

Chapter 2

Partial Differential Equations in Engineering

Many of the PDEs used in Engineering and Physics are the result of applying physical laws of conservation or balance to systems involving *fields*, that is, quantities defined over a continuous background of two or more dimensions, such as space and time. Under suitable continuity and differentiability conditions, a generic balance law in both global (integral) and local (differential) forms can be derived and applied to various contexts of practical significance, such as Traffic flow, Solid Mechanics, Fluid Mechanics and Heat Conduction.

2.1 Introduction

Partial differential equations arise quite naturally when we apply the laws of nature to systems of continuous extent. We speak then of *field theories*. Thus, whereas the analysis of the vibrations of a chain of masses interconnected by springs gives rise to a system of ODEs, the dynamic analysis of a bar, where the mass is smeared out continuously over the length of the bar, gives rise to a PDE. From this simple example, it would appear that PDEs are a mere generalization of their ordinary counterparts, whereby a few details need to be taken care of. This false impression is exacerbated these days by the fact that numerical procedures, that can be implemented as computer codes with relative ease, do actually approximate the solutions of PDEs by means of discrete systems of algebraic equations. This is clearly a legitimate thing to do, but one must bear in mind that, unless one possesses a basic knowledge of the qualitative aspects of the behaviour of the continuous system, the discrete approximation may not be amenable to a correct interpretation.

One hardly needs to defend the study of PDEs on these grounds, since they stand alone as one of the greatest intellectual achievements of the human race in its attempt to understand the physical world. Need one say more than the fact that from solid and fluid mechanics all the way to quantum mechanics and general relativity, the

language of nature has so far been transcribed into PDEs? There has been recently a trend to declare the emergence of a “new science”, in which the prevalent language will be that of cellular automata and other tools that represent the behaviour of complex systems as the result of simple interactions between a very large (but finite) number of discrete sites of events.¹ These models are particularly powerful in applications where the underlying phenomena are too intricate to capture in any degree of detail by means of PDEs. Such is the case in multi-scale phenomena that appear in many modern applications in a variety of fields (biology, environmental engineering, nanomechanics, and so on). It is too early to predict the demise of Calculus, however. As many times in the past (think of quantum mechanics, chaos, economics), it appears that in one way or another the usefulness of mathematical limits (differentiation, integration) is not entirely dependent on whether or not the actual physical system can “in reality” attain those limits. Calculus and differential equations are here to stay just as trigonometry and Euclidean geometry are not likely to go away.

2.2 What is a Partial Differential Equation?

A partial differential equation for a function u of the independent variables x_1, x_2, \dots, x_n ($n > 1$) is a relation of the form

$$F\left(x_i, u, u_{,i}, u_{,ij}, \dots, u_{,\underbrace{ijk\dots}_{m \text{ indices}}}\right) = 0, \quad i, j, k, \dots = 1, \dots, n, \quad (2.1)$$

where F is a function and where we have introduced the following notation: A subscript preceded by a comma, indicates a partial derivative with respect to the corresponding independent variable. If more than one index follows a comma, it is understood that successive derivatives have been taken. Thus, for instance,

$$u_{,i} = \frac{\partial u}{\partial x_i} \quad u_{,ijk} = \frac{\partial^3 u}{\partial x_k \partial x_j \partial x_i}. \quad (2.2)$$

By abuse of notation, we have listed in Eq. (2.1) just a generic term for each order of differentiation, understanding that all the values of the indices are to be considered. For example, when we write the argument $u_{,ij}$ we mean the n^2 entries in the square matrix of second partial derivatives.² The requirement $n > 1$ is essential, otherwise (if $n = 1$) we would have an ordinary differential equation. The highest order of

¹This point of view is advocated in [4] with particular force by Stephen Wolfram, a physicist and the creator of the Mathematica code (which, ironically, is one of the best tools in the market for the solution of differential equations).

²On the other hand, recalling the equality of mixed partial derivatives (under assumptions that we assume to be fulfilled), the number of independent entries of this matrix is actually only $n(n+1)/2$.

differentiation appearing, which in our case we have indicated by m , is called the *order* of the differential equation.

By a *solution* of the PDE (2.1) we mean a function $u = u(x_i) = u(x_1, \dots, x_n)$ which, when substituted in Eq. (2.1), satisfies it identically within a given domain of the independent variables x_1, \dots, x_n . Clearly, in this version of the theory, the proposed solution must necessarily be differentiable at least m times (otherwise we wouldn't even be able to check that the equation is satisfied). The function F , which actually characterizes the particular differential equation being studied and represents the physical laws at hand, is also subject to conditions of continuity and differentiability which we will not stipulate at this point, but we will assume that as many derivatives of this function exist as we need. A PDE is said to be *linear* if F depends linearly on the unknown function u and all its derivatives. The coefficients of a linear PDE may still depend on the independent variables. If they don't, we have a case of a linear equation with constant coefficients. Even such a simple situation is not amenable to a straightforward treatment, like in the case of ODEs. If the function F is linear in the highest derivatives only (namely, on all the derivatives of order m), the PDE is said to be *quasi-linear*. Otherwise (if it depends non-linearly on at least one of the highest derivatives) the equation is *nonlinear*.

A relatively simple case is that for which the number of independent variables is equal to 2. In this case, a solution can be visualized as a surface in the 3-dimensional space with coordinates x_1, x_2, u . It follows from this intuitive picture that the analog of the notion of integral curve is that of an *integral surface* and perhaps that the analog of the initial conditions at a point is the specification of initial conditions along a whole curve through which the integral surface must pass. More difficult is to visualize at this point what might be the analog of the vector field which, as we know, is associated with a system of ODEs. Leaving this issue aside for the moment, let us remark that just as we have systems of ODEs we can also have systems of PDEs. The question of the equivalence of a single PDE of higher order to a system of PDEs of order 1 is somewhat more delicate than its ODE counterpart.

2.3 Balance Laws

One of the primary sources of PDEs in Engineering and Physics is the stipulation of *conservation laws*. Conservation laws or, more generally, *balance laws*, are the result of a complete accounting of the variation in time of the content of an extensive physical quantity in a certain domain. A simple analogy is the following. Suppose that you are looking at a big farming colony (the domain of interest) and you want to focus attention on the produce (the physical quantity of interest). As time goes on, there is a variation in the quantity of food contained in the domain. At any given instant of time, you want to account for the rate of change of this food content. There are some internal *sources* represented in this case by the rate at which the land yields new produce (so and so many tons per week, say). There are also *sinks* (or negative sources) represented by the internal consumption of food by workers and cattle,

damage caused by hail and pests, etcetera. We will call these sources and sinks the *production* of the quantity in question. It is measured in units of the original quantity divided by the unit of time. In addition to these internal factors, there is also another type of factors that can cause a change in content. We are referring to exchanges of food through the boundary of the colony. These include the buying and selling of produce that takes place at the gates, the perhaps illegal activities of some members or visitors that personally take some food away to other destinations, etcetera. At any given instant of time, we can estimate the rate at which these exchanges take place at the boundary. We will call these transactions the *flux* of the quantity in question. We may have a flux arising also from the fact that the boundary of the domain of interest is changing (encroached by an enemy or by natural causes, etcetera). Assuming that we have accounted for every one of these causes and that we believe in the principles of causality and determinism (at least as far as the material world is concerned), we may write the generic equation of balance as

$$\frac{d \text{ content}}{dt} = \text{production} + \text{flux}, \quad (2.3)$$

where t is the time variable.

In physically meaningful examples (balance of energy, momentum, mass, electric charge, and so on), it is often the case that the content, the production and the flux are somehow distributed (smeared) over the volume (in the case of the content and the production) or over the area of the boundary (in the case of the flux). In other words, these magnitudes are given in terms of *densities*, which vary (continuously, say) from point to point and from one instant to the next. It is precisely this property (whether real or assumed) that is responsible for the fact that we can express the basic equation of balance (2.3) in terms of differential equations. Indeed, the differential equations are obtained by assuming that Eq. (2.3) applies to any sub-domain, no matter how small.

2.3.1 The Generic Balance Equation

Let U represent an extensive quantity for which we want to write the equation of balance. We assume this quantity to be scalar, such as mass, charge or energy content.³ Consider a spatial region ω fixed in \mathbb{R}^3 and representing a subset of the region of interest. Our four independent variables are the natural coordinates x_1, x_2, x_3 of \mathbb{R}^3 and the time variable t .⁴ When we say that U is an *extensive* quantity, we mean that we can assign a value of U (the *content*) to each such subset ω . On physical grounds we further assume that this set function is *additive*. By this we understand

³Vector quantities, such as linear and angular momentum, can be treated in a similar way by identifying U alternatively with each of the components in a global Cartesian frame of reference.

⁴Consequently, we will not strictly adhere to the notational convention (2.2).

that the total content in two disjoint subsets is equal to the sum of the contents in each separate subset. Under suitable continuity conditions, it can be shown that the content of an extensive quantity U is given by a density $u = u(x_1, x_2, x_3, t)$ in terms of an integral, namely,

$$U = \int_{\omega} u \, d\omega. \quad (2.4)$$

It is clear that this expression satisfies the additivity condition. The units of u are the units of U divided by a unit of volume.

Similarly, the *production* P is assumed to be an extensive quantity and to be expressible in terms of a production density $p = p(x_1, x_2, x_3, t)$ as

$$P = \int_{\omega} p \, d\omega. \quad (2.5)$$

The units of p are the units of U divided by a unit of volume and by the time unit. We adopt the sign convention that a positive p corresponds to creation (source) and a negative p corresponds to annihilation (sink).

The *flux* \mathcal{F} represents the change in content per unit time flowing through the boundary $\partial\omega$, separating the chosen subset ω from the rest of the region of interest. In other words, the flux represents the contact interaction between adjacent subsets. A remarkable theorem of Cauchy shows that under reasonable assumptions the flux is governed by a vector field, known as the *flux vector* \mathbf{f} . More precisely, the inflow per unit area and per unit time is given by

$$d\mathcal{F} = (\mathbf{f} \cdot \mathbf{n}) da, \quad (2.6)$$

where \mathbf{n} is the exterior unit normal to da . The significance of this result can be summarized as follows:

1. The ‘principle of action and reaction’ is automatically satisfied. Indeed at any given point an element of area da can be considered with either of two possible orientations, corresponding to opposite signs of the unit normal \mathbf{n} . Physically, these two opposite vectors represent the exterior unit normals of the sub-bodies on either side of da . Thus, what comes out from one side must necessarily flow into the other.
2. All boundaries $\partial\omega$ that happen to have the same common tangent plane at one point transmit, at that point, exactly the same amount of flux. Higher order properties, such as the curvature, play no role whatsoever in this regard. In fact, this is the main postulate needed to prove Cauchy’s theorem.
3. The fact that the amount of flux depends *linearly* on the normal vector (via the dot product) conveys the intuitive idea that the intensity and the angle of incidence of the flowing quantity are all that matter. If you are sun-tanning horizontally at

high noon in the same position as two hours later, you certainly are taking in more radiation per unit area of skin in the first case.

We are now in a position of implementing all our hypotheses and conclusions into the basic balance Eq. (2.3). The result is

$$\frac{d}{dt} \int_{\omega} u \, d\omega = \int_{\omega} p \, d\omega + \int_{\partial\omega} \mathbf{f} \cdot \mathbf{n} \, da. \quad (2.7)$$

This equation represents the *global balance equation* for the volume ω . It should be clear that this equation is valid under relatively mild conditions imposed on the functions involved. Indeed, we only need the density u to be differentiable with respect to time and otherwise we only require that the functions be integrable. This remark will be of great physical significance when we study the propagation of shocks. In the case of a content u and a flux vector \mathbf{f} which are also space-wise differentiable, we can obtain a *local* version of the generic balance equation. This local ('infinitesimal') version is a partial differential equation. To derive it, we start by observing that, due to the fact that the volume ω is fixed (that is, independent of time), the order of differentiation and integration on the left-hand side of Eq. (2.7) can be reversed, that is,

$$\frac{d}{dt} \int_{\omega} u \, d\omega = \int_{\omega} \frac{\partial u}{\partial t} \, d\omega. \quad (2.8)$$

Moreover, the surface integral on the right-hand side of Eq. (2.7) is the flux of a vector field on the boundary of a domain and is, therefore, amenable to be treated by means of the divergence theorem according to Eq. (1.18), namely,

$$\int_{\partial\omega} \mathbf{f} \cdot \mathbf{n} \, da = \int_{\omega} \operatorname{div} \mathbf{f} \, d\omega. \quad (2.9)$$

Collecting all the terms under a single integral we obtain the global balance equation in the form

$$\int_{\omega} \left(\frac{\partial u}{\partial t} - p - \operatorname{div} \mathbf{f} \right) d\omega = 0. \quad (2.10)$$

This equation is satisfied identically for any arbitrary sub-domain ω . If the integrand is continuous, however, it must vanish identically. For suppose that the integrand is, say, positive at one point within the domain of integration. By continuity, it will also be positive on a small ball B around that point. Applying the identity (2.10) to this sub-domain B , we arrive at a contradiction. We conclude, therefore, that a necessary and sufficient condition for the global balance equation to be satisfied identically for arbitrary sub-domains is the identical satisfaction of the partial differential equation

$$\frac{\partial u}{\partial t} - p - \operatorname{div} \mathbf{f} = 0. \quad (2.11)$$

This is the generic equation of balance in its local (differential) form. It is a single PDE for a function of 4 variables, x_1, x_2, x_3 and $x_4 = t$.

2.3.2 The Case of Only One Spatial Dimension

There are several reasons to present an independent derivation of the generic law of balance for the case of a single spatial dimension. The first reason is that in the case of just one dominant spatial dimension (waves or heat flow in a long bar, current in a wire, diffusion of pollutants in a tube, etcetera), the divergence theorem mercifully reduces to the statement of the fundamental theorem of calculus of one variable (roughly speaking: “differentiation is the inverse of integration”). Notice that we still are left with two independent variables, one for the spatial domain (x) and one for the time dependence (t). Another important reason has to do with the peculiar nature of a domain in \mathbb{R} as compared with domains in higher dimensions. If the spatial domain is two-dimensional, such as a membrane, its boundary is the perimeter curve, while the upper and lower faces of the membrane are identified with the interior points. For a three-dimensional domain, the boundary is the whole bounding surface. On the other hand, a closed connected domain in \mathbb{R} is just a closed interval $[a, b]$, with $a < b$. Its boundary consists of just two distinct points, as shown in Fig. 2.1. Moreover, the exterior normal to the boundary is defined at those points only, as a unit vector at a pointing in the negative direction and a unit vector at b pointing in the positive direction of the real line. The flux vector \mathbf{f} and the velocity vector \mathbf{v} have each just one component and can be treated as scalars. Physically, we may think of U as the content of some extensive quantity in a wire or a long cylindrical bar. It is important to realize that the lateral surface of this wire does not exist, in the sense that it is not part of the boundary. On the contrary, the points on this

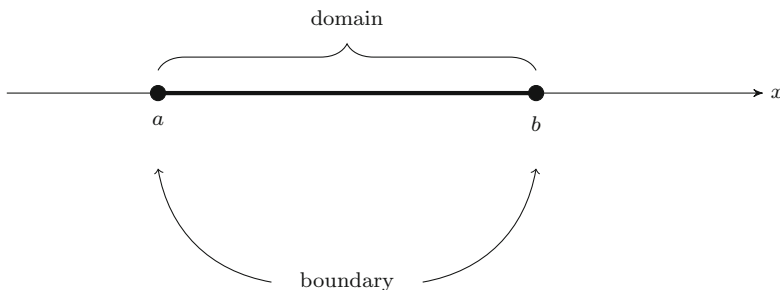


Fig. 2.1 The boundary of a domain in \mathbb{R} consists of two points

(vanishingly small) lateral surface are identified precisely with the interior points of the wire.

If we assume that the quantity $U = U(t)$ is continuously distributed throughout the domain, we can express it in terms of a density $u = u(x, t)$ per unit length of the bar as

$$U = \int_a^b u \, dx. \quad (2.12)$$

Similarly, the production $P = P(t)$ can be expressed in terms of a density $p = p(x, t)$ per unit length and per unit time as

$$P = \int_a^b p \, dx. \quad (2.13)$$

As a sign convention, we assume that a positive p corresponds to creation (source) and a negative sign to annihilation (sink).

The flux term requires some further discussion. Clearly, if we cut the bar into two pieces and we focus attention on one of these pieces alone, as far as the quantities U and P are concerned, all we have to do is change the limits of integration in Eqs. (2.12) and (2.13). On the other hand, by the process of cutting, we have created a new boundary point at each of the sub-bodies and a corresponding flux. If we assume (following an idea similar to Newton's law of action and reaction) that whatever flow enters through the new boundary into one of the parts must necessarily be coming out of the new boundary of the other part, we realize that the flux is best represented by the dot product of a *flux vector* $\mathbf{f} = \mathbf{f}(x, t)$ with the unit vector \mathbf{n} at the boundary *pointing in the outward direction of the part under study*. This situation is illustrated in Fig. 2.2.

If the flux vector points to the right, the flux through the cross section, as shown in the figure, will be positive (an inflow) for the left part of the bar and negative

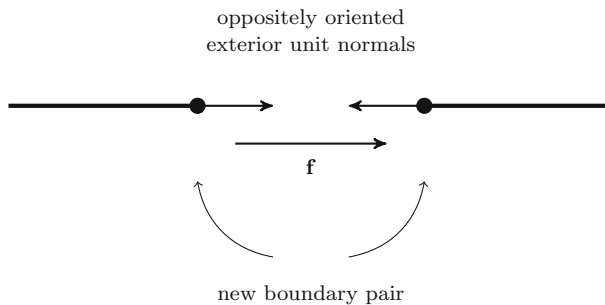


Fig. 2.2 The flux vector

(and of equal absolute value) for the right part of the bar. Notice that in this simple one-dimensional case, the flux vector has only one component, which we denote by $f = f(x, t)$. Nevertheless, the concept of flux vector is very important and can be used directly in two- and three- dimensional spatial contexts. For the actual boundary of the body, the flux vector may be specified as a *boundary condition*, depending on the specific problem being solved.

Introducing our specific expressions for content, production and flux in the generic balance equation (2.3), we obtain

$$\frac{d}{dt} \int_a^b u(x, t) dx = \int_a^b p(x, t) dx + f(b, t) - f(a, t). \quad (2.14)$$

As far as sign convention for the flux is concerned, we have assumed that a positive scalar flux is inwards, into the body. What this means is that if the flux vector points outwards, the scalar flux is actually inwards. If this convention is found unnatural, all one has to do is reverse the sign of the last two terms in Eq. (2.14).⁵

This is the equation of balance in its *global form*. It is not yet a partial differential equation. It is at this point that, if we wish to make the passage to the local form of the equation, we need to invoke the fundamental theorem of calculus (or the divergence theorem in higher dimensional contexts). Indeed, we can write

$$f(b, t) - f(a, t) = \int_a^b \frac{\partial f(x, t)}{\partial x} dx. \quad (2.15)$$

We obtain, therefore,

$$\frac{d}{dt} \int_a^b u(x, t) dx = \int_a^b p(x, t) dx + \int_a^b \frac{\partial f}{\partial x} dx. \quad (2.16)$$

If we consider that the integral limits a and b , though arbitrary, are independent of time, we can exchange in the first term of the equation the derivative with the integral, namely,

$$\int_a^b \frac{\partial u}{\partial t} dx = \int_a^b p(x, t) dx + \int_a^b \frac{\partial f}{\partial x} dx, \quad (2.17)$$

identically for all possible integration limits. We claim now that this identity is possible only if the integrands themselves are balanced, namely, if

⁵This common policy is adopted in [3]. This is an excellent introductory text, which is highly recommended for its clarity and wealth of examples.

$$\frac{\partial u}{\partial t} = p(x, t) + \frac{\partial f}{\partial x}. \quad (2.18)$$

The truth of this claim can be verified by collecting all the integrands in Eq. (2.17) under a single integral and then arriving at a combined integrand whose integral must vanish no matter what limits of integration are used. Clearly, if the integrand is continuous and does not vanish at some point in the domain of integration, it will also not vanish at any point in a small interval containing that point (by continuity). It will, therefore, be either strictly positive or strictly negative therein. Choosing, then, that small interval as a new domain of integration, we would arrive at the conclusion that the integral does not vanish, which contradicts the assumption that the integration must vanish *for all values* of the limits. We conclude that Eq. (2.18) must hold true.

2.3.3 The Need for Constitutive Laws

When we look at the local form of the balance equation, (2.18), we realize that we have a single equation containing partial derivatives of two unknown functions, u and f . What this is telling us from the physical point of view is that the equations of balance are in general not sufficient to solve a physical problem. What is missing? If we think of the problem of heat transfer through a wire (which is an instance of the law of balance of energy), we realize that the *material properties* have played no role whatsoever in the formulation of the equation of balance. In other words, at some point we must be able to distinguish (in these macroscopic phenomenological models in which matter is considered as a continuum) between different materials. Copper is a better heat conductor than wood, but the law of balance of energy is the same for both materials! The missing element, namely the element representing the response of a specific medium, must be supplied by means of an extra equation (or equations) called the *constitutive law* of the medium. Moduli of elasticity, heat conductivities, piezoelectric and viscosity constants are examples of the type of information that may be encompassed by a constitutive law. And what is that the constitutive equation can stipulate? Certainly not the production, since this is a matter of sources and sinks, which can be controlled in principle regardless of the material at hand. Instead, it is the flux vector within the body that will differ from material to material according to the present state of the system. The state of a system is given in terms of some local variables of state s_1, s_2, \dots, s_k (positions, temperatures, velocity gradients, and so on), so that both the flux f and the content density u may depend on them. The constitutive law is then expressed by equations such as

$$u = \hat{u}(s_1(x, t), \dots, s_k(x, t), x) \quad (2.19)$$

and

$$f = \hat{f}(s_1(x, t), \dots, s_k(x, t), x). \quad (2.20)$$

The reason that we have included a possible explicit dependence on x is that the properties of the medium may change from point to point (as is the case in the so-called functionally graded bodies, for instance). In principle, these properties could also change in time, as is the case in processes of aging (biological or otherwise). In some cases, a single variable of state is enough to characterize the system, so that ultimately Eq. (2.18) becomes a PDE for the determination of this variable of state as a function of space and time. Sometimes, it is possible to adopt the density u itself as a single variable of state, so that the constitutive law simply reads

$$f = \hat{f}(u(x, t), x). \quad (2.21)$$

In this case, substituting the constitutive law into (2.18), we obtain

$$\frac{\partial u}{\partial t} = p + \frac{\partial \hat{f}}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \hat{f}}{\partial x}. \quad (2.22)$$

It is often convenient to adopt a subscript notation for partial derivatives of the unknown field variable $u = y(x, t)$, such as

$$u_x = \frac{\partial u}{\partial x} \quad u_t = \frac{\partial u}{\partial t} \quad u_{xx} = \frac{\partial^2 u}{\partial x^2} \quad u_{xt} = \frac{\partial^2 u}{\partial t \partial x} \dots \quad (2.23)$$

Notice that, since there is no room for confusion, we don't place a comma before the subscripts indicating derivatives, as we did in (2.2). In this compact notation, Eq. (2.22) reads

$$u_t - \frac{\partial \hat{f}}{\partial u} u_x = p + \frac{\partial \hat{f}}{\partial x}. \quad (2.24)$$

We have purposely left the partial derivatives of the constitutive function \hat{f} unaffected by the subscript notation. The reason for this is that the constitutive function \hat{f} is not an unknown of the problem. On the contrary, it is supposed to be known as that part of the problem statement that identifies the material response. Its partial derivatives are also known as some specific functions of u and x . Notice that in the case of a homogeneous material, the last term in Eq. (2.24) vanishes.

In the terminology introduced in the previous section, Eq. (2.24) is a first order, quasi-linear PDE. If the constitutive function \hat{f} happens to be a linear function of u , the PDE becomes linear. The linearity of constitutive laws is still one of the most common assumptions in many branches of engineering (for example: Hooke's law, Ohm's law, Fourier's law, Darcy's law, etcetera, are not actual laws of nature but constitutive assumptions that are useful linear approximations to the behaviour of some materials within certain ranges of operation). Notice that in the examples just mentioned, the constitutive laws are expressed in terms of space derivatives of state variables (respectively, displacement, electric potential, temperature and pressure). As a result, the equation of balance combined with the constitutive law yields a second order PDE. The theory of a single first-order PDE is comparable in its precision and

implementation to the theory of systems of ODEs. This is not the case for higher order PDEs or for systems of first order PDEs, as we shall see later. At this point, however, we are only interested in illustrating the emergence of PDEs of any order and type from well-defined engineering contexts, without much regard for their possible solutions. Accordingly, in the next section, we will display several instances of balance laws, which constitute a good (but by no means the only) source of PDEs in applications.

2.4 Examples of PDEs in Engineering

2.4.1 Traffic Flow

A comprehensive review of models for traffic flow is beyond our present scope. Instead, we present here a simplified version of the fundamental equation, based on the assumptions that the road is of a single lane and that (within the portion of road being analyzed) there are no entrances or exits. The quantity we want to balance is the content of cars. We, therefore, interpret $u = u(x, t)$ as the car density at the point x along the road at time t . Since we have assumed no entrances or exits, the production term p vanishes identically. The flux term f has the following physical interpretation: At any given cross section of the road and at a given instant of time, it measures the number of cars per unit time that pass through that cross section or, more precisely, the number of cars per unit time that enter one of the portions of road to the right or left of the cross section. With our (counter-intuitive) sign convention, a positive value of f corresponds to an inflow of cars. We have seen that the flux is actually governed by a flux vector \mathbf{f} . Denoting by $\mathbf{v} = \mathbf{v}(x, t)$ the car-velocity field, we can write

$$\mathbf{f} = -u \mathbf{v}. \quad (2.25)$$

In other words, if the velocity points in the direction of the exterior normal to the boundary (so that the dot product is positive) the term $u \mathbf{v}$ measures the number of cars that in a unit of time are coming out through that boundary. Since in our case everything is one-dimensional, the velocity vector is completely defined by its component v along the axis of the road, so that we can write

$$f = -u v. \quad (2.26)$$

The local balance equation (2.24) for this traffic flow problem reads, therefore,

$$u_t + \frac{\partial(u v)}{\partial u} u_x = 0. \quad (2.27)$$

The time has come now to adopt some constitutive law. Clearly, the velocity of the cars may depend on a large number of factors, including the time of day, the weather, the traffic density, etcetera. In the simplest model, the velocity will depend only on the traffic density, with larger densities giving rise to smaller speeds. From practical considerations, since cars have a finite length, there will be an upper bound u_{max} for the density, and it is sensible to assume that when this maximum is attained the traffic comes to a stop. On the other hand, we may or may not wish to consider an upper limit v_{max} for the speed, when the traffic density tends to zero. If we do, a possible constitutive equation that we may adopt is

$$v = v_{max} \left(1 - \frac{u}{u_{max}} \right). \quad (2.28)$$

If, on the other hand, we do not want to impose a speed limit in our model, a possible alternative constitutive law is

$$v = k \ln \frac{u_{max}}{u}, \quad (2.29)$$

where k is a positive constant (which perhaps varies from road to road, as it may take into consideration the quality of the surface, the width of the lane, and so on).

Introducing the constitutive law (2.28) into our balance law (2.27), we obtain the quasi-linear first-order PDE

$$u_t + \left(1 - 2 \frac{u}{u_{max}} \right) v_{max} u_x = 0. \quad (2.30)$$

In the extreme case when the speed is independent of the density and equal to a constant, we obtain the *advection equation*

$$u_t + v_{max} u_x = 0. \quad (2.31)$$

2.4.2 Diffusion

Diffusive processes are prevalent in everyday life. They occur, for example, whenever a liquid or gaseous substance spreads within another (sneezing, pouring milk into a cup of coffee, industrial pollution, etc.). The process of heat flow through a substance subjected to a temperature gradient is also a diffusive process. All these processes are characterized by thermodynamic irreversibility (the drop of milk poured into the coffee will never collect again into a drop).

Consider a tube filled with water at rest in which another substance (the pollutant) is present with a variable concentration $u = u(x, t)$. Let $p = p(x, t)$ be the production of pollutant per unit length and per unit time. This production can be the result of industrial exhaust into the tube, coming from its lateral surface at various points,

or of a similar process of partial clean-up of the tube. If there is any influx through the ends of the tube, it will have to be considered as part of the boundary conditions (which we have not discussed yet), rather than of the production term. The flux, just as in the case of traffic flow, represents the amount of pollutant traversing a given cross section per unit time. In the case of traffic flow, we introduced as a variable of state the speed of the traffic, which we eventually related to the car density by means of a constitutive law. In the case of diffusion of a pollutant, on the other hand, it is possible to formulate a sensible, experimentally based, constitutive law directly in terms of the pollutant concentration. The most commonly used law, called Fick's law, states that the flux vector is proportional to the gradient of the concentration, namely,

$$f = D u_x, \quad (2.32)$$

where the constant D is the *diffusivity* of the pollutant in water. A moment's reflection reveals that, with our sign convention, if we want the pollutant to flow in the direction of smaller concentrations, the diffusivity must be positive. Introducing these results into the general balance equation (2.18), we obtain

$$u_t - D u_{xx} = p. \quad (2.33)$$

This second-order linear PDE is known as the (non-homogeneous)⁶ diffusion equation. It is also known as the one-dimensional *heat equation*, in which case u stands for the temperature and the constant D is a combination of the heat capacity and the conductivity of the material.

2.4.3 Longitudinal Waves in an Elastic Bar

Assuming that the particles in a thin cylindrical bar are constrained to move in the axial direction, the law of balance of momentum (Newton's second law) can be seen as a scalar equation. The momentum density (momentum per unit length) is given by $\rho A v$, ρ being the mass density, A the cross-section area and v the component of the velocity vector. The production term in this case consists of any applied force per unit length (such as the weight, if the bar is held vertically). We will assume for now that there are no applied external forces, so that the production term vanishes identically. The flux associated with the momentum is what we call the stress tensor, which in this case can be represented by a single component σ (perpendicular to the normal cross sections). The balance of momentum⁷ reads

⁶The adjective *non-homogeneous*, in this case, refers to the fact that there are sources or sinks, that is, p does not vanish identically. *Material inhomogeneity*, on the other hand, would be reflected in a variation of the value of the diffusivity D throughout the tube.

⁷Neglecting convective terms.

$$\frac{\partial(\rho A v)}{\partial t} = \frac{\partial(\sigma A)}{\partial x}. \quad (2.34)$$

Assuming constant cross section and density, we obtain

$$\rho v_t = \sigma_x. \quad (2.35)$$

This balance equation needs to be supplemented with a constitutive law. For a linearly elastic material, the stress is proportional to the strain (ε), that is,

$$\sigma = E \varepsilon, \quad (2.36)$$

where E is Young's modulus. Adopting the (axial) displacement $u = u(x, t)$ as a state variable, we have

$$v = u_t, \quad (2.37)$$

by definition of velocity, and

$$\varepsilon = u_x, \quad (2.38)$$

by the kinematic relations of the infinitesimal theory of strain. Putting all these results together we obtain the second-order linear PDE

$$u_{tt} = c^2 u_{xx}, \quad (2.39)$$

where the constant c is given by

$$c = \sqrt{\frac{E}{\rho}}. \quad (2.40)$$

Equation (2.39) is known as the *one-dimensional wave equation*. The constant c will be later interpreted as the speed of propagation of waves in the medium. A similar equation can be derived for the problem of small transverse vibrations of a string (such as that of a guitar) under tension. In this case, the constant c is given by the square root of the ratio between the tension in the string and its mass per unit length.⁸

2.4.4 Solitons

In many phenomena not governed by the wave equation it is possible to observe the propagation of highly concentrated pulses traveling undistorted at a constant speed. These traveling waves are known as *solitons*. They were first observed in shallow water, where gravity drives the propagation. The equation governing this phenomenon was first derived by Korteweg and de Vries in 1895 and is now known

⁸See Sect. 8.1.

as the KdV equation. We will not present its derivation. It reads

$$u_t + u u_x + u_{xxx} = 0. \quad (2.41)$$

Here, $u = u(x, t)$ represents a measure of the height of the water in a long channel of constant cross section. The KdV equation is a third-order quasi-linear PDE. It can be brought to the form of a conservation law (2.18) by setting $p = 0$ and

$$f = -\left(\frac{1}{2}u^2 + u_{xx}\right). \quad (2.42)$$

2.4.5 Time-Independent Phenomena

If in the examples just presented we eliminate the dependence of the variables on time, namely if $u = u(x)$, we obtain in each case an ODE representing a configuration of steady state or equilibrium of the system. On the other hand, if we were to extend the spatial domain from one to two dimensions (rather than just one, as we have been doing so far), the steady-state equation would still be a PDE in two independent variables. As an example, we will consider the equilibrium configuration of a membrane which has been initially stretched by applying a high uniform tension T (per unit length, say) in all directions and then attached to a rigid plane frame along its perimeter. The membrane thus prepared is then subjected to a transverse load (perpendicular to the plane of the frame) of magnitude $q(x, y)$, where x, y is a system of Cartesian coordinates in the plane of the unloaded membrane. We are interested in calculating the transverse deflection $w = w(x, y)$ corresponding to an equilibrium configuration.

We have already remarked that the transverse vibrations of a tensed string are given by the wave equation (2.39). A careful derivation of the analogous two-dimensional membrane counterpart, as depicted in Fig. 2.3, would lead to the dynamical equation

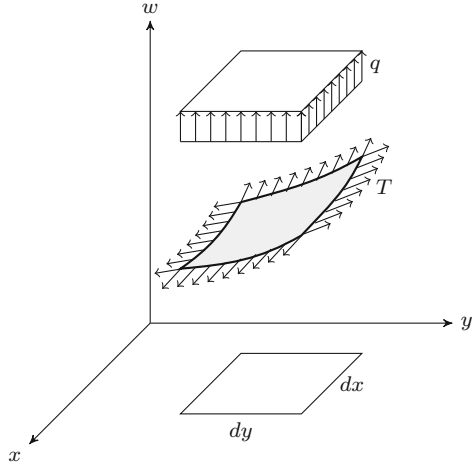
$$w_{xx} + w_{yy} = -\frac{q}{T} + \frac{\rho h}{T} w_{tt}, \quad (2.43)$$

where h is the thickness of the membrane. In the absence of the external loading term q , this is the *two-dimensional wave equation*. If, on the other hand, we seek an equilibrium position under the action of a time-independent load, we obtain the second-order linear PDE

$$w_{xx} + w_{yy} = -\frac{q}{T}. \quad (2.44)$$

This is the *Poisson equation*. If the right-hand side vanishes (no load, but perhaps a slightly non-planar frame) we obtain the *Laplace equation*. These equations appear in many other engineering applications, including fluid mechanics, acoustics, electrostatics and gravitation.

Fig. 2.3 Balance of forces in a membrane



2.4.6 Continuum Mechanics

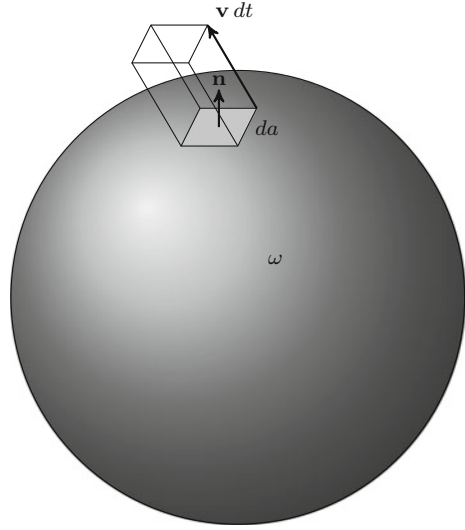
In Continuum Mechanics [1] the field variables are always associated with a continuous *material body* as the carrier of contents, sources and fluxes. The material body, made up of *material points*, manifests itself through its *configurations* in the physical space \mathbb{R}^3 . In this brief presentation, we will adhere strictly to the *Eulerian formulation*, which adopts as its theatre of operations the current configuration of the body in space. The domains ω used in the formulation of the generic balance equation must, accordingly, be subsets of the current configuration. In other words, they must be made of spatial points occupied at the current time by material particles. The generic equation of balance in its global form (2.7) or in its local form (2.11), is still applicable. In Continuum Mechanics, however, it is convenient to identify two distinct parts of the total flux \mathcal{F} through the boundary $\partial\omega$ which we call the *convected flux* \mathcal{F}_c and the *physical flux* \mathcal{F}_p , that is,

$$\mathcal{F} = \mathcal{F}_c + \mathcal{F}_p. \quad (2.45)$$

The convected flux appears as a natural consequence of having adopted a *fixed spatial volume* ω for the analysis. Since the material particles are moving with a velocity field $\mathbf{v} = \mathbf{v}(x_1, x_2, x_3, t)$, they are in general entering into or exiting from ω . In so doing, they import or export a certain amount of content per unit time. Figure 2.4 makes it clear what this amount $d\mathcal{F}_c$ is at any given elemental area da lying on the boundary of ω . We have

$$d\mathcal{F}_c = -u(\mathbf{v} \cdot \mathbf{n})da, \quad (2.46)$$

Fig. 2.4 The amount of content flowing out of the volume ω through the area element da during a time dt is proportional to the volume of the parallelepiped shown



where \mathbf{n} is the *exterior unit normal* to da as part of $\partial\omega$. The negative sign indicates that we have assumed an influx as positive. Naturally, if the particle velocity happens to be tangential to the boundary at a point, the convected flux at that point vanishes.

Remark 2.1 Had we assumed a *material volume* as the point of departure (that is, a volume that follows the particles in their motion in space), the corresponding convected flux would have automatically vanished. This simplification would have resulted, however, in the need to use a compensatory *transport theorem* for the calculation of the time variation of an integral on a moving domain. The convected flux is, literally, in the eyes of the beholder.

The second part of the flux, \mathcal{F}_p has the clear physical meaning of the flow of content through the boundary due to causes other than mere motion or rest of the control volume. Thus, for instance, if the content in question is the internal energy of a rigid body at rest, the flux through the boundary represents the conductive heat flux. It is important to notice once more that the physical flux takes place at each internal boundary (not just the external boundary of the body) separating a sub-body from the rest. Cauchy's theorem implies that the physical flux is governed by a flux vector field \mathbf{f}_p , as before. Thus, we obtain the global form

$$\frac{d}{dt} \int_{\omega} u \, d\omega = \int_{\omega} p \, d\omega + \int_{\partial\omega} (-u\mathbf{v} + \mathbf{f}_p) \cdot \mathbf{n} \, da. \quad (2.47)$$

Applying the divergence theorem and following our previous localization argument we obtain the generic law of balance of Continuum Mechanics in its local form as

$$\frac{\partial u}{\partial t} - p - \operatorname{div}(-u\mathbf{v} + \mathbf{f}_p) = 0. \quad (2.48)$$

In terms of the *material derivative*, discussed in Box 2.1, this equation can also be written as

$$\frac{Du}{Dt} - p + u \operatorname{div} \mathbf{v} - \operatorname{div} \mathbf{f}_p = 0. \quad (2.49)$$

Box 2.1 The material derivative

The partial time-derivative $\frac{\partial u}{\partial t}$ appearing in Eq. (2.48) describes the rate of change of the density u at a *fixed spatial position* (x_1, x_2, x_3) . If we imagine an observer sitting at that position and recording events as time goes on, this partial derivative is the slope of the graph of u versus time as produced by that observer. Thus, if the regime happens to be *steady*, this slope will vanish identically. If we imagine, instead, a hypothetical observer riding with a *fixed material particle*, this second observer will record a different result! The slope of the graph recorded by this moving observer describes the rate of change of u at that particle. This is called the *material derivative* of u , denoted by Du/Dt . In the case of a steady state which is spatially non-constant, clearly the material derivative will not vanish in general. What is the relation between the partial derivative and the material derivative? The particle that at time t passes through the spatial position (x_1, x_2, x_3) will occupy at time $t + dt$ the position $(x_1 + v_1 dt, x_2 + v_2 dt, x_3 + v_3 dt)$, where (v_1, v_2, v_3) are the local components of the velocity. At this point the value of u has, to a first degree of approximation, the value

$$\begin{aligned} & u(x_1 + v_1 dt, x_2 + v_2 dt, x_3 + v_3 dt, t + dt) \\ &= u(x_1, x_2, x_3, t) + \frac{\partial u}{\partial x_1} v_1 dt + \frac{\partial u}{\partial x_2} v_2 dt + \frac{\partial u}{\partial x_3} v_3 dt + \frac{\partial u}{\partial t} dt. \end{aligned}$$

Accordingly, we obtain

$$\begin{aligned} \frac{Du}{Dt} &= \lim_{dt \rightarrow 0} \frac{u(x_1 + v_1 dt, x_2 + v_2 dt, x_3 + v_3 dt, t + dt) - u(x_1, x_2, x_3, t)}{dt} \\ &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1} v_1 + \frac{\partial u}{\partial x_2} v_2 + \frac{\partial u}{\partial x_3} v_3 \\ &= \frac{\partial u}{\partial t} + \nabla u \cdot \mathbf{v}, \end{aligned}$$

where ∇u is the gradient of u .

2.4.6.1 Conservation of Mass: The Continuity Equation

A balance law in Continuum Mechanics is said to be a *conservation law* if the production p and the physical flux \mathbf{f}_p vanish identically.⁹ The content u in this case is to be identified with the *mass density* $\rho = \rho(x_1, x_2, x_3, t)$. Applying the generic equation of balance (2.48) we obtain

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \quad (2.50)$$

or, using the material derivative,

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{v} = 0. \quad (2.51)$$

This equation is known in Fluid Mechanics as the *continuity equation*.

2.4.6.2 Balance of Linear Momentum

In this case, according to Newton's second law, the content density is the vector of linear momentum, namely, $\rho \mathbf{v}$. The production is the *body force* \mathbf{b} per unit spatial volume and the physical flux is the *surface traction vector* \mathbf{t} . We will implement the generic balance equation component by component in a global Cartesian inertial frame. For each component t_i of the traction vector, according to Cauchy's theorem, there exists a flux vector with components σ_{ij} ($j = 1, 2, 3$). We have thus a matrix representing, in the given coordinate system, the components of the *Cauchy stress tensor* $\boldsymbol{\sigma}$. The surface traction $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$ is best expressed in components as

$$t_i = \sigma_{ij} n_j, \quad (2.52)$$

where the *summation convention* for repeated indices has been enforced, as explained in Box 2.2. The equation of balance (2.48) for ρv_i reads

$$\frac{\partial \rho v_i}{\partial t} - b_i - (-\rho v_i v_j + \sigma_{ij})_{,j} = 0. \quad (2.53)$$

On the other hand, the continuity equation (2.50) is written in Cartesian components (always enforcing the summation convention) as

$$\frac{\partial \rho}{\partial t} + (\rho v_j)_{,j} = 0. \quad (2.54)$$

⁹This is the case of the conservation of mass in conventional Continuum Mechanics. In the context of *growing bodies* (such as is the case in some biological materials) mass is not necessarily conserved.

Combining the last two results, we obtain

$$\rho \frac{\partial v_i}{\partial t} + \rho v_{i,j} v_j = b_i + \sigma_{ij,j}. \quad (2.55)$$

Using the material derivative, we can write the local form of the balance of linear momentum as

$$\rho \frac{Dv_i}{Dt} = b_i + \sigma_{ij,j}. \quad (2.56)$$

The material derivative of the velocity is, not surprisingly, the acceleration. Having thus enforced the conservation of mass, the form of Newton's second law for a continuum states that the mass density times the acceleration equals the body force plus the net contact force over the boundary of an elementary volume element.

Box 2.2 The summation convention in Cartesian coordinates

Attributed to Albert Einstein, the summation convention is a notational device which, in addition to being compact and elegant, is often revealing of possible mistakes in computations. In an expression or equation made up of sums of monomials, where each monomial consists of products of indexed quantities, the following rules apply:

1. In each monomial an index (subscript) can appear at most twice. If repeated in a monomial, an index is said to be *dummy*. Otherwise, if it appears just once, it is called *free*.
2. The free indices must be *balanced* in all monomials. In other words, every monomial in an expression must have exactly the same free indices.
3. The dummy indices in every monomial indicate a summation over that index in the range 1 to n , where n is the dimension of the Cartesian space under consideration.

As an example, the divergence of a vector field \mathbf{v} is given by

$$\text{div} \mathbf{v} = v_{i,i},$$

where, as in Eq. (2.2), commas stand for partial derivatives with respect to the coordinates designated by the indices following the comma. An equation such as

$$B_i = A_{ijkjk}$$

stands for

$$B_i = \sum_{k=1}^n \sum_{j=1}^n A_{ijkjk}.$$

Expressions such as

$$C = D_{kkk} \quad C_i = D_{jkk} \quad C_{ij} = D_{ikk}$$

are considered wrong, unless the summation convention has been explicitly suspended.

2.4.6.3 Balance of Angular Momentum

For a continuous deformable medium, unlike the case of a rigid body, the balance of angular momentum is an independent postulate. It establishes that the angular momentum with respect to any point attached to an inertial reference frame is equal to the sum of the moments of the external forces about that point. The angular momentum density is the (pseudo-)vector $\mathbf{r} \times (\rho \mathbf{v})$, where, without any loss of generality, we have identified the fixed point as the origin of coordinates, so that \mathbf{r} is the standard position vector. Assuming the absence of applied couples, and enforcing both mass conservation and balance of linear momentum, the final result is purely algebraic, namely,

$$\sigma_{ij} = \sigma_{ji}. \quad (2.57)$$

Put in words, the Cauchy stress tensor is *symmetric*.

2.4.6.4 Balance of Energy: The First Law of Thermodynamics in a Continuous Medium

The *total energy density* e in a continuum is given by

$$e = \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} + \rho \epsilon. \quad (2.58)$$

In this expression, the first term on the right-hand side represents the *kinetic energy density* while ϵ is the *internal energy* per unit mass, postulated to exist as a function of state by the first law of Thermodynamics. The mechanical source density is stipulated by the same law as the *power of the body force*, that is $\mathbf{b} \cdot \mathbf{v}$, while the thermal (that is, non-mechanical) source is provided by sources of heat distributed with a density r per unit volume. Similarly, the physical mechanical flux is given by the power of the traction, that is, $\mathbf{t} \cdot \mathbf{v}$ while the physical heat flux is defined by means of a *heat flux vector* \mathbf{q} such that the non-mechanical influx per unit area and per unit time is given by $-\mathbf{q} \cdot \mathbf{n}$. The balance of energy equates the rate of change of the total energy

with the sum of the mechanical and thermal contributions. The final result can be expressed as

$$\rho \frac{D\epsilon}{Dt} = r - q_{i,i} + \sigma_{ij} v_{i,j}, \quad (2.59)$$

where the previous balance equations have been enforced.

Exercises

Exercise 2.1 Derive Eq. (2.18) by applying the equation of balance to a small (infinitesimal) slice of the bar, that is, for a slice contained between the cross sections at x and at $x + dx$.

Exercise 2.2 Carry out all the steps leading to Eq. (2.57) that establishes the symmetry of the Cauchy stress tensor.

Exercise 2.3 Carry out all the steps leading to Eq. (2.59) that establishes the balance of energy in a continuous medium.

Exercise 2.4 (*The Navier-Stokes equations*) The *rate of deformation tensor* \mathbf{D} is defined as the symmetric part of the velocity gradient. In components

$$D_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i}).$$

The *Newtonian* compressible viscous fluid has the constitutive equation

$$\sigma = -p(\rho)\mathbf{I} + \lambda \operatorname{tr}\mathbf{D} + 2\mu\mathbf{D}.$$

In this equation, $p = p(\rho)$ is some increasing function of the density and λ and μ are constant *viscosity coefficients*. The symbol \mathbf{I} stands for the spatial identity tensor and the operator tr is the *trace*, namely, $\operatorname{tr}\mathbf{D} = D_{ii}$ is the sum of the diagonal entries in the matrix representing \mathbf{D} in a Cartesian frame. Use this constitutive equation in the equation of balance of linear momentum to obtain the *Navier-Stokes equations* of Fluid Mechanics [2]. These are three PDEs for the density and the components of the velocity field. The continuity equation completes the system.

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