UDPS1203 Marking Scheme(May2017):

Q1.

(a)

Separating variables and integrating we have

$$\frac{dy}{y^2} = -2(t+1)\,dt$$

$$-\frac{1}{y} = -(t+1)^2 + c$$

$$y = \frac{1}{(t+1)^2 + c_1} \quad \leftarrow \text{letting } -c = c_1$$

The initial condition $y(0) = -\frac{1}{8}$ implies $c_1 = -9$, so a solution of the initial-value problem is

$$y = \frac{1}{(t+1)^2 - 9} \qquad \text{or} \qquad y = \frac{1}{t^2 + 2t - 8},$$

where $-4 < t < 2$.

(b)

(i)

By separation of variables and partial fractions,

$$\ln\left|\frac{T - T_m}{T + T_m}\right| - 2\tan^{-1}\left(\frac{T}{T_m}\right) = 4T_m^3 kt + c.$$

(ii)

When $T - T_m$ is small compared to T_m, every term in the expansion after the first two can be ignored, giving

$$\frac{dT}{dt} \approx k_1(T - T_m), \quad \text{where} \quad k_1 = 4kT_m^3.$$

Q2.

(a)

Taking the Laplace transform of the differential equation we obtain

$$\mathscr{L}\{y\} = \frac{1}{(s-1)^2(s^2 - 8s + 20)}$$

$$= \frac{6}{169}\frac{1}{s-1} + \frac{1}{13}\frac{1}{(s-1)^2} - \frac{6}{169}\frac{s-4}{(s-4)^2 + 2^2} + \frac{5}{338}\frac{2}{(s-4)^2 + 2^2}$$

so that

$$y = \frac{6}{169}e^t + \frac{1}{13}te^t - \frac{6}{169}e^{4t}\cos 2t + \frac{5}{338}e^{4t}\sin 2t.$$

(b)

The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{(s+1)^2}e^{-s}$$

so that

$$y = (t-1)e^{-(t-1)}\,\mathscr{U}(t-1)\,.$$

(c)

The Laplace transform of the given equation is

$$s\mathscr{L}\{y\} - y(0) + 6\mathscr{L}\{y\} + 9\mathscr{L}\{1\}\,\mathscr{L}\{y\} = \mathscr{L}\{1\}\,.$$

Solving for $\mathscr{L}\{f\}$ we obtain $\mathscr{L}\{y\} = \dfrac{1}{(s+3)^2}$. Thus, $y = te^{-3t}$.

Q3.
(a)

and

$$c_1 = \frac{3}{4}c_0, \qquad c_2 = \frac{9}{56}c_0, \qquad c_3 = \frac{9}{560}c_0,$$

and so on. For $r = 1/3$ the recurrence relation is

$$c_k = \frac{3c_{k-1}}{3k^2 - k}\,, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = \frac{3}{2}c_0, \qquad c_2 = \frac{9}{20}c_0, \qquad c_3 = \frac{9}{160}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1 x^{2/3}\left(1 + \frac{3}{4}x + \frac{9}{56}x^2 + \frac{9}{560}x^3 + \cdots\right) + C_2 x^{1/3}\left(1 + \frac{3}{2}x + \frac{9}{20}x^2 + \frac{9}{160}x^3 + \cdots\right).$$

(b)

and so on. For $c_0 = 0$ and $c_1 = 1$ we find

$$c_2 = 0, \quad c_3 = -\frac{1}{6}, \quad c_4 = 0, \quad c_5 = -\frac{1}{60}, \quad c_6 = 0$$

and so on. Thus, two solutions are

$$y_1 = 1 - \frac{1}{2}x^2 + \frac{1}{720}x^6 + \cdots \qquad \text{and} \qquad y_2 = x - \frac{1}{6}x^3 - \frac{1}{60}x^5 + \cdots .$$
