + \big[\rho\, V(\rho)\big]_x \;=\; 0 \qquad\qquad (\;=\;\varepsilon\rho_{xx}\;)$$

$$\rho_t + \left[\rho\left(V(\rho) - \varepsilon\frac{\rho_x}{\rho}\right)\right]_x \;=\; 0$$

effective velocity of cars: $v \;=\; V(\rho) - \varepsilon\dfrac{\rho_x}{\rho}$

can be negative, at the beginning of a queue



Second order models

$v = V(\rho) \quad \Longrightarrow \quad$ velocity is instantly adjusted to the density

Models with acceleration

$$\begin{cases} \rho_t + (\rho v)_x &= 0 \\ v_t + v\, v_x &= a(\rho, v, \rho_x) \end{cases} \qquad a = \text{acceleration}$$

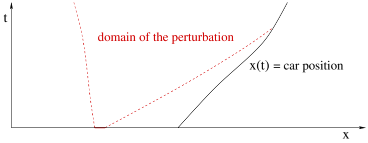
$$\begin{cases} \rho_t + (\rho v)_x &= 0 \\ v_t + v\, v_x &= \frac{1}{\tau}(V(\rho) - v) - \frac{p'(\rho)}{\rho}\,\rho_x \end{cases} \qquad \text{(Payne - Witham, 1971)}$$

[relaxation] $+$ [pressure term] $\qquad p = \rho^\gamma, \quad \gamma > 0$

$$\begin{cases} \rho_t + (\rho v)_x & = & 0 \\ v_t + v\, v_x + \frac{p'(\rho)}{\rho}\, \rho_x & = & \frac{1}{\tau}(V(\rho) - v) \end{cases}$$

$$\begin{pmatrix} \rho_t \\ v_t \end{pmatrix} + \begin{pmatrix} v & \rho \\ p'(\rho)/\rho & v \end{pmatrix} \begin{pmatrix} \rho_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

eingenvalues $=$ characteristic speeds: $v \pm \sqrt{p'(\rho)}$



Wrong predictions: • negative speeds

• perturbations travel faster than the speed of cars

Idea: replace the partial derivative of the pressure $\partial_x p$ with the *convective derivative* $(\partial_t + v\partial_x)p$

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) &= 0 \\ \partial_t(v + p(\rho)) + v\partial_x(v + p(\rho)) &= 0 \end{cases} \qquad \text{(Aw - Rascle)}$$

$$\begin{pmatrix} \rho_t \\ v_t \end{pmatrix} + \begin{pmatrix} v & \rho \\ 0 & v - \rho p'(\rho) \end{pmatrix} \begin{pmatrix} \rho_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

strictly hyperbolic for $\rho > 0$, positive speed: $v + p(\rho) \geq 0$

eigenvalues: $\lambda_1 = v - \rho p'(\rho), \quad \lambda_2 = v$

Properties of the Aw-Rascle model

- system is strictly hyperbolic (away from vacuum)
- the density ρ and the velocity v remain bounded and non-negative
- characteristic speeds (= eigenvalues) are smaller than car speed
  $\Longrightarrow$ drivers are not influenced by what happens behind them.
- maximum speed of cars on an empty road depends on initial data

An improved model (R. M. Colombo, 2002)

$$\text{Aw - Rascle:} \qquad \begin{cases} \partial_t \rho + \partial_x (v\rho) & = & 0 \\ \partial_t q + \partial_x (vq) & = & 0 \end{cases}$$

$$q \; = \; v\rho + \rho p(\rho) \; = \; \text{"momentum"}$$

$$\text{Colombo:} \qquad \begin{cases} \partial_t \rho + \partial_x (v\rho) & = & 0 \\ \partial_t q + \partial_x \big(v(q - q_{max})\big) & = & 0 \end{cases}$$

$$v \; = \; \left(\frac{1}{\rho} - \frac{1}{\rho_{max}} \right) q$$

ρ_{max} = maximum density $\qquad$ q_{max} = "maximum momentum"

$\Longrightarrow$ velocity can vanish only when $\rho = \rho_{max}$, and remains uniformly bounded

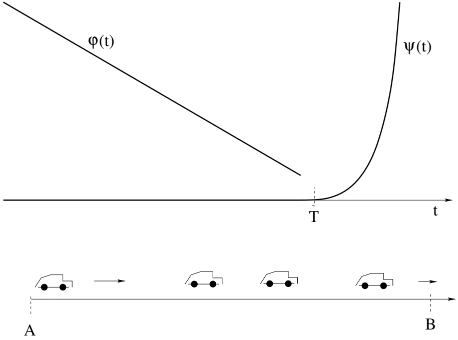
Concluding remarks

Number of vehicles on a road $<<$ number of molecules in a gas

- microscopic models (solving an ODE for each car) are within computational reach
- kinetic models and macroscopic models are realistic on longer stretches of road, for densities away from vacuum
- optimization problems, dependence of solution on parameters, are better understood by studying macroscopic models
- Simple ODE models, delay models are popular among engineers.
  Scalar conservation laws are OK.
  Kinetic models, second order models, are a hard sell.

Part 3 - Optimization problems for traffic flow

- Car drivers starting from a location A (a residential neighborhood) need to reach a destination B (a working place) at a given time T.
- There is a cost $\varphi(\tau_d)$ for departing early and a cost $\psi(\tau_a)$ for arriving late.



Elementary solution

L = length of the road, v = speed of cars

$$\tau_a \;=\; \tau_d + \frac{L}{v}$$

Optimal departure time:

$$\tau_d^{\rm opt} \;=\; \operatorname*{argmin}_t \left\{ \varphi(t) + \psi\left(t + \frac{L}{v}\right)\right\}.$$

If everyone departs exactly at the same optimal time,
a traffic jam is created and this strategy is not optimal anymore.

An optimization problem for traffic flow

Problem: choose the departure rate $\bar{u}(t)$ in order to minimize the total cost to all drivers.

$$u(t,x) \doteq \rho(t,x) \cdot v(\rho(t,x)) = \text{flux of cars}$$

$$\text{minimize:} \qquad \int \varphi(t) \cdot u(t,0)\, dt + \int \psi(t) u(t,L)\, dt$$

for a solution of

$$\begin{cases} \rho_t + [\rho\, v(\rho)]_x = 0 & x \in [0,L] \\ \rho(t,0) v(\rho(t,0)) = \bar{u}(t) & \end{cases}$$

Choose the optimal departure rate $\bar{u}(t)$, subject to the constraint

$$\int \bar{u}(t)\, dt = \kappa = [\text{total number of drivers}]$$

Equivalent formulations

Boundary value problem for the density ρ:

conservation law: $\rho_t + [\rho v(\rho)]_x = 0, \qquad (t, x) \in \mathbb{R} \times [0, L]$

control (on the boundary data): $\rho(t, 0)v(\rho(t, 0)) = \bar{u}(t)$

Cauchy problem for the flux u:

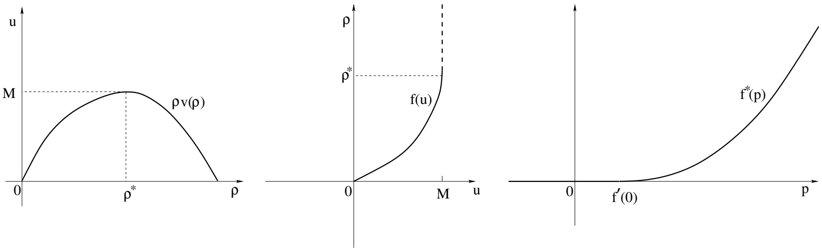
conservation law: $u_x + f(u)_t = 0, \qquad u = \rho\, v(\rho), \qquad f(u) = \rho$

control (on the initial data): $u(t, 0) = \bar{u}(t)$

$$\text{Cost:} \quad J(u) = \int_{-\infty}^{+\infty} \varphi(t)u(t, 0)\, dt + \int_{-\infty}^{+\infty} \psi(t)u(t, L)\, dt$$

$$\text{Constraint:} \quad \int_{-\infty}^{+\infty} \bar{u}(t)\, dt = \kappa$$

The flux function and its Legendre transform



$$u \;=\; \rho\, v(\rho)\,, \qquad\qquad \rho \;=\; f(u)$$

Legendre transform: $\qquad f^*(p) \;\doteq\; \max_u \Big\{ pu - f(u) \Big\}$

Solution to the conservation law is provided by the Lax formula

The globally optimal (Pareto) solution

minimize: $$J(u) \;=\; \int \varphi(x)\cdot u(0,x)\,dx + \int \psi(x)\,u(T,x)\,dx$$

subject to: $$\begin{cases} u_t + f(u)_x \;=\; 0 \\ u(0,x) \;=\; \bar u(x)\,, \qquad \displaystyle\int \bar u(x)\,dx \;=\; \kappa \end{cases}$$

(A1) The flux function $f : [0, M] \mapsto \mathbb{R}$ is continuous, increasing, and strictly convex. It is twice continuously differentiable on the open interval $]0, M[$ and satisfies

$$f(0) = 0\,, \quad \lim_{u\to M-} f'(u) = +\infty, \qquad f''(u) \geq b > 0 \quad \text{for} \quad 0 < u < M$$

(A2) The cost functions φ, ψ satisfy $\varphi' \;<\; 0, \quad \psi, \psi' \;\geq\; 0,$

$$\lim_{x\to-\infty} \varphi(x) \;=\; +\infty\,, \qquad \lim_{x\to+\infty} \Big(\varphi(x) + \psi(x)\Big) \;=\; +\infty$$

Existence and characterization of the optimal solution

Theorem (A.B. and K. Han, 2011). Let **(A1)-(A2)** hold. Then, for any given T, κ, there exists a unique admissible initial data $\bar{u}$ minimizing the cost $J(\cdot)$. In addition,

1. No shocks are present, hence $u = u(t,x)$ is continuous for $t > 0$. Moreover

$$\sup_{t\in[0,T],\, x\in\mathbb{R}} u(t,\, x) \;<\; M$$

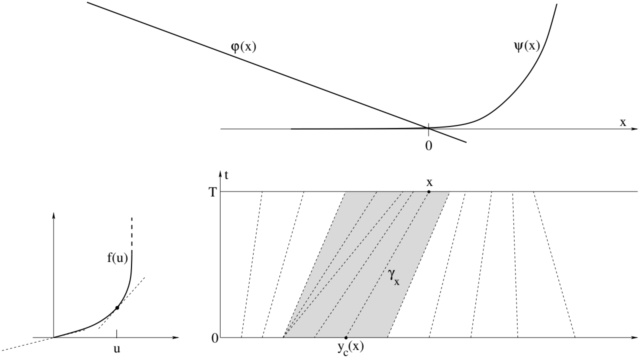
2. For some constant $c = c(\kappa)$, this optimal solution admits the following characterization: For every $x \in \mathbb{R}$, let $y_c(x)$ be the unique point such that

$$\varphi(y_c(x)) + \psi(x) \;=\; c$$

Then, the solution $u = u(t,x)$ is constant along the segment with endpoints $(0, y_c(x))$, (T, x).

Indeed, either $f'(u) \;\equiv\; \frac{x-y_c(x)}{T}$, or $u \equiv 0$

Necessary conditions



$$\varphi(y_c(x)) + \psi(x) \;=\; c$$

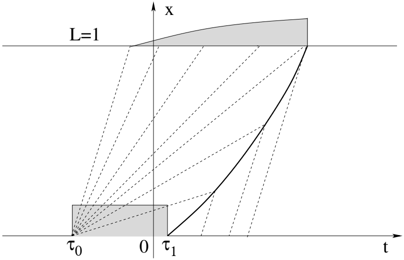
$$f'(u) \;=\; \frac{x - y_c(x)}{T} \qquad \text{on the characteristic segment} \;\; \gamma_x$$

An Example

Cost functions: $\varphi(t) = -t, \qquad \psi(t) = \begin{cases} 0, & \text{if } t \le 0 \\ t^2, & \text{if } t > 0 \end{cases}$

$$L = 1, \quad u = \rho(2-\rho), \quad M = 1, \quad \kappa = 3.80758$$

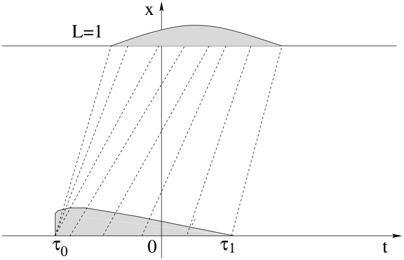
Bang-bang solution



$\tau_0 = -2.78836, \qquad \tau_1 = 1.01924$

total cost $= 5.86767$

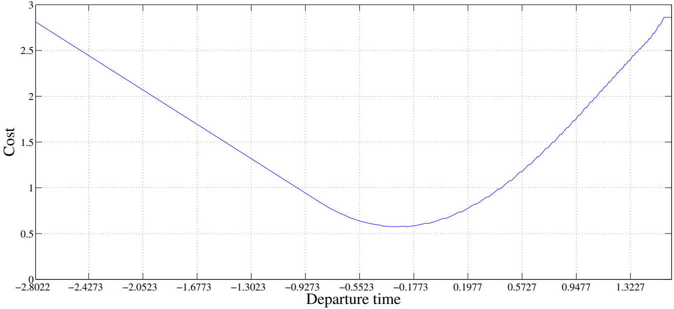
Pareto optimal solution



$\tau_0 = -2.8023, \qquad \tau_1 = 1.5976$

total cost $= 5.5714$

Does everyone pay the same cost?



Departure time vs. cost in the Pareto optimal solution

The Nash equilibrium solution

A solution $u = u(t,x)$ is a Nash equilibrium if no driver can reduce his/her own cost by choosing a different departure time.
This implies that all drivers pay the same cost.

To find a Nash equilibrium, write the conservation law $u_t + f(u)_x = 0$ in terms of a Hamilton-Jacobi equation

$$U_t + f(U_x) = 0 \qquad\qquad U(0,x) = Q(x)$$

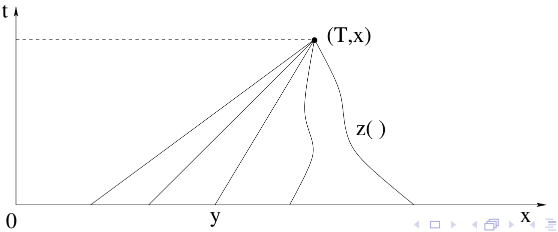
$$U(t,x) \doteq \int_{-\infty}^{x} u(t,y)\,dy$$

A representation formula

$$U_t + f(U_x) \;=\; 0 \qquad\qquad U(0,x) \;=\; Q(x)$$

$$U(T,x) \;=\; \inf_{z(\cdot)} \left\{ \int_0^T f^*(\dot z(s))\,ds + Q(z(0))\,; \qquad z(T) = x \right\}$$

$$= \; \min_{y\in\mathbb{R}} \left\{ T\, f^*\left(\frac{x-y}{T}\right) + Q(y) \right\}$$



No constraint can be imposed on the departing rate, so a queue can form at the entrance of the highway.

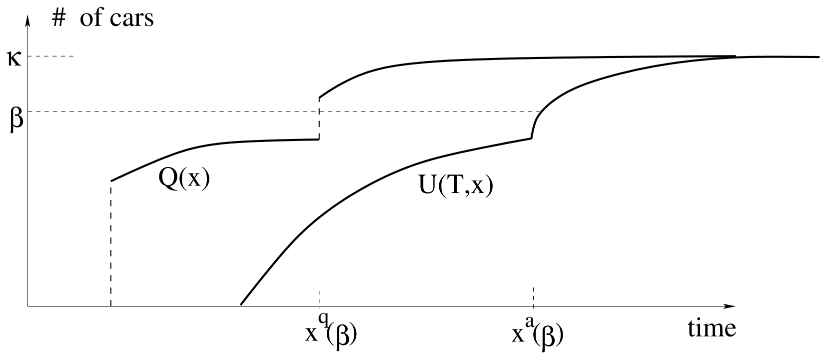
$x \mapsto Q(x) =$ number of drivers who have started their journey before time x (joining the queue, if there is any).

$$Q(-\infty) = 0, \qquad\qquad Q(+\infty) = \kappa$$

$x \mapsto U(T,x) =$ number of drivers who have reached destination within time x

$$U(T,\, x) = \min_{y\in\mathbb{R}} \left\{ T\, f^* \left(\frac{x-y}{T} \right) + Q(y) \right\}$$

Characterization of a Nash equilibrium



$\beta \in [0, \kappa]$ = Lagrangian variable labeling one particular driver

$x^q(\beta)$ = time when driver β departs (possibly joining the queue)

$x^a(\beta)$ = time when driver β arrives at destination

Existence and Uniqueness of Nash equilibrium

Departure and arrival times are implicitly defined by

$$Q(x^q(\beta)-) \;\leq\; \beta \;\leq\; Q(x^q(\beta)+)\,, \qquad\qquad U(T, x^a(\beta)) \;=\; \beta$$

$$\text{Nash equilibrium} \qquad \Longrightarrow \qquad \varphi(x^q(\beta)) + \psi(x^a(\beta)) \;\equiv\; c$$

Theorem (A.B. - K. Han, *SIAM J. Math. Anal.* 2012).

Let the flux f and cost functions φ, ψ satisfy the assumptions (A1)-(A2). Then, for every $\kappa > 0$, the Hamilton-Jacobi equation

$$U_t + f(U_x) \;=\; 0$$

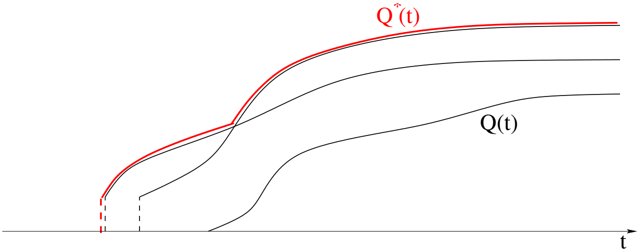
admits a unique Nash equilibrium solution with total mass κ

1. For a given cost c, let $\mathcal{Q}_c^-$ be the set of all initial data $Q(\cdot)$ for which every driver has a cost $\leq c$:

$$\varphi(\tau^q(\beta)) + \psi(\tau^a(\beta)) \ \leq \ c \qquad \text{for a.e. } \beta \in [0, \ Q(+\infty)]\,.$$

2. Claim: $\quad Q^*(t) \ \doteq \ \sup\Big\{Q(t)\,; \quad Q \in \mathcal{Q}_c^-\Big\}$

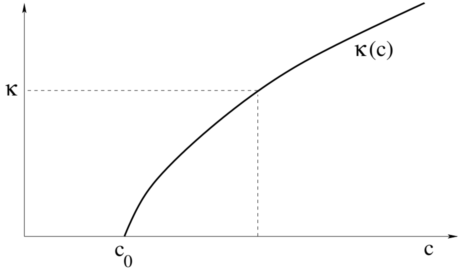
is the initial data for a Nash equilibrium with common cost c.



3. For each c, the Nash equilibrium solution where each driver has a cost $= c$ is unique. Define $\kappa(c) \ \doteq$ total number of drivers in this solution.

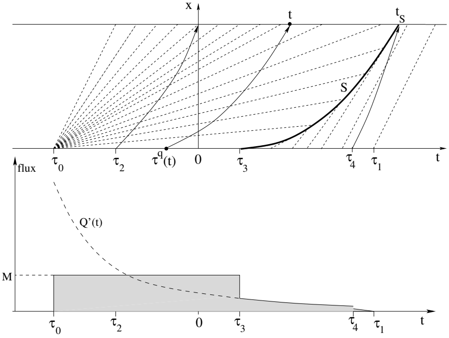
4. There exists a minimum cost c_0 such that $\kappa(c) = 0$ for $c \leq c_0$.

The map $c \mapsto \kappa(c)$ is strictly increasing and continuous from $[c_0\,, +\infty[$ to $[0, +\infty[$.



Numerical results

$$L = 1, \quad u(\rho) = \rho(2-\rho), \quad M = 1, \quad \kappa = 3.80758, \quad c = 2.7$$



$\tau_0 = -2.7$ $\quad \tau_2 = -0.9074$

$\tau_3 = 0.9698$ $\quad \tau_4 = 1.52303$

$\tau_1 = 1.56525$ $\quad t_S = 2.0550$

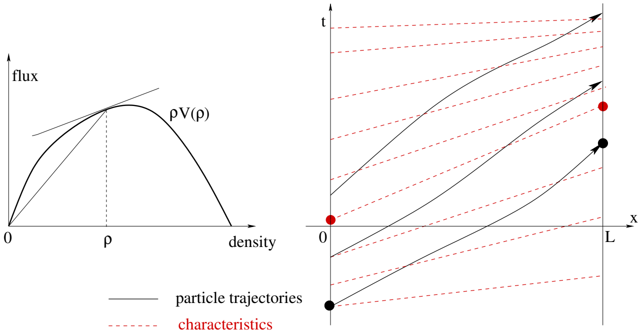
$\delta_0 = 1.79259$

total cost $=$ 10.286

$$Q(t) = 1.7 + \sqrt{t+2.7} + 1/(4(\sqrt{t+2.7} + 2.7))$$

$$Q'(t) = \Big(1 - 1/(4(\sqrt{t+2.7} + 2.7)^2)\Big)/(2\sqrt{t+2.7})$$

Globally optimal solution vs. Nash equilibrium



Globally optimal solution:

starting cost + arrival cost = constant for all characteristics

Nash equilibrium solution:

starting cost + arrival cost = constant for all car trajectories

A comparison

Total cost of the Pareto optimal solution: $J^{opt} = 5.5714$

Total cost of the Nash equilibrium solution: $J^{Nash} = 10.286$

Price of anarchy: $J^{Nash} - J^{opt} \approx 4.715$

Can one eliminate this inefficiency,
yet allowing freedom of choice to each driver ?

(goal of non-cooperative game theory: devise incentives)

Optimal pricing

Scientific American, Dec. 2010: Ten World Changing Ideas

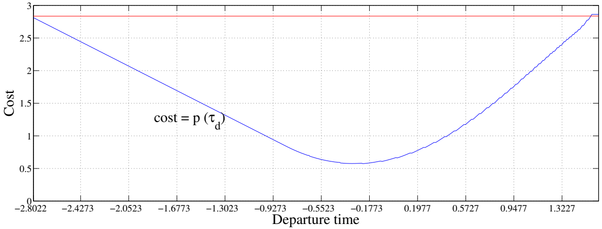
"Building more roads won't eliminate traffic. Smart pricing will."

Suppose a fee $b(t)$ is collected at a toll booth at the entrance of the highway, depending on the departure time.

$$\text{New departure cost:} \qquad \tilde{\varphi}(t) \;=\; \varphi(t) + b(t)$$

Problem: We wish to collect a total revenue R.

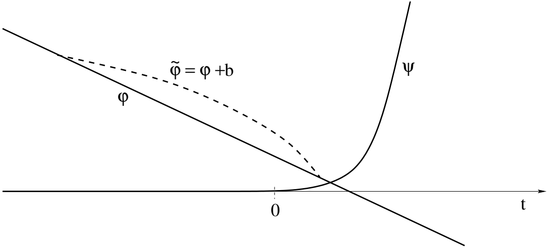
How do we choose $t \mapsto b(t) \geq 0$ so that the Nash solution with departure and arrival costs $\tilde{\varphi}, \psi$ yields the minimum total cost to each driver?



$p(t)$ = cost to a driver starting at time t, in the globally optimal solution

Optimal pricing: $b(t) = p_{max} - p(t) + C$

choosing the constant C so that [total revenue] $= R$.



Continuous dependence of the Nash solution

$\varphi_1(x),\ \varphi_2(x)$ costs for departing at time x

$\psi_1(x),\ \psi_2(x)$ costs for arriving at time x

$v_1(\rho),\ \ v_2(\rho)$ speeds of cars, when the density is $\rho \geq 0$

$Q_1(x),\ Q_2(x) =$ number of cars that have departed up to time x, in the corresponding Nash equilibrium solutions (with zero total cost to all drivers)

Theorem (A.B., C.J.Liu, and F.Yu, *Quarterly Appl. Math.* 2012)

Assume all cars depart and arrive within the interval $[a, b]$, and the maximum density is $\leq \rho^*$. Then

$$\begin{aligned}&\|Q_1(x) - Q_2(x)\|_{\mathbf{L}^1([a,b])}\\ &\leq\ C \cdot \left(\|\varphi_1 - \varphi_2\|_{\mathbf{L}^\infty([a,b])} + \|\psi_1 - \psi_2\|_{\mathbf{L}^\infty([a,b])} + \|v_1 - v_2\|^{1/2}_{\mathbf{L}^\infty([0,\rho^*])} \right)\end{aligned}$$

A min-max property of Nash equilibrium solutions

Fix: $\kappa =$ total number of drivers

For any departure distribution

$t \mapsto Q(t) =$ number of drivers who have departed within time t

(possibly joining the queue at the entrance of the highway)

Define: $\Phi(Q) \doteq$ maximum cost, among all drivers

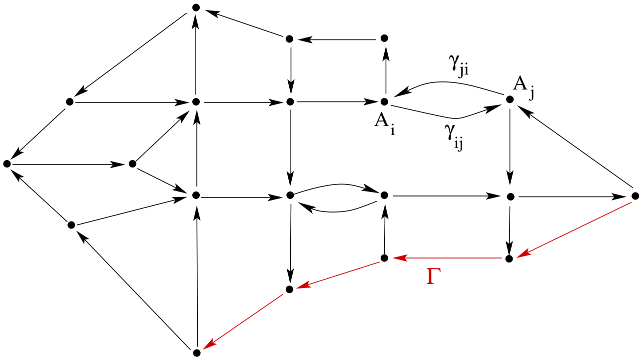
Theorem (A.B., C.J.Liu, and F.Yu, *Quarterly Appl. Math.* 2012)

The starting distribution $Q^*(\cdot)$ for the Nash equilibrium solution yields a global minimum of Φ.

Traffic Flow on a Network

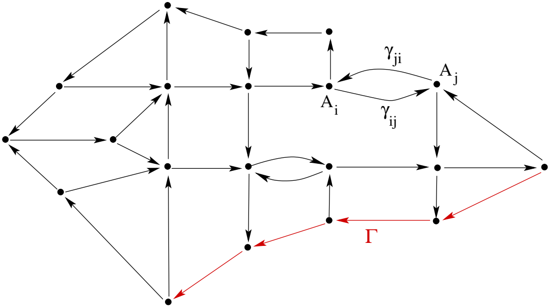
Nodes: $A_1, \ldots, A_m$ arcs: γ_{ij}

L_{ij} = length of the arc γ_{ij}



A viable path Γ is a concatenation of viable arcs

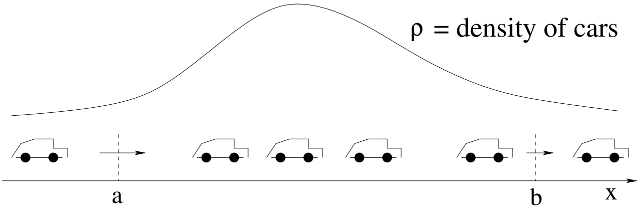
Given the departure times of N drivers, and the paths $\Gamma_1, \ldots, \Gamma_N$ along which they travel, describe the overall traffic pattern.



Delay Model: If a drivers enters the arc γ_{ij} at time t,
he will exit form that arc at time $\quad t + D_{ij}(n)$

$n \;=\;$ number of cars present along the arc γ_{ij} at time t

Conservation law model

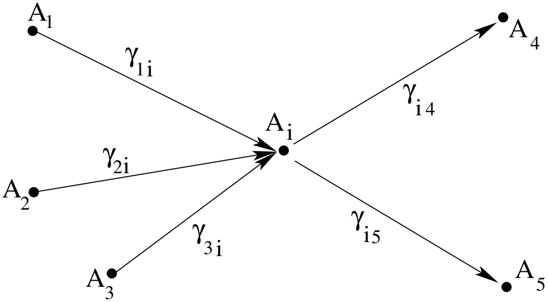


Along the arc γ_{ij}, the density of cars satisfies the conservation law

$$\rho_t + [\rho v_{ij}(\rho)]_x = 0$$

$v_{ij}(\rho)$ = velocity of cars, depending on the density

Boundary conditions at nodes

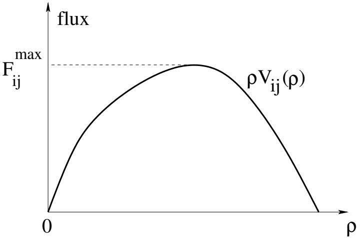


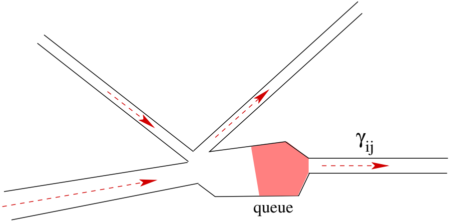
Need: junction conditions

given the flux from incoming arcs, determine the flux along outgoing arcs

A queue at the entrance of each arc

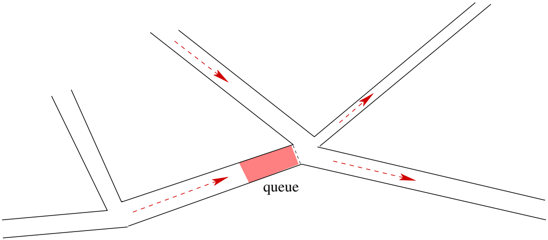
Simplest model: a queue is formed at the entrance of each outgoing arc if the flux is too large





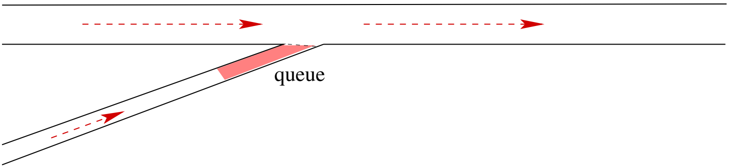
An upper bound on the flow is imposed (by a crosslight) at the end of each incoming arc.

A queue is formed, if the flux is too large (with possible spill-over)



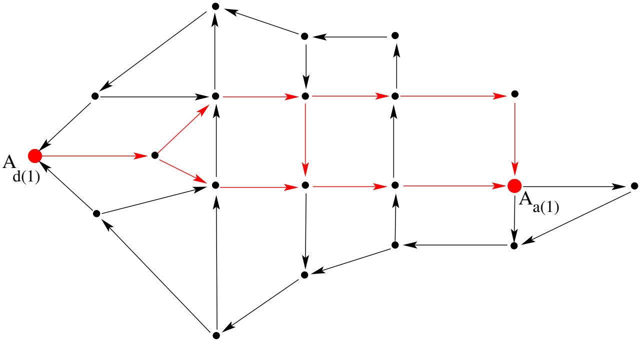
Cars from the incoming road having priority pass instantly through the intersection

Cars from the access ramp wait in a queue

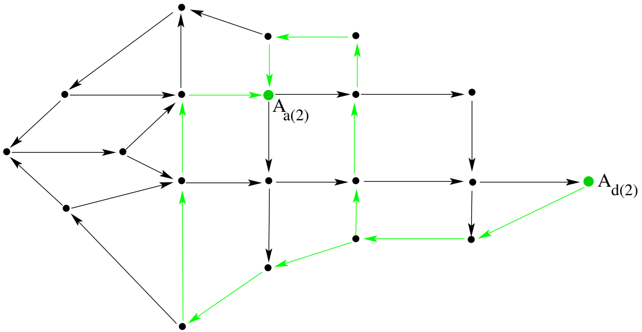


n groups of drivers with different origins and destinations, and different costs

$$k\text{-drivers: } \begin{cases} \text{depart from } A_{d(k)} \text{ and arrive to } A_{a(k)} \\[1em] \text{departure cost: } \varphi_k(t), \qquad \text{arrival cost: } \psi_k(t). \end{cases}$$



Traffic Flow on a Network



drivers can use different paths $\Gamma_1, \Gamma_2, \ldots$ to reach destination

Does there exist a globally optimal solution, and a Nash equilibrium solution for traffic flow on a network ?

Admissible departure rates

G_k = total number of drivers in the k-th group, $\quad k = 1, \ldots, n$

Γ_p = viable path (concatenation of viable arcs γ_{ij}), $\quad p = 1, \ldots, N$

$t \mapsto \bar{u}_{k,p}(t)$ = departure rate of k-drivers traveling along the path Γ_p

The set of departure rates $\{\bar{u}_{k,p}\}$ is **admissible** if

$$\bar{u}_{k,p}(t) \geq 0\,, \qquad \sum_p \int_{-\infty}^{\infty} \bar{u}_{k,p}(t)\,dt \;=\; G_k \qquad k = 1, \ldots, n$$

Let $\tau_p(t)$ = arrival time for a driver starting at time t, traveling along Γ_p

Main assumptions

(A1) Along each arc γ_{ij} the flux function $\rho \mapsto \rho\, v_{ij}(\rho)$ is twice continuously differentiable and concave down.

$$v_{ij}(0) \;>\; 0, \qquad\qquad v_{ij}(\rho_{max}) \;=\; 0$$

(A2) The cost functions φ, ψ satisfy $\quad \varphi' \;<\; 0, \quad \psi, \psi' \;\geq\; 0,$

$$\lim_{x\to-\infty} \varphi(x) \;=\; +\infty\,, \qquad\qquad \lim_{x\to+\infty} \Big(\varphi(x) + \psi(x)\Big) \;=\; +\infty$$

Global optima and Nash equilibria on networks

An admissible family $\{\bar{u}_{k,p}\}$ *of departure rates is* **globally optimal** *if it minimizes the sum of the total costs of all drivers*

$$J(\bar{u}) \doteq \sum_{k,p} \int \Big(\varphi_k(t) + \psi_k(\tau_p(t))\Big) \bar{u}_{k,p}(t)\, dt$$

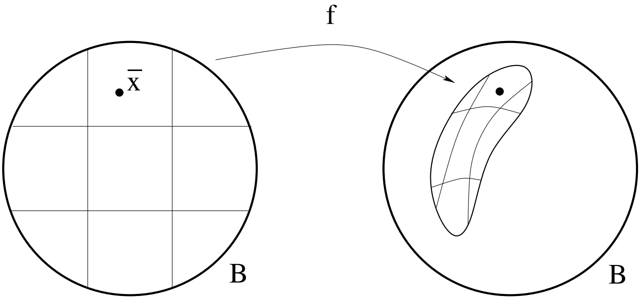
An admissible family $\{\bar{u}_{k,p}\}$ *of departure rates is a* **Nash equilibrium solution** *if no driver of any group can lower his own total cost by changing departure time or switching to a different path to reach destination.*

Theorem. *(A.B. - Ke Han, Networks & Heterogeneous Media, 2012).*

On a general network of roads, there exists at least one globally optimal solution, and at least one Nash equilibrium solution.

Theorem (Luitzen Egbertus Jan Brouwer, 1912)

Let $B \subset \mathbb{R}^n$ be a closed ball.
Every continuous map $f : B \mapsto B$ admits a fixed point.



$$f(\bar{x}) = \bar{x}$$

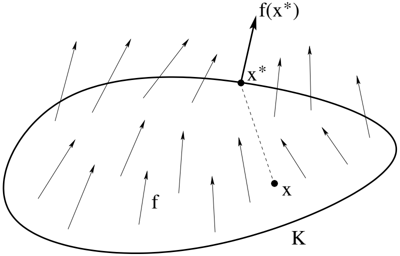
A variational inequality

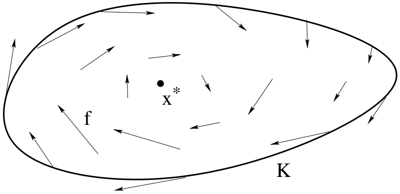
$K \subset \mathbb{R}^n$ closed, bounded convex set, $\quad f : K \mapsto \mathbb{R}^n$ continuous

Then there exists $x^* \in K$ such that

$$\langle x - x^*\,,\ f(x^*)\rangle \ \le\ 0 \qquad \text{for all } x \in K$$

Either $f(x^*) = 0$, or $f(x^*)$ is an outer normal vector to K at x^*

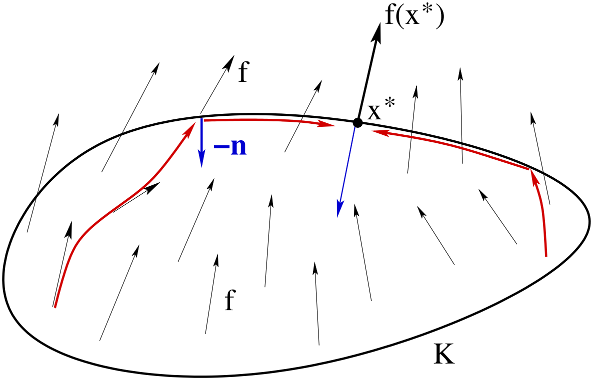




If $f(x)$ is tangent, or points inward at every boundary point of K, then $f(x^*) = 0$

A constrained evolution

Trajectories of $\dot{x} = f(x)$ are constrained to remain in K by a frictionless barrier

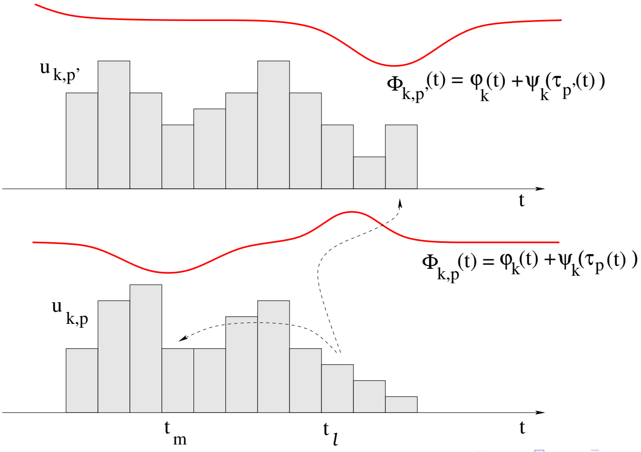


There exists a point $x^* \in K$ that does not move.

Finite dimensional approximations

On a family $\mathcal{K}$ of admissible piecewise constant departure rates $u = (u_{k,p})$, define an evolution equation

$$\frac{d}{d\theta} u \;=\; \Psi(u)$$



Existence of a Nash equilibrium on a network

The map $\Psi : \mathcal{K} \mapsto \mathbb{R}^N$ is continuous and inward-pointing

hence it admits a zero: $\Psi(\bar{u}) = 0$

The departure rates $\bar{u} = (\bar{u}_{k,p})$ represent a Galerkin approximation to a Nash equilibrium

Letting the discretization step Δt approach zero, taking subsequences:

departure rates:	$\bar{u}^{\nu}_{k,p}(\cdot) \rightharpoonup \bar{u}_{k,p}(\cdot)$	weakly
arrival times:	$\tau^{\nu}_{p}(\cdot) \to \tau_{p}(\cdot)$	uniformly

The departure rates $\bar{u}_{k,p}(\cdot)$ provide a Nash equilibrium

Work in progress

- More general conditions at junctions (K. Han, B. Piccoli)
- Necessary conditions for globally optimal solutions on networks
  No queues ? No shocks ?

Stability of Nash equilibrium ?

To justify the practical relevance of a Nash equilibrium, we need to

- analyze a suitable dynamic model
- check whether the rate of departures asymptotically converges to the Nash equilibrium

Assume: drivers can change their departure time on a day-to-day basis, in order to decrease their own cost (one group of drivers, one single road)

Introduce an additional variable θ counting the number of days on the calendar.

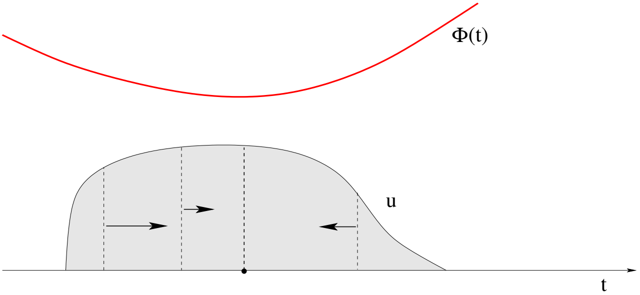
$$\bar{u}(t,\theta) \doteq \text{rate of departures at time } t, \text{ on day } \theta$$

$$\Phi(t,\theta) \doteq \text{cost to a driver starting at time } t, \text{ on day } \theta$$

A conservation law with non-local flux

Model 1: drivers gradually change their departure time, drifting toward times where the cost is smaller.
If the rate of change is proportional to the gradient of the cost, this leads to the conservation law

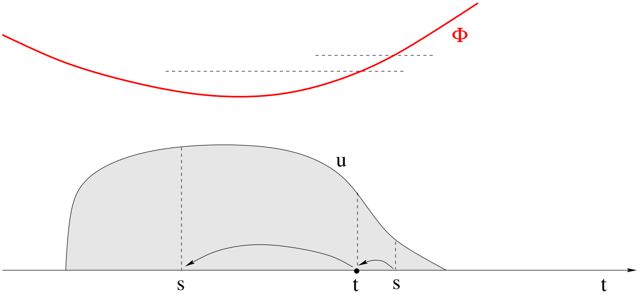
$$\bar{u}_\theta + [\Phi_t \, \bar{u}]_t \;=\; 0$$



An integral evolution equation

Model 2: drivers jump to different departure times having a lower cost.
If the rate of change is proportional to the difference between the costs, this yields

$$\frac{d}{d\theta}\bar{u}(t) \;=\; \int \bar{u}(s)\Big[\Phi(s)-\Phi(t)\Big]_+ \,ds - \int \bar{u}(t)\Big[\Phi(t)-\Phi(s)\Big]_+ \,ds$$



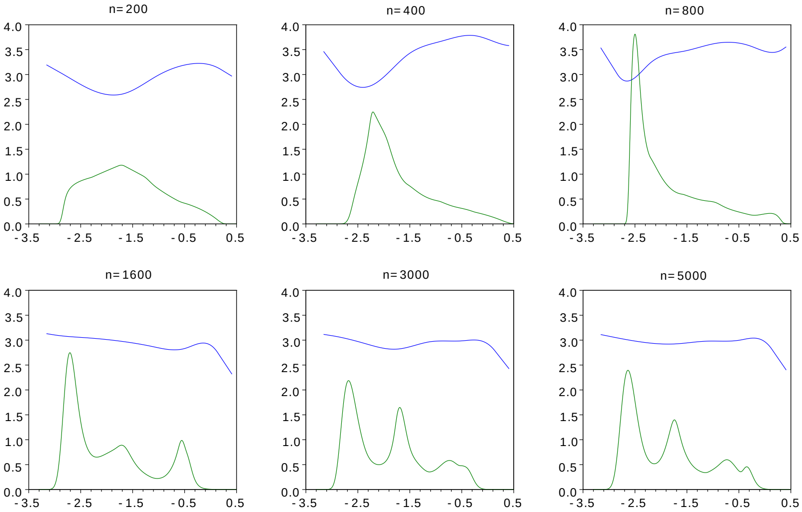
Numerical experiments (Wen Shen, 2011)

Question: as $\theta \to \infty$, does the departure rate $\overline{u}(t, \theta)$ approach the unique Nash equilibrium?

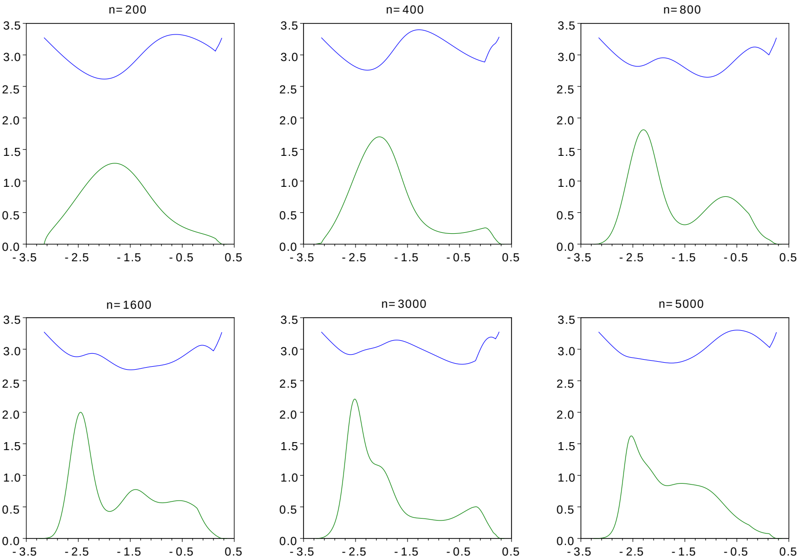
Flux function: $f(\rho) \;=\; \rho\,(2 - \rho)$

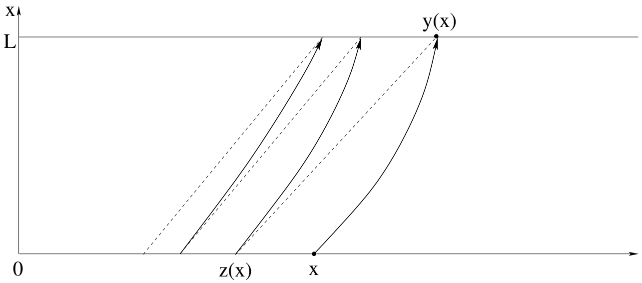
Departure and arrival costs: $\varphi(t) \;=\; -\,t\,, \qquad \psi(t) \;=\; e^t$

Numerical simulation: Model 1



Numerical simulation: Model 2





main difficulty: non-local dependence

linearized equation: $$\frac{d}{d\theta}Y(x) \;=\; \Big[\alpha(x)\Big(\beta(x)Y(x) - Y(z(x))\Big)\Big]_x$$