UDPS1203 Answer (May2017):

Q1.
(a) Q=-1/25t^4+1/5t^4lnt+c/t

(b) 4.6

(c) y=0.5cos x+ 0.5sin x +0.5 sec x

(d)(i) y^2=x^2(2-2lnx)

(ii) 0<x<e

Q2.
(a) Not in syllabus

(b)

We have

$$\mathbf{X}_c = c_1\begin{pmatrix}1\\-1\end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix}1\\-1\end{pmatrix}te^{2t} + \begin{pmatrix}1\\0\end{pmatrix}e^{2t}\right].$$

Then

$$\Phi = \begin{pmatrix}e^{2t} & te^{2t}+e^{2t}\\ -e^{2t} & -te^{2t}\end{pmatrix}, \quad \Phi^{-1} = \begin{pmatrix}-te^{-2t} & -te^{-2t}-e^{-2t}\\ e^{-2t} & e^{-2t}\end{pmatrix},$$

and

$$\mathbf{U} = \int \Phi^{-1}\mathbf{F}\,dt = \int \begin{pmatrix}t-1\\-1\end{pmatrix}dt = \begin{pmatrix}\frac{1}{2}t^2-t\\-t\end{pmatrix},$$

so that

$$\mathbf{X}_p = \Phi\mathbf{U} = \begin{pmatrix}-1/2\\1/2\end{pmatrix}t^2e^{2t} + \begin{pmatrix}-2\\1\end{pmatrix}te^{2t}.$$

Q3.
(a) x=0 is regular singular point

(b) y=x-1/4x^3+1/16x^4+…

(c)

$$y = -6\left[1-\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3+\frac{1}{12}(x-1)^4+\cdots\right]$$
$$+3\left[(x-1)-\frac{1}{2}(x-1)^2-\frac{1}{6}(x-1)^3+\frac{1}{6}(x-1)^4+\cdots\right].$$

Q4.
(a)

$$y = -\frac{2}{125} - \frac{2}{25}t - \frac{1}{5}t^2 + \frac{127}{125}e^{5t} - \left[-\frac{37}{125} - \frac{12}{25}(t-1) - \frac{1}{5}(t-1)^2 + \frac{37}{125}e^{5(t-1)}\right]\mathscr{U}(t-1).$$

(b)
From $q'' + 10^4 q = 100\sin 50t$, $q(0) = 0$, and $q'(0) = 0$ we obtain $q_c = c_1\cos 100t + c_2\sin 100t$, $q_p = \frac{1}{75}\sin 50t$, and

(i) $q = -\frac{1}{150}\sin 100t + \frac{1}{75}\sin 50t$,

(ii) $i = -\frac{2}{3}\cos 100t + \frac{2}{3}\cos 50t$, and

(iii) $q = 0$ when $\sin 50t(1 - \cos 50t) = 0$ or $t = n\pi/50$ for $n = 0, 1, 2, \ldots$.

Q5.

(a) f(t)=+/- 6t

(b) (i)
Since $C(0) = 0$ we have

$$c_2 = \frac{\lambda_1}{\lambda_1 + \lambda_2}K_0 \qquad \text{and} \qquad C(t) = \frac{\lambda_1}{\lambda_1 + \lambda_2}K_0\left[1 - e^{-(\lambda_1+\lambda_2)t}\right].$$

Finally

$$\frac{dA}{dt} = \lambda_2 K = \lambda_2 K_0 e^{-(\lambda_1+\lambda_2)t} \qquad \text{so} \qquad A(t) = -\frac{\lambda_1}{\lambda_1 + \lambda_2}K_0 e^{-(\lambda_1+\lambda_2)t} + c_3.$$

Since $A(0) = 0$

$$c_3 = \frac{\lambda_2}{\lambda_1 + \lambda_2}K_0 \qquad \text{and} \qquad A(t) = \frac{\lambda_2}{\lambda_1 + \lambda_2}K_0\left[1 - e^{-(\lambda_1+\lambda_2)t}\right].$$

(ii)
Since $\lambda_1 + \lambda_2 = 5.34 \times 10^{-10}$ we have

$$K(t) = K_0 e^{-0.000000000534t} = \frac{1}{2}K_0 \quad \text{and} \quad t = \frac{\ln\frac{1}{2}}{-0.000000000534} \approx 1.3 \times 10^9 \text{ years.}$$

(iii)
Since

$$\lim_{t\to\infty} C(t) = \frac{\lambda_1}{\lambda_1+\lambda_2}\lim_{t\to\infty} K_0\big[1-e^{-(\lambda_1+\lambda_2)t}\big] = \frac{\lambda_1}{\lambda_1+\lambda_2}K_0$$

and

$$\lim_{t\to\infty} A(t) = \frac{\lambda_1}{\lambda_1+\lambda_2}\lim_{t\to\infty} K_0\big[1-e^{-(\lambda_2+\lambda_2)t}\big] = \frac{\lambda_2}{\lambda_1+\lambda_2}K_0,$$

we have

$$\lim_{t\to\infty} C(t) = \frac{4.7526\times 10^{-10}}{5.34\times 10^{-10}}K_0 = 0.89K_0 \quad \text{or} \quad 89\%$$

and

$$\lim_{t\to\infty} A(t) = \frac{0.5874\times 10^{-10}}{5.34\times 10^{-10}}K_0 = 0.11K_0 \quad \text{or} \quad 11\%.$$